

DIFFERENTIAL EQUATIONS AND OPTIMAL CHOICE PROBLEMS

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Asymptotic forms for the optimal payoff and optimal stop rule for a generalized class of "secretary" problems are obtained by the analysis of a related family of ordinary differential equations.

1. Introduction. The stop rule problems to be considered in this paper are generalizations of the "secretary problem" for bounded payoffs. A subclass of such problems is considered in Gusein-Zade [6].

Let q map the positive integers into $[0, 1]$ with $q(k) \downarrow 0$. Let a probability $(N!)^{-1}$ be attached to each permutation σ of the first N integers, and let $\{X_k\}$ $k = 1, \dots, N$ be the sequence of independent random variables where X_k is the rank of $\sigma(k)$ among $\sigma(1), \sigma(2), \dots, \sigma(k)$. The optimal choice problem consists in determining v_N, t_N where

$$(1.1) \quad v_N = \max_t Eq(\sigma(t)) = Eq(\sigma(t_N))$$

as t runs through stop rules on the sequence X_1, X_2, \dots, X_N . This paper is concerned with asymptotic forms for v_N and t_N . Our principal result is expressed in the following theorem.

THEOREM 1.1. *Let*

$$R_k(\alpha) = \sum_{l=k}^{\infty} q(l) \binom{l-1}{k-1} \alpha^k (1-\alpha)^{l-k}, \quad \alpha \in (0, 1].$$

Then the differential equation

$$g'(\alpha) = -\alpha^{-1} \sum_{l=1}^{\infty} (R_l(\alpha) - g(\alpha))^+; \quad g(1) = 0$$

has a unique solution on $[0, 1]$ and

$$|v_N - g(0)| \leq C_1 [\ln N / N^{\frac{1}{2}} + q(C_2 \ln N)]$$

where C_1 and C_2 are determined constants.

Analogous results for optimal choice problems involving unbounded functions, q , are treated in [8], and will be the subject of a forthcoming paper. The latter results generalize Chow, Moriguti, Robbins and Samuels [2].

2. Basic recursions and difference equations. Let q be a function on the positive integers, I , such that

- (a) $q(1) = 1$,
- (b) $q(l) \downarrow 0$.

We call q a payoff function. Let σ be any permutation of the first N integers; we assume all $N!$ permutations to be equally likely. We define X_r to be the

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relative rank of $\sigma(r)$ among $\sigma(1), \sigma(2), \dots, \sigma(r)$. The random variables, X_r , $r = 1, 2, \dots, N$, called observations, are independent with distributions

$$P(X_r = k) = \frac{1}{r} \quad \text{if } 1 \leq k \leq r \\ = 0 \quad \text{otherwise.}$$

Let us set

$$Q(r, k) = E(q(\sigma(r)) | X_r = k) = \sum_k^{N-(r-k)} q(l) \frac{\binom{l-1}{k-1} \binom{N-l}{r-k}}{\binom{N}{r}}.$$

Our objective is to find asymptotic forms and estimates for the values v_N, t_N defined by

$$(2.1) \quad v_N = \max_t Eq(\sigma(t)) = \max_t EQ(t, X_t) = EQ(t_N, X_{t_N})$$

where t runs through all stop rules on the sequence X_r , $r = 1, \dots, N$. We call v_N the optimal payoff and t_N the optimal stop rule. We sometimes call v_N a utility. A standard recursive technique for generating v_N and t_N is the following.

Set

$$v_N(N) = Eq(\sigma(N)) \\ v_N(N, k) = q(k), \quad k = 1, 2, \dots, N \\ v_N(r, k) = \max(Q(r, k), Ev_N(r+1, X_{r+1})) \\ v_N(r) = Ev_N(r, X_r) = \frac{1}{r} \sum_1^r v_N(r, k).$$

We then have

$$(2.2) \quad v_N(r) = E \max(Q(r, X_r), v_N(r+1)) \\ = \frac{1}{r} \sum_1^r \max(Q(r, k), v_N(r+1)).$$

Since $v_N(r) = \sup_{t \geq r} Eq(\sigma(t))$, we see that $v_N(r) \geq v_N(r+1)$ and $v_N = v_N(1)$. This recursive technique, called a "backwards recursion," defines the optimal stop rule, t_N , by the prescription that one stop with the observation $X_r = k$ unless $v_N(r, k) > Q(r, k)$, i.e., unless it pays more to continue.

We will presently prove that

$$Q(r, k) \geq Q(r, k+1) \\ Q(r+1, k) \geq Q(r, k).$$

We can therefore characterize our optimal stop rule t_N , (which is the collection of pairs (r, k) such that $Q(r, k) \geq v_N(r+1)$) as a tuple (r_1, r_2, \dots, r_N) where $r_1 \leq r_2 \leq \dots \leq r_N$ and where one stops with observation $X_r = k$ provided $r \geq r_k$.

The procedure outlined above is a rewording of the procedures found in [4], [6], and [2].

We now state

PROPOSITION 2.1. (*Basic Recursion.*)

$$Q(r, k) = \frac{k}{r+1} Q(r+1, k+1) + \left(1 - \frac{k}{r+1}\right) Q(r+1, k).$$

PROOF. Straightforward calculation.

This recursion will be used often in this paper. A first application is:

COROLLARY 2.1. (a) $Q(r, k) \geq Q(r, k+1)$

(b) $Q(r+1, k) \geq Q(r, k)$.

PROOF. (a) Note first: $Q(N, k+1) = q(k+1) \leq q(k) = Q(N, k)$. Then backwards, assuming the result for $r+1$, we have

$$\begin{aligned} Q(r, k+1) &= \frac{k+1}{r+1} Q(r+1, k+2) + \left(1 - \frac{k+1}{r+1}\right) Q(r+1, k+1) \\ &= \frac{k}{r+1} Q(r+1, k+2) + \left(1 - \frac{k}{r+1}\right) Q(r+1, k+1) \\ &\quad + \frac{1}{r+1} [Q(r+1, k+2) - Q(r+1, k+1)] \\ &\leq \frac{k}{r+1} Q(r+1, k+2) + \left(1 - \frac{k}{r+1}\right) Q(r+1, k+1) \\ &\leq \frac{k}{r+1} Q(r+1, k+1) + \left(1 - \frac{k}{r+1}\right) Q(r+1, k) = Q(r, k). \end{aligned}$$

(b) Use the same procedure as for (a). \square

Let us now normalize $v_N(\cdot)$ by setting $v_N(r) = f_N(r/N)$. We have then:

$$\begin{aligned} f_N(1) &= \frac{1}{N} \sum_1^N g(l) = \frac{s(N)}{N} \\ f_N\left(\frac{r}{N}\right) &= \frac{1}{r} \sum_1^r \max\left(Q(r, k), f_N\left(\frac{r+1}{N}\right)\right). \end{aligned}$$

We extend f_N by linear interpolation to the entire unit interval and we rewrite this recursion in the form

$$\begin{aligned} f_N\left(\frac{r}{N}\right) - f_N\left(\frac{r+1}{N}\right) &= \frac{1}{r} \sum_1^r \left(Q(r, k) - f_N\left(\frac{r+1}{N}\right)\right)^+ \\ f_N(1) &= \frac{s(N)}{N}. \end{aligned}$$

We note that $v_N = f_N(0)$.

These last equations resemble

$$\begin{aligned} g(1) &= 0 \\ g_N\left(\frac{r}{N}\right) - g_N\left(\frac{r+1}{N}\right) &= \frac{1}{r+1} \sum_1^{r+1} \left(R_k\left(\frac{r+1}{N}\right) - g_N\left(\frac{r+1}{N}\right)\right)^+ \end{aligned}$$

where

$$R_k(\alpha) = \sum_k^\infty q(l) \binom{l-1}{k-1} \alpha^k (1-\alpha)^{l-k} \quad \alpha \in [0, 1].$$

Now g_N , extended by linear interpolation to $[0, 1]$, can be shown to be a piecewise differentiable function which in the limit tends to the function g which uniquely satisfies

$$g(1) = 0$$

$$g'(\alpha) = -\frac{1}{\alpha} \sum_1^\infty (R_k(\alpha) - g(\alpha))^+ \quad \alpha \in [0, 1].$$

One employs the standard Picard method for ordinary differential equations to establish the existence of g and the fact that $\lim g_N = g$. The details, lengthy but straightforward, can be found in [8]. Using estimates for the difference $R_k(r/N) - Q(r, k)$, one deduces

THEOREM 2.1.

$$\|f_N - g\| \leq 10 \frac{\log N}{N^{\frac{1}{2}}} + 30q \left(\left[\frac{\alpha_*}{2} \log N \right] \right)$$

provided $\log N \geq \max(4, 12\alpha_*)$ where $R_1(\alpha_*) \leq .25$.

REMARK 2.1.

(i) Since $|v_N - g(0)| = |f_N(0) - g(0)| \leq \sup_{0 \leq \alpha \leq 1} |f_N(\alpha) - g(\alpha)| = \|f_N - g\|$, we see that $v_N \rightarrow g(0)$.

(ii) If q is truncated, i.e., if there exists m such that $q(l) = 0$ all $l > m$, then $\|f_N - g\| < 10 \log N/N^{\frac{1}{2}}$. Actually, much more can be said. We have the stronger inequality:

$$\|f_N - g\| \leq C_m/N \quad \text{for large } N$$

where C_m is a constant that depends only on m . Again, the details are spelled out in [8].

(iii) Note that g is non-increasing.

(iv) The functions R_k play an important role in this paper. We list some properties:

- (a) $R_k'(\alpha) = k\alpha^{-1}(R_k(\alpha) - R_{k+1}(\alpha)) = k \sum_k^\infty (q(l) - q(l+1)) \binom{l}{k} \alpha^{k-1} \times (1-\alpha)^{l-k}$.
- (b) If $q(k) > 0$, then R_k is strictly increasing and $R_k(\alpha) > R_{k+1}(\alpha)$, all $\alpha \in (0, 1]$.
- (c) $\sum_1^m R_l(\alpha) = \alpha s(m) + m\alpha \int_\alpha^1 t^{-2} R_{m+1}(t) dt$.
- (d) $R_k(0) = 0$; $R_k(1) = q(k)$.

(v) The properties listed above imply that $g' \equiv 0$ on $[0, \alpha_1]$ where α_1 is the unique solution for the equation $R_1(\alpha_1) = g(\alpha_1)$. Thus, $\lim_{N \rightarrow \infty} v_N = R_1(\alpha_1)$.

We described t_N , the optimal stop rule, as an N -tuple of increasing integers (r_1, r_2, \dots, r_N) where one stops on observation $x_r = k$ provided $r \geq r_k$. The value r_k was the minimal integer for which $Q(r, k) > v_N(r+1)$. This condition, in the limit as $N \rightarrow \infty$, leads to

THEOREM 2.2. The optimal stop rule $t_N = (r_1, r_2, \dots, r_N)$ satisfies

$$\lim_{N \rightarrow \infty} r_k/N = \alpha_k$$

where $0 < \alpha_1 < \alpha_2 < \dots$ is a sequence uniquely determined by the equations:

$$R_k(\alpha_k) = g(\alpha_k).$$

There is a converse of sorts to the last theorem, namely, that if \bar{v}_N is the utility associated with the stop rule $\bar{i}_N = ([\alpha_1 N], [\alpha_2 N], \dots, N)$, then $\lim_{N \rightarrow \infty} \bar{v}_N = \lim_{N \rightarrow \infty} v_N = g(0)$. We will sketch a proof of this result in the last section of this paper.

3. Applications. Our objective in this section is to determine the sequence $\{\alpha_n\}$ and the utility $v = R_1(\alpha_1)$. Our differential equation

$$(3.1) \quad g' = -\frac{1}{\alpha} \sum_1^\infty (R_k - g)^+; \quad g(1) = 0$$

reduces on the interval $[\alpha_n, \alpha_{n+1}]$ to the equation

$$(3.2) \quad g' = -\frac{1}{\alpha} \sum_1^n (R_k - g)$$

with boundary conditions

$$(3.3) \quad \begin{aligned} g(\alpha_n) &= R_n(\alpha_n) \\ g(\alpha_{n+1}) &= R_{n+1}(\alpha_{n+1}). \end{aligned}$$

Equivalently, we can write (3.2) in the form $(g(\alpha)/\alpha^n)' = -\alpha^{-(n+1)} \sum_1^n R_k(\alpha)$ on $[\alpha_n, \alpha_{n+1}]$. Now for $n \geq 2$, $[\alpha^{-n} \sum_1^{n-1} R_k(\alpha)]' = -\alpha^{-(n+1)}(n-1) \sum_1^{n-1} R_k(\alpha)$ follows immediately from the properties of R_k' . Thus, on $[\alpha_n, \alpha_{n+1}]$ we have $(g(\alpha)/\alpha^n)' = (n-1)^{-1}[\alpha^{-n} \sum_1^{n-1} R_k]'$ for $n \geq 2$. Using our boundary conditions (3.3) we conclude

$$(3.4) \quad \frac{1}{\alpha_n^n} \sum_1^{n-1} (R_k(\alpha_n) - R_n(\alpha_n)) = \frac{1}{\alpha_{n+1}^{n+1}} \sum_1^{n-1} (R_k(\alpha_{n+1}) - R_{n+1}(\alpha_{n+1})).$$

When $n = 1$, we see that (3.2) becomes $(g(\alpha)/\alpha)' = -R_1(\alpha)/\alpha^2$ or, using (3.3),

$$(3.5) \quad \frac{R_2(\alpha_2)}{\alpha_2} - \frac{R_1(\alpha_1)}{\alpha_1} = -\int_{\alpha_1}^{\alpha_2} \frac{R_1(\alpha)}{\alpha^2} d\alpha.$$

Now

$$\frac{d}{d\alpha} \sum_1^\infty \frac{s(k)}{k} (1 - \alpha)^k = -\sum_1^\infty s(k)(1 - \alpha)^{k-1} = -\frac{R_1(\alpha)}{\alpha^2},$$

the integrand in (3.5). So we can rewrite (3.5) in the form:

$$(3.6) \quad \sum_1^\infty \frac{s(k)}{k} (1 - \alpha_1)^k - \frac{R_1(\alpha_1)}{\alpha_1} = \sum_1^\infty \frac{s(k)}{k} (1 - \alpha_2)^k - \frac{R_2(\alpha_2)}{\alpha_2}.$$

DEFINITION 3.1. Set

$$H_1(\alpha) = \sum_1^\infty \frac{s(k)}{k} (1 - \alpha)^k - \frac{R_1(\alpha)}{\alpha}$$

$$G_1(\alpha) = H_1(\alpha) + \frac{1}{\alpha} (R_1(\alpha) - R_2(\alpha))$$

and for $n \geq 2$

$$H_n(\alpha) = \frac{1}{\alpha^n} \sum_{i=1}^{n-1} (R_i(\alpha) - R_n(\alpha))$$

$$G_n(\alpha) = \frac{1}{\alpha^n} \sum_{i=1}^{n-1} (R_i(\alpha) - R_{n+1}(\alpha)).$$

From (3.4) and (3.6) we see that our sequence $\{\alpha_n\}$ must satisfy

$$(3.7) \quad H_n(\alpha_n) = G_n(\alpha_{n+1}).$$

Let us now list some properties for H_n and G_n which will be useful for examining (3.7).

REMARK 3.1.

(A) $0 \leq H_n(\alpha) \leq G_n(\alpha) \leq H_{n+1}(\alpha)$ for $n \geq 2$.

(B) $H_n'(\alpha) = -(n-1)H_{n+1}(\alpha)$ for $n \geq 2$.

Thus, H_n is non-increasing.

(C) $G_n(\alpha) = H_n(\alpha) + \alpha^{-n}(n-1)(R_n - R_{n+1})$, $n \geq 2$.

(D) $G_n'(\alpha) \leq -(n-1)H_{n+1}(\alpha)$, $n \geq 2$.

Thus, G_n is non-increasing.

(E) $H_n(\alpha) = \sum_{k=0}^{\infty} \binom{n-2+k}{k} [s(n+k-1) - (n+k-1)q(n+k)](1-\alpha)^k$, $n \geq 2$. For by repeated application of (B), we get $H_n^{(k)} = (-1)^k k! \binom{n-2k+1}{k} H_{n+k}$ and further $H_{n+k}(1) = s(n+k-1) - (n+k-1)q(n+k)$, so we simply take the Taylor expansion of H_n around $\alpha = 1$.

(F) $\lim_{\alpha \rightarrow 0^+} H_n(\alpha) = \infty$. $n \geq 2$. For we can write

$$(3.8) \quad H_2(\alpha) = \alpha^{-1} R_1'(\alpha) = \alpha^{-1} \sum_{l=1}^{\infty} l(q(l) - q(l+1))(1-\alpha)^{l+1},$$

and clearly $\lim_{\alpha \rightarrow 0^+} H_2(\alpha) = \infty$. Now use the fact that $H_n \geq H_2$ for $n \geq 2$. We clearly have also that $\lim_{\alpha \rightarrow 0^+} G_n(\alpha) = \infty$, $n \geq 2$.

(G) $H_n(1) = s(n-1) - (n-1)q(n)$, $n \geq 2$.

$$G_n(1) = s(n-1) - (n-1)q(n+1), \quad n \geq 2.$$

(H) If $q(m+1) = 0$, then $k \geq 1$ implies $H_{m+k}(\alpha) = H_{m+1}(\alpha) = s(m)/\alpha^m$, $G_{m+k}(\alpha) = G_{m+1}(\alpha) = s(m)/\alpha^m$, since $R_{m+k} \equiv 0$ and $\sum_{i=1}^m R_i(\alpha) = \alpha s(m)$.

(I) H_n is strictly decreasing on $(0, 1]$ for $n \geq 2$. G_n is strictly decreasing on $(0, 1]$ for $n \geq 2$.

(J) $H_1'(\alpha) = -H_2(\alpha)$, $G_1'(\alpha) \leq -H_2(\alpha)$. In particular, H_1 and G_1 are strictly decreasing on $(0, 1]$.

(K) $\lim_{\alpha \rightarrow 0^+} H_1(\alpha) = \infty$.

(L) Given any $\alpha \in (0, 1]$, there exists a unique $\bar{\alpha} \in (0, 1]$, $\bar{\alpha} \leq \alpha$ such that $H_n(\bar{\alpha}) = G_n(\alpha)$. This result follows from the facts that $G_n \geq H_n$, H_n is strictly decreasing, and $H_n(0^+) = \infty$.

PROPOSITION 3.1. Let q be truncated at m . Then the sequence $\{\alpha_n\}$, $n = 1, \dots, m$ which defined the limiting optimal stop rule is generated recursively by the formulas:

$$\alpha_{m+1} = 1, \quad H_n(\alpha_n) = G_n(\alpha_{n+1}) \quad n = 1, \dots, m.$$

EXAMPLE 3.1. (a) Let

$$q(l) = 1 \quad \text{if } l = 1 \\ = 0 \quad \text{otherwise.}$$

Then $R_1(\alpha) = \alpha$, $R_k(\alpha) = 0$ for $k > 1$. $H_1(\alpha) = \sum_{l=1}^{\infty} k^{-l}(1-\alpha)^k - 1 = -\log \alpha - 1$, $G_1(\alpha) = -\log \alpha$. Now $\alpha_2 = 1$, and α_1 is the solution to $-\log \alpha_1 - 1 = -\log \alpha_2 = -\log 1 = 0$, i.e., $\log \alpha_1 = -1$ so $\alpha_1 = 1/e$.

(b) Let

$$q(l) = 1 \quad \text{if } l = 1, 2 \\ = 0 \quad \text{otherwise.}$$

Then $R_1(\alpha) = 2\alpha - \alpha^2$, $R_2(\alpha) = \alpha^2$. So $H_1(\alpha) = 2\alpha - 2 \log \alpha - 2$, $H_2(\alpha) = 2\alpha^{-1} - 2$, $G_1(\alpha) = -2 \log \alpha$, $G_2(\alpha) = 2\alpha^{-1} - 1$. Thus, starting with $\alpha_3 = 1$, we have $2\alpha_2^{-1} - 2 = \frac{2}{1} - 2 = 0$, i.e., $\alpha_2 = \frac{2}{3}$ and $2\alpha_1 - 2 \log \alpha_1 - 2 = -\log \frac{2}{3}$, i.e., $\alpha_1 - \log \alpha_1 \sim 1.40$ so $\alpha_1 \sim .35$. Let us note that in case (a), our utility v satisfies $v = R_1(1/e) = 1/e$, and in case (b) we have $v = R_1(.35) = 2(.35) - (.35)^2 \sim .58$.

We also note that for any q truncated at m we have: $H_m(\alpha) = s(m)\alpha^{-(m-1)} - mq(m)$. Since $G_m(1) = s(m-1)$, we have $\alpha_m = [s(m)/\{s(m-1) + mq(m)\}]^{(m-1)^{-1}}$; a generalization of a formula found in [6].

REMARK 3.2. It may happen that, although q is not truncated, we can still find $(\alpha_1, \alpha_2, \dots)$. The idea is to set up the recursion in a form which expresses α_m as a limit depending in an essential manner on the fact that $\lim_{n \rightarrow \infty} \alpha_n = 1$, which is proven in [8]. Let us illustrate this.

A special class of payoffs. Let $\beta \in (0, 1)$. We set

$$(3.9) \quad q_{\beta}(k) = 1 \quad \text{if } k = 1 \\ = \frac{\beta(\beta + 1), \dots, (\beta + k - 2)}{(k - 1)!} \quad \text{if } k \geq 2.$$

We can write this payoff in the form $q_{\beta}(k) = (-1)^{k-1} \binom{-\beta}{k-1}$. We note that

- (a) $q_{\beta}(k+1)/q_{\beta}(k) = 1 - (1-\beta)/k \leq 1$,
- (b) $q_{\beta}(k) = \prod_{l=1}^{k-1} (1 - (1-\beta)/l)$.

Thus, for large k , we have $\log q_{\beta}(k) \leq -(1-\beta) \sum_{l=1}^{k-1} l^{-1} \rightarrow -\infty$, i.e., $q_{\beta}(k) \downarrow 0$. In fact, let us note that $\log q_{\beta}(k) \leq -(1-\beta) \int_1^k s^{-1} ds = -(1-\beta) \log k$, so $q_{\beta}(k) \leq k^{-(1-\beta)}$ while $\log q_{\beta}(k) \geq -(1-\beta)\{1 + \int_1^k s^{-1} ds\}$, from which $q_{\beta}(k) \geq e^{-(1-\beta)} k^{-(1-\beta)} \geq e^{-2} \cdot k^{-(1-\beta)}$. This last inequality shows that $\sum_{k=1}^{\infty} q_{\beta}(k) = \infty$ for all $\beta \in (0, 1]$.

- (c) $\beta \geq \tilde{\beta}$, $q_{\beta} \geq q_{\tilde{\beta}}$.
- (d) $R_1(\alpha) = \alpha^{1-\beta}$.
- (e) $R_n(\alpha) = q_{\beta}(n)\alpha^{1-\beta}$. This follows immediately from the recursion $R_{n+1} = R_n - \alpha n^{-1} R_n'$. In what follows, $\beta < 1$.
- (f) $H_2(\alpha) = (1-\beta)\alpha^{-1-\beta}$.

So $H_1(\alpha) = (1-\beta)\beta^{-1}\alpha^{-\beta} - \beta^{-1}$ since $H_1' = -H_2$ and $H_1(1) = -1$.

(g) $H_n(\alpha) = (1 - \beta)\beta^{-1}(n - 1)q_\beta(n)\alpha^{-\beta-(n-1)}$, $n \geq 2$. Here we simply use the recursion: $H_n' = -(n - 1)H_{n+1}$.

(h) $G_1(\alpha) = (\beta^{-1} - \beta)\alpha^{-\beta} - \beta^{-1}$.

(i) $G_n(\alpha) = (1 - \beta)\beta^{-1}(n + \beta)n^{-1}(n - 1)q_\beta(n)\alpha^{-\beta-(n-1)}$, $n \geq 2$.

These formulas allow a straightforward calculation for the sequence $\{\alpha_n\}$.

We have the recursion $H_n(\alpha_n) = G_n(\alpha_{n+1})$ reducing to $\alpha_n^{-\beta-(n-1)} = (n + \beta)n^{-1}\alpha_{n+1}^{-\beta-(n-1)}$, i.e.,

$$\frac{\alpha_n}{\alpha_{n+1}} = \left(1 + \frac{\beta}{n}\right)^{1/(-\beta-(n-1))}$$

for $n \geq 2$. Now since $\alpha_n \rightarrow 1$, we have

$$\alpha_m = \prod_n^\infty \left(1 + \frac{\beta}{n}\right)^{-1/(\beta+(n-1))}, \quad m \geq 2.$$

Further, the recursion $H_1(\alpha_1) = G_1(\alpha_2)$ reduces to

$$\alpha_1/\alpha_2 = (1 + \beta)^{-1/\beta}.$$

So

$$\alpha_1 = \prod_1^\infty (1 + \beta/l)^{-1/(\beta+(l-1))}.$$

Finally

$$(3.10) \quad v = g(\alpha_1) = R_1(\alpha_1) = \prod_1^\infty \left(1 + \frac{\beta}{l}\right)^{\frac{1}{1-(l/(1-\beta))}}$$

$$g(\alpha_n) = R_n(\alpha_n) = q_\beta(n) \prod_n^\infty \left(1 + \frac{\beta}{l}\right)^{\frac{1}{1-(l/(1-\beta))}}.$$

REMARK 3.3. Note that

$$\begin{aligned} \ln \frac{1}{\alpha_1} &= \sum_1^\infty \frac{1}{\beta + (l - 1)} \log \left(1 + \frac{\beta}{l}\right) \\ &\leq \sum_1^\infty \frac{1}{\beta + (l - 1)} \frac{\beta}{l} \\ &\leq 1 + \beta \sum_2^\infty \frac{1}{l(l - 1)} = 1 + \beta. \end{aligned}$$

Thus $\alpha_1^{-1} \leq e^{1+\beta}$ or $\alpha_1 \geq e^{-1-\beta} \geq e^{-2}$. So α_1 is bounded away from zero uniformly in β .

On the other hand,

$$\begin{aligned} \log \frac{1}{\alpha_1} &= \sum_1^\infty \frac{1}{\beta + (l - 1)} \log \left(1 + \frac{\beta}{l}\right) \\ &\geq \sum_1^\infty \frac{1}{\beta + (l - 1)} \frac{1}{1 + \beta/l} \frac{\beta}{l} \\ &= \beta \sum_1^\infty \frac{1}{(\beta + l)(\beta + l - 1)} = \beta \sum_1^\infty \left\{ \frac{1}{\beta + l - 1} - \frac{1}{\beta + l} \right\} \\ &= \beta(1/\beta) = 1; \end{aligned}$$

therefore $\alpha_1 \leq 1/e$.

PROPOSITION 3.2. Let $\beta \in (0, 1)$, q_β as defined above. Then $1/e^2 \leq \alpha_1 \leq 1/e$.

4. Asymptotically optimal stop rules. Our objective in this section is to sketch a proof of the following:

THEOREM 4.1. Let $\bar{t}_N = ([\alpha_1 N], [\alpha_2 N], \dots, N)$, $\bar{v}_N = Eq(\sigma \bar{t}_N)$. Then

$$\lim_{N \rightarrow \infty} \bar{v}_N = g(0).$$

We begin by defining:

$$\begin{aligned} p_1 &= 1 \\ p_{l,l+1} &= p(x_l > l, [\alpha_l N] \leq t < [\alpha_{l+1} N]) \\ p_k &= \prod_1^{k-1} p_{l,l+1} && k > 1 \\ u_k &= \sum_{[\alpha_k N]}^{[\alpha_{k+1} N]-1} \sum_{l=1}^k p(x_t = l; x_s > k \text{ for } [\alpha_l N] \leq s < t) \cdot Q(t, l). \end{aligned}$$

Then, for any fixed m we have

$$\bar{v}_N \geq \sum_1^m p_k m_k \quad \text{for } N \geq m.$$

Using the equation

$$p_{l,l+1} = \prod_{[\alpha_l N]}^{[\alpha_{l+1} N]-1} (1 - l/t)$$

and letting $N \rightarrow \infty$, we get

$$p_{l,l+1} \rightarrow \left(\frac{\alpha_l}{\alpha_{l+1}} \right)^l$$

and

$$u_k \rightarrow \alpha_k^k \int_{\alpha_k}^{\alpha_{k+1}} \frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda) d\lambda$$

so that

$$\liminf v_N \geq \sum_{k=1}^m (\prod_1^k \alpha_l) \int_{\alpha_k}^{\alpha_{k+1}} \frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda) d\lambda.$$

Now for $k \geq 2$ we have

$$\frac{1}{\lambda^{k+1}} \sum_1^k R_l(\lambda) = -\frac{1}{k-1} \left(\frac{1}{\lambda^k} \sum_1^{k-1} R_l(\lambda) \right)'$$

So we can write

$$\begin{aligned} \liminf \bar{v}_N &\geq \alpha_1 \int_{\alpha_1}^{\alpha_2} \frac{R_1(\lambda)}{\lambda^2} d\lambda - \sum_2^m \frac{1}{k-1} (G_k(\alpha_{k+1}) - H_k(\alpha_k)) \prod_1^k \alpha_l \\ &\quad - \sum_2^m \left(\frac{R_{k+1}(\alpha_{k+1})}{\alpha_{k+1}^k} - \frac{R_k(\alpha_k)}{\alpha_k^k} \right) \prod_1^k \alpha_l. \end{aligned}$$

The second term is zero while the third term reduces to

$$\begin{aligned} \frac{\alpha_1}{\alpha_2} R_2(\alpha_2) - \frac{\prod_1^m \alpha_l}{\alpha_{m+1}^m} R_{m+1}(\alpha_{m+1}) &\geq \frac{\alpha_1}{\alpha_2} R_2(\alpha_2) - R_{m+1}(\alpha_{m+1}) \\ &\geq \frac{\alpha_1}{\alpha_2} R_2(\alpha_2) - q(m+1). \end{aligned}$$

Now

$$\int_{\alpha_1}^{\alpha_2} \frac{R_1(\lambda)}{\lambda^2} d\lambda = \frac{R_1(\alpha_1)}{\alpha_1} - \frac{R_2(\alpha_2)}{\alpha_2},$$

so we have

$$\liminf \bar{v}_N \geq R_1(\alpha_1) - q(m+1).$$

Letting $m \rightarrow \infty$ we have $\liminf \bar{v}_N \geq R_1(\alpha_1)$. Since $v_N \geq \bar{v}_N$ and $v_N \rightarrow R_1(\alpha_1) = g(0)$, we have

$$\lim \bar{v}_N = g(0).$$

Note. To be completely rigorous,

$$\lim u_k = \alpha_k^k \int_{\alpha_k}^{\alpha_{k+1}} \frac{1}{\lambda^{k+1}} \sum_1^k R_1(\lambda) d\lambda$$

is true only if q is truncated. One can get around this restriction quite easily, however, and the end result is the same. Again, details can be found in [8].

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