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DIFFERENTIAL EQUATIONS ON CLOSED SUBSETS OF A BANACH SPACE

BY

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ABSTRACT. The problem of existence of solutions to the initial value problem $x' = f(t, x)$, $x(t_0) = x_0 \in F$, where $f \in C[[t_0, t_0 + a] \times F, E]$, F is a locally closed subset of a Banach space E is considered. Nonlinear comparison functions and dissipative type conditions in terms of Lyapunov-like functions are employed. A new comparison theorem is established which helps in surmounting the difficulties that arise in this general setup.

1. Introduction. Recently, in an interesting paper, Martin [4] considers the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0,$$

in a closed subset of a Banach space employing a dissipative type condition in terms of a generalized pairing, namely,

$$(f(t, x) - f(t, y), x - y)_+ \leqslant L\|x - y\|^2. \tag{1.1}$$

The results obtained in [4] crucially depend on the properties of $(x, y)_+$ and the linearity of the comparison function $g(t, u) = Lu$ and are technical. Even to replace the right-hand side of (1.1) by a nonlinear comparison function poses a difficult problem.

In this paper, we extend the existence results in [4], [5] using nonlinear comparison functions and dissipative type conditions in terms of Lyapunov-like functions. A new comparison result (Theorem 2.4) needed to surmount the difficulties is established and employed to prove the general results. The use of Lyapunov-like functions instead of the norm also raises nontrivial problems. For recent results dealing with similar problems see the references in [2], [3], [4].

2. Preliminary results. Let E be a real Banach space and let $\|\cdot\|$ denote the norm on E. Let $F \subset E$ be a locally closed set, that is, for each $x_0 \in F$, there exists a $b > 0$ such that $F_0 = F \cap B(x_0, b)$ is closed in E where $B(x_0, b) = [x \in E: \|x - x_0\| \leqslant b]$. Let R^+ denote the nonnegative real line and let

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$t_0 \in R^+$. We consider the differential equation

$$x' = f(t, x), \quad x(t_0) = x_0 \in F. \tag{2.1}$$

We list the following assumptions which we use frequently.

(A_1) $f \in C[[t_0, t_0 + a] \times F, E]$ and the numbers $a, b > 0, M > 1$ are chosen such that $\|f(t, x)\| \leq M - 1$, on $[t_0, t_0 + a] \times F_0$.

(A_2) $\lim_{h \to 0} h^{-1} d[x + hf(t, x), F] = 0, (t, x) \in [t_0, t_0 + a] \times F$, where $d(x, F) = \inf[\|x - y\| : y \in F]$.

(A_3) $g \in C[[t_0, t_0 + a] \times R^+, R], g(t, 0) \equiv 0$ and $u \equiv 0$ is the unique solution of

$$u' = g(t, u), \quad u(t_0) = 0, \tag{2.2}$$

on $[t_0, t_0 + a]$.

(A_4) $V \in C[[t_0, t_0 + a] \times B(x_0, b) \times B(x_0, b), R^+], V(t, x, x) = 0$, $V(t, x, y) > 0$ if $x \neq y$,

$$|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq L[\|x - x_1\| + \|y - y_1\|],$$

if $\{x_n\}$, $\{y_n\}$ are sequences in $B(x_0, b)$ such that $\lim_{n \to \infty} V(t, x_n, y_n) = 0$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$ and for $t \in [t_0, t_0 + a], x, y \in F_0$,

$$\begin{aligned} D^+ V(t, x, y) &= \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x + hf(t, x), y + h(t, y)) - V(t, x, y)] \\ &\leq g(t, V(t, x, y)). \end{aligned} \tag{2.3}$$

(A_5) $V \in C[B(x_0, b), R^+], V(0) = 0, V(x) > 0, x \neq 0$, if $\lim_{n \to \infty} V(x_n) = 0$ with $x_n \in B(x_0, b)$, then $\lim_{n \to \infty} \|x_n\| = 0$ and there exists a mapping $M: B(x_0, b) \times E \to R$ such that:

(a) $M[x, y]$ is upper semicontinuous, i.e., if $\lim_{n \to \infty}(x_n, y_n) = (x, y)$ then $\limsup_{n \to \infty} M[x_n, y_n] \leq M[x, y]$;

(b) $V(x + y) - V(x) \leq M[x, y] + o(\|y\|), x, x + y \in B(x_0, b)$;

(c) $M[x, \lambda y] \leq \lambda M[x, y], \lambda \geq 0, x \in B(x_0, b), y \in E$;

(d) $M[x, y_1 + y_2] \leq M[x, y_1] + N\|x\|\|y_2\|, N > 0, x \in B(x_0, b), y_1, y_2 \in E$.

(e) $M[x - y, f(t, x) - f(t, y)] \leq g(t, V(x - y)), x, y \in F_0, t \in [t_0, t_0 + a]$.

The construction of a sequence of approximate solutions for (2.1) and the proof that the limit function, when it exists, is a solution of (2.1) is assured by the following results.

LEMMA 2.1. *Suppose that assumptions* (A_1) *and* (A_2) *hold. Let* $\{\epsilon_n\}$ *be a sequence of numbers such that* $\epsilon_n \in (0, 1)$ *and* $\lim_{n \to \infty} \epsilon_n = 0$. *Then for each* *positive integer n, problem* (2.1) *has an ϵ_n-approximate solution x_n from $[t_0, t_0 + a]$ into $B(x_0, b)$ in the following sense: There exists a nondecreasing sequence $\{t_i\}$ in $[t_0, t_0 + a]$ such that*

(i) $t_0^n = t_0, t_i^n < t_{i+1}^n$ *if* $t_i^n < t_0 + a$, $t_{i+1}^n - t_i^n \leq \epsilon_n$ *and* $\lim t_i = t_0 + a$;

(ii) $x_n(t_0) = x_0$ *and* $\|x_n(t) - x_n(s)\| \leq M|t - s|, t, s \in [t_0, t_0 + a]$;

(iii) $x_n(t_i^n) \in F_0$ *and $x_n(t)$ is linear on* $[t_i, t_{i+1}]$ *for each i*;

(iv) *if* $t_i^n < t_0 + a$ *and* $t \in (t_i^n, t_{i+1}^n)$, *then* $\|x_n'(t) - f(t_i^n, x_n(t_i^n))\| \leq \epsilon_n$;

(v) *if* $(t, y) \in [t_i^n, t_{i+1}^n] \times F$ *with* $\|y - x_n(t_i^n)\| \leq (t_{i+1}^n - t_i^n)M$, *then* $\|f(t, y) - f(t_i^n, x_n(t_i^n))\| \leq \epsilon_n$.

LEMMA 2.2. *Suppose that the assumptions of Lemma* 2.1 *hold and that* $\lim_{n\to\infty} x_n(t) = x(t)$ *for each* $t \in [t_0, t_0 + a]$. *Then $x(t)$ is a solution for problem* (2.1) *on* $[t_0, t_0 + a)$.

For a proof of Lemmas 2.1 and 2.2 see [4].

LEMMA 2.3. *Let* $g \in C[R^+ \times R^+, R]$ *and let the maximal solution $r(t)$ of the scalar differential equation*

$$u' = g(t, u), \quad u(t_0) = u_0 \geq 0, \qquad t_0 \in R^+, \tag{2.4}$$

exist on $[t_0, \infty)$. *Suppose that* $[t_0, t_1] \subset R^+$. *Then there exists an* $\epsilon_0 > 0$ *such that for* $0 < \epsilon < \epsilon_0$, *the maximal solution $r(t, \epsilon)$ of*

$$u' = g(t, u) + \epsilon, \quad u(t_0) = u_0 + \epsilon,$$

exists on $[t_0, t_1]$ *and* $\lim_{\epsilon\to\infty} r(t, \epsilon) = r(t)$ *uniformly on* $[t_0, t_1]$.

THEOREM 2.1. *Let* $g \in C[R^+ \times R^+, R], m \in C[R^+, R^+]$ *and* $Dm(t) \leq g(t, m(t)), t \in [t_0, \infty) - S$, *where S is a countable subset of* $[t_0, \infty)$ *and D is any one of the Dini derivatives. Suppose that the maximal solution $r(t)$ of* (2.4) *exists on* $[t_0, \infty)$ *and* $m(t_0) \leq u_0$. *Then* $m(t) \leq r(t), t \geq t_0$.

For a proof of Lemma 2.3 and Theorem 2.1 see [1, pp. 13, 15].

THEOREM 2.2. *Let* $m(t) \geq 0$ *be right continuous on* $[t_0, \infty)$ *with isolated discontinuities at* $t_k, k = 1, 2, \ldots, t_k > t_0$, *such that* $|m(t_k) - m(t_k^-)| \leq \lambda_k$, *where* $\Sigma_{k=1}^\infty \lambda_k$ *is convergent. Let* $g \in C[R^+ \times R^+, R], g(t, u)$ *nondecreasing in u for each t, and*

$$Dm(t) \leq g(t, m(t)), \quad t \in [t_k, t_{k+1}], \qquad k = 0, 1, 2, \ldots$$

Then $m(t_0) \leq u_0$ *implies that*

$$m(t) \leq r\left(t, t_0, u_0 + \sum_{k=1}^{\infty} \lambda_k\right), \qquad t \geq t_0,$$

where $r(t, t_0, u_0)$ *is the maximal solution of* (2.4) *existing on* $[t_0, \infty)$.

PROOF. By Theorem 2.1, we have

$$m(t) \leq r_0(t), \quad t \in [t_0, t_1), \tag{2.5}$$

and $m(t) \leq r_1(t)$, $t \in [t_1, t_2)$, where $r_0(t)$ and $r_1(t)$ are the maximal solutions of

$$u' = g(t, u), \tag{2.6}$$

starting at $(t_0, m(t_0))$ and $(t_1, m(t_1))$ respectively. Since $m(t_1) \leq (t_1^-) + \lambda_1$ and $r_1'(t) \leq g(t, r_1(t))$, by Theorem 2.1, it follows that $m(t) \leq r_{11}(t)$, $t \in [t_1, t_2)$, where $r_{11}(t)$ is the maximal solution of (2.6) through $(t_1, m(t_1^-) + \lambda_1)$. By (2.5), $m(t_1^-) \leq r_0(t_1)$ and therefore, again applying Theorem 2.1, we get $m(t) \leq r_{12}(t)$, $t \in [t_1, t_2)$, where $r_{12}(t)$ is the maximal solution of (2.6) through $(t_1, r_0(t_1) + \lambda_1)$. Define a function $\rho_0(t)$ as follows:

$$\rho_0(t) = \begin{cases} r_0(t) + \lambda_1, & t \in [t_0, t_1], \\ r_{12}(t), & t \in [t_1, t_2). \end{cases}$$

Note that $\rho_0(t)$ is well defined. Now, by the monotonicity of $g(t, u)$,

$$\rho_0'(t) = r_0'(t) = g(t, r_0(t)) \leq g(t, r_0(t) + \lambda_1) = g(t, \rho_0(t)), \quad t \in [t_0, t_1],$$

and

$$\rho_0'(t) = r_{12}'(t) = g(t, r_{12}(t)) = g(t, \rho_0(t)), \quad t \in [t_1, t_2).$$

Hence $\rho_0'(t) \leq g(t, \rho_0(t))$, $t \in [t_0, t_2)$, which yields, by Theorem 2.1,

$$\rho_0(t) \leq R_0(t), \quad t \in [t_0, t_2),$$

where $R_0(t)$ is the maximal solution of (2.6) with $R_0(t_0) = m(t_0) + \lambda_1$. Clearly $m(t) \leq \rho_0(t) \leq R_0(t)$, $t \in [t_0, t_2)$. Proceeding in the same way and arguing as before, we obtain $m(t) \leq R_1(t)$, $t \in [t_0, t_3)$, where $R_1(t)$ is the maximal solution of (2.6) through $(t_0, m(t_0) + \lambda_1 + \lambda_2)$. It therefore follows, repeating the arguments successively,

$$m(t) \leq r\left(t, t_0, m(t_0) + \sum_{k=1}^{\infty} \lambda_k\right), \quad t \geq t_0,$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.4) and the proof is complete.

To prove an existence result in a general case, it becomes necessary to construct an appropriate sequence of right continuous functions with isolated discontinuities and then employ Theorem 2.2. The next lemma serves this purpose but employs (A_5) since (A_4) is not strong enough.

LEMMA 2.4. *Let m, n be positive integers and let the sequence* $\{t_k\}$ *be the minimal refinement of the sequences* $\{t_i^n\}$, $\{t_j^m\}$. *Assume that* (A_1), (A_2), (A_3), *and* (A_5) *hold. Then there exists a sequence of functions* $\{y_p\}$ *from* $[t_0, t_0 + a]$ *into* $B(x_0, b)$ *satisfying the following properties*:

(i) $\|y_p(t) - x_0\| \leq M(t - t_0)$ *and* $\|y_p(t) - y_p(s)\| \leq M|t - s|$, $t, s \in [t_k, t_{k+1}]$, $m \leq p \leq n$;

(ii) $y_p(t_{k+1}) \in F_0$, $p = m, n$;

(iii) *for all but a countable number of* $t \in [t_k, t_{k+1})$, $y_p'(t)$ *exists such that*

$$M[y_n(t) - y_m(t), y_n'(t) - y_m'(t)] \leq g(t, V(y_n(t) - y_m(t))) + (1 + N\|y_n(t) - y_m(t)\|)(\epsilon_n + \epsilon_m);$$

(iv) *if* $t_i^n, t_j^m \leq t_k \leq t_{k+1} \leq t_{i+1}^n, t_{j+1}^m$, *then*

$$\text{(a)}\qquad \begin{aligned} y_n(t_{k+1}) &= x_n(t_{k+1}) \quad \text{if } t_{k+1} = t_{i+1}^n, \\ y_n(t_{k+1}) &= y_n(t_{\bar{k}+1}) \quad \text{if } t_{k+1} < t_{i+1}^n, \end{aligned}$$

and $\|y_n(t) - x_n(t)\| \leq 3(t - t_i^n)\epsilon_n$, $t \in [t_k, t_{k+1})$;

$$\text{(b)}\qquad \begin{aligned} y_m(t_{k+1}) &= x_m(t_{k+1}) \quad \text{if } t_{k+1} = t_{j+1}^m, \\ y_m(t_{k+1}) &= y_m(t_{\bar{k}+1}) \quad \text{if } t_{k+1} < t_{j+1}^m, \end{aligned}$$

and $\|y_m(t) - x_m(t)\| \leq 3(t - t_j^m)\epsilon_m$, $t \in [t_k, t_{k+1})$;

(v) $\|y_p(t_{i+1}^p) - y_p(t_{\bar{i}+1}^p)\| \leq 3(t_{i+1}^p - t_i^p)\epsilon_p$, $p = m, n$, *i being an integer.*

This lemma is an extension of Lemma 1 in [4] and consequently we only indicate changes in the proof.

PROOF. Let k be a nonnegative integer and suppose that $y_p(t)$, $p = n, m$, is defined on $[t_0, t_k]$ satisfying the properties on $[t_0, t_k]$. To show that $y_p(t)$ can be suitably extended to $[t_0, t_{k+1}]$, we define inductively the sequence $\{s_q\}$ and $y_p(t)$ on $[t_k, s_{q+1}]$ as follows: if $s_q = t_{k+1}$, then $s_{q+1} = t_{k+1}$ and if $s_q < t_{k+1}$, we let $s_{q+1} = s_q + \gamma_q$ where $\gamma_q \geq 0$ is such that

$$(1)\qquad s_q + \gamma_q \leq t_{k+1};$$

$$(2)\qquad d[y_p(s_q) + \gamma_q f(s_q, y_p(s_q)), F] \leq \gamma_q \epsilon_p/2, \qquad p = m, n;$$

$$(3)\qquad M[(y_n(s_q) + x) - (y_m(s_q) + y), f(s_q, y_n(s_q)) - f(s_q, y_m(s_q))] \leq g(s_q + \sigma, V(y_n(s_q) + x - y_m(s_q) - y)) + \epsilon_n + \epsilon_m,$$

whenever $\|x\|, \|y\| \leq \gamma_q M$ and $0 < \sigma \leq \gamma_q$.

(4, γ_q is the largest number satisfying (1) to (3).

The condition (A_2), (a) and (e) of (A_5) imply that $\gamma_q > 0$. Using (2), for $p = m, n$, let $y_p(s_{q+1}) \in F$ such that

$$(2.7)\qquad \|y_p(s_q) + (s_{q+1} - s_q)f(s_q, y_p(s_q)) - y_p(s_{q+1})\| \leq (s_{q+1} - s_q)\epsilon_p,$$

and for $t \in (s_q, s_{q+1})$ define

$$(2.8) \qquad y_p(t) = [(t - s_q)y_p(s_{q+1}) + (s_{q+1} - t)y_p(s_q)](s_{q+1} - s_q)^{-1}.$$

It is easy to see that $y_p(s_{q+1}) \in F_0$, $\|y_p(t) - y_p(s)\| \leq M|t - s|$, and $\|y_p(t) - x_0\| \leq M(t - t_0)$, $t, s \in [t_k, s_{q+1}]$. Thus (1) holds for $t \in [t_k, s_{q+1})$. Moreover, by (2.7) and (2.8), it follows that

$$(2.9) \qquad \|y_p'(t) - f(s_q, y_p(s_q))\| \leq \epsilon_p, \qquad t \in (s_q, s_{q+1}).$$

Since $\|y_p(t) - y_p(s_q)\| \leq \gamma_q M$ and $|t - s_q| \leq \gamma_q$, we get from (2.9), (3) and (d) of (A_5),

$$\begin{aligned} M[y_n(t) - y_n(t), y_n'(t) - y_m'(t)] &\\ \leq M[y_n(t) - y_m(t), f(s_q, y_n(s_q)) - f(s_q, y_m(s_q))] &\\ + N\|y_n(t) - y_m(t)\|(\epsilon_n + \epsilon_m) &\\ \leq g(t, V(y_n(t) - y_m(t))) + (1 + N\|y_n(t) - y_m(t)\|)(\epsilon_n + \epsilon_m), \end{aligned}$$

for $t \in (s_q, s_{q+1})$. Hence (iii) is true for $t \in [t_k, s_{q+1})$. The rest of the proof is very much the same as the proof of Lemma 1 in [4] with appropriate changes. In particular $(x, y)_+$ has to be replaced by $M[x, y]$. We therefore omit the remaining details.

Finally we need the following result which relates $M[x(t), x'(t)]$ to $D_+ V(x(t))$ for any differentiable function $x(t)$. This is required relative to condition (A_5).

LEMMA 2.5. *Assume* (A_5). *Let $x(t)$ be any differentiable function on $[t_0, t_0 + a]$ into $B(x_0, b)$. Then*

$$D_+ V(x(t)) \leq M[x(t), x'(t)], \qquad t \in [t_0, t_0 + a].$$

For a proof of this lemma and use of assumptions like (A_5) see [6, pp. 142, 144].

3. **Main existence results.** We begin with a simple but illustrative existence result which is in the spirit of Ważewski [5].

THEOREM 3.1. *Assume that conditions* (A_1),(A_2) *and* (A_3) *hold. Suppose further that for $t \in [t_0, t_0 + a]$, $x, y \in F_0$,*

$$(3.1) \qquad \|f(t, x) - f(t, y)\| \leq g(t, \|x - y\|),$$

and $g(t, u)$ is nondecreasing in u for each t. Then problem (2.1) *has a unique solution on $[t_0, t_0 + a]$.*

PROOF. Let n, m be positive integers and let $m(t) = \|x_n(t) - x_m(t)\|$, $t \in [t_0, t_0 + a]$. If $t \in (t_i^n, t_{i+1}^n) \cap (t_j^m, t_{j+1}^m)$, then

$$D^+m(t) \leqslant \|x_n'(t) - x_m'(t)\|$$
$$\leqslant \|f(t, x_n(t_i^n)) - f(t, x_m(t_j^m))\| + \|f(t, x_n(t_i^n)) - f(t_i^n, x_n(t_i^n))\|$$
$$+ \|f(t, x_m(t_j^m)) - f(t_j^m, x_m(t_j^m))\| + \|x_n'(t) - f(t_i^n, x_n(t_i^n))\|$$
$$+ \|x_m'(t) - f(t_j^m, x_m(t_j^m))\|.$$

By (iv) of Lemma 2.1 and (3.1), we get

$$D^+m(t) \leqslant g(t, \|x_n(t_i^n) - x_m(t_j^m)\|) + 2(\epsilon_n + \epsilon_m). \tag{3.2}$$

Now using (i) and (ii) of Lemma 2.1, we see that

$$\begin{aligned} \|x_n(t_i^n) - x_m(t_j^m)\| &\leqslant \|x_n(t_i^n) - x_n(t)\| + \|x_n(t) - x_m(t)\| \\ &\quad + \|x_m(t_j^m) - x_m(t)\| \\ &\leqslant M(\epsilon_n + \epsilon_m) + \|x_n(t) - x_m(t)\|. \end{aligned} \tag{3.3}$$

Inequality (3.2) yields, in view of the monotony of $g(t, u)$ in u and (3.3),

$$D^+m(t) \leqslant g(t, m(t) + \beta_{m,n}) + \eta_{m,n},$$

where $\beta_{m,n} = M(\epsilon_n + \epsilon_m)$ and $\eta_{m,n} = 2(\epsilon_n + \epsilon_m)$. Setting $v(t) = m(t) + \beta_{m,n}$, we have $D^+v(t) \leqslant g(t, v(t)) + \eta_{m,n}$. This inequality holds for all but a countable number of $t \in [t_0, t_0 + a]$. Also, $v(t_0) = \beta_{m,n}$. Hence by Theorem 2.1, we have

$$m(t) \leqslant v(t) \leqslant r_{m,n}(t), \qquad t \in [t_0, t_0 + a],$$

where $r_{m,n}(t)$ is the maximal solution of $u' = g(t, u)$, $u(t_0) = \beta_{m,n}$. Since $\beta_{m,n}, \eta_{m,n} \to 0$ as $n, m \to \infty$, we see, by Lemma 2.3, that $\lim_{n,m\to\infty} r_{m,n}(t) = r(t)$ uniformly on $[t_0, t_0 + a]$, where $r(t)$ is the maximal solution of (2.2). By (A_3) it follows that $m(t) \equiv 0$ on $[t_0, t_0 + a]$, as $n, m \to \infty$. Thus the sequence $\{x_n(t)\}$ is uniformly Cauchy on $[t_0, t_0 + a]$ and the existence of a solution for problem (2.1) follows by Lemma 2.2. The proof of uniqueness of solutions is standard. Hence the proof is complete.

The improvement of (3.1) even to

$$\limsup_{h\to 0^+} \frac{1}{h}[\|x - y + [f(t, x) - f(t, y)]\| - \|x - y\|] \leqslant g(t, \|x - y\|),$$

for $x, y \in F_0$ and $t \in [t_0, t_0 + a]$, creates several difficulties that demand additional assumptions. In view of this, we wish to utilize better candidates than $\|x - y\|$. Consequently, the results that follow employ Lyapunov-like functions and the theory of differential inequalities in a variety of ways.

THEOREM 3.2. *Suppose that assumptions* (A_1) *to* (A_4) *hold. Then the convexity of* F_0 *implies that problem* (2.1) *has a unique solution on* $[t_0, t_0 + a]$.

PROOF. The convexity of F_0 implies by (iii) of Lemma 2.1 that $x_n(t) \in F_0$ for all $t \in [t_0, t_0 + a]$. Let m, n be positive integers and let

$$m(t) = V(t, x_n(t), x_m(t)), t \in [t_0, t_0 + a].$$

If $t \in (t_i^n, t_{i+1}^n) \cap (t_j^m, t_{j+1}^m)$, then using the Lipschitzian character of V, we have

$$\begin{aligned} D^+ m(t) \leqslant \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x_n(t) + hf(t, x_n(t)), x_m(t) \\ + hf(t, x_m(t))) - V(t, x_n(t), x_m(t))] \\ + L[\|x_n'(t) - f(t, x_n(t, x_n)))\| + \|x_m'(t) - f(t, x_m(t))\|]. \end{aligned} \tag{3.4}$$

Since $\|x_n(t) - x_n(t_i^n)\| \leqslant M(t - t_i^n)$, by (iv) and (v) of Lemma 2.1,

$$\begin{aligned} \|x_n'(t) - f(t, x_n(t))\| &\leqslant \|x_n'(t) - f(t_i^n, x_n(t_i^n))\| \\ &\quad + \|f(t_i^n, x_n(t_i^n)) - f(t, x_n(t))\| \\ &\leqslant 2\epsilon_n. \end{aligned}$$

Similarly $\|x_m'(t) - f(t, x_m(t))\| \leqslant 2\epsilon_m$. Consequently, we get, because of (2.3) and (3.4), the different inequality

$$D^+ m(t) \leqslant g(t, m(t)) + 2L(\epsilon_n + \epsilon_m),$$

which is true for all but a countable number of $t \in [t_0, t_0 + a]$. Since $m(t_0) = 0$, Theorem 2.1 gives

$$m(t) \leqslant r_{n,m}(t, t_0, 0), \qquad t \in [t_0, t_0 + a],$$

where $r_{n,m}(t, t_0, 0)$ is the maximal solution of $u' = g(t, u) + 2L(\epsilon_n + \epsilon_m)$, $u(t_0) = 0$. By Lemma 2.3, $\lim_{n,m \to \infty} r_{n,m}(t, t_0, 0) = r(t, t_0, 0)$ uniformly on $[t_0, t_0 + a]$, where $r(t, t_0, 0)$ is the maximal solution of (2.2). But by (A_3), $r(t, t_0, 0) \equiv 0$ on $[t_0, t_0 + a]$. It therefore follows that

$$\lim_{n,m \to \infty} V(t, x_n(t), x_m(t)) = 0$$

and consequently by (A_4), the sequence $\{x_n(t)\}$ is uniformly Cauchy on $[t_0, t_0 + a]$. Hence problem (2.1) has a solution on $[t_0, t_0 + a)$.

To prove uniqueness, if $x(t)$ and $y(t)$ are two solutions of (2.1), we let $m(t) = V(t, x(t), y(t))$ to obtain

$$D^+ m(t) \leqslant g(t, m(t)), \qquad t \in [t_0, t_0 + a].$$

The fact $m(t_0) = 0$ implies, by Theorem 2.1, that $m(t) \leqslant r(t, t_0, 0)$, $t \in [t_0, t_0 + a]$ where $r(t, t_0, 0)$ is the maximal solution of (2.2) which is identically zero by (A_3). Thus $m(t) \equiv 0$ on $[t_0, t_0 + a]$ and this completes the proof of Theorem 3.2.

The assumption F_0 is convex was crucial in the proof of Theorem 3.2. In general, we have to use different approaches.

THEOREM 3.3. *Let assumptions* (A_1) *to* (A_4) *hold except that condition* (2.3) *in* (A_4) *is replaced by*

$$\limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x+hf(t, x_1), y+hf(x, y_1)) - V(t, x, y)] \leqslant g(t, V(t, x, y)) + P(|t-s| + \|x-x_1\| + \|y-y_1\|), \tag{3.5}$$

for $s, t \in [t_0, t_0+a]$, $x, y \in F$ *and* $x_1, y_1 \in F_0$, *where* $p\colon R^+ \to R^+$ *is nondecreasing and* $\lim_{u\to 0} p(u) = 0$. *Then problem* (2.1) *has a unique solution on* $[t_0, t_0+a)$.

PROOF. Let n, m be positive integers and let $m(t) = V(t, x_n(t), x_m(t))$, $t \in [t_0, t_0+a]$. If $t \in (t_i^n, t_{i+1}^n) \cap (t_j^m, t_{j+1}^m)$, using the Lipschitzian character of V, Lemma 2.1(iv), and (3.5), we get

$$\begin{aligned} D^+m(t) \leqslant \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x_n(t) + hf(t_i^n, x_n(t_i^n)), x_m(t) \\ + hf(t_j^m, x_m(t_j^m)) - V(t, x_n(t), x_m(t)))] + L(\epsilon_n + \epsilon_m) \\ \leqslant g(t, m(t)) + p[|t_i^n - t_j^m| + \|x_n(t) - x_n(t_i^n)\| + \|x_m(t) - x_m(t_j^m)\|] \\ + L(\epsilon_n + \epsilon_m). \end{aligned}$$

Since $|t_i^n - t_j^m| \leqslant \epsilon_n + \epsilon_m$, $\|x_n(t) - x_n(t_i^n)\| \leqslant M\epsilon_n$ and $\|x_n(t) - x_m(t_j^m)\| \leqslant M\epsilon_m$, and $p(u)$ is nondecreasing in u, we have

$$D^+m(t) \leqslant g(t, m(t)) + \beta_{m,n},$$

where $\beta_{m,n} = p[(1+M)(\epsilon_n + \epsilon_m)] + L(\epsilon_n + \epsilon_m)$. Notice that $\lim_{n,m\to\infty} \beta_{n,m} = 0$ in view of the property of $p(u)$, we proceed as in the corresponding part of the proof of Theorem 3.2, to complete the proof as before.

THEOREM 3.4. *Let assumptions* (A_1), (A_2), (A_3) *and* (A_5) *hold and let* $g(t, u)$ *be nondecreasing in* u *for each* $t \in [t_0, t_0+a]$. *Then problem* (2.1) *has a unique solution on* $[t_0, t_0+a]$.

PROOF. Let m, n be positive integers and let $m(t) = V(y_n(t) - y_m(t))$ for $t \in [t_0, t_0+a]$, where $y_p(t)$ are the functions constructed in Lemma 2.4. Using Lemma 2.4(iii) and Lemma 2.5, we have

$$\begin{aligned} D_+m(t) &\leqslant M[y_n(t) - y_m(t), y_n'(t) - y_m'(t)] \\ &\leqslant g(t, m(t)) + (1 + N\|y_n(t) - y_m(t)\|)(\epsilon_n + \epsilon_m), \end{aligned}$$

for $t \in [t_k, t_{k+1})$. Also, $\|y_n(t) - y_m(t)\| \leqslant 2(b + \|x_0\|) \equiv L$. Moreover, for each $k \geqslant 1$,

$$|m(t_k) - m(t_{\bar{k}})| \leqslant L[\|y_n(t_k) - y_n(t_{\bar{k}})\| + \|y_m(t_k) - y_m(t_{\bar{k}})\|].$$

Hence, in view of (iv) and (v) of Lemma 2.4, we obtain

$$\sum_{k=1}^{\infty} |m(t_k) - m(t_{\bar{k}})| \leq 3L\left[\sum_{i=0}^{\infty}(t_{i+1}^n - t_i^n)\epsilon_n + \sum_{j=0}^{\infty}(t_{j+1}^m - t_j^m)\epsilon_m\right]$$
$$\leq 3La(\epsilon_n + \epsilon_m) \equiv \eta_{m,n}.$$

Furthermore, $m(t_0) = 0$. An application of Theorem 2.2 yields

$$m(t) \leq r_{m,n}(t, t_0, \eta_{m,n}), \qquad t \in [t_0, t_0 + a],$$

where $r_{m,n}(t, t_0, u_0)$ is the maximal solution of

$$u' = g(t, u) + (L + 1)(\epsilon_m + \epsilon_n), \quad u(t_0) = \eta_{m,n}.$$

As before, we can conclude by Lemma 2.1, (A_3) and (A_5),

$$\lim_{n,m\to\infty} \|y_n(t) - y_m(t)\| = 0$$

and this implies by Lemma 2.4(iv) that $\lim_{n,m\to\infty}\|x_n(t) - x_m(t)\| = 0$. Hence $\{x_n(t)\}$ is uniformly Cauchy on $[t_0, t_0 + a]$ and the proof is complete.

REMARKS. In the boundary condition (A_2), "lim" may be replaced by "lim inf". Similarly, in (A_4), one could employ other generalized derivatives $D^- V$, $D_- V$ and $D_+ V$ in place of $D^+ V$. The proofs work without any difficulty. However, we need conditions (A_2) and (A_5) as stated, both of which are used in Theorem 3.4.

Consider the special case $V(t, x, y) = \|x - y\|$. Condition (2.3) becomes

$$\limsup_{h\to 0^+} \frac{1}{h}[\|x - y + h[f(t, x) - f(t, y)]\| - \|x - y\|] \leq g(t, \|x - y\|),$$

which is clearly satisfied when one assumes Perron's type uniqueness condition (3.1) in Theorem 3.1.

Let E^* be the dual space of E and let $J\colon E \to 2^{E^*}$ be the duality map defined by

$$J(x) = [x^* \in E^*\colon \|x^*\| = \|x\| \text{ and } x^*(x) = \|x\|^2].$$

For each $x, y \in E$ define the generalized pairings

$$(x, y)_- = \inf[x^*(x)\colon x^* \in J(y)] \quad \text{and} \quad (x, y)_+ = \sup[x^*(x)\colon x^* \in J(y)].$$

If $x, y, z \in E$, we have

$$(x + y, z)_\pm \leq (x, z)_\pm + \|y\|\|z\|.$$

Also, if $x(t)$ is a differentiable function on $[t_0, t_0 + a]$ and $m(t) = \|x(t)\|^2$, then $D^-m(t) \leq 2(x'(t), x(t))_-$ and $D^+m(t) \leq 2(x'(t), x(t))_+$. Consequently, the assumptions $(f(t, x) - f(t, y), x - y)_\pm \leq L\|x - y\|^2$ imply that the choice $V(t, x, y) = \|x - y\|e^{-2Lt}$ is admissible in Theorems 3.2 and 3.3 with $g(t, u) \equiv 0$

and $D^{\pm} V(t, x, y)$. For Theorem 3.4 we take $V(x) = \|x\|^2$ and $M[x, y] = (x, y)_+$ so that (A_5) is satisfied since $(x, y)_+$ is upper semicontinuous. These considerations show that our results contain Theorems 1, 2 and 3 in [4] which in turn include many earlier results.

REFERENCES

1. V. Lakshmikantham and S. Leela, *Differential and integral inequalities.* Vols. I, II, Academic Press, New York, 1969.

2. V. Lakshmikantham, *Stability and asymptotic behaviour of solutions of differential equations in a Banach space,* Lecture notes, CIME, Italy, 1974.

3. ———, *Existence and comparison results for differential equations in a Banach space,* (Proc. Internat. Conf. Differential Equations, Univ. of Southern California, Los Angeles, 1974), Academic Press, New York, 1975, pp. 459–473.

4. R. H. Martin, *Differential equations on closed subsets of a Banach space,* Trans. Amer. Math. Soc. 179 (1973), 399–414. MR 47 #7537.

5. T. Ważewski, *Sur l'existence et l'unicité des intégrales des équations différentielles ordinaires au cas de l'espace de Banach,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), 301–305. MR 24 #A892.

6. G. Ladas and V. Lakshmikantham, *Differential equations in abstract spaces,* Academic Press, New York, 1972.

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