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Differentiable 7-manifolds with a certain homotopy type

By Itiro TAMURA

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J. Milnor [**10**] has determined the so-called J-equivalence (h-cobordism) classes of oriented differentiable 7-manifolds having the homotopy type of the 7-sphere, and S. Smale [**13**] has proved that such manifolds are homeomorphic to the 7-sphere and the J-equivalence classes are the same as the diffeomorphic classes in this case. Thus compact unbounded oriented differentiable 7-manifolds which are homotopy spheres were completely determined. There exist precisely 28 such differentiable 7-manifolds which form a cyclic group Θ^7 under the connected sum.

In this note we shall consider compact unbounded 2-connected oriented differentiable 7-manifolds whose third homology groups are cyclic of order 3, having trivial Steenrod operations. We shall show that there exist precisely 56 differentiable 7-manifolds of this homotopy type and that they are obtained from the standard one by connected sums of elements of Θ^7 and the orientation-reversing.

1. Let M^7 be the compact unbounded 2-connected oriented (C^∞-) differentiable 7-manifold such that $H_3(M^7; Z) \approx Z_3$ and that the Steenrod operation $\mathcal{P}_3^1: H^3(M^7; Z_3) \to H^7(M^7; Z_3)$ is trivial, namely, for $u \in H^3(M^7; Z_3)$

$$(P) \qquad \mathcal{P}_3^1(u) = 0 .$$

LEMMA 1. *The condition* (P) *is equivalent to* $p_1(M^7) = 0$, *where* $p_1(M^7)$ *is the first Pontrjagin class of* M^7.

PROOF. This lemma follows from the formula given by Hirzebruch [**6**]:

$$p_1(M^7) \cup u = \mathcal{P}_3^1(u) \qquad \mod 3$$

for $u \in H^3(M^7; Z_3)$.

LEMMA 2. M^7 *is a* π*-manifold.*

PROOF. Suppose that M^7 is imbedded in a high dimensional Euclidean space R^{7+N}. Denote by ν^N the normal bundle of M^7. Let K be a triangulation of M^7. Let us define a (continuous) field of normal N-frames on M^7 by stepwise extensions on the skeletons $K^{(q)}$ $(q = 0, 1, \cdots, 7)$ of K using the obstruction theory in the well-known manner. Since $H^q(M^7; Z) = 0$ $(q = 1, 2, 3)$ and $\pi_2(SO(N)) = 0$, we can define a field f of normal N-frames on $K^{(3)}$. Let $c(f) \in Z^4(M^7; Z)$ be the obstruction cocycle to extend f in $K^{(4)}$, Then the first

Pontrjagin class $p_1(\nu^N)$ of ν^N is $\{2c(f)\}$ (Milnor-Kervaire [**12**]). Therefore Lemma 1 and the product theorem for Pontrjagin classes yield $\{c(f)\}=0$. The next obstruction is in dimension 7 with values in $\pi_6(SO(N))=0$. Thus ν^N is trivial. This completes the proof.

LEMMA 3. *M^7 bounds a compact 3-connected oriented π-manifold.*

PROOF. Since the cokernel of the J-homomorphism $J_7: \pi_7(SO(N))\rightarrow\pi_{7+N}(S^N)$ is zero, this lemma follows from [**10**; Theorem 6.7 (b)].

LEMMA 4. *M^7 bounds a compact 3-connected oriented π-manifold.*

PROOF. Since M^7 bounds a compact oriented π-manifold, we obtain a compact 3-connected oriented π-manifold with boundary M^7 by performing a series of surgeries (spherical modifications) (Milnor [**10**], [**11**]).

Let W^8 be the compact 3-connected oriented π-manifold with boundary M^7. The exactness of the homology sequence of (W^8, M^7)

$$\cdots\longrightarrow H_q(M^7;Z)\longrightarrow H_q(W^8;Z)\longrightarrow H_q(W^8,M^7;Z)\longrightarrow H_{q-1}(M^7;Z)\longrightarrow\cdots$$

and the Poincaré-Lefschetz duality

$$H_q(W^8,M^7;Z)\approx H^{8-q}(W^8;Z)$$

imply that $H_q(W^8;Z)=0$ $(q=5,6,7)$ and that $H_4(W^8;Z)$ has no torsion.

Let ϕ denote the quadratic form over the group $H_4(W^8;Z)$ defined by the formula $x\rightarrow x\circ x$, where $x\circ y$ is the intersection number of two homology classes $x,y\in H_4(W^8;Z)$. The index (signature) of this form ϕ will be denoted by $I(W^8)$.

LEMMA 5. *The index $I(W^8)$ modulo $2^5\cdot 7$ is a diffeomorphy invariant of M^7.*

PROOF. Suppose that M^7 is the boundary of two compact 3-connected oriented π-manifolds W_1^8 and W_2^8. Let V^8 be the compact unbounded oriented differentiable 8-manifold obtained from W_1^8 and $-W_2^8$ by pasting together the common boundary. The exactness of the Mayer-Vietoris cohomology sequence

$$\begin{aligned}\cdots\longrightarrow H^{q-1}(M^7;Z)\longrightarrow H^q(V^8;Z)\xrightarrow{\iota^*} H^q(W_1^8;Z)+H^q(W_2^8;Z)\\ \longrightarrow H^q(M^7;Z)\longrightarrow\cdots\end{aligned}$$

implies that V^8 is 3-connected and that

$$\iota^*: H^4(V^8;Z)\longrightarrow H^4(W_1^8;Z)+H^4(W_2^8;Z)$$

is injective. Since $\iota^*p_1(V^8)=0$, we have $p_1(V^8)=0$. Therefore the index theorem $I(V^8)=\frac{1}{45}(7p_2(V^8)-p_1^2(V^8))[V^8]$ (Hirzebruch [**7**]) implies

$$45\, I(V^8)=7\, p_2(V^8)[V^8],$$

$$I(V^8)\equiv 0 \qquad \text{mod } 7,$$

and the integrality of $\hat{A}$-genus $\hat{A}(V^8)=\frac{1}{2^7\cdot 45}(-4\, p_2(V^8)+7\, p_1^2(V^8))[V^8]$ (Atiyah and Hirzebruch [**1**], Borel and Hirzebruch [**2**]) implies

$$p_2(V^8)\equiv 0 \qquad \text{mod } 2^5\cdot 45.$$

Thus we have

$$I(V^8) \equiv 0 \qquad \bmod 2^5 \cdot 7 .$$

Since $I(V^8) = I(W_1^8) - I(W_2^8)$, we have

$$I(W_1^8) \equiv I(W_2^8) \qquad \bmod 2^5 \cdot 7 .$$

This completes the proof.

DEFINITION. The residue class of $I(W^8)$ mod $2^5 \cdot 7$ will be denoted by $\bar{\lambda}(M^7)$.

LEMMA 6. *The determinant of the matrix of the quadratic form* ϕ *is* ± 3.

PROOF. Since $H_4(W^8; Z)$ has no torsion, the Poincaré-Lefschetz duality theorem implies $H_4(W^8, M^7; Z) \approx \mathrm{Hom}(H_4(W^8; Z), Z)$. The natural homomorphism

$$H_4(W^8; Z) \longrightarrow H_4(W^8, M^7; Z) \approx \mathrm{Hom}(H_4(W^8; Z), Z)$$

is determined by the matrix of intersection numbers of $H_4(W^8; Z)$. Thus the lemma follows from the exactness of the homology sequence of (W^8, M^7) and $H_3(M^7; Z) \approx Z_3$.

Let C and U denote matrices

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

LEMMA 7. *The index* $I(W^8)$ *is equal to* ± 2 *modulo* 8.

PROOF. Let

$$G_q(\phi) = \sum_{(q)} e^{\frac{2\pi i}{q}\phi(\alpha, \alpha)}$$

denote the Gauss sum of the quadratic form ϕ, where the sum is extended over all residue classes of $H_4(W^8; Z)$ mod q. Then the index $I(W^8)$ satisfies

$$G_{8\cdot 27}(\phi) = e^{\frac{\pi i}{4} I(W^8)} (2 \cdot 8 \cdot 27)^{\frac{r}{2}} \sqrt{3} ,$$

where r denotes the 4th Betti number of W^8 (Braun [**3**; §1, (ε)]). We shall prove that $G_{8\cdot 27}(\phi)$ is purely imaginary.

Every diagonal entry of the matrix of the quadratic form 27ϕ is even (Milnor [**9**]). Thus the matrix of the quadratic form 27ϕ is equivalent to diag $(U, \cdots, U)$ or diag $(C, U, \cdots, U)$ over the 2-adic integers [**8**; Theorem 33a], which implies that $G_8(27\phi)$ is a positive integer. (Compare Milnor [**9**].)

Since $G_{8\cdot 27}(\phi) = G_8(27\phi) G_{27}(8\phi)$ (Braun [**3**; §2, (4)]), it is sufficient to prove that $G_{27}(8\phi)$ is purely imaginary. According to [**8**; Theorem 25], there exists a basis $\alpha_1, \alpha_2, \cdots, \alpha_r$ of $H_4(W^8; Z)$ mod 27 such that

$$8\phi\left(\sum_{i=1}^{r} x_i \alpha_i, \sum_{i=1}^{r} x_i \alpha_i\right) \equiv \sum_{i=1}^{r} a_i x_i^2 \qquad \bmod 27 .$$

Thus we have

$$G_{27}(8\phi)=\sum e^{\frac{2\pi i}{27}\sum_j a_j x_j^2}=\sum\prod_j e^{\frac{2\pi i}{27}a_j x_j^2}$$
$$=\prod_j\sum_{x=0}^{26} e^{\frac{2\pi i}{27}a_j x^2}.$$

It is easy to see that if $a\not\equiv 0 \bmod 3$, $\sum_{x=0}^{26} e^{\frac{2\pi i}{27}ax^2}$ is purely imaginary and that if $a\equiv 0 \bmod 3$, $\not\equiv 0 \bmod 9$, $\sum_{x=0}^{26} e^{\frac{2\pi i}{27}ax^2}$ is real. Since Lemma 6 implies

$$\prod_j a_j\equiv 0 \mod 3, \qquad \not\equiv 0 \mod 9,$$

it follows that if r is an even integer $G_{27}(8\phi)$ is purely imaginary and if r is an odd integer $G_{27}(8\phi)$ is real. Therefore, in either case, the index $I(W^8)$ is an even integer, which shows that r is even. This completes the proof.

LEMMA 8. *If the index $I(W^8)$ is equal to 2, then the matrix of the quadratic form ϕ, with respect to a suitable basis, is*

$$\operatorname{diag}(C, U, \cdots, U)=\begin{pmatrix} C & & & \\ & U & & \\ & & \ddots & \\ & & & U\end{pmatrix}.$$

PROOF. Choosing a basis of $H_4(W^8; Z)$, let us denote $\phi=\sum_{i,j}a_{ij}x_ix_j$ $(a_{ij}=a_{ji})$. The determinant of the matrix $A=(a_{ij})$ is ± 3 (Lemma 6). Every diagonal entry of the matrix A is even (Milnor [**9**]). If ϕ is a form of rank 2, then ϕ is positive definite. Therefore, according to [**8**; Theorem 76], the matrix A is equivalent to C. If ϕ is a form of rank $\geqq 4$, then ϕ is indefinite. According to [**8**; Theorem 36], the matrices A and $\operatorname{diag}(C, U, \cdots, U)$ are equivalent over the p-adic integers for every prime $p\neq 3$. The Hasse symbols $c_p(A)$ and $c_p(\operatorname{diag}(C, U, \cdots, U))$ are equal for every prime $p\neq 3$ [**8**; Theorem 12], which implies $c_3(A)=c_3(\operatorname{diag}(C, U, \cdots, U))$ [**8**; § 12, 2]. The matrices A and $\operatorname{diag}(C, U, \cdots, U)$ are equivalent to the matrix $\operatorname{diag}(\pm 3, \pm 1, 1, \cdots, 1)$ over the 3-adic integers [**8**; Theorems 35, 36b], where signs are determined by the Hasse symbol. Thus the matrices A and $\operatorname{diag}(C, U, \cdots, U)$ have the same genus.

There exists a matrix X with rational elements such that ${}^tXAX=\operatorname{diag}(C, U, \cdots, U)$ [**8**; Theorem 28]. Let L denote the lattice $H_4(W^8; Z)$ in $H_4(W^8; Q)$ and let L' denote the lattice L transformed by X, where Q is the field of rational numbers. The lattices L and L' are both maximal [**4**; Sätze 9.3, 12.3]. Thus Eichler's theorem ([**4**; Satz 15.2], [**5**; Satz 3]) implies that the matrix A is equivalent to the matrix $\operatorname{diag}(C, U, \cdots, U)$. (See Milnor [**9**]). This completes the proof.

Let T be a closed tubular neighborhood of the diagonal S^4 in $S^4\times S^4$, the product of two copies of S^4 with a fixed orientation. Then T is a compact parallelizable oriented differentiable 8-manifold-with-boundary. The self-intersection number of S^4 in T is 2. Let (T, S^4, D^4, π) denote the D^4-bundle over S^4. Let W^8_e be the parallelizable 3-connected oriented differentiable 8-manifold-with-boundary obtained by straightening the angle of the quotient space of two copies $'T$ and $''T$ of T under an identification of $'\pi^{-1}('\sigma^4)$ with $''\pi^{-1}(''\sigma^4)$ in such a way that the images of base spaces $'S^4$ and $''S^4$ in W^8_e have intersection number 1, where $'\sigma^4$ and $''\sigma^4$ are 4-cells of $'S^4$ and $''S^4$ respectively. Denote by M^7_e the boundary of W^8_e with the orientation compatible with that of W^8_e. Then M^7_e is a compact unbounded 2-connected oriented differentiable 7-manifold such that $H_3(M^7_e; Z) \approx Z_3$ and that $p_1(M^7_e)=0$. The invariant $\bar{\lambda}$ of M^7_e is 2.

LEMMA 9. *If the index $I(W^8)$ equals 2, then M^7 is diffeomorphic to M^7_e.*

PROOF. By Lemma 8, there exists a basis $\alpha, \beta, \alpha_1, \beta_1, \cdots, \alpha_s, \beta_s$ of $H_4(W^8; Z)$ such that

$$\alpha \circ \alpha = \beta \circ \beta = 2\,, \qquad \alpha \circ \beta = 1\,,$$
$$\alpha \circ \alpha_i = \alpha \circ \beta_j = \beta \circ \alpha_i = \beta \circ \beta_j = 0\,,$$
$$\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0\,, \qquad \alpha_i \circ \beta_j = \delta_{ij}\,.$$

By performing a series of surgeries (spherical modifications) on W^8 (Milnor [**10**; Theorem 5.6], [**11**; Theorem 4]), we obtain a compact parallelizable 3-connected oriented differentiable 8-manifold W'^8 with boundary M^7 such that α and β are generators of $H_4(W'^8; Z) \approx Z+Z$. Let

$$f: S^4 \longrightarrow W'^8\,, \qquad g: S^4 \longrightarrow W'^8$$

be differentiable imbeddings which represent homology classes α, β respectively. Since $\alpha \circ \beta = 1$, making use of the method of Whitney [**15**; Theorem 4], we may assume that $f(S^4)$ and $g(S^4)$ intersect regularly at one point. Let N_f, N_g be tubular neighborhoods of $f(S^4)$, $g(S^4)$ respectively. The self-intersection number of base space and the first Pontrjagin classes characterize a D^4-bundle over S^4 (see [**14**]). Since N_f and N_g are parallelizable, it follows that N_f and N_g are diffeomorphic to T. Thus we may assume that $N_f \cup N_g$ is diffeomorphic to W^8_e. The exactness of the Mayer-Vietoris homology sequence of a proper triad $(W'^8; N_f \cup N_g, W'^8 - \mathrm{Int}(N_f \cup N_g))$

$$\cdots \longrightarrow H_{q+1}(W'^8; Z) \longrightarrow H_q(\partial(N_f \cup N_g); Z)$$
$$\longrightarrow H_q(N_f \cup N_g; Z) + H_q(W'^8 - \mathrm{Int}(N_f \cup N_g); Z) \longrightarrow H_q(W'^8; Z) \longrightarrow \cdots$$

implies that $\partial(N_f \cup N_g)$ is a deformation retract of $W'^8 - \mathrm{Int}(N_f \cup N_g)$. The exactness of the homology sequence of a triple $(W'^8, W'^8 - \mathrm{Int}(N_f \cup N_g), M^7)$

$$\cdots \longrightarrow H_q(W'^8 - \mathrm{Int}(N_f \cup N_g), M^7; Z) \longrightarrow H_q(W'^8, M^7; Z)$$
$$\longrightarrow H_q(W'^8, W'^8 - \mathrm{Int}(N_f \cup N_g); Z) \longrightarrow H_{q-1}(W'^8 - \mathrm{Int}(N_f \cup N_g), M^7; Z) \longrightarrow \cdots$$

and the Poincaré-Lefschetz duality

$$H_q(W'^8, M^7; Z) \approx H^{8-q}(W'^8; Z)\,,$$
$$H_q(W'^8, W'^8 - \mathrm{Int}(N_f \cup N_g); Z) \approx H^{8-q}(N_f \cup N_g; Z)$$

imply

$$H_q(W'^8-\mathrm{Int}(N_f\cup N_g), M^7; Z)=0 \qquad q=0,1,\cdots,8,$$

which shows that M^7 is a deformation retract of $W'^8-\mathrm{Int}(N_f\cup N_g)$. Therefore $W'^8-\mathrm{Int}(N_f\cup N_g)$ defines a J-equivalence (h-cobordism) between M^7 and $\partial(N_f\cup N_g)$. By a result of Smale [**13**; Theorem I], M^7 is diffeomorphic to $\partial(N_f\cup N_g)$. This completes the proof.

REMARK. Since $\bar{\lambda}(-M^7)=-\bar{\lambda}(M^7)$, M^7 with $\bar{\lambda}(M^7)=-2$ is diffeomorphic to $-M_e^7$.

Let M_0^7 denote the oriented differentiable 7-manifold homeomorphic to S^7 which bounds the compact parallelizable 3-connected oriented differentiable 8-manifold W_0^8 with $I(W_0^8)=8$ (Milnor [**10**; §4]). M_0^7 is a generator of the group Θ^7.

LEMMA 10. *If* $\bar{\lambda}(M^7)=2+8s$, *then* M^7 *is diffeomorphic to* $M_e^7\# M_0^7\#\cdots\# M_0^7$ (*s-fold connected sum of* M_0^7). *If* $\bar{\lambda}(M^7)=-2+8s$, *then* M^7 *is diffeomorphic to* $(-M_e^7)\# M_0^7\#\cdots\# M_0^7$ (*s-fold connected sum of* M_0^7).

PROOF. Suppose that $\bar{\lambda}(M^7)=2+8s$. There exists a compact parallelizable 3-connected oriented differentiable 8-manifold W^8 with boundary M^7 such that $I(W^8)$ equals $2+8s+2^5\cdot 7t$. We form the sum $W^8+(-W_0^8)+\cdots+(-W_0^8)$ of W^8 with the $(s+28t)$-fold sum of $(-W_0^8)$ in the following sense. The sum $W_1^n+W_2^n$ will mean the compact oriented differentiable manifold-with-boundary obtained from the disjoint union of compact oriented differentiable manifolds W_1^n and W_2^n by identifying $f_1(x)$ with $f_2(x)$ $(x\in D^{n-1})$, where $f_1: D^{n-1}\to\partial W_1$ (resp. $f_2: D^{n-1}\to\partial W_2$) is an orientation-preserving (resp. orientation-reversing) imbedding of $(n-1)$-disk D^{n-1}. Then $W^8+(-W_0^8)+\cdots+(-W_0^8)$ is compact parallelizable 3-connected oriented differentiable 8-manifold. Since the index of $W^8+(-W_0^8)+\cdots+(W_0^8)$ equals 2, it follows that $\partial(W^8+(-W_0^8)+\cdots+(-W_0^8))=M^7\#(-M_0^7)\#\cdots\#(-M_0^7))$ $(s+28t)$-fold sum of $-M_0^7$) is diffeomorphic to M_e^7 (Lemma 9). Thus M^7 is diffeomorphic to $M_e^7\# M_0^7\#\cdots\# M_0^7$ (s-fold connected sum of M_0^7). This completes the proof for the case of $\bar{\lambda}(M^7)=2+8s$. For the case of $\bar{\lambda}(M^7)=-2+8s$, the proof is similar.

From Lemma 7 and Lemma 10 we have

THEOREM. *There exist precisely* 56 *distinct compact unbounded* 2*-connected oriented differentiable* 7*-manifolds whose third homology groups are cyclic of order* 3, *satisfying the condition* (*P*). *The invariant* $\bar{\lambda}$ *characterizes these manifolds. All these manifolds are homeomorphic to each other.*

2. Let $(\bar{B}_{3m,3}^8, S^4, D^4, \bar{\pi})$ be the D^4-bundle over S^4 with the characteristic map $3m\rho+3\sigma$. (For notations in this section, see [**14**].) Let α_4 be a generator of $H_4(\bar{B}_{3m,3}^8; Z)\approx Z$. We choose the orientation of $\bar{B}_{3m,3}^8$ in such a way that $\alpha_4\circ\alpha_4$ is positive. Let $B_{3m,3}^7$ denote the boundary of $\bar{B}_{3m,3}^8$ with the orientation compatible with that of $\bar{B}^8_{3m,3}$. $B^7_{3m,3}$ is a compact unbounded 2-connected oriented differentiable 7-manifold such that $H_3(B^7_{3m,3}; Z) \approx Z_3$ and that $p_1(B^7_{3m,3}) = 0$ (see [**14**]).

Let us compute the invariant $\bar{\lambda}$ of $B^7_{3m,3}$. Suppose that $B^7_{3m,3}$ bounds a compact parallelizable 3-connected oriented differentiable 8-manifold W^8. Let V^8 be the compact unbounded 2-connected oriented differentiable 8-manifold obtained from the disjoint union of $\bar{B}^8_{3m,3}$ and $-W^8$ by identifying $\partial \bar{B}^8_{3m,3}$ with ∂W^8. The index theorem $I(V^8) = \frac{1}{45}(7p_2(V^8) - p_1^2(V^8))[V^8]$ implies

$$45(1 - I(W^8)) = 7p_2(V^8)[V^8] - 2^2 \cdot 3^3(2m+1)^2,$$
$$I(W^8) \equiv 4m(m+1) + 2 \qquad \text{mod } 7.$$

The integrality of $\hat{A}$-genus $\hat{A}(V^8) = \frac{1}{2^7 \cdot 45}(-4p_2(V^8) + 7p_1^2(V^8))[V^8]$ implies

$$p_2(V^8)[V^8] \equiv 3^3 \cdot 7(2m+1)^2 \qquad \text{mod } 2^5 \cdot 45.$$

Hence

$$I(W^8) \equiv 4m(m+1) - 26 \qquad \text{mod } 2^5.$$

Therefore *the invariant* $\bar{\lambda}$ *of* $B^7_{3m,3}$ *is equal to* $4m(m+1) - 26$.

University of Tokyo

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