

DIFFERENTIABLE PINCHING THEOREMS FOR SUBMANIFOLDS VIA RICCI FLOW

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Abstract. Two differentiable pinching theorems are verified via the Ricci flow and stable currents. We first prove a differentiable sphere theorem for positively pinched submanifolds in a space form. Moreover, we obtain a differentiable sphere theorem for submanifolds in the sphere \mathbb{S}^{n+p} under extrinsic restriction.

1. Introduction. The sphere theorem is an important topic in the study of curvature and topology of Riemannian manifolds. Recently, S. Brendle and R. Schoen [6] proved a remarkable differentiable sphere theorem for Riemannian manifolds whose sectional curvatures lie in the interval $(1/4, 1]$ by developing the theory and techniques of Ricci flow introduced by R. Hamilton [9]. This improves the topological sphere theorem for quarter-pinched Riemannian manifolds, which was firstly taken up by H. E. Rauch [16] in 1951, and solved by M. Berger [2] and W. Klingenberg [10] around 1960. In 1977, K. Grove and K. Shiohama [8] proved a topological sphere theorem for complete and connected Riemannian manifolds whose sectional curvature K and diameter $d(M)$ satisfy $K \geq 1$ and $d(M) > \pi/2$. Some other sphere theorems for Riemannian manifolds have also been obtained, see [1, 14, 15, 17, 25], etc.

In 1973, H. B. Lawson and J. Simons [11] proved a topological sphere theorem for closed submanifolds in a unit sphere by using the nonexistence for stable currents on compact submanifolds in a sphere. Let $\mathbb{F}^{n+p}(c)$ denote the complete simply connected space form with constant sectional curvature c . K. Shiohama and H. W. Xu [18] improved Lawson-Simons' result and proved a topological sphere theorem for complete submanifolds in $\mathbb{F}^{n+p}(c)$ with $c \geq 0$. Recently, H. W. Xu and J. R. Gu [20] obtained an optimal differentiable sphere theorem for complete submanifolds with pinched scalar curvature in $\mathbb{F}^{n+p}(c)$ with $c \geq 0$. Let K_M denote the sectional curvature of a Riemannian manifold M . Under the pinching condition of sectional curvature, H. W. Xu and G. X. Yang [23] proved a topological sphere theorem for compact submanifolds in the spherical space form.

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THEOREM 1.1. *Let M be a compact, oriented $n(\geq 4)$ -dimensional submanifold in the space form $\mathbb{F}^{n+p}(c)$ with $c > 0$. If the sectional curvature of M satisfies*

$$K_M > \frac{c}{2} + \frac{n}{8}H^2,$$

then M is homeomorphic to the standard n -sphere \mathbb{S}^n .

Let h be the second fundamental form of a compact submanifold in a Riemannian manifold. In 1986, H. Gauchman [7] proved that if M is an n -dimensional closed minimal submanifold in \mathbb{S}^{n+d} , and if $\|h(u, u)\|^2 < \frac{1}{3}$ for any unit vector $u \in UM$, where UM is the unit tangent bundle over M , then M is a totally geodesic sphere. H. W. Xu, W. Fang and F. Xiang [21] generalized this rigidity result to the case where M is an n -dimensional closed submanifold with parallel mean curvature in \mathbb{S}^{n+d} . On the other hand, P. F. Leung [12] proved that if $\|h(u, u)\|^2 < \frac{1}{3}$ holds for any unit tangent vector u at any point on the submanifold, then it is a homotopy sphere. The result was improved to be a topological sphere theorem for complete submanifolds in a sphere by H. W. Xu and W. Fang [19], and further improved to be a differentiable sphere theorem by H. W. Xu and E. T. Zhao [24].

THEOREM 1.2. *Let M be an n -dimensional complete submanifold in the unit sphere \mathbb{S}^{n+p} . If*

$$\|h(u, u)\|^2 < \frac{1}{3}, \quad \text{for all } u \in UM,$$

then M is diffeomorphic to the standard n -sphere \mathbb{S}^n .

Let \mathring{h} be the trace free second fundamental form of a submanifold in a Riemannian manifold. In [22], H. W. Xu, F. Huang and F. Xiang investigated the pinching theorem of \mathring{h} for the submanifolds with parallel mean curvature in a sphere, which extends Gauchman's theorem in [7].

In this paper, using the convergence result of Ricci flow by S. Brendle [5], we obtain several differentiable sphere theorems for complete submanifolds which generalize Theorems 1.1 and 1.2. We first prove the following

THEOREM 1.3. *Let M be an $n(\geq 4)$ -dimensional oriented complete submanifold in $\mathbb{F}^{n+p}(c)$ with $c > 0$. Assume that the sectional curvature of M satisfies*

$$(1) \quad K_M > \frac{(n-2)c}{n} + \frac{nH^2}{8},$$

then M is diffeomorphic to the standard n -sphere \mathbb{S}^n .

Next, we investigate the differentiable pinching problem of \mathring{h} and prove the following

THEOREM 1.4. *Let M be an n -dimensional complete submanifold in the unit sphere \mathbb{S}^{n+p} . If*

$$\|\mathring{h}(u, u)\|^2 < \frac{1}{3}, \quad \text{for all } u \in UM,$$

then M is diffeomorphic to the standard n -sphere \mathbb{S}^n .

In fact, Theorem 1.4 is a consequence of the following theorem and Lemma 3.2 in Section 3.

THEOREM 1.5. *Let M be an n -dimensional complete submanifold in an $(n + p)$ -dimensional point-wise $\delta (> 1/4)$ -pinched Riemannian manifold N^{n+p} . Set $\overline{K}_{\max}(x) := \max_{\pi \subset T_x N} \overline{K}(x, \pi)$, where $\overline{K}(x, \pi)$ is the sectional curvature of N^{n+p} for $x \in N$ and 2-plane $\pi \in T_x N$. If*

$$\|\mathring{h}(u, u)\|^2(x) < \frac{4}{9} \overline{K}_{\max}(x) \left(\delta - \frac{1}{4} \right), \quad \text{for all } u \in U_x M, \quad x \in M,$$

where $\inf_{x \in N} \overline{K}_{\max}(x) > 0$, then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to the standard n -sphere \mathbb{S}^n .

2. Preliminaries. Let M^n be an n -dimensional complete submanifold isometrically immersed into an $(n + p)$ -dimensional Riemannian manifold N^{n+p} . The following conventions of indices are used throughout.

$$\begin{aligned} 1 \leq A, B, C, \dots \leq n + p, \\ 1 \leq i, j, k, \dots \leq n, \\ n + 1 \leq \alpha, \beta, \gamma, \dots \leq n + p. \end{aligned}$$

Choose a local orthonormal frame field $\{e_A\}$ in N^{n+p} such that e'_i s are tangent to M . Let $\{\omega_A\}$ be the dual frame field of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of N^{n+p} . Restricting these forms to M , we have

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

The curvature tensor of N and M are denoted by \overline{R}_{ABCD} and R_{ijkl} , respectively. The second fundamental form of M is denoted by h and the mean curvature normal field by ξ . Set $H = \|\xi\|$ and $c_\alpha = \frac{1}{n} \sum_i h_{ii}^\alpha$. Then we have

$$h = \sum_{\alpha, i, j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

$$\xi = \sum_\alpha c_\alpha e_\alpha,$$

$$H = \sqrt{\sum_\alpha c_\alpha^2},$$

$$(2) \quad R_{ijkl} = \overline{R}_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha).$$

The trace free second fundamental form of M is defined by

$$\mathring{h} = \sum_{i, j, \alpha} \mathring{h}_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

where $\mathring{h}_{ij}^\alpha = h_{ij}^\alpha - c_\alpha \delta_{ij}$. Let UM denote the unit tangent bundle on M and $U_x M$ its fiber over $x \in M$. Then $UM = \bigcup_{x \in M} U_x M$, where $U_x M = \{u \in T_x M : \|u\| = 1\}$.

LEMMA 2.1. *Let M^n be a submanifold in a Riemannian manifold N^{n+p} . Then*

$$\|\mathring{h}(u, v)\| \leq \mathring{h}_x,$$

for all unit vectors $u, v \in U_x M$ at each point $x \in M$, where $\mathring{h}_x := \max_{w \in U_x M} \|\mathring{h}(w, w)\|$.

PROOF. If $u \neq \pm v$, set $y = \frac{u+v}{\|u+v\|}$ and $z = \frac{u-v}{\|u-v\|}$. Then

$$\begin{aligned} \mathring{h}(u, v) &= \frac{1}{4}[\mathring{h}(u+v, u+v) - \mathring{h}(u-v, u-v)] \\ &= \frac{1}{4}[\|u+v\|^2 \mathring{h}(y, y) - \|u-v\|^2 \mathring{h}(z, z)]. \end{aligned}$$

Using the triangle inequality, we get

$$\begin{aligned} \|\mathring{h}(u, v)\| &\leq \frac{1}{4} \mathring{h}_x (\|u+v\|^2 + \|u-v\|^2) \\ (3) \qquad \qquad &\leq \mathring{h}_x. \end{aligned}$$

If $u = \pm v$, then $\|\mathring{h}(u, v)\| = \|\mathring{h}(u, u)\| \leq \mathring{h}_x$. This completes the proof. \square

LEMMA 2.2. *Let M^n be a submanifold in a Riemannian manifold N^{n+p} . Then*

$$\langle h(e_i, e_i), h(e_j, e_j) \rangle \geq -\mathring{h}_x^2,$$

for any orthonormal frame field $\{e_i\}$ at each point $x \in M$.

PROOF.

$$\begin{aligned} &\|h(e_i, e_i)\|^2 + \|h(e_j, e_j)\|^2 - 2\langle h(e_i, e_i), h(e_j, e_j) \rangle \\ &= \|\mathring{h}(e_i, e_i)\|^2 + \|\mathring{h}(e_j, e_j)\|^2 + 2\langle \mathring{h}(e_i, e_i), \xi \rangle + 2\langle \mathring{h}(e_j, e_j), \xi \rangle + 2H^2 \\ &\quad - 2(\langle \mathring{h}(e_i, e_i), \mathring{h}(e_j, e_j) \rangle + \langle \mathring{h}(e_i, e_i), \xi \rangle + \langle \mathring{h}(e_j, e_j), \xi \rangle + H^2) \\ &= \|\mathring{h}(e_i, e_i)\|^2 + \|\mathring{h}(e_j, e_j)\|^2 - 2\langle \mathring{h}(e_i, e_i), \mathring{h}(e_j, e_j) \rangle \\ &\leq 4\mathring{h}_x^2, \end{aligned}$$

where Lemma 2.1 is used in the last inequality. Hence we have

$$(4) \qquad \langle h(e_i, e_i), h(e_j, e_j) \rangle \geq \frac{1}{2}(\|h(e_i, e_i)\|^2 + \|h(e_j, e_j)\|^2) - 2\mathring{h}_x^2.$$

Thus,

$$\begin{aligned} \langle h(e_i, e_i), h(e_j, e_j) \rangle &= \frac{1}{2} \langle h(e_i, e_i), h(e_j, e_j) \rangle + \frac{1}{2} \langle h(e_i, e_i), h(e_j, e_j) \rangle \\ &\geq \frac{1}{2} \left[\frac{1}{2} (\|h(e_i, e_i)\|^2 + \|h(e_j, e_j)\|^2) - 2\mathring{h}_x^2 \right] \\ &\quad - \frac{1}{2} \|h(e_i, e_i)\| \cdot \|h(e_j, e_j)\| \\ &= \frac{1}{4} \left[\|h(e_i, e_i)\|^2 + \|h(e_j, e_j)\|^2 \right] \end{aligned}$$

$$\begin{aligned} & -2\|h(e_i, e_i)\| \cdot \|h(e_j, e_j)\| - \mathring{h}_x^2 \\ & \geq -\mathring{h}_x^2. \end{aligned}$$

□

S. Brendle [5] obtained the following useful result.

LEMMA 2.3. *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$. Assume that*

$$(5) \quad R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0,$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)} + \frac{2}{n} r_{g(t)} g(t),$$

exists for all time and converges to a positive constant sectional curvature metric as $t \rightarrow \infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

Inequality (5) is closely related to the positivity of the isotropic curvature. We refer the readers to [13] for isotropic curvature. As a matter of fact, $R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$ holds for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$ if and only if $M \times \mathbb{R}$ has positive isotropic curvature (see [5, 6]). It follows from Berger's inequality that every manifold with positively pointwise 1/4-pinched sectional curvatures satisfies the curvature condition (5) in Lemma 2.3. In fact, we have the following Berger's inequality [3]

$$|R_{ijkl}| \leq \frac{2}{3} (K_{\max} - K_{\min}),$$

for all distinct indices i, j, k, l . Here K_{\max} and K_{\min} are the maximum and minimum of the sectional curvatures at a point of M . The curvature assumption implies that $K_{\max} < 4K_{\min}$. So we have

$$\begin{aligned} & R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\ & \geq 2(1 + \lambda^2)K_{\min} - \frac{4}{3}|\lambda|(K_{\max} - K_{\min}) \\ & \geq 2(1 + \lambda^2)K_{\min} - 4|\lambda|K_{\min} \\ & = 2(1 - |\lambda|)^2 K_{\min} \\ & \geq 0. \end{aligned}$$

Observe that at least one of the second and last inequalities is strict, which implies $R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$. In fact, the 1/4-pinched differentiable sphere theorem is an immediate consequence of Lemma 2.3.

We also need the following topological lemma due to H. B. Lawson and J. Simons [11].

LEMMA 2.4. *Let M^n be a compact submanifold in a unit sphere \mathbb{S}^{n+p} . Let k and l be positive integers with $k + l = n$. If the following inequality*

$$\sum_{j=k+1}^n \sum_{i=1}^k (2\|h(e_i, e_j)\|^2 - \langle h(e_i, e_i), h(e_j, e_j) \rangle) < k \cdot l,$$

holds for any orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of tangent space $T_x M$ at any point $x \in M$, then there does not exist any stable k -current, and

$$H_k(M, \mathbb{Z}) = H_l(M, \mathbb{Z}) = 0,$$

where $H_i(M, \mathbb{Z})$ is the i -th homology group of M with integer coefficients.

3. Proof of theorems. We first give the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Since $K_M > \frac{(n-2)c}{n} + \frac{nH^2}{8} > 0$, we obtain that M is a compact submanifold by Myers' Theorem.

Suppose that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and $\lambda \in \mathbb{R}$. We extend $\{e_1, e_2, e_3, e_4\}$ to an orthonormal frame $\{e_1, \dots, e_n\}$. From the Gauss equation (2) we get

$$R_{ijij} = c + \sum_{\alpha} h_{ii}^{\alpha} h_{jj}^{\alpha} - \sum_{\alpha} (h_{ij}^{\alpha})^2.$$

From the assumption we obtain

$$\sum_{\alpha} h_{ii}^{\alpha} h_{jj}^{\alpha} - \sum_{\alpha} (h_{ij}^{\alpha})^2 > \frac{nH^2}{8} - \frac{2c}{n}.$$

This together with (1) implies

$$\begin{aligned} & R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\ & > (1 + \lambda^2) \left(\frac{2(n-2)}{n}c + \frac{nH^2}{4} \right) - 2\lambda \sum_{\alpha} (h_{13}^{\alpha} h_{24}^{\alpha} - h_{14}^{\alpha} h_{23}^{\alpha}) \\ & \geq (1 + \lambda^2) \left(\frac{2(n-2)}{n}c + \frac{nH^2}{4} \right) - |\lambda| \sum_{\alpha} \sum_{i=3}^n [(h_{1i}^{\alpha})^2 + (h_{2i}^{\alpha})^2] \\ & \geq (1 + \lambda^2) \left(\frac{2(n-2)}{n}c + \frac{nH^2}{4} \right) - |\lambda| \left[\sum_{\alpha} (h_{11}^{\alpha} + h_{22}^{\alpha}) \left(\sum_{i=3}^n h_{ii}^{\alpha} \right) \right. \\ & \quad \left. - \left(\frac{n(n-2)H^2}{4} - \frac{4(n-2)c}{n} \right) \right] \\ & \geq (1 + \lambda^2) \left(\frac{2(n-2)}{n}c + \frac{nH^2}{4} \right) - |\lambda| \frac{n^2}{4} H^2 + |\lambda| \left(\frac{n(n-2)H^2}{4} - \frac{4(n-2)c}{n} \right) \\ & = \frac{2(n-2)}{n} (1 - |\lambda|)^2 c + \frac{n}{4} (1 - |\lambda|)^2 H^2 \\ & \geq 0. \end{aligned}$$

Hence, the inequality (5) holds for all $\lambda \in [-1, 1]$, which implies that M is diffeomorphic to a spherical space form by Lemma 2.3. On the other hand, since $K_M > \frac{(n-2)c}{n} + \frac{nH^2}{8} \geq \frac{c}{2} + \frac{nH^2}{8}$, from Theorem 1.1 we know that M is homeomorphic to \mathbb{S}^n . Combining the results above, we get that M is diffeomorphic to \mathbb{S}^n . \square

In the following we consider the submanifolds with restriction on \mathring{h} .

LEMMA 3.1. *Let M be an $n(\geq 4)$ -dimensional submanifold in an $(n+p)$ -dimensional point-wise $\delta(> 1/4)$ -pinched Riemannian manifold N^{n+p} . If*

$$(6) \quad \|\mathring{h}(u, u)\|^2(x) < \frac{4}{9}\overline{K}_{\max}(x)\left(\delta - \frac{1}{4}\right), \quad \text{for all } u \in U_x M, \quad x \in M,$$

where $\inf_{x \in N} \overline{K}_{\max}(x) > 0$, then inequality (5) is satisfied for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and $\lambda \in [-1, 1]$.

PROOF. For any point $x \in M$, let $\mathring{h}_x := \max_{w \in U_x M} \|\mathring{h}(w, w)\|$. We suppose $\{e_1, e_2, e_3, e_4\}$ to be an orthonormal four-frame and $\lambda \in [-1, 1]$. By the Gauss equation we have

$$\begin{aligned} &R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\ &= \overline{R}_{1313} + \langle h(e_1, e_1), h(e_3, e_3) \rangle - \|h(e_1, e_3)\|^2 \\ &\quad + \lambda^2 [\overline{R}_{1414} + \langle h(e_1, e_1), h(e_4, e_4) \rangle - \|h(e_1, e_4)\|^2] \\ &\quad + \overline{R}_{2323} + \langle h(e_2, e_2), h(e_3, e_3) \rangle - \|h(e_2, e_3)\|^2 \\ &\quad + \lambda^2 [\overline{R}_{2424} + \langle h(e_2, e_2), h(e_4, e_4) \rangle - \|h(e_2, e_4)\|^2] \\ &\quad - 2|\lambda|[\overline{R}_{1234} + \langle h(e_1, e_3), h(e_2, e_4) \rangle - \langle h(e_1, e_4), h(e_2, e_3) \rangle]. \end{aligned}$$

Applying Lemma 2.1, Lemma 2.2 and $h(e_i, e_j) = \mathring{h}(e_i, e_j), i \neq j$, we obtain

$$\begin{aligned} &R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\ &\geq \overline{R}_{1313} - 2\mathring{h}_x^2 + \lambda^2 [\overline{R}_{1414} - 2\mathring{h}_x^2] \\ &\quad + \overline{R}_{2323} - 2\mathring{h}_x^2 + \lambda^2 [\overline{R}_{2424} - 2\mathring{h}_x^2] \\ &\quad - 2|\lambda|[\overline{R}_{1234} + 2\mathring{h}_x^2]. \end{aligned}$$

Since we have Berger's inequality $|\overline{R}_{1234}| \leq \frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min})$, the assumption (6) implies that

$$\begin{aligned} &R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\ &> (2 + 2\lambda^2) \left[\overline{K}_{\min} - \frac{8}{9}\overline{K}_{\max} \left(\delta - \frac{1}{4} \right) \right] - 2|\lambda| \left[\frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min}) + \frac{8}{9}\overline{K}_{\max} \left(\delta - \frac{1}{4} \right) \right] \\ &\geq (2 + 2\lambda^2) \left[\delta \overline{K}_{\max} - \frac{8}{9}\overline{K}_{\max} \left(\delta - \frac{1}{4} \right) \right] - 2|\lambda| \left[\frac{2}{3}(1 - \delta)\overline{K}_{\max} + \frac{8}{9}\overline{K}_{\max} \left(\delta - \frac{1}{4} \right) \right] \\ &= \frac{2}{9}(1 + \lambda^2 - 2|\lambda|)(\delta + 2)\overline{K}_{\max} \\ &\geq 0. \end{aligned}$$

This completes the proof of the Lemma 3.1. □

Now we give the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. Since $\|\mathring{h}(u, u)\|^2(x) < \frac{4}{9}(\delta - \frac{1}{4})\overline{K}_{\max}$ for any $u \in U_x M$, from the Gauss equation, we know that the sectional curvature of M is bounded from bellow by $\frac{2+\delta}{9}\overline{K}_{\max} \geq \frac{2+\delta}{9} \inf_{x \in M} \overline{K}_{\max}(x) > 0$. By Myers' Theorem, M is a compact submanifold.

When $n = 2$, it's easy to see that M is diffeomorphic to \mathbb{S}^2 or $\mathbb{R}\mathbb{P}^2$. When $n = 3$, Hamilton's Theorem [9] says that M is diffeomorphic to a spherical space form. When $n \geq 4$, by Lemma 2.3 and Lemma 3.1, we see that M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M must be diffeomorphic to the standard unit n -sphere \mathbb{S}^n . This completes the proof of Theorem 1.5. □

LEMMA 3.2. *Let M^n be an n -dimensional complete submanifold in the unit sphere \mathbb{S}^{n+p} . If*

$$\|\mathring{h}(u, u)\|^2 < \frac{1}{3}, \quad \text{for all } u \in UM,$$

then M is simply connected.

PROOF. It is easy to see that the sectional curvature of M is bounded by $\frac{1}{3}$ from below. Hence M is compact by Myers' Theorem.

Let $\{e_1, e_2, \dots, e_n\}$ be any orthonormal basis of the tangent space at any point $x \in M$. Using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} \|h(e_i, e_j)\| &= \|\mathring{h}(e_i, e_j)\| < \frac{1}{3}, \\ \langle h(e_i, e_i), h(e_j, e_j) \rangle &> -\frac{1}{3}, \end{aligned}$$

for any unit vector field e_i and e_j ($i \neq j$). Hence for any $1 \leq k \leq n - 1$,

$$\begin{aligned} &\sum_{j=k+1}^n \sum_{i=1}^k (2\|h(e_i, e_j)\|^2 - \langle h(e_i, e_i), h(e_j, e_j) \rangle) \\ &< \sum_{j=k+1}^n \sum_{i=1}^n \left(\frac{2}{3} + \frac{1}{3}\right) \\ &= kl. \end{aligned}$$

By Lemma 2.4, there does not exist any stable integral current in M .

Suppose that $\pi_1(M) \neq 0$. Since M is compact, it follows from a classical theorem due to Cartan and Hadamard (see [4]) that there exists a minimal closed geodesic in any nontrivial homotopy class in $\pi_1(M)$. Then we get a contradiction. Therefore, $\pi_1(M) = 0$ and M is simply connected. □

Theorem 1.4 is an immediate consequence of Lemma 3.2 and Theorem 1.5.

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