Differential equations driven by Hölder continuous functions of order greater than 1/2

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Summary. We derive estimates for the solutions to differential equations driven by a Hölder continuous function of order $\beta > 1/2$. As an application we deduce the existence of moments for the solutions to stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

1 Introduction

We are interested in the solutions of differential equations on $\mathbb{R}^d$ of the form

$$x_t = x_0 + \int_0^t f(x_r) dy_r, \tag{1}$$

where the driving force $y : [0, \infty) \to \mathbb{R}^m$ is a Hölder continuous function of order $\beta > 1/2$. If the function $f : \mathbb{R}^d \to \mathbb{R}^{md}$ has bounded partial derivatives which are Hölder continuous of order $\lambda > \frac{1}{\beta} - 1$, then there is a unique solution $x : \mathbb{R}^d \to \mathbb{R}$, which has bounded $\frac{1}{\beta}$-variation on any finite interval. These results have been proved by Lyons in [2] using the p-variation norm and the technique introduced by Young in [8]. The integral appearing in (1) is a Riemann-Stieltjes integral.

In [9] Zähle has introduced a generalized Stieltjes integral using the techniques of fractional calculus. This integral is expressed in terms of fractional derivative operators and it coincides with the Riemann-Stieltjes integral $\int_0^T f dg$, when the functions f and g are Hölder continuous of orders λ and μ, respectively and $\lambda + \mu > 1$ (see Proposition 1 below). Using this formula for the Riemann-Stieltjes integral, Nualart and Răşcanu have obtained in [3] the existence of a unique solution for a class of general differential equations that includes (1). Also they have proved that the solution of (1) is bounded on a finite interval $[0, T]$ by $C_1 \exp(C_2 \|y\|_{0,T,\beta}^{\kappa})$, where $\kappa > \frac{1}{\beta}$ if f is bounded and $\kappa > \frac{1}{1-2\beta}$ is f has linear growth. Here $\|y\|_{0,T,\beta}$ denotes the β-Hölder norm of y on the time interval $[0, T]$. These estimates are based on a suitable application of Gronwall's lemma. It turns out that the estimate in the linear growth case is unsatisfactory because κ tends to infinity as β tends to $1/2$.

The main purpose of this paper is to obtain sharper estimates for the solution x_t in the case where f is bounded or has linear growth using a direct approach based on formula (8). In the case where f is bounded we estimate $\sup_{0\le t\le T}|x_t|$ by

$$C\left(1+\|y\|_{0,T,\beta}^{\frac{1}{\beta}}\right)$$

and if f has linear growth we obtain the estimate

$$C_1\exp\left(C_2\|y\|_{0,T,\beta}^{\frac{1}{\beta}}\right).$$

In Theorem 2 we provide explicit dependence on f and T for the constants C, C_1 and C_2. We also establish estimates for the solution of a linear equation with rough time dependent coefficient (Theorem 3.2).

Another novelty of this paper is that we establish stability type of results for the solution x_t to (1) on the initial condition x_0, the driving control y and the coefficient f (Theorem 3.2).

As an application we deduce the existence of moments for the solutions to stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H>\frac{1}{2}$. We also discuss the regularity of the solution in the sense of Malliavin Calculus, improving the results of Nualart and Saussereau [4], and we apply the techniques of the Malliavin calculus to establish the smoothness of the density of the solution under suitable non-degeneracy conditions. More precisely, Theorem 3.2 allows us to show that the solution of a stochastic differential equation

2 Fractional integrals and derivatives

Let $a,b\in\mathbb{R}$ with $a<b$. Let $f\in L^1(a,b)$ and $\alpha>0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x\in(a,b)$ by

$$I_{a+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s)\,ds$$

and

$$I_{b-}^{\alpha}f(t)=\frac{(-1)^{-\alpha}}{\Gamma(\alpha)}\int_t^b(s-t)^{\alpha-1}f(s)\,ds,$$

respectively, where $(-1)^{-\alpha}=e^{-i\pi\alpha}$ and $\Gamma(\alpha)=\int_0^\infty r^{\alpha-1}e^{-r}dr$ is the Euler gamma function. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p(a,b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f\in I_{a+}^{\alpha}(L^p)$ (resp. $f\in I_{b-}^{\alpha}(L^p)$) and $0<\alpha<1$ then the Weyl derivatives are defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right) \tag{1}$$

and

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right) \tag{2}$$

where $a \le t \le b$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a, b)$ if $p = 1$ and moreover in L^p-sense if $1 < p < \infty$).

For any $\lambda \in (0, 1)$, we denote by $C^{\lambda}(a, b)$ the space of λ-Hölder continuous functions on the interval $[a, b]$. We will make use of the notation

$$\|x\|_{a,b,\beta} = \sup_{a \le \theta < r \le b} \frac{|x_r - x_{\theta}|}{|r - \theta|^{\beta}},$$

and

$$\|x\|_{a,b,\infty} = \sup_{a \le r \le b} |x_r|,$$

where $x : \mathbb{R}^d \to \mathbb{R}$ is a given continuous function.

Recall from [6] that we have:

- If $\alpha < \frac{1}{p}$ and $q = \frac{p}{1-\alpha p}$ then
$$I_{a+}^{\alpha}(L^p) = I_{b-}^{\alpha}(L^p) \subset L^q(a, b).$$
- If $\alpha > \frac{1}{p}$ then
$$I_{a+}^{\alpha}(L^p) \cup I_{b-}^{\alpha}(L^p) \subset C^{\alpha - \frac{1}{p}}(a, b).$$

The following inversion formulas hold:

$$I_{a+}^{\alpha}\left(D_{a+}^{\alpha} f\right) = f, \qquad \forall f \in I_{a+}^{\alpha}(L^p) \tag{3}$$

$$I_{a-}^{\alpha}\left(D_{a-}^{\alpha} f\right) = f, \qquad \forall f \in I_{a-}^{\alpha}(L^p) \tag{4}$$

and

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha} f\right) = f, \quad D_{a-}^{\alpha}\left(I_{a-}^{\alpha} f\right) = f, \quad \forall f \in L^1(a, b). \tag{5}$$

On the other hand, for any $f, g \in L^1(a, b)$ we have

$$\int_a^b I_{a+}^{\alpha} f(t) g(t) dt = (-1)^{\alpha} \int_a^b f(t) I_{b-}^{\alpha} g(t) dt, \tag{6}$$

and for $f \in I_{a+}^{\alpha}(L^p)$ and $g \in I_{a-}^{\alpha}(L^p)$ we have

$$\int_a^b D_{a+}^{\alpha} f(t) g(t) dt = (-1)^{-\alpha} \int_a^b f(t) D_{b-}^{\alpha} g(t) dt. \tag{7}$$

Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Then, from the classical paper by Young [8], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives (see [9]).

Proposition 1. *Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Let $\lambda > \alpha$ and $\mu > 1 - \alpha$. Then the Riemann Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f\,(t)\, D_{b-}^{1-\alpha} g_{b-}\,(t)\, dt, \tag{8}$$

where $g_{b-}\,(t) = g\,(t) - g\,(b)$.

3 Estimates for the solutions of differential equations

Suppose that $y : [0, \infty) \to \mathbb{R}^m$ is a Hölder continuous function of order $\beta > 1/2$. Fix an initial condition $x_0 \in \mathbb{R}^d$ and consider the following differential equation

$$x_t = x_0 + \int_0^t f(x_r) dy_r, \tag{1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^{md}$ is given function. Lyons has proved in [2] that Equation (1) has a unique solution if f is continuously differentiable and it has a derivative f' which is bounded and locally Hölder continuous of order $\lambda > \frac{1}{\beta} - 1$.

Our aim is to obtain estimates on x_t which are better than those given by Nualart and Răşcanu in [3].

Theorem 2. *Let f be a continuously differentiable function such that f' is bounded and locally Hölder continuous of order $\lambda > \frac{1}{\beta} - 1$.*

(i) Assume $\|f'\|_\infty > 0$. There is a constant k depending only on β, such that for all T,

$$\sup_{0 \le t \le T} |x_t| \le 2^{1 + kT\left[\|f'\|_\infty \vee |f(0)|\right]^{1/\beta} \|y\|_{0,T,\beta}^{1/\beta}} \left(|x_0| + 1\right). \tag{2}$$

(ii) Assume that f is bounded. Then, there is a constant k, which depends only on β, such that for all T,

$$\sup_{0 \le t \le T} |x_t| \le |x_0| + k\|f\|_\infty \left(T^{\beta} \|y\|_{0,T,\beta} \vee T \|f'\|_\infty^{\frac{1-\beta}{\beta}} \|y\|_{0,T,\beta}^{\frac{1}{\beta}} \right). \tag{3}$$

Proof. Without loss of generality we assume that $d = m = 1$. Set $\|y\|_\beta = \|y\|_{0,T,\beta}$. We can assume that $\|y\|_\beta > 0$, otherwise the inequalities are obvious. Let $\alpha < 1/2$ such that $\alpha > 1-\beta$. Henceforth k will denote a generic constant depending only on β.

Step 1. Assume first that f is bounded. It suffices to assume that $\|f'\|_\infty > 0$. First we use the fractional integration by parts formula given in Proposition 1 to obtain for all $s,t \in [0,T]$,

$$|\int_s^t f(x_r)dy_r| \leq \int_s^t |D^\alpha_{s+} f(x_r)\, D^{1-\alpha}_{t-} y_{t-}(r)|dr.$$

From (1) and (2) it is easy to see

$$|D^{1-\alpha}_{t-} y_{t-}(r)| \leq k\|y\|_{r,t,\beta}(t-r)^{\alpha+\beta-1} \leq k\|y\|_\beta(t-r)^{\alpha+\beta-1} \tag{4}$$

and

$$|D^\alpha_{s+} f(x_r)| \leq k\left[\|f\|_\infty (r-s)^{-\alpha} + \ \|f'\|_\infty \|x\|_{s,t,\beta}(r-s)^{\beta-\alpha}\right]. \tag{5}$$

Therefore

$$\begin{aligned}
|\int_s^t f(x_r)dy_r| &\leq k\|y\|_\beta \int_s^t \left[\|f\|_\infty (r-s)^{-\alpha}(t-r)^{\alpha+\beta-1}\right.\\
&\qquad \left. + \|f'\|_\infty \|x\|_{s,t,\beta}(r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta-1}\right] dr\\
&\leq k\|y\|_\beta \left[\|f\|_\infty (t-s)^\beta + \|f'\|_\infty \|x\|_{s,t,\beta}(t-s)^{2\beta}\right] .
\end{aligned}$$

Consequently, we have

$$\|x\|_{s,t,\beta} \leq k\|y\|_\beta \left[\|f\|_\infty + \|f'\|_\infty \|x\|_{s,t,\beta}(t-s)^\beta\right] .$$

Choose Δ such that

$$\Delta = \left(\frac{1}{2k\,\|f'\|_\infty \|y\|_\beta}\right)^{\frac{1}{\beta}} .$$

Then, for all s and t such that $t-s \leq \Delta$ we have

$$\|x\|_{s,t,\beta} \leq 2k\|y\|_\beta \|f\|_\infty . \tag{6}$$

Therefore,

$$\|x||_{s,t,\infty} \leq |x_s| + \|x\|_{s,t,\beta}(t-s)^\beta \leq |x_s| + 2k\|y\|_\beta \|f\|_\infty \Delta^\beta. \tag{7}$$

If $\Delta \geq T$ we obtain the estimate

$$\|x||_{0,T,\infty} \leq |x_0| + 2k\|y\|_\beta \|f\|_\infty T^\beta. \tag{8}$$

Assume $\Delta < T$. Then, from (7) we get

$$\|x||_{s,t,\infty} \leq |x_s| + \|f\|_\infty \|f'\|_\infty^{-1} . \tag{9}$$

Divide the interval $[0,T]$ into $n=[T/\Delta]+1$ subintervals (where $[a]$ denotes the largest integer bounded by a). Applying the inequality (9) for $s=0$ and $t=\Delta$ we obtain

$$\sup_{0\le t\le \Delta}|x_t|\le |x_0|+\|f\|_\infty \|f'\|_\infty^{-1}.$$

Then, applying the inequality (9) on the intervals $[\Delta,2\Delta]$, ..., $[(n-1)\Delta,n\Delta]$ recursively, we obtain

$$\begin{aligned}\sup_{0\le t\le T}|x_t| &\le |x_0|+n\|f\|_\infty\|f'\|_\infty^{-1}\le |x_0|+\Delta^{-1}(T+\Delta)\|f\|_\infty\|f'\|_\infty^{-1}\\ &\le |x_0|+Tk\|f\|_\infty \|f'\|_\infty^{\frac{1-\beta}{\beta}}\|y\|_\beta^{\frac{1}{\beta}}. \qquad (10)\end{aligned}$$

The inequality (3) follows from (8) and (10).

Step 2. In the general case, assuming $\|f'\|_\infty>0$, instead of (5) we have

$$|D^\alpha_{s+}f(x_r)|\le k\left[(|f(0)|+\|f'\|_\infty|x_r|)(r-s)^{-\alpha}+\|f'\|_\infty\|x\|_{s,t,\beta}(r-s)^{\beta-\alpha}\right].$$

As a consequence,

$$\|x\|_{s,t,\beta}\le k\|y\|_\beta\left[|f(0)|+\|f'\|_\infty\|x\|_{s,t,\infty}+\|f'\|_\infty\|x\|_{s,t,\beta}(t-s)^\beta\right].$$

Suppose that Δ satisfies

$$\Delta\le\left(\frac{1}{3k\|f'\|_\infty\|y\|_\beta}\right)^{\frac{1}{\beta}}. \qquad (11)$$

Then, for all s and t such that $t-s\le\Delta$ we have

$$\|x\|_{s,t,\beta}\le\frac{3}{2}k\|y\|_\beta\left(|f(0)|+\|f'\|_\infty\|x\|_{s,t,\infty}\right).$$

Therefore,

$$|x_t|\le|x_s|+\frac{3}{2}k\|y\|_\beta\left(|f(0)|+\|f'\|_\infty\|x\|_{s,t,\infty}\right)\Delta^\beta,$$

and

$$\|x\|_{s,t,\infty}\le|x_s|+\frac{3}{2}k\|y\|_\beta\left(|f(0)|+\|f'\|_\infty\|x\|_{s,t,\infty}\right)\Delta^\beta.$$

Using again (11) we get

$$\|x\|_{s,t,\infty}\le 2|x_s|+2k\|y\|_\beta|f(0)|\Delta^\beta.$$

Assume also that

$$\Delta\le\left(\frac{1}{k\,|f(0)|\,\|y\|_\beta}\right)^{\frac{1}{\beta}}. \qquad (12)$$

Then

$$\|x\|_{s,t,\infty} \le 2\left(|x_s|+1\right).$$

Hence,

$$\sup_{0\le r\le t} |x_r| \le 2\left(\sup_{0\le r\le s} |x_r| + 1\right). \tag{13}$$

As before, divide the interval $[0,T]$ into $n=[T/\Delta]+1$ subintervals, and use the estimate (13) in every interval to obtain

$$\sup_{0\le t\le T} |x_t| \le 2^n\left(|x_0|+1\right) . \tag{14}$$

Choose

$$\Delta = \left(k\|y\|_\beta\left(\|f'\|_\infty \vee |f(0)|\right)\right)^{-\frac{1}{\beta}},$$

in such a way that (11) and (12) hold. Then, (14) implies

$$\sup_{0\le t\le T} |x_t| \ \le 2^{1+kT\left[\|f'\|_\infty \vee |f(0)|\right]^{1/\beta}\|y\|_\beta^{1/\beta}}\left(|x_0|+1\right).$$

The proof of the theorem is now complete.

Consider now the following system of equations

$$x_t = x_0 + \int_0^t f(x_r)dy_r,$$
$$z_t = z_0 + \int_0^t g(x_r)z_r dy_r,$$

where $y:[0,\infty)\to\mathbb{R}^m$ is a Hölder continuous function of order $\beta>1/2$. $f:\mathbb{R}^d\to\mathbb{R}^{md}$ and $g:\mathbb{R}^d\to\mathbb{R}^{M^2d}$ are given functions and $x_0\in\mathbb{R}^m$, $z_0\in\mathbb{R}^M$. We make the following assumptions:

H1) f is bounded with a bounded derivative f' which is locally Hölder continuous of order $\lambda > \frac{1}{\beta}-1$.
H2) g is bounded with bounded derivative.

Theorem 3. *Assume conditions H1) and H2). Then, there is a constant k depending only on β, such that for all T,*

$$\sup_{0\le t\le T} |z_t| \le 2^{1+kT\left[\|f'\|_\infty \vee \left(\|g\|_\infty + \sqrt{\|g'\|_\infty\|f\|_\infty}\right)\right]^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta}}|z_0|. \tag{15}$$

Proof. Without loss of generality we assume that $d=m=M=1$. Set $\|y\|_\beta=\|y\|_{0,T,\beta}$. We can assume that $\|y\|_\beta>0$, otherwise the inequality is obvious. Let $\alpha<1/2$ such that $\alpha>1-\beta$.

If we choose Δ such that

$$\Delta^\beta \le \frac{1}{2k\,\|f'\|_\infty\,\|y\|_\beta},$$

by (6) for all s and t such that $t-s\le\Delta$ we have

$$\|x\|_{s,t,\beta} \le 2k\|y\|_\beta\,\|f\|_\infty\,. \tag{16}$$

On the other hand, using the fractional integration by parts formula we obtain for all $s,t\in[0,T]$,

$$|\int_s^t g(x_r)z_r dy_r| \le \int_s^t |D^\alpha_{s+}\,\left(g(x_r)z_r\right)\,D^{1-\alpha}_{t-}y_{t-}(r)|dr. \tag{17}$$

From (1) we get

$$\left|D^\alpha_{s+}\left(g(x_r)z_r\right)\right| \le k\,\left(\|g\|_\infty\,\|z\|_{s,r,\infty}(r-s)^{-\alpha}+\int_s^r\frac{|g(x_r)z_r-g(x_\theta)z_\theta|}{|r-\theta|^{\alpha+1}}d\theta\right).$$

Now if $0\le s\le r\le t\le T$, then

$$\begin{aligned}\int_s^r\frac{|g(x_r)z_r-g(x_\theta)z_\theta|}{|r-\theta|^{\alpha+1}}d\theta &\le k\,\|g\|_\infty\int_s^r\|z\|_{s,r,\beta}|r-\theta|^{\beta-\alpha-1}d\theta\\ &\quad+k\,\|g'\|_\infty\int_s^r\|z\|_{s,r,\infty}\|x\|_{s,r,\beta}|r-\theta|^{\beta-\alpha-1}d\theta\\ &\le k\left(\|g\|_\infty\,\|z\|_{s,t,\beta}+\|g'\|_\infty\,\|z\|_{s,r,\infty}\|x\|_{s,r,\beta}\right)|r-s|^{\beta-\alpha}.\end{aligned}$$

Therefore

$$\begin{aligned}&\left|D^\alpha_{s+}\left(g(x_r)z_r\right)\right| \le k(\|g\|_\infty\,\|z\|_{s,r,\infty}(r-s)^{-\alpha}\\ &+\left(\|g\|_\infty\,\|z\|_{s,t,\beta}+\|g'\|_\infty\,\|z\|_{s,r,\infty}\|x\|_{s,r,\beta}\right)|r-s|^{\beta-\alpha}).\end{aligned} \tag{18}$$

Substituting (18) and (4) into (17) yields

$$\begin{aligned}|\int_s^t g(x_r)z_r dy_r| \le k\|y\|_\beta\Bigg(&\|g\|_\infty\,\|z\|_{s,t,\infty}(t-s)^\beta\\ &+\left(\|g\|_\infty\,\|z\|_{s,t,\beta}+\|g'\|_\infty\,\|z\|_{s,t,\infty}\|x\|_{s,t,\beta}\right)(t-s)^{2\beta}\Bigg).\end{aligned}$$

Consequently, is $t-s\le\Delta$, applying (16) yields

$$\begin{aligned}\|z\|_{s,t,\beta} \le k\|y\|_\beta\Big\{&\|g\|_\infty\,\|z\|_{s,t,\infty}\\ &+\left(\|g\|_\infty\,\|z\|_{s,t,\beta}+\|g'\|_\infty\,\|z\|_{s,t,\infty}\|x\|_{s,t,\beta}\right)\Delta^\beta\Big\}\\ \le k\|y\|_\beta\Big\{&\|g\|_\infty\,\|z\|_{s,t,\infty}\\ &+\left(\|g\|_\infty\,\|z\|_{s,t,\beta}+\|g'\|_\infty\,\|f\|_\infty\,\|y\|_\beta\|z\|_{s,t,\infty}\right)\Delta^\beta\Big\}.\end{aligned}$$

Supposse that Δ is sufficiently small such that

$$\Delta \leq \left(\frac{1}{2k\|g\|_\infty \|y\|_\beta} \right)^{\frac{1}{\beta}} . \tag{19}$$

Then we have

$$\|z\|_{s,t,\beta} \leq 2k\|y\|_\beta \|z\|_{s,t,\infty} \left(\|g\|_\infty + \|g'\|_\infty \|f\|_\infty \|y\|_\beta \Delta^\beta \right) .$$

This implies that

$$\|z\|_{s,t,\infty} \leq |z_s| + k\|y\|_\beta \Delta^\beta \|z\|_{s,t,\infty} \left(\|g\|_\infty + \|g'\|_\infty \|f\|_\infty \|y\|_\beta \Delta^\beta \right) .$$

If Δ satisfies

$$\|g\|_\infty \Delta^\beta + \|g'\|_\infty \|f\|_\infty \|y\|_\beta \Delta^{2\beta} \leq \frac{1}{2k\|y\|_\beta} \tag{20}$$

then we have

$$\|z\|_{s,t,\infty} \leq 2|z_s|$$

Hence,

$$\sup_{0 \leq r \leq t} |z_r| \leq 2 \sup_{0 \leq r \leq s} |z_r|. \tag{21}$$

As before, divide the interval $[0, T]$ into $n = [T/\Delta] + 1$ subintervals, and use the estimate (21) in every interval to obtain

$$\|z\|_{0,T,\infty} \leq 2^n |z_0| . \tag{22}$$

Notice that for (20) to hold it suffices that

$$\begin{aligned} \Delta^\beta \|y\|_\beta &\leq \frac{\sqrt{\|g\|_\infty^2 + \frac{2}{k}\|g'\|_\infty \|f\|_\infty} - \|g\|_\infty}{2\|g'\|_\infty \|f\|_\infty} \\ &= \frac{1}{k\left(\sqrt{\|g\|_\infty^2 + \frac{2}{k}\|g'\|_\infty \|f\|_\infty} + \|g\|_\infty \right)}. \end{aligned}$$

If we choose

$$\Delta = \left[k\|y\|_\beta \max\left(\|f'\|_\infty , \|g\|_\infty + \sqrt{\|g'\|_\infty \|f\|_\infty} \right) \right]^{-\frac{1}{\beta}} ,$$

then (22) yields

$$\|z\|_{0,T,\infty} \leq 2^{1+kT\left[\|f'\|_\infty \vee \left(\|g\|_\infty + \sqrt{\|g'\|_\infty \|f\|_\infty} \right) \right]^{1/\beta} \|y\|_{0,T,\beta}^{1/\beta}} |z_0| .$$

The proof is now complete.

Suppose now that we have two differential equations of the form

$$x_t = x_0 + \int_0^t f(x_s) dy_s,$$

and

$$\tilde{x}_t = \tilde{x}_0 + \int_0^t \tilde{f}(\tilde{x}_s) \tilde{y}_s \,,$$

where y and $\widetilde{y}$ are Hölder continuous functions of order $\beta > 1/2$, and f and $\widetilde{f}$ are two functions which are continuously differentiable with locally Hölder continuous derivatives of order $\lambda > \frac{1}{\beta} - 1$. Then, we have the following estimate.

Theorem 4. *Suppose in addition that f is twice continuously differentiable and f'' is bounded. Then there is a constant k such that*

$$\sup_{0\le r\le T} |x_r - \tilde{x}_r| \le k 2^{k D^{1/\beta} \|y\|_{0,T,\beta}^{1/\beta} T} \times \Big\{ |x_0 - \tilde{x}_0| + \|y\|_{0,T,\beta} \left[\|f - \tilde{f}\|_\infty + \|x\|_{0,T,\beta} \|f' - \tilde{f}'\|_\infty \right] + \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|x\|_{0,T,\infty} \right] \|y - \tilde{y}\|_{0,T,\beta} \Big\},$$

where

$$D = \|f'\|_\infty \vee \left(\|f'\|_\infty + \|f''\|_\infty (\|x\|_{0,T,\beta} + \|\tilde{x}\|_{0,T,\beta}) T^\beta \right).$$

Remark 5. The above inequality is valid only when each term appeared on the right hand side is finite.

Proof. Fix $s, t \in [0, T]$. Set

$$x_t - \tilde{x}_t - (x_s - \tilde{x}_s) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_s^t \left[f(x_r) - f(\tilde{x}_r) \right] dy_r$$

$$I_2 = \int_s^t \left[f(\tilde{x}_r) - \tilde{f}(\tilde{x}_r) \right] dy_r$$

$$I_3 = \int_s^t \tilde{f}(\widetilde{x}_r) d\left[y_r - \tilde{y}_r \right].$$

The terms I_2 and I_3 can be estimated easily. In fact, we have

$$|I_2| \le k \|y\|_\beta \left[\|f - \tilde{f}\|_\infty (t-s)^\beta + \|f' - \tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta} (t-s)^{2\beta} \right]$$

and

$$|I_3| \leq k\|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty (t-s)^\beta + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right] ,$$

where $\|y\|_\beta = \|y\|_{0,T,\beta}$ and $\|y-\tilde{y}\|_\beta = \|y-\tilde{y}\|_{0,T,\beta}$. The term I_1 is a little more complicated.

$$\begin{aligned}
|I_1| &\leq \int_s^t |D^\alpha_{s+}\left[f(x_r)-f(\tilde{x}_r)\right]||D^{1-\alpha}_{t-}y_{t-}(r)|dr \\
&\leq k\int_s^t \|y\|_{s,t,\beta}(t-r)^{\alpha+\beta-1}\left[|f(x_r)-f(\tilde{x}_r)|(r-s)^{-\alpha}\right. \\
&\quad +\|f'\|_\infty\|x-\tilde{x}\|_{s,r,\beta}(r-s)^{\beta-\alpha} \\
&\quad \left.+\|f''\|_\infty\|x-\tilde{x}\|_{s,r,\infty}\left[\|x\|_{s,r,\beta}+\|\tilde{x}\|_{s,r,\beta}\right](r-s)^{\beta-\alpha}\right]dr \\
&\leq k\|y\|_\beta\left\{\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\infty}(t-s)^\beta+\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\beta}(t-s)^{2\beta}\right. \\
&\quad \left.+\|f''\|_\infty\|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^{2\beta}\right\} .
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x-\tilde{x}\|_{s,t,\beta} \leq k\|y\|_\beta & \left\{\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\infty}+\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right. \\
&+\|f''\|_\infty\|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^\beta \\
&\left.+\|f-\tilde{f}\|_\infty+\|f'-\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right\} \\
&+k\|y-\tilde{y}\|_\beta\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right].
\end{aligned}$$

Rearrange it to obtain

$$\begin{aligned}
\|x-\tilde{x}\|_{s,t,\beta} \leq k(1-k\|f'\|_\infty\|y\|_\beta(t-s)^\beta)^{-1} & \Bigg\{\|y\|_\beta\Bigg[\|f'\|_\infty\|x-\tilde{x}\|_{s,t,\infty} \\
&+\|f''\|_\infty\|x-\tilde{x}\|_{s,t,\infty}\left[\|x\|_{s,t,\beta}+\|\tilde{x}\|_{s,t,\beta}\right](t-s)^\beta \\
&+\|f-\tilde{f}\|_\infty+\|f'-\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\Bigg] \\
&+k\|y-\tilde{y}\|_\beta\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{s,t,\beta}(t-s)^\beta\right]\Bigg\}.
\end{aligned}$$

Set $\Delta = t-s$, and $A = k\|f'\|_\infty\|y\|_\beta$. Then

$$
\begin{aligned}
\|x-\tilde{x}\|_{s,t,\infty} &\le |x_s-\tilde{x}_s| + \|x-\tilde{x}\|_{s,t,\beta}(t-s)^\beta \\
&\le |x_s-\tilde{x}_s| + k(1-A\Delta^\beta)^{-1}\Delta^\beta \Bigg\{ \|y\|_\beta \Bigg[\|f'\|_\infty \|x-\tilde{x}\|_{s,t,\infty} \\
&\quad + \|f''\|_\infty \|x-\tilde{x}\|_{s,t,\infty} \left[\|x\|_{s,t,\beta} + \|\tilde{x}\|_{s,t,\beta}\right] \Delta^\beta \\
&\quad + \|f-\tilde{f}\|_\infty + \|f'-\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \Bigg] \\
&\quad + k\|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \Bigg\}.
\end{aligned}
$$

Denote

$$
B = k\|y\|_\beta \left(\|f'\|_\infty + \|f''\|_\infty (\|x\|_{0,T,\beta} + \|\tilde{x}\|_{0,T,\beta}) T^\beta \right) .
$$

Then

$$
\begin{aligned}
\|x-\tilde{x}\|_{s,t,\infty} &\le \left(1-(1-A\Delta^\beta)^{-1}\Delta^\beta B\right)^{-1} \\
&\quad \times \Bigg\{ |x_s-\tilde{x}_s| + k(1-A\Delta^\beta)^{-1}\Delta^\beta \\
&\quad \times \Bigg[\|y\|_\beta \left[\|f-\tilde{f}\|_\infty + \|f'-\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \\
&\quad + \|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \Bigg] \Bigg\}.
\end{aligned}
$$

Let Δ satisfy

$$
A\Delta^\beta \le 1/3 , \quad B\Delta^\beta \le 1/3
$$

Namely, we take

$$
\Delta = \left(\frac{1}{3\,(A \vee B)} \right)^{1/\beta} .
$$

Then

$$
\|x-\tilde{x}\|_{s,t,\infty} \le 2\left[|x_s-\tilde{x}_s| + C\Delta^\beta \right] ,
$$

where

$$
\begin{aligned}
C = \frac{3}{2}k \Bigg[& \|y\|_\beta \left[\|f-\tilde{f}\|_\infty + \|f'-\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \\
& + \|y-\tilde{y}\|_\beta \left[\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty \|\tilde{x}\|_{s,t,\beta}\Delta^\beta \right] \Bigg].
\end{aligned}
$$

Applying the above estimate recursively we obtain

$$\sup_{0\le r\le T}|x_r-\tilde{x}_r|\le 2^n\left[|x_0-\tilde{x}_0|+C\Delta^\beta\right],$$

where $n=[T/\Delta]+1$. Or we have

$$\sup_{0\le r\le T}|x_r-\tilde{x}_r|\le k2^{k(\|f'\|_\infty\vee(\|f'\|_\infty+\|f''\|_\infty(\|x\|_{0,T,\beta}+\|\tilde{x}\|_{0,T,\beta})T^\beta))^{1/\beta}\|y\|_{0,T,\beta}^{1/\beta}T}$$
$$\times\left\{|x_0-\tilde{x}_0|+\|y\|_{0,T,\beta}\left[\|f-\tilde{f}\|_\infty+\|\tilde{x}\|_{0,T,\beta}\|f'-\tilde{f}'\|_\infty\right]\right.$$
$$\left.+\left[\|\tilde{f}\|_\infty+\|\tilde{f}'\|_\infty\|\tilde{x}\|_{0,T,\infty}\right]\|y-\tilde{y}\|_{0,T,\beta}\right\}.$$

4 Stochastic differential equations driven by a fBm

Let $B=\{B_t,t\ge 0\}$ be an m-dimensional fractional Brownian motion (fBm) with Hurst parameter $H>1/2$. That is, B is a Gaussian centered process with the covariance function $E(B_t^iB_s^j)=R_H(t,s)\delta_{ij}$, where

$$R_H(t,s)=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

Consider the stochastic differential equation on $\mathbb{R}^d$

$$X_t=X_0+\int_0^t\sigma(X_s)dB_s, \qquad (1)$$

where X_0 is a fixed d-dimensional random variable and the stochastic integral is is a path-wise Riemann-Stieltjes integral. ([1]). This equation has a unique solution (see [2] and [3]) provided σ is continuously differentiable, and σ' is bounded and Hölder continuous of order $\lambda>\frac{1}{H}-1$.

Then, using the estimate (2) in Theorem 2 we obtain the following estimate for the solution of Equation (1), if we choose $\beta\in\left(\frac{1}{2},H\right)$. Notice that $\frac{1}{\beta}<2$.

$$\sup_{0\le t\le T}|X_t|\le 2^{1+kT\left(\|\sigma'\|_\infty\vee|\sigma(0)|\right)\|B\|_{0,T,\beta}^{1/\beta}}\left(|X_0|+1\right). \qquad (2)$$

If σ is bounded and $\|\sigma'\|\ne 0$ we can make use of the estimate (3) and we obtain

$$\sup_{0\le t\le T}|X_t|\le|X_0|+k\|\sigma\|_\infty\left(T^\beta\|B\|_{0,T,\beta}^{\frac{1}{\beta}}\vee T\|\sigma'\|_\infty^{\frac{1-\beta}{\beta}}\|B\|_{0,T,\beta}^{\frac{1}{\beta}}\right). \qquad (3)$$

These estimates improve those obtained by Nualart and Răşcanu in [3] based on a suitable version of Gronwall's lemma. The estimates (2) and (3) allow us to establish the following integrability properties for the solution of Equation (1).

Theorem 6. *Consider the stochastic differential equation (1), and assume that $E(|X_0|^p) < \infty$ for all $p \geq 2$. If σ' is bounded and Hölder continuous of order $\lambda > \frac{1}{H} - 1$, then*

$$E\left(\sup_{0\leq t\leq T} |X_t|^p\right) < \infty \tag{4}$$

for all $p \geq 2$. If furthermore σ is bounded and $E\left(\exp(\lambda|X_0|^\gamma)\right) < \infty$ for any $\lambda > 0$ and $\gamma < 2H$, then

$$E\left(\exp \lambda\left(\sup_{0\leq t\leq T} |X_t|^\gamma\right)\right) < \infty \tag{5}$$

for any $\lambda > 0$ and $\gamma < 2H$.

In [4] Nualart and Saussereau have proved that the random variable X_t belongs locally to the space $\mathbb{D}^\infty$ if the function σ is infinitely differentiable and bounded together with all its partial derivatives. As a consequence, they have derived the absolute continuity of the law of X_t for any $t > 0$ assuming that the initial condition is constant and the vector space spanned by $\{(\sigma^{ij}(x_0))_{1\leq i\leq d}, 1 \leq j \leq m\}$ is $\mathbb{R}^d$.

Applying Theorem 3.2 we can show that the derivatives of X_t possess moments of all orders, and we can then derive the C^∞ property of the density. Define the matrix

$$\alpha(x) = \left(\sum_{l=1}^m \sigma^{il}(x)\sigma^{jl}(x)\right)_{1\leq i,j\leq d}.$$

Theorem 7. *Consider the stochastic differential equation (1), with constant initial condition x_0. Suppose that $\sigma(x)$ is bounded infinitely differentiable with bounded derivatives of all orders, and $\alpha(x)$ is uniformly elliptic. Then, for any $t > 0$ the probability law of X_t has an C^∞ density.*

Proof. Let us first show that X_t belongs to the space $\mathbb{D}^\infty$. From Equation (34) of [4] we have

$$D_r^j X_t^i = \sigma^{ij}(X_r) + \sum_{k=1}^d \int_0^t \sum_{l=1}^m \partial_k \sigma^{il}(X_u) D_r^j X_u^k dB_u^l. \tag{6}$$

As a consequence, (15) applied to the system formed by the equations (1) and (6) yields

$$\left|D_r^j X_t^i\right| \leq 2^{1+kT\left[\|\sigma\|_\infty \vee \left(\|\sigma'\|_\infty + \sqrt{\|\sigma''\|_\infty \|\sigma\|_\infty}\right)\right]^{\frac{1}{\beta}} \|B\|_{0,T,\beta}^{1/\beta}} \|\sigma\|_\infty.$$

This implies that for all $p \geq 2$

$$E\left(\left|\sum_{j=1}^m \int_0^t \int_0^t D_s^j X_t^i D_r^j X_t^i |r-s|^{2H-2} dsdr\right|^p\right) < \infty,$$

and the random variable X_t^i belongs to the Sobolev space $\mathbb{D}^{1,p}$ for all $p \geq 2$. In a similar way, writing down the linear equations satisfied by the iterated derivatives, one can show that X_t^i belongs to the Sobolev space $\mathbb{D}^{k,p}$ for all $p \geq 2$ and $k \geq 2$.

In order to show the nongeneracy of the density we use the notation of [4] and follow the idea of [7]. By Itô's formula we have

$$\begin{aligned} D_r^j X_t^i D_{r'}^j X_t^{i'} &= \sigma^{ij}(X_r)\sigma^{i'j}(X_{r'}) + \sum_{k=1}^d \sum_{l=1}^m \int_0^t \partial_k \sigma^{i'l}(X_u) D_r^j X_u^i D_{r'}^j X_u^k dB_u^l \\ &+ \sum_{k=1}^d \sum_{l=1}^m \int_0^t \partial_k \sigma^{il}(X_u) D_{r'}^j X_u^{i'} D_r^j X_u^k dB_u^l \,. \end{aligned}$$

Denote

$$\begin{aligned} \beta_l(X_u) &= \left(\partial_k \sigma^{i'l}(X_u)\right)_{1\leq i',k\leq d} \\ \Gamma_t &= \left(\sum_{j=1}^m \int_0^t \int_0^t |r-r'^{2H-2} D_r^j X_t^i D_{r'}^j X_t^{i'} dr dr'\right)_{1\leq i,i'\leq d} . \end{aligned}$$

Then $H(2H-1)\Gamma_t$ is the Malliavin covariance matrix of the random vector X_t, and we need to show that Γ_t^{-1} is in L_p for any $p \geq 1$ and for all $t > 0$. We have

$$\Gamma_t = \alpha_0 + \sum_{l=0}^m \int_0^t \left(\beta_l(X_u)\Gamma_u + \Gamma_u \beta_l^T(X_u)\right) dB_u^l,$$

where

$$\alpha_0 = \sum_{j=1}^m \int_0^t \int_0^t |r-r'|^{2H-2} \sigma_{ij}(X_r)\sigma_{i'j}(X_{r'}) dr dr'.$$

By using Itô formula again we have

$$\Gamma_t^{-1} = \alpha_0^{-1} - \sum_{l=0}^m \int_0^t \left(\Gamma_u^{-1}\beta_l(X_u) + \beta_l^T(X_u)\Gamma_u^{-1}\right) dB_u^l. \tag{7}$$

By the estimate (15) applied to the equations (1) and (7) , we see that Γ_t^{-1} is in L_p for any $p \geq 1$. This proves the theorem (see [5]).

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