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Differential equations in Banach spaces

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Let H be a fixed Hilbert space and $B(H, H)$ be the Banach space of bounded linear operators from H to H with the uniform operator topology. Oscillation criteria are obtained for the operator differential equation

$$\frac{d}{dt}\left(A(t)\,\frac{dY}{dt}\right) + C(t, Y)Y = 0 \ , \quad t \geq 0 \ ,$$

where the coefficients A, C are linear operators from $B(H, H)$ to $B(H, H)$, for each $t \geq 0$. A solution $Y : R^+ \to B(H, H)$ is said to be oscillatory if there exists a sequence of points $t_i \in R^+$, so that $t_i \to \infty$ as $i \to \infty$, and $Y(t_i)$ fails to have a bounded inverse. The main theorem states that a solution Y is oscillatory if an associated scalar differential equation is oscillatory.

1. Introduction

In this paper we study those aspects of the qualitative behaviour of solutions of second order differential equations in Banach spaces involving the notions of oscillation and non-oscillation. The basic tool in the analysis is a generalization of Picone's identity [7], [8].

Some other papers pertinent to the subject under consideration here are [4], [5], [6].

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bounded linear operators from H to H with the uniform operator topology. Consider the differential equation

$$TY = \frac{d}{dt}\left(A(t)\frac{dY}{dt}\right) + C(t, Y)Y = 0$$

for $t \in R^+ = \{t : t > 0\}$. Solutions $Y(t)$ are functions from R^+ into $B(H, H)$. We assume that $C(t, Y)$ is a linear map from $B(H, H)$ to $B(H, H)$, for each $t \in R^+$, and further, that $A(t)$, $C(t, Y)$ are continuous in the uniform operator topology as functions from R^+ to $B(H, H)$, and from $R^+ \times B(H, H)$ to $B(H, H)$ respectively. We also assume for each $t \geq 0$ and each $Y \in B(H, H)$ that $C(t, Y)$ is self-adjoint $(C = C^*)$, and $A(t)$ is positive definite, that is, $(A(t)e, e) > 0$ for all $t > 0$ and for all $0 \neq e \in H$, where $(\;,\;)$ is the inner product in H .

Derivatives of $Y(t)$ are computed in the uniform operator topology, that is,

$$\lim_{\delta \to 0}\left\|\frac{Y(t+\delta)-Y(t)}{\delta} - Y'(t)\right\| = 0 .$$

Under the above assumptions the initial value problem:

$$TY = 0 \qquad t \in R^+ ,$$

$$Y(t_0) = Y_0 , \quad \frac{d}{dt}Y(t_0) = Y_1 , \quad t_0 \in R^+$$

has a unique solution [4]. Motivated by the finite dimensional case we introduce the following definitions.

DEFINITION. A solution $Y(t)$ is said to be *nonsingular* at t_0 if $Y(t_0)$ has a bounded inverse. $Y(t)$ is said to be *oscillatory* if there exists a sequence $\{t_i\} \in R^+$ such that $t_i \to \infty$ as $i \to \infty$, and each t_i is a singular point of Y . This definition is a natural extension of the finite dimensional case, where it is customary to say that matrix solutions of (1) are nonsingular at t_0 if the determinant of $Y(t_0)$ is not zero.

DEFINITION. A solution $Y(t)$ of equation (1) is said to be *prepared* if:

(1) $Y^*(t)A(t)\frac{d}{dt}Y(t) = \left(\frac{d}{dt}Y(t)\right)^* A(t)Y(t)$ for all $t \le 0$;

(2) there exists a common nonzero vector e which belongs to the ranges of the operators $\{Y(t) : t$ is a nonsingular point$\}$.

This definition ensures, as pointed out by Noussair and Swanson [6], for example, that every solution of the scalar equation $Y'' + Y = 0$ is oscillatory, as is well known [3]. However, the nonprepared matrix solution

$$Y(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

is obviously non-oscillatory. Accordingly, the prepared hypothesis on Y is needed in order that an analog of the classical theory of oscillation $[H = R^1]$ can be developed for operator equations.

Condition (2) above is satisfied in the finite dimensional case since a nonsingular operator is onto. We could replace condition (2) by requiring that $Y(t)$ has a bounded inverse defined on the whole space for every nonsingular point t . However, this will restrict the class of oscillatory solutions as the following example shows.

EXAMPLE 1. Let $H = l^2$ the Hilbert space of square-summable sequences $x = \{x_i\}_{i=1}^{\infty}$. Let A be the right shift operator on l^2 , that is $A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$. In (1) take $A(t) = C(t, Y) = I$, the identity operator, $Y(0) = 0$ and $\frac{d}{dt}Y(0) = A$. Then the solution to (1) is $Y(t) = (\sin t)A$. Now if $\sin t$ is not zero then $Y(t)$ has a bounded inverse. But the range of $Y(t)$ is a proper closed subspace of l^2 . However, $Y(t)$ is a prepared solution according to the definition above, as can be easily verified.

2. Oscillation theorems

Let I be the closed interval $[t_0, t_1]$, and let D_0 denote the set of all vector functions $u \in C^1(I)$, with range in H such that $u(t_0) = u(t_1) = 0$.

THEOREM 1. *If*

(1) $Y(t)$ *is a prepared solution of* $Y^*TY \le 0$, *and* $e \in H$ *is in the range of* $Y(t)$ *for all* $t \in I$; *and*

(2) *there exists a nontrivial function* $\phi(t) \in C^1(I)$ *such that*

$$\int_{t_0}^{t_1} \left[(A(t), e, e)\left(\frac{d\phi}{dt}\right)^2 - (C(t, Y)e, e)\phi^2\right]dt \le 0,$$

then

(a) $Y(t)$ *has a singular point in* I; *and*

(b) either $Y(t)$ *has a singular point in the open interval* (t_0, t_1), *or* $\phi(t)e = Y(t)c$ *on* I *for some constant vector* $c \ne 0$.

Proof. If $Y(t)$ has no singular point in I, there exists a unique vector $\omega \in D_0$ satisfying $\phi(t)e = Y(t)\omega(t)$ identically in I. The following identity, a generalization of Picone's identity [8], can be easily verified by differentiation:

$$(2)\quad -\frac{d}{dt}\left(A(t)\left(\frac{dY}{dt}\right)\omega, Y\omega\right) = -(A(t)e, e)\left(\frac{d\phi}{dt}\right)^2 - (C(t, Y)e, e)\phi^2 - (Y^*TY\omega, \omega)$$
$$- \left(\left(Y^*A(t)\frac{dY}{dt} - \left(\frac{dY}{dt}\right)^* A(t)Y\right)\frac{d\omega}{dt}, \omega\right) + \left(A(t)Y\frac{d\omega}{dt}, Y\frac{d\omega}{dt}\right).$$

Since Y is a prepared solution of $Y^*TY \le 0$ and $\omega(t_0) = \omega(t_1) = 0$, integration of (2) over I and use of Green's formula yields

$$(3)\quad \int_{t_0}^{t_1} \left[(A(t)e, e)\left(\frac{d\phi}{dt}\right)^2 - (C(t, Y)e, e)\phi^2\right]dt \ge \int_{t_0}^{t_1} \left(A(t)Y\frac{d\omega}{dt}, Y\frac{d\omega}{dt}\right)dt.$$

Since $A(t)$ is positive-definite, by the assumptions made on the coefficients of the operator T, inequality (3) and hypothesis (2) imply that $\frac{d\omega}{dt} = 0$ identically in I, that is, $\phi(t)e = Y(t)\omega(t) = Y(t)c$

identically in I, for some constant vector c, with $c \neq 0$ since $\phi(t)$ is nontrivial by the hypothesis (2). Since $\phi(t)$ and Y are continuous in I and $\phi(t_0) = \phi(t_1) = 0$, the equality $\frac{d\omega}{dt} = 0$ implies that t_0, t_1 are singular points of Y. This proves conclusion *(a)* of the theorem. The proof of the "strong conclusion" *(b)* is similar to the proof in the finite dimensional case [5].

THEOREM 2. *Every prepared solution of* $Y^*TY \leq 0$ *is oscillatory in* R^+ *if the scalar equation*

$$(4) \qquad lu = \frac{d}{dt}\left[(A(t)e,\ e)\frac{du}{dt}\right] + (C(t,\ Y)e,\ e)u = 0$$

is oscillatory in R^+ *for every unit vector* $e \in H$, *and for every nonsingular operator* $Y \in B(H,\ H)$.

Proof. Let $Y(t)$ be a prepared solution of $Y^*TY \leq 0$. If $Y(t)$ is not oscillatory then there exists a number $r > 0$ such that $Y(t)$ has no singular points for $t \geq r$. Since Y is prepared, there exists a vector e in the range of $Y(t)$ for all $t \geq 0$.

By hypothesis, equation (4) is oscillatory in $[r,\ \infty)$. Hence there exist points $r < t_0 < t_1$, a function $\phi(t)$ defined for $t_0 \leq t \leq t_1$ such that $\phi(t_0) = \phi(t_1) = 0$ and $l\phi = 0$ in $[t_0,\ t_1]$. Then

$$(5) \qquad \int_{t_0}^{t_1} (A(t)e,\ e)\left(\frac{d\phi}{dt}\right)^2 - (C(t,\ Y)e,\ e)\phi^2 dt = 0 .$$

Theorem 2 is therefore a consequence of Theorem 1.

Theorem 2 extends all oscillation criteria for ordinary differential equations to equation (1).

COROLLARY 3. *Every prepared solution of equation* (1) *is oscillatory in* $[a,\ \infty)$ *if, for every unit vector* $e \in H$ *and for all operators* Y *such that* $Y(t)$ *has a bounded inverse for sufficiently large* t, *one of the following criteria is satisfied:*

$$(1) \quad \int_0^\infty \frac{1}{(A(t)e,e)}\, dt = \infty , \quad \int_0^\infty (C(t,\ Y(t))e,\ e)dt = \infty ,$$

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(2) $(A(t)e,\ e) \le K$, $(C(t,\ Y(t))e,\ e) \ge 0$ *for large* t , *and*

$$\limsup_{t\to\infty} t\int_t^\infty (C(t,\ Y(t))e,\ e)dt > 1 ,$$

(3) $(A(t)e,\ e) \le K$, $4t^2(C(t,\ Y(t))e,\ e) > (A(t)e,\ e)$ *for sufficiently large* t , *and*

$$\int_0^\infty t\left|(C(t,\ Y(t))e,\ e) - \frac{(A(t)e,e)}{4t^2}\right|dt = \infty .$$

Proof. The proof follows from Theorem 2 by applying known oscillation criteria, [2], [3], [5], for the ordinary equation (4).

A recent result of Hayden and Howard, [1], is criterion (1) of Corollary 3 when $A(t) = I$, the identity operator.

When H is finite dimensional, the following stronger version of Theorem 2 is valid, and gives a new oscillation theorem for matrix differential inequalities.

THEOREM 4. *Every prepared solution of the matrix differential inequality* $Y^*TY \le 0$ *is oscillatory in* R^+ *if there exists a non-zero vector* $e \in H$ *such that*

$$lu = \frac{d}{dt}\left[(A(t)e,\ e)\frac{du}{dt}\right] + (C(t,\ Y)e,\ e) = 0 \tag{4}$$

is oscillatory in R^+ , *for every nonsingular matrix* Y *in* $B(H,\ H)$.

Proof. If $Y(t)$ is a prepared solution of $Y^*TY \le 0$ which has no singular point for $t \ge r$, for some r , then $Y(t)$ has a bounded inverse defined on the whole space H for $t \ge r$. In particular e belongs to the range of Y for all $t \ge r$.

The rest of the proof is identical to the proof of Theorem 2.

We shall give now a simple application of Theorem 1. Consider the equation

$$TY = \frac{d^2Y}{dt^2} + C(t)Y(t) = 0 , \tag{5}$$

where $C(t)$ is a self-adjoint operator for each $t \ge 0$. Let P and Q be two bounded operators on H such that P^*Q is self-adjoint. Let Y

be the solution of the initial value problem

$$(6) \qquad TY = 0 \text{ in } R^+ ,$$
$$Y(t_0) = P , \quad \frac{dY}{dt}(t_0) = Q , \quad t_0 \in R^+ .$$

Then $Y^*(t_0) \frac{d}{dt} Y(t_0) = \left(\frac{d}{dt} Y^*(t_0)\right) Y(t_0)$. But, from equation (5), it is easy to see that

$$Y^*(t) \frac{d}{dt} Y(t) - \left(\frac{d}{dt} Y^*(t)\right) Y(t) = K ,$$

where K is a constant operator. Hence

$$Y^*(t) \frac{d}{dt} Y(t) - \left(\frac{d}{dt} Y^*(t)\right) Y(t) \equiv 0 \quad \text{for all} \quad t \geq t_0 .$$

THEOREM 5. *If there exists a nonzero vector* $e \in H$ *such that the scalar equation*

$$lu = \frac{d^2u}{dt^2} + (C(t)e, e)u = 0$$

is oscillatory in R^+ *, then the solution* Y *of the initial value problem* (6) *has the property that there exists a sequence* $\{t_i\}$ *,* $t_i \to \infty$ *, such that either:*

(a) $Y(t_i)$ *has no bounded inverse, or*

(b) the range of $Y(t_i)$ *is not the whole space.*

Proof. If such sequence doesn't exist, then there is a $t_1 > 0$ such that $Y(t)$ has a bounded inverse which is defined on the whole space for $t \geq t_1$. Hence $e \in$ range of $Y(t)$ for all $t \geq t_1$. Since lu is oscillatory in $[t_1, \infty)$ by hypothesis, there exist points $t'' > t' \geq t_1$ and a function $\phi \in C'(t', t'')$ such that $l\phi = 0$ and $\phi(t') = \phi(t'') = 0$. Hence hypotheses (1) and (2) of Theorem 1 are satisfied and we obtain the contradiction that $Y(t)$ has a singular point in $[t_1, \infty)$. This completes the proof.

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