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Differentiability of convex envelopes

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For an extended real-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ we denote by f^c its convex envelope defined as

$$f^c(x) = \sup\{g(x) : g \le f,\, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \text{ convex}\}.$$

We give a surprisingly simple proof of the following theorem, whose first part was previously established under linear growth conditions from below, see [1]. The second part improves similar results for superlinear growth from [2].

Theorem *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be continuous, let it be differentiable on its effective domain* $\mathrm{dom}_e(f) = \{x : f(x) < +\infty\}$ *and assume*

$$f(x) \to +\infty \text{ as } |x| \to +\infty. \tag{1}$$

Then f^c is a C^1 function on the (open) set $\mathrm{dom}_e(f^c)$. Moreover, if for some $\alpha \in (0,1]$ the function f is $C^{1,\alpha}_{\mathrm{loc}}$ on $\mathrm{dom}_e(f)$, then the same holds true for f^c on $\mathrm{dom}_e(f^c)$.

The proof uses the following three elementary facts about convex functions.

(I) The representation formula for the convex envelope:

$$f^c(x) = \inf\left\{\sum_{i=1}^{n+1}\lambda_i f(x_i) \;:\; \lambda_i \ge 0, \sum_{i=1}^{n+1}\lambda_i = 1 \text{ and } \sum_{i=1}^{n+1}\lambda_i x_i = x\right\},$$

has a proof similar to that of Carathéodory's theorem, see e.g. Corol. 17.1.5 in [3].

(II) The (local) Lipschitz constant can be estimated in terms of the oscillation: If $g : B(x,2r) \to \mathbb{R}$ is convex, then

$$\mathrm{lip}(g, B(x,r)) \le \mathrm{osc}(g, B(x,2r))/r.$$

To prove this assume that the right-hand side is finite and fix any $y, z \in B(x,r)$. Suppose $g(z) \ge g(y)$ and choose u to be the intersection of $\partial B(x,2r)$ with the ray from y through z. Then $z \in \mathrm{conv}\{y,u\}$ and $|y-u| > r$. Now the desired estimate follows, since convexity implies that

$$(g(z) - g(y))/|z-y| \le (g(u) - g(y))/|u-y| \le \mathrm{osc}(g, B(x,2r))/r.$$

(III) Criterion for differentiability: If g is convex, f differentiable in x, $g \le f$ and $g(x) = f(x)$, then g is differentiable at x and $\nabla f(x) = \nabla g(x)$. The proof is straightforward and is left to the interested reader.

Proof of the Theorem: The effective domain $\mathrm{dom}_e(f^c)$ is open, since by Fact (I) and the continuity of f, f^c is upper semicontinuous (and in fact could be shown to be continuous). To show that f^c is C^1 on $\mathrm{dom}_e(f^c)$ suppose that $\mathrm{dom}_e(f^c) \ne \emptyset$ and note that if f^c is differentiable on $\mathrm{dom}_e(f^c)$, then it is continuously differentiable there. Indeed, fix any point $x \in \mathrm{dom}_e(f^c)$. Consider the function $h(y) = f^c(y) - f^c(x) - \langle \nabla f^c(x), y-x\rangle$; it is convex and ∇f^c is continuous at x if $\nabla h(y) \to 0$ as $y \to x$. This, however, is a consequence of Fact (II) and the differentiability of f^c at x. We

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are left to show that f^c is differentiable at each point x where it is finite. Referring to Fact (I) we take a minimizing sequence $\{(\lambda_i^{(k)}, x_i^{(k)})_{1\le i\le n+1}\}_{k=1}^{\infty}$, such that $\lambda_1^{(k)} \ge \lambda_2^{(k)} \ge \cdots \ge \lambda_{n+1}^{(k)} \ge 0$, $\sum_{i=1}^{n+1}\lambda_i^{(k)} = 1$, $\sum_{i=1}^{n+1}\lambda_i^{(k)} x_i^{(k)} = x$ and

$$\sum_{i=1}^{n+1}\lambda_i^{(k)} f(x_i^{(k)}) \to f^c(x) \text{ as } k \to \infty. \tag{2}$$

Observe that $\lambda_1^{(k)} \ge 1/(n+1)$. Due to continuity and (1), f is bounded from below, say by 0. Hence, by (2) and (1) both $\{f(x_1^{(k)})\}_k$ and $\{x_1^{(k)}\}_k$ are bounded (at least from a certain step $k \ge k_0$). Therefore, we can extract a subsequence (for convenience not relabelled), such that $\lambda_1^{(k)} \to \lambda_1$ and $x_1^{(k)} \to x_1$ as $k \to \infty$. Again by continuity and since $f \ge 0$ we see that $f(x_1) \le (n+1)f^c(x)$; consequently, by assumption, f is differentiable at x_1. Next we observe that for $h \in \mathbb{R}^n$ and each k,

$$x + h = \lambda_1^{(k)}(x_1^{(k)} + (h/\lambda_1^{(k)})) + \sum_{i=2}^{n+1}\lambda_i^{(k)} x_i^{(k)}.$$

Thus, by convexity we have for k sufficiently large

$$f^c(x+h) - f^c(x) \le \lambda_1^{(k)}(f(x_1^{(k)} + (h/\lambda_1^{(k)})) - f(x_1^{(k)})) + \Big(\sum_{i=1}^{n+1}\lambda_i^{(k)} f(x_i^{(k)}) - f^c(x)\Big)$$

and passing to the limit as $k \to \infty$ we obtain

$$f^c(x+h) - f^c(x) \le \lambda_1(f(x_1 + (h/\lambda_1)) - f(x_1)) \text{ for all } h \in \mathbb{R}^n. \tag{3}$$

Since the left hand side is a convex function, Fact (III) implies as required that f^c is differentiable at x with $\nabla f^c(x) = \nabla f(x_1)$.

As concerns Hölder continuity of the derivatives let O be an open bounded set with closure contained in $\mathrm{dom}_e(f^c)$. Observe that (3) together with (1), the upper bound, $f(x_1) \le (n+1)f^c(x)$, and the Hölder continuity of ∇f on compact subsets of $\mathrm{dom}_e(f)$, imply that for some $c = c(O) < +\infty$, $0 \le f^c(x+h) - f^c(x) - \langle \nabla f^c(x), h\rangle \le c|h|^{1+\alpha}$, whenever x, $x+h \in O$. Using Fact (II), we conclude that $\nabla f^c \in C^{\alpha}_{\mathrm{loc}}(\mathrm{dom}_e(f^c))$ and our proof is finished.

Finally, we would like to mention that for the C^1-regularity, as well as for the $C^{1,\alpha}$-regularity, it is sufficient to assume the existence of a 'superdifferential' $a \in \mathbb{R}^n$, i.e. it suffices that the positive part of $f(x+h) - f(x) - \langle a, h\rangle$ vanishes in a prescribed way as $h \to 0$. We also like to remark that even without the assumption (1) our method proves smoothness in all points x satisfying $f^c(x) < \liminf_{|y|\to\infty} f(y)$. The Example 4.1 in [1], i.e. the function $(x,y) \to \sqrt{x^2 + \exp(-y^2)}$, shows that this growth condition is the weakest possible (of this general kind).

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