DIFFERENTIABILITY OF DISTANCE FUNCTIONS IN NORMED LINEAR SPACES WITH UNIFORMLY GATEAUX DIFFERENTIABLE NORM

Brett Davis

Abstract

Consider a non-empty proper closed subset K of a Banach space $(X,\|\cdot\|)$ where the norm is uniformly Gateaux (uniformly Frechet) differentiable. Then the associated distance function d is guaranteed to be Gateaux differentiable on a dense subset D of $X\backslash K$. Furthermore, Gateaux (Frechet) differentiability of the distance function at a point $x \in X\backslash K$ can be characterised in terms of the weak* (norm) convergence of sequences $\{d'(x_n)\}$ where $\{x_n\}$ is a sequence in D converging to x.

1980 A.M.S. Classification (1985 Revision): 41A65.

Consider a non-empty proper closed subset K of a normed linear space $(X,\|\cdot\|)$ and the associated distance function $d : X\backslash K \to \mathbb{R}$ defined by

$$d(x) = \inf\{\|x-k\| : k \in K\}.$$

d is Lipschitz-1 and is a convex function if and only if K is convex.

In a normed linear space with uniformly Gateaux differentiable norm the distance function is not necessarily convex. However, we show that in such spaces the distance function has many of the differentiability properties of convex functions. The following results are taken from my Master's Thesis which was submitted to the University of Newcastle in July 1988.

Consider a locally Lipschitz function ϕ on an open subset A of a normed linear space $(X,\|\cdot\|)$. Clarke defined a generalised subdifferential $\partial\phi^0(x)$ for ϕ at $x \in A$ by

$$\partial\phi^0(x) = \{f \in X^* : f(y) \le \phi^0(x)(y) \quad \text{for all} \quad y \in X\}$$

where

$$\phi^0(x)(y) := \limsup_{\substack{z \to x \\ t \to 0^+}} \frac{\phi(z+ty)-\phi(z)}{t} .$$

The mapping $y \mapsto \phi^0(x)(y)$ is sublinear so, by the Hahn Banach Dominated Extension Theorem, $\partial\phi^0(x)$ is non-empty. Furthermore, when ϕ is a continuous convex function $\partial\phi^0(x)$ equals the subdifferential of ϕ at x.

Theorem 1. Borwein, Fitzpatrick, Giles 1987

Consider the distance function d *associated with a non-empty proper closed subset* K *of a normed linear space* $(X,\|\cdot\|)$ *where the norm is uniformly Gateaux differentiable. Then, for each* $x \in X\backslash K$, d *has a right-hand Gateaux derivative at* x. *Furthermore,*

$$-d'_+(x)(y) = (-d)^0(x)(y) \quad \text{for all} \quad y \in X.$$

Theorem 2. Borwein, Fitzpatrick, Giles 1987

Consider the distance function d *associated with a non-empty proper closed subset* K *of a normed linear space* $(X,\|\cdot\|)$ *where the norm is uniformly Gateaux differentiable. Then, for each* $x \in X\backslash K$, $\partial d^0(x)$ *is the weak*-closed convex hull of the set of all weak*-cluster points of the set*

$$\{\|z-u\|' : \|z-u\| \to d(x) \quad as \quad z \to x,\ u \in K\}.$$

We now discuss some applications of these results. For our first application we use a consequence of the Borwein-Preiss Smooth Variational Principle.

Proposition 3. Borwein-Preiss 1987

Consider a Banach space $(X,\|\cdot\|)$ *where the norm is Gateaux differentiable on* $X\backslash\{0\}$. *Suppose* $\theta : X \to \mathbb{R}$ *is bounded below and lower semi-continuous on* $(X,\|\cdot\|)$. *Then* θ *is Gateaux subdifferentiable on a dense subset of* $(X,\|\cdot\|)$.

Theorem 4.

Consider the distance function d *associated with a non-empty proper closed subset* K *of a Banach space* $(X,\|\cdot\|)$ *where the norm is uniformly Gateaux differentiable. Then* d *is Gateaux differentiable on a dense subset* D *of* $X\backslash K$.

Proof.

By Proposition 3, d is Gateaux subdifferentiable on a dense subset D of $X\backslash K$. That is, for each $x \in D$, there is a continuous linear functional f_x on $(X,\|\cdot\|)$ such that, for $\|y\| = 1$, given $\epsilon > 0$, there exists $\delta(\epsilon,y) > 0$ where

$$\frac{d(x+ty)-d(x)}{t} > f_x(y) - \epsilon \quad \text{when} \quad 0 < t < \delta.$$

But, by Theorem 1, d has a right-hand Gateaux derivative at x. So

$$d'_+(x)(y) \geq f_x(y) \quad \text{for all} \quad y \in X.$$

But, also by Theorem 1, $-d'_+(x)(y)$ is sublinear in y. Hence,

$$d'_+(x)(y) = f_x(y) \quad \text{for all} \quad y \in X. \qquad \square$$

Our next application gives characterisations of Gateaux and Frechet differentiability for the distance function. The Gateaux differentiable case follows immediately from Theorem 5, and the Frechet differentiable case needs an independent argument. However, both are generalisations of similar characterisation results for convex functions.

Theorem 5. Giles 1989

Consider the distance function d *associated with a non-empty proper closed subset* K *of a Banach space* $(X, \|\cdot\|)$ *where the norm is uniformly Gateaux differentiable. Then* d *is Gateaux differentiable on a dense subset* D *of* $X\backslash K$ *and, for* $x \in X\backslash K$, $\partial d^0(x)$ *is the weak*-closed convex hull of the set of weak*-cluster points of*

$$\{d'(x_n) : n \in N\}$$

where $\{x_n\}$ *is any sequence in* D *converging to* x.

Proof.

We note that since d is Gateaux differentiable at x_n then it follows from Theorem 1 that $\partial d^0(x_n)$ is a singleton and

$$\partial d^0(x_n) = \{d'(x_n)\}.$$

So, by Theorem 2,

$$d'(x_n)(y) = \lim\{\|z-u\|'(y) : z \to x_n, \|z-u\| \to d(x_n), u \in K\}.$$

Now applying Theorem 2 to $\partial d^0(x)$ we have the result. □

Theorem 6. Giles 1989

Consider the distance function d associated with a non-empty proper closed subset K of a Banach space $(X, \|\cdot\|)$ where the norm is uniformly Gateaux differentiable (uniformly Frechet differentiable). Then d is Gateaux differentiable on a dense subset D of $X \backslash K$. Furthermore, d is Gateaux (Frechet) differentiable at $x \in X \backslash K$ if and only if, for every sequence $\{x_n\}$ in D converging to x, $\{d'(x_n)\}$ is weak-convergent (norm convergent).*

References

1. J.M. Borwein, S.P. Fitzpatrick, J.R. Giles, "The differentiability of real functions on normed linear spaces using generalised subgradients", J. Math. Anal. Appl. *128* (1987) 512-534.

2. J.M. Borwein and D. Preiss, "A smooth variational principle with applications to differentiability of convex functions", Trans. Amer. Math. Soc. *Vol. 303 No. 3* (1987) 517-528.

3. F.H. Clarke, "Generalised gradients and applications", Trans. Amer. Math. Soc. *205* (1975) 247-262.

4. J.R. Giles, "A distance function property implying convexity", Bull. Austral. Math. Soc. *39* (1989) 59-71.

Department of Mathematics,
Monash University,
Clayton, Victoria. 3168