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DIFFERENTIAL GEOMETRIC STRUCTURES ON PRINCIPAL TOROIDAL BUNDLES

BY

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ABSTRACT. Under an assumption of regularity a manifold with an f-structure satisfying certain conditions analogous to those of a Kähler structure admits a fibration as a principal toroidal bundle over a Kähler manifold. In some natural special cases, additional information about the bundle space is obtained. Finally, curvature relations between the bundle space and the base space are studied.

Let M^{2n+s} be a C^∞ manifold of dimension $2n+s$. If the structural group of M^{2n+s} is reducible to $U(n)\times O(s)$, then M^{2n+s} is said to have an *f-structure of rank $2n$*. If there exists a set of 1-forms $\{\eta^1,\cdots,\eta^s\}$ satisfying certain properties described in §1, then M^{2n+s} is said to have an *f-structure with complemented frames*. In [1] it was shown that a principal toroidal bundle over a Kähler manifold with a certain connection has an f-structure with complemented frames and $d\eta^1=\cdots=d\eta^s$ as the fundamental 2-form. On the other hand, the following theorem is proved in §2 of this paper.

Theorem 1. *Let M^{2n+s} be a compact connected manifold with a regular normal f-structure. Then M^{2n+s} is the bundle space of a principal toroidal bundle over a complex manifold N^{2n} $(=M^{2n+s}/\mathfrak{M})$. Moreover, if M^{2n+s} is a K-manifold, then N^{2n} is a Kähler manifold.*

After developing a theory of submersions in §3, we discuss in §4 further properties of this fibration in the cases where $d\eta^x=0$, $x=1,\cdots,s$ and $d\eta^x=\alpha^xF$, F being the fundamental 2-form of the f-structure.

Finally in §5 we study the relation between the curvature of M^{2n+s} and N^{2n}.

Since $U(n)\times O(s)\subset O(2n+s)$, M^{2n+s} is a new example of a space in the class provided by Chern in his generalization of Kähler geometry [4]. S. I. Goldberg's paper [5] also suggests the study of framed manifolds as bundle spaces over Kähler manifolds with parallelisable fibers.

1. Normal f-structures.

Let M^{2n+s} be a $2n+s$-dimensional manifold with an f-structure. Then there is a tensor field f of type $(1,1)$ on M^{2n+s} that is of rank

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$2n$ everywhere and satisfies

(1) $$f^3 + f = 0.$$

If there exist vector fields ξ_x, $x = 1, \cdots, s$ on M^{2n+s} such that

(2) $$f\xi_x = 0, \quad \eta^x(\xi_y) = \delta^x_y, \quad \eta^x \circ f = 0, \quad f^2 = -I + \eta^y \otimes \xi_y,$$

we say M^{2n+s} has an *f-structure with complemented frames.* Further we say that the f-structure is *normal* if

(3) $$[f, f] + d\eta^x \otimes \xi_x = 0,$$

where $[f, f]$ is the Nijenhuis torsion of f. It is a consequence of normality that $[\xi_x, \xi_y] = 0$. Moreover it is known that there exists a Riemannian metric g on M^{2n+s} satisfying

(4) $$g(X, Y) = g(fX, fY) + \sum_x \eta_x(X)\eta_x(Y),$$

where X and Y are arbitrary vector fields on M^{2n+s}. Define a 2-form F on M^{2n+s} by

(5) $$F(X, Y) = g(X, fY).$$

A normal f-structure for which F is closed will be called a *K-structure* and a K-structure for which there exist functions $\alpha^1, \cdots, \alpha^s$ such that $\alpha^x F = d\eta^x$ for $x = 1, \cdots, s$ will be called an *S-structure.*

Lemma 1. *If M^{2n+s}, $n > 1$, has an S-structure, then the α^x are all constant.*

Proof. $\alpha^x F = d\eta^x$ so that $d\alpha^x \wedge F = 0$ since $dF = 0$. However $F \neq 0$ so $d\alpha^x = 0$ and hence α^x is constant.

The special case where the α^x are all 0 or all 1 has been studied in [1]. Also, the following were proved.

Lemma 2. *If M^{2n+s} has a K-structure, the ξ_x are Killing vector fields and $d\eta^x(X, Y) = -2(\tilde{\nabla}_Y \eta^x)(X)$. Here $\tilde{\nabla}$ is the Riemannian connection of g on M^{2n+s}.*

From Lemma 2, we can see that in the case of an S-structure $\alpha^x fY = -2\tilde{\nabla}_Y \xi_x$.

Lemma 3. *If M^{2n+s} has a K-structure, then*

$$(\tilde{\nabla}_X F)(Y, Z) = \frac{1}{2} \sum_x (\eta^x(Y) d\eta^x(fZ, X) + \eta^x(Z) d\eta^x(X, fY)).$$

2. **Proof of Theorem 1.** In Chapter 1 of [9] R. S. Palais discusses quotient manifolds defined by foliations. In particular, a cubical coordinate system $\{U, (u^1, \cdots, u^n)\}$ on an n-dimensional manifold is said to be *regular* with respect

to an involutive m-dimensional distribution if $\{\partial(m)/\partial u^x\}$, $x = 1, \cdots, m$, is a basis of $\mathfrak{M}_m$ for every $m \in U$ and if each leaf of $\mathfrak{M}$ intersects U in at most one m-dimensional slice of $\{U, (u^1, \cdots, u^n)\}$. We say $\mathfrak{M}$ is *regular* if every leaf of $\mathfrak{M}$ intersects the domain of a cubical coordinate system which is regular with respect to $\mathfrak{M}$.

In [9] it is proven that if $\mathfrak{M}$ is regular on a compact connected manifold M, then every leaf of $\mathfrak{M}$ is compact and that the quotient $M/\mathfrak{M}$ is a compact differentiable manifold. Moreover the leaves of $\mathfrak{M}$ are the fibers of a C^∞ fibering of M with base manifold $M/\mathfrak{M}$ and the leaves are all C^∞ isomorphic.

We now note that the distribution $\mathfrak{M}$ spanned by the vector fields $\xi_1, \cdots, \xi_s$ of a normal f-structure is involutive. In fact we have by normality

$$0 = [f, f](\xi_y, \xi_z) + d\eta^x(\xi_y, \xi_z)\xi_x = f^2[\xi_y, \xi_z] - \eta^x([\xi_y, \xi_z])\xi_x = -[\xi_y, \xi_z]$$

from which it easily follows that $\mathfrak{M}$ is involutive. If $\mathfrak{M}$ is regular and the vector fields ξ_x are regular we say that the normal f-structure is *regular.* Thus from the results of [9] we see that if M^{2n+s} is compact and has a regular normal f-structure, then M^{2n+s} admits a C^∞ fibering over the $(2n)$-dimensional manifold $N^{2n} = M^{2n+s}/\mathfrak{M}$ with compact, C^∞ isomorphic, fibers.

Since the distribution $\mathfrak{M}$ of a regular normal f-structure consists of s 1-dimensional regular distributions each given by one of the ξ_x's, if M^{2n+s} is compact, the integral curves of ξ_x are closed and hence homeomorphic to circles S^1. The ξ_x's being independent and regular show that the fibers determined by the distribution $\mathfrak{M}$ are homeomorphic to tori T^s.

Now define the *period function* λ_X of a regular closed vector field X by

$$\lambda_X(m) = \inf\{t > 0 \mid (\exp tX)(m) = m\}.$$

For brevity we denote λ_{ξ_x} by λ_x. W. M. Boothby and H. C. Wang [3] proved that $\lambda_x(m)$ is a differentiable function on M^{2n+s}. We now prove the following

Lemma 4. *The functions λ_x are constants.*

The proof of the lemma makes use of the following theorem of A. Morimoto [7].

Theorem (Morimoto [7]). *Let M be a complex manifold with almost complex structure tensor J. Let X be an analytic vector field on M such that X and JX are closed regular vector fields. Set $p(m) = \lambda_X(m) + \sqrt{-1}\lambda_{JX}(m)$. Then p is a holomorphic function on M.*

Proof of lemma. For s even,

$$\tilde{f} = f + \sum_{i=1}^{s/2} (\eta^i \otimes \xi_{i^*} - \eta^{i^*} \otimes \xi_i), \qquad i = 1, \cdots, s/2,\ i^* = i + s/2,$$

defines a complex structure on $M = M^{2n+s}$ (cf. [6]). It is clear from the normality that ξ_x is a holomorphic vector field. For s odd, a normal almost contract structure $(\tilde{f}, \xi_0, \eta_0)$ is defined where ξ_0 and η_0 generically denote one of the ξ_x's and η_x's respectively [6]. It is well known that this structure induces a complex structure J on $M = M^{2n+s} \times S^1$. Moreover, by the normality, ξ_0 considered as a vector field on M is analytic. Then $p(m) = \lambda_x(m) + \sqrt{-1}\lambda_{x^*}(m)$ or $p((m, q)) = \lambda_{\xi_0}((m, q)) + \sqrt{-1}\lambda_{J\xi_0}((m, q))$, $q \in S^1$, for s odd, is a holomorphic function on M by the theorem of Morimoto. Since M is compact, p must be constant. Thus λ_x is constant on M and since $\lambda_x((m, q)) = \lambda_x(m)$, λ_x is constant on M^{2n+s}.

Let $C_x = \lambda_x(m)$, then the circle group S^1_x of real numbers modulo C_x acts on M^{2n+s} by $(t, m) \longrightarrow (\exp t\xi_x)(m)$, $t \in R$. Now the only element in $T^s = S^1_1 \times \cdots \times S^1_s$ with a fixed point in M^{2n+s} is the identity and since M^{2n+s} is a fiber space over N^{2n}, we need only show that M^{2n+s} is locally trivial [3]. Let $\{U_\alpha\}$ be a cover of N^{2n} such that each U_α is the projection of a regular neighborhood on M^{2n+s} and let $s_\alpha: U_\alpha \longrightarrow M^{2n+s}$ be the section corresponding to $u^1 =$ constant, $\cdots$, $u^s =$ constant. Then the maps $\Psi_\alpha: U_\alpha \times T^s \longrightarrow M^{2n+s}$ defined by

$$\Psi_\alpha(p, t_1, \cdots, t_s) = (\exp(t_1\xi_1 + \cdots + t_s\xi_s))(s_\alpha(p))$$

give coordinate maps for M^{2n+s}.

Finally (cf. [1]) we note that $\gamma = (\eta^1, \cdots, \eta^s)$ defines a Lie algebra valued connection form on M^{2n+s} and we denote by $\tilde{\pi}$ the horizontal lift with respect to γ. Define a tensor field J of type (1, 1) on N^{2n} by $JX = \pi_* f\tilde{\pi}X$. Then, since the distribution $\mathfrak{L}$ complementary to $\mathfrak{M}$ is horizontal with respect to γ,

$$J^2X = \pi_* f\tilde{\pi}\,\pi_* f\tilde{\pi}X = \pi_* f^2\tilde{\pi}X = -X.$$

Moreover

$$\begin{aligned}
[J, J](X, Y) &= -[X, Y] + [\pi_* f\tilde{\pi}X, \pi_* f\tilde{\pi}Y] - \pi_* f\tilde{\pi}[\pi_* f\tilde{\pi}X, Y] - \pi_* f\tilde{\pi}[X, \pi_* f\tilde{\pi}Y] \\
&= -\pi_*[\tilde{\pi}X, \tilde{\pi}Y] + \pi_*[f\tilde{\pi}X, f\tilde{\pi}Y] - \pi_* f\tilde{\pi}\,\pi_*[f\tilde{\pi}X, \tilde{\pi}Y] - \pi_* f\tilde{\pi}\,\pi_*[\tilde{\pi}X, f\tilde{\pi}Y] \\
&= \pi_*(f^2[\tilde{\pi}X, \tilde{\pi}Y] - \eta^x([\tilde{\pi}X, \tilde{\pi}Y]), \xi_x) + \pi_*[f\tilde{\pi}X, f\tilde{\pi}Y] - \pi_* f[f\tilde{\pi}X, \tilde{\pi}Y] - \pi_* f[\tilde{\pi}X, f\tilde{\pi}Y] \\
&= \pi_*([f, f](\tilde{\pi}X, \tilde{\pi}Y) + d\eta^x(\tilde{\pi}X, \tilde{\pi}Y)\xi_x) \\
&= 0.
\end{aligned}$$

Thus we see that N^{2n} is a complex manifold.

We define an Hermitian metric G on N^{2n} by $G(X, Y) = g(\tilde{\pi}X, \tilde{\pi}Y)$. Indeed

$$\begin{aligned}
G(JX, JY) &= g(\tilde{\pi}\,\pi_* f\tilde{\pi}X, \tilde{\pi}\,\pi_* f\tilde{\pi}Y) = g(f\tilde{\pi}X, f\tilde{\pi}Y) \\
&= g(\tilde{\pi}X, \tilde{\pi}Y) - \sum \eta^x(\tilde{\pi}X)\eta^x(\tilde{\pi}Y) = G(X, Y).
\end{aligned}$$

Now define the fundamental 2-form Ω by $\Omega(X, Y) = G(X, JY)$. Then for vector fields $\tilde{X}, \tilde{Y}$ on M^{2n+s} we have

$$\pi^*\Omega(\tilde{X}, \tilde{Y}) = \Omega(\pi_*\tilde{X}, \pi_*\tilde{Y}) = G(\pi_*\tilde{X}, J\pi_*\tilde{Y})$$
$$= g(\tilde{\pi}\pi_*\tilde{X}, \tilde{\pi}J\pi_*\tilde{Y}) = g(-f^2\tilde{X}, \tilde{\pi}\pi_* f\tilde{Y}) = g(-f^2\tilde{X}, f\tilde{Y}) = g(\tilde{X}, f\tilde{Y}) = F(\tilde{X}, \tilde{Y}).$$

Thus $F = \pi^*\Omega$. If now $dF = 0$, then $0 = d\pi^*\Omega = \pi^* d\Omega$ and hence $d\Omega = 0$ since π^* is injective. Thus the manifold N^{2n} is Kählerian.

3. Submersions. Let $\tilde{\nabla}$ denote the Riemannian connection of g on M^{2n+s}. Since the ξ_x's are Killing, g is projectable to the metric G on N^{2n}. Then following [8] the horizontal part of $\tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y$ is $\tilde{\pi}\nabla_X Y$ where as we shall see ∇ is the Riemannian connection of G. Now for an S-structure we have seen that $\tilde{\nabla}_{\tilde{X}}\xi_x = \alpha^x f\tilde{X}$ for any vector field $\tilde{X}$ on M^{2n+s}. By normality f is projectable ($\mathfrak{L}_{\xi_x} f = 0$) and the α^x's are constants; thus we can write

$$\tilde{\nabla}_{\tilde{\pi}X}\xi_x = -\tilde{\pi}H_x X,$$

where H_x is a tensor field of type (1, 1) on N^{2n}.

We can now find the vertical part of $\tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y$.

$$g(\tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y, \xi_x) = -g(\tilde{\pi}Y, \tilde{\nabla}_{\tilde{\pi}X}\xi_x) = g(\tilde{\pi}Y, \tilde{\pi}H_x X).$$

Thus we can write

$$\tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y = \tilde{\pi}\nabla_X Y + h^x(X, Y)\xi_x$$

where each h^x is a tensor field of type (0, 2) and

$$G(H_x X, Y) = h^x(X, Y).$$

Lemma 5. $\mathfrak{L}_{\xi_x}(\tilde{\pi}X) = 0$ *for any vector field* X *on* N^{2n}, *where* $\mathfrak{L}_{\xi_x}$ *is the operator of Lie differentiation in the* ξ_x *direction.*

Proof. We have that $g(\xi_y, \tilde{\pi}X) = 0$ for $y = 1, \cdots, s$. By Lemma 2, the ξ_x are Killing, that is $\mathfrak{L}_{\xi_x} g = 0$. From the normality of f, $\mathfrak{L}_{\xi_x}\xi_y = 0$. Hence, we have that

$$g(\xi_y, \mathfrak{L}_{\xi_x}(\tilde{\pi}X)) = 0, \qquad y = 1, \cdots, s,$$

and so $\mathfrak{L}_{\xi_x}(\tilde{\pi}X)$ is horizontal. However,

$$\pi_*\mathfrak{L}_{\xi_x}(\tilde{\pi}X) = \pi_*[\xi_x, \tilde{\pi}X] = [\pi_*\xi_x, \pi_*\tilde{\pi}X] = 0$$

and so $\mathfrak{L}_{\xi_x}(\tilde{\pi}X)$ is vertical.

Using the lemma we see that $\tilde{\nabla}_{\xi_x}\tilde{\pi}X = \tilde{\nabla}_{\tilde{\pi}X}\xi_x$ for any vector field X on N^{2n}. Since ξ_x is Killing, we have

$$0 = g(\tilde{\nabla}_{\tilde{\pi}X}\xi_x, \tilde{\pi}X) = -g(\xi_x, \tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}X) = -g(\xi_x, h^y(X, X)\xi_y) = -h^x(X, X)$$

for all X. That is to say $h^x(X, Y) = -h^x(Y, X)$ for all X and Y. Now we have that

$$\begin{aligned} 0 &= \tilde{\nabla}_{\tilde{\pi}X}(\tilde{\pi}Y) - \tilde{\nabla}_{\tilde{\pi}Y}(\tilde{\pi}X) - [\tilde{\pi}X, \tilde{\pi}Y] \\ &= \tilde{\pi}(\nabla_X Y - \nabla_Y X - [X, Y]) + (h^x(X, Y) - h^x(Y, X) + d\eta^x(\tilde{\pi}X, \tilde{\pi}Y))\xi_x \qquad (6) \\ &= \tilde{\pi}(\nabla_X Y - \nabla_Y X - [X, Y]) + (2h^x(X, Y) + d\eta^x(\tilde{\pi}X, \tilde{\pi}Y))\xi_x, \end{aligned}$$

where we have used the following lemma.

Lemma 6. $[\tilde{\pi}X, \tilde{\pi}Y] = \tilde{\pi}[X, Y] - d\eta^x(\tilde{\pi}X, \tilde{\pi}Y)\xi_x$.

Proof. Since $\pi_*[\tilde{\pi}X, \tilde{\pi}Y] = [\pi_*\tilde{\pi}X, \pi_*\tilde{\pi}Y] = [X, Y]$ we see that $\tilde{\pi}[X, Y]$ is the horizontal part of $[\tilde{\pi}X, \tilde{\pi}Y]$. By Lemma 2, we have

$$d\eta^x(\tilde{\pi}X, \tilde{\pi}Y) = -2(\tilde{\nabla}_{\tilde{\pi}Y}\eta^x)(\tilde{\pi}X) = -2g(\tilde{\nabla}_{\tilde{\pi}Y}\xi_x, \tilde{\pi}X) = +2g(\xi_x, \tilde{\nabla}_{\tilde{\pi}Y}\tilde{\pi}X).$$

Also $d\eta^x(\tilde{\pi}X, \tilde{\pi}Y) = -d\eta^x(\tilde{\pi}Y, \tilde{\pi}X) = -2g(\xi_x, \tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y)$. Thus

$$2d\eta^x(\tilde{\pi}X, \tilde{\pi}Y) = 2g(\xi_x, \tilde{\nabla}_{\tilde{\pi}Y}\tilde{\pi}X - \tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y)$$

or

$$d\eta^x(\tilde{\pi}X, \tilde{\pi}Y)\xi_x = \sum_x g(\xi_x, [\tilde{\pi}X, \tilde{\pi}Y])\xi_x = \text{vertical part of } [\tilde{\pi}X, \tilde{\pi}Y].$$

From (6) we see $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ and $h^x(X, Y) = -\frac{1}{2}d\eta^x(\tilde{\pi}X, \tilde{\pi}Y)$. Furthermore,

$$\begin{aligned} XG(Y, Z) &= \tilde{\pi}Xg(\tilde{\pi}Y, \tilde{\pi}Z) = g(\tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y, \tilde{\pi}Z) + g(\tilde{\pi}Y, \tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Z) \\ &= g(\tilde{\pi}\nabla_X Y, \tilde{\pi}Z) + g(\tilde{\pi}Y, \tilde{\pi}\nabla_X Z) = G(\nabla_X Y, Z) + G(Y, \nabla_X Z). \end{aligned}$$

Thus, we have the following proposition.

Proposition. *∇ is the Riemannian connection of G on N^{2n}.*

4. The S-structure case. Let M^{2n+s}, $n > 1$, be a manifold with an S-structure. Then, as we have seen, there exist constants α^x, $x = 1, \cdots, s$, such that $\alpha^x F = d\eta^x$. We will consider two cases, namely $\Sigma_x(\alpha^x)^2 = 0$ and $\Sigma_x(\alpha^x)^2 \neq 0$.

In the first case each $d\eta_x = 0$ and by Lemma 2 each ξ_x is Killing, hence the regular vector fields $\xi_1, \cdots, \xi_s$ are parallel on M^{2n+s}. Moreover the complementary distribution $\mathfrak{L}$ (projection map is $-f^2 = I - \eta^x \otimes \xi_x$) is parallel. If now the distribution $\mathfrak{L}$ is also regular, we have a second fibration of M^{2n+s} with fibers the integral submanifolds L^{2n} of $\mathfrak{L}$ and base space an s-dimensional manifold N^s. Thus by a result of A. G. Walker [10] we see that although M^{2n+s} is not necessarily reducible (even though it is locally the product of N^{2n} and T^s) it is a covering space of $N^{2n} \times N^s$ and is covered by $L^{2n} \times T^s$. In summary we have

Theorem 2. *If M^{2n+s} is as in Theorem 1 with $d\eta^x = 0$, $x = 1, \cdots, s$, and $\mathfrak{L}$ regular, then M^{2n+s} is a covering space of $N^{2n} \times N^s$, where N^s is the base space of the fibration determined by $\mathfrak{L}$.*

Now as in Theorem 1, since the ξ_x's, $x = 1, \cdots, s$, are regular, we could fibrate by any $s - t$ of them to obtain a fibration of M^{2n+s} as a principal T^{s-t} bundle over a manifold P^{2n+t}. By normality the remaining t vector fields are projectable to P^{2n+t}. Moreover they are regular on P^{2n+t}; for if not, their integral curves would be dense in a neighborhood U over which M^{2n+s} is trivial with compact fiber T^{s-t} contradicting their regularity on M^{2n+s}. Thus P^{2n+t} is a principal T^t bundle over N^{2n}.

Theorem 3. *If M^{2n+s}, $n > 1$, is as in Theorem 1 with $d\eta^x = \alpha^x F$ and $\Sigma_x(\alpha^x)^2 \neq 0$, then M^{2n+s} is a principal T^{s-1} bundle over a principal circle bundle P^{2n+1} over N^{2n} and the induced structure on P^{2n+1} is a normal contact metric (Sasakian) structure.*

Proof. Without loss of generality we suppose $\alpha^s \neq 0$. Then fibrating as above by $\xi_1, \cdots, \xi_{s-1}$ we have that M^{2n+s} is a principal T^{s-1} bundle over a principal circle bundle P^{2n+1} over N^{2n}. Let $p: M^{2n+s} \longrightarrow P^{2n+1}$ denote the projection map. By normality f, ξ_s, η^s are projectable, so we define ϕ, ξ, η on P^{2n+1} by

$$\phi X = p_* f \tilde{p} X, \quad \xi = p_* \xi_s, \quad \eta(X) = \eta^s(\tilde{p} X)$$

where $\tilde{p}$ denotes the horizontal lift with respect to the connection $(\eta^1, \cdots, \eta^{s-1})$ considered as a Lie algebra valued connection form as in the proof of Theorem 1. Then by a straight-forward computation we have

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \phi^2 = -I + \xi \otimes \eta, \quad [\phi, \phi] + \xi \otimes d\eta = 0,$$

that is, (ϕ, ξ, η) is a normal almost contact structure on P^{2n+1}. Defining a metric $\dot{g}$ by $\dot{g}(X, Y) = g(\tilde{p}X, \tilde{p}Y)$ we have $\dot{g}(X, \xi) = \eta(X)$ and $\dot{g}(\phi X, \phi Y) = \dot{g}(X, Y) - \eta(X)\eta(Y)$. Moreover setting $\Phi(X, Y) = \dot{g}(X, \phi Y)$ we obtain $F = p^*\Phi$. Thus since

$d\eta^s = \alpha^s F$, $p^*\Phi = d\eta^s/\alpha^s$ and

$$\Phi(X, Y) = g(\tilde{p}X, \tilde{p}\phi Y) = d\eta^s(\tilde{p}X, \tilde{p}Y)/\alpha^s$$
$$= (X\eta(Y) - Y\eta(X) - \eta^s([\tilde{p}X, \tilde{p}Y]))/\alpha^s = d\eta(X, Y)/\alpha^s$$

since η^s is horizontal. Thus we have that $\eta_\wedge(d\eta)^n = \eta_\wedge(\alpha^s\Phi)^n \neq 0$ and hence that P^{2n+1} has a normal contact metric structure with ξ regular.

Remark 1. While it is already clear that P^{2n+1} is a principal circle bundle over N^{2n}, it now also follows from the well-known Boothby-Wang and Morimoto fibrations.

Remark 2. Under the hypotheses of Theorem 3, it is possible to assume without loss of generality that α^x equals 0 or $1/\sqrt{t}$ where t is the number of nonzero α^x and hence there exist constants β^x_q, $q = 1, \cdots, s-1$, such that $\bar{\eta}^q = \Sigma_x \beta_q{}^x \eta^x$ and $\bar{\eta}^s = \Sigma_x \alpha^x \eta^x$ are 1-forms with $d\bar{\eta}^q = 0$ and $d\bar{\eta}^s = F$. Then f, $\bar{\eta}^x$ and the dual vector fields $\bar{\xi}_x$ again define a K-structure on M^{2n+s}. If now this K-structure is regular, then, since the distribution spanned by $\bar{\xi}_1, \cdots, \bar{\xi}_{s-1}$ and its complement are parallel, M^{2n+s} is a covering of the product of P^{2n+1} and a manifold P^{s-1} as in the proof of Theorem 2.

Remark 3. In [1] one of the authors gave the following example of an S-manifold as a generalization of the Hopf-fibration of the odd-dimensional sphere over complex projective space, $\pi'\colon S^{2n+1} \to PC^n$. Let Δ denote the diagonal map and define a space H^{2n+s} by the diagram

$$\begin{array}{ccc} H^{2n+s} & \xrightarrow{\hat{\Delta}} & S^{2n+1} \times \cdots \times S^{2n+1} \\ \downarrow & & \downarrow \pi' \times \cdots \times \pi' \\ PC^n & \xrightarrow{\Delta} & PC^n \times \cdots \times PC^n \end{array}$$

that is $H^{2n+s} = \{(P_1, \cdots, P_s) \in S^{2n+1} \times \cdots \times S^{2n+1} \mid \pi'(P_1) = \cdots = \pi'(P_s)\}$ and thus H^{2n+s} is diffeomorphic to $S^{2n+1} \times T^{s-1}$. Further properties of the space H^{2n+s} are given in [1], [2].

If however the $d\eta^x$'s are independent then there can be no intermediate bundle P^{2n+t} over N^{2n} such that M^{2n+s} is trivial over P^{2n+t}.

Remark 4. If M^{2n+s} is as in Theorem 1 with the $d\eta^x$'s independent, then there is no fibration by $s - t$ of the ξ_x's yielding a principal toroidal bundle P^{2n+t} over N^{2n} such that $M^{2n+s} = P^{2n+t} \times T^{s-t}$. For suppose P^{2n+t} is such an intermediate bundle, then it is necessary that $\tilde{\nabla}_{\tilde{\pi}X}\xi_x = 0$ (see e.g. [8]) and thus the η^x's are parallel contradicting the independence of the $d\eta^x$'s.

5. Curvature. Let $\tilde{R}$ and R denote the curvature tensors of $\tilde{\nabla}$ and ∇ respectively. Then

$$g(\tilde{R}_{\tilde{\pi}X\tilde{\pi}Y}\tilde{\pi}Z, \tilde{\pi}W) = g(\tilde{\nabla}_{\tilde{\pi}X}\tilde{\nabla}_{\tilde{\pi}Y}\tilde{\pi}Z - \tilde{\nabla}_{\tilde{\pi}Y}\tilde{\nabla}_{\tilde{\pi}X}\tilde{\pi}Z - \tilde{\nabla}_{[\tilde{\pi}X,\tilde{\pi}Y]}\tilde{\pi}Z, \tilde{\pi}W)$$

$$= g(\tilde{\nabla}_{\tilde{\pi}X}(\tilde{\pi}\nabla_Y Z + h^x(Y, Z)\xi_x) - \tilde{\nabla}_{\tilde{\pi}Y}(\tilde{\pi}\nabla_X Z + h^x(X, Z)\xi_x)$$

$$- \tilde{\nabla}_{\tilde{\pi}[X,Y]-d\eta^x(\tilde{\pi}X,\tilde{\pi}Y)\xi_x}\tilde{\pi}Z, \tilde{\pi}W)$$

$$= g(\tilde{\pi}\nabla_X\nabla_Y Z - h^x(Y, Z)\tilde{\pi}(H_x X) - \tilde{\pi}\nabla_Y\nabla_X Z + h^x(X, Z)\tilde{\pi}(H_x Y)$$

$$- \tilde{\pi}\nabla_{[X,Y]}Z - d\eta^x(\tilde{\pi}X, \tilde{\pi}Y)\tilde{\pi}(H_x Z), \tilde{\pi}W)$$

$$= G(R_{XY}Z, W) - \sum_x (h^x(Y, Z)h^x(X, W) - h^x(X, Z)h^x(Y, W) + d\eta^x(\tilde{\pi}X, \tilde{\pi}Y)h^x(Z, W))$$

$$= G(R_{XY}Z, W) - \sum_x (h^x(Y, Z)h^x(X, W) - h^x(X, Z)h^x(Y, W) - 2h^x(X, Y)h^x(Z, W)).$$

In [1], one of the present authors developed a theory of manifolds with an f-structure of constant f-sectional curvature. This is the analogue of a complex manifold of constant holomorphic curvature. A plane section of M^{2n+s} is called an f-section if there is a vector X orthogonal to the distribution spanned by the ξ_x's such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of this section is called an *f-sectional curvature* and is of course given by $g(\tilde{R}_{XfX}X, fX)$. M^{2n+s} is said to be of constant *f-sectional curvature* if the f-sectional curvatures are constant for all f-sections. This is an absolute constant. We then have the following theorem.

Theorem 5. *If M^{2n+s} is a compact, connected manifold with a regular S-structure of constant f-sectional curvature c, then N^{2n} is a Kähler manifold of constant holomorphic curvature.*

Proof. That N^{2n} is Kähler follows from Theorem 1. By definition there exist $\alpha^1, \cdots, \alpha^s$, necessarily constant such that $\alpha^x F = d\eta^x$. If X is a unit vector on N^{2n}, then we have

$$G(R_{XJX}JX, X) = g(\tilde{R}_{\tilde{\pi}X\tilde{\pi}JX}\tilde{\pi}JX, \tilde{\pi}X)$$

$$+ \sum_x (\tfrac{1}{2}\alpha^x F(\tilde{\pi}JX, \tilde{\pi}JX)\tfrac{1}{2}\alpha^x F(\tilde{\pi}X, \tilde{\pi}X)$$

$$- \tfrac{1}{2}\alpha^x F(\tilde{\pi}X, \tilde{\pi}JX)\tfrac{1}{2}\alpha^x F(\tilde{\pi}JX, \tilde{\pi}X)$$

$$-2(\tfrac{1}{2})\alpha^x F(\tilde{\pi}X, \tilde{\pi}JX)\tfrac{1}{2}\alpha^x F(\tilde{\pi}JX, \tilde{\pi}X))$$

$$= c + \tfrac{3}{4}\sum_x (\alpha^x)^2 (F(\tilde{\pi}X, f\tilde{\pi}X))^2$$

$$= c + \tfrac{3}{4}\sum_x (\alpha^x)^2, \quad \text{which is constant.}$$

Remark. This agrees with the results in [1] on H^{2n+s}. H^{2n+s} is a principal toroidal bundle over PC^n and PC^n is of constant holomorphic curvature equal to 1. Also, $\alpha^x = 1$ for $x = 1, \cdots, s$ and H^{2n+s} was found to be of constant f-sectional curvature equal to $1 - 3s/4$.

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