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Sorin Dragomir
Giuseppe Tomassini

Differential Geometry and Analysis on CR Manifolds

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Sorin Dragomir
Università degli Studi della Basilicata
Dipartimento di Matematica
Contrada Macchia Romana
85100 Potenza
Italy

Giuseppe Tomassini
Scuola Normale Superiore
Classe di Scienze
Piazza dei Cavalieri 7
65126 Pisa
Italy

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Preface

A *CR manifold* is a C^∞ differentiable manifold endowed with a complex subbundle $T_{1,0}(M)$ of the complexified tangent bundle $T(M) \otimes \mathbf{C}$ satisfying $T_{1,0}(M) \cap \overline{T_{1,0}(M)} = (0)$ and the Frobenius (formal) integrability property

$$\left[\Gamma^\infty(T_{1,0}(M))\,,\ \Gamma^\infty(T_{1,0}(M))\right] \subseteq \Gamma^\infty(T_{1,0}(M)).$$

The bundle $T_{1,0}(M)$ is the *CR structure* of M, and C^∞ maps $f : M \to N$ of CR manifolds preserving the CR structures (i.e., $f_* T_{1,0}(M) \subseteq T_{1,0}(N)$) are *CR maps*. CR manifolds and CR maps form a category containing that of complex manifolds and holomorphic maps. The most interesting examples of CR manifolds appear, however, as real submanifolds of some complex manifold. For instance, any real hypersurface M in $\mathbf{C}^n$ admits a CR structure, naturally induced by the complex structure of the ambient space

$$T_{1,0}(M) = T^{1,0}(\mathbf{C}^n) \cap [T(M) \otimes \mathbf{C}].$$

Let $(z^1, \dots, z^n)$ be the natural complex coordinates on $\mathbf{C}^n$. Locally, in a neighborhood of each point of M, one may produce a frame $\{L_\alpha : 1 \le \alpha \le n-1\}$ of $T_{1,0}(M)$. Geometrically speaking, each L_α is a (complex) vector field tangent to M. From the point of view of the theory of PDEs, the L_α's are purely tangential first-order differential operators

$$L_\alpha = \sum_{j=1}^{n} a_\alpha^j(z) \frac{\partial}{\partial z^j}, \quad 1 \le \alpha \le n-1,$$

and $T_{1,0}(M)$ may be thought of as a bundle-theoretic recasting of the first-order PDE system with complex-valued C^∞ coefficients

$$\overline{L}_\alpha u(z) = 0, \quad 1 \le \alpha \le n-1,$$

called the *tangential Cauchy–Riemann equations*. These may be equally thought of as being induced on M by the Cauchy–Riemann equations in $\mathbf{C}^n$. *CR functions* are solutions $u(z)$ to the tangential Cauchy–Riemann equations, and any holomorphic function defined on a neighborhood of M will restrict to a CR function on M.

These introductory remarks lead to two fundamental problems in CR geometry and analysis. Given an (abstract) CR manifold, is it possible to realize it as a CR submanifold of $\mathbf{C}^n$ (of some complex manifold)? This is known as the *embeddability problem*, introduced to mathematical practice by J.J. Kohn [246]. The second problem is whether a given CR function $u : M \to \mathbf{C}$ extends to a holomorphic function defined on some neighborhood of M (the *CR extension problem*). Both these problems have local and global aspects, present many intricacies, and involve scientific knowledge from many mathematical fields. The solution to the local embeddability problem is due to A. Andreotti and C.D. Hill [13] in the real analytic category. Partial solutions in the C^∞ category are due to L. Boutet de Monvel [77], M. Kuranishi [263], and T. Akahori [2]. As to the CR extension problem, it is the object of intense investigation, cf. the monographs by A. Boggess [70] and M.S. Baouendi, P. Ebenfelt, and L.P. Rothschild [31] for an account of the present scientific achievements in this direction.

It should become clear from this discussion that CR manifolds and their study lie at the intersection of three main mathematical disciplines: the theory of partial differential equations, complex analysis in several variables, and differential geometry. While the analysis and PDE aspects seem to have captured most of the interest within the mathematical community, there has been, over the last ten or fifteen years, some effort to understand the differential-geometric side of the subject as well. It is true that A. Bejancu's discovery [55] of CR submanifolds signaled the start of a large number of investigations in differential geometry, best illustrated by the monographs by K. Yano and M. Kon [446], A. Bejancu [56], and S. Dragomir and L. Ornea [125]. Here by a *CR submanifold* we understand a real submanifold M of a Hermitian manifold (X, J, g), carrying a distribution $H(M)$ that is J-invariant (i.e., $JH(M) = H(M)$) and whose g-orthogonal complement is J-anti-invariant (i.e., $JH(M)^\perp \subseteq T(M)^\perp$, where $T(M)^\perp \to M$ is the normal bundle of M in X). The notion (of a CR submanifold of a Hermitian manifold) unifies concepts such as invariant, anti-invariant, totally real, semi-invariant, and generic submanifolds. Also, the observation (due to D.E. Blair and B.Y. Chen [64]) that proper CR submanifolds, in the sense of A. Bejancu, are actually CR manifolds shows that these investigations have the same central object, the CR category, as defined at the beginning of this preface, or by S. Greenfield [187]. The study of CR submanifolds in Hermitian manifolds, in the sense of A. Bejancu, has led to the discovery of many refined differential-geometric properties (e.g., K. Yano and M. Kon's classification of CR submanifolds of a complex projective space, with semiflat normal connection, parallel f-structure in the normal bundle, and the covariant derivative of the second fundamental form of constant length [445]) and will surely develop further within its own borders. It should be remarked nevertheless that as confined to Riemannian geometry (i.e., to the theory of submanifolds in Riemannian manifolds, cf., e.g., [91]), the above-mentioned study is perhaps insufficiently related to the (pseudo) convexity properties of submanifolds in complex manifolds, as understood in analysis in several complex variables. To be more precise, if M is a real hypersurface in $\mathbf{C}^n$ then the first and second fundamental forms of the given immersion describe the way M is shaped, both intrinsically (Riemannian curvature) and extrinsically, yet do not describe a priori the intrinsic properties of M as related to its Levi form. As an extreme case, M may be Levi flat yet will always exhibit, say, curvature properties arising from its first fundamental form. Or to give a nondegenerate example, the boundary of the Siegel domain $\Omega_n = \{z = (z', z_n) \in \mathbf{C}^{n-1} \times \mathbf{C} : \mathrm{Im}(z_n) > \|z'\|^2\}$ (i.e., the *Heisenberg group*) admits a contact form θ with a positive definite Levi form G_θ, and hence $g_\theta = \pi_H G_\theta + \theta \otimes \theta$ (the *Webster metric*) is a Riemannian metric, yet none of the metrics $g_{\lambda\theta},\ \lambda \in C^\infty(M),\ \lambda > 0$, coincides with the metric induced on M by the flat Kähler metric of $\mathbf{C}^n$.

Central to the present monograph is the discovery, around 1977–78, of a canonical linear connection ∇ on each nondegenerate CR manifold M of hypersurface type (the *Tanaka–Webster connection*) due to independent investigations by N. Tanaka [398] and S. Webster [422]. ∇ parallelizes both the Levi form and the complex structure of the Levi, or maximally complex, distribution of M, resembles both the Levi-Civita connection of a Riemannian manifold, and the Chern connection of a Hermitian manifold, and is a foundational tool for the *pseudo-Hermitian geometry* of a (nondegenerate) CR manifold, which is the main subject of this book. Now the curvature properties of ∇ are indeed tied to the CR structure: for instance, the Chern curvature tensor $C_\beta{}^\alpha{}_{\lambda\overline{\sigma}}$, a CR invariant of M, is computable in terms of the curvature of ∇ (and its contractions, such as the *pseudo-Hermitian Ricci tensor* and the *pseudo-Hermitian scalar curvature*) and $C_\beta{}^\alpha{}_{\lambda\overline{\sigma}} = 0$ if and only if M is locally CR equivalent to the standard sphere in $\mathbf{C}^{n+1},\ \ n > 1$ (cf. S.S. Chern and J. Moser [99]). Variants of the Tanaka–Webster connection are known already in different contexts, e.g., on CR manifolds of higher CR codimension (R. Mizner [312]) or on contact Riemannian manifolds (S. Tanno [401]), whose almost CR structure is not integrable, in general.

After a detailed exposition of the basic facts of pseudo-Hermitian geometry of nondegenerate CR manifolds in Chapter 1, the present monograph introduces the main geometric object, the *Fefferman metric*, both a tool and object of investigation of the first magnitude. It is due to C. Fefferman [138], who first devised it as a (Lorentz) metric on $(\partial\Omega) \times S^1$, for a given strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$, in connection with the boundary behavior of the Bergman kernel of Ω and the solution to the Dirichlet problem for the (inhomogeneous) complex Monge–Ampére equation

$$\begin{cases} (-1)^{n+1} \det \begin{pmatrix} u & \partial u/\partial \overline{z}^k \\ \partial u/\partial z^j & \partial^2 u/\partial z^j \partial \overline{z}^k \end{pmatrix} = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(the existence, uniqueness, and regularity of the solution are due to S.Y. Cheng and S.T. Yau [97]). See Chapter 2 of this book. By the mathematical creation of F. Farris [137], and J.M. Lee [271], an intrinsic description of the Fefferman metric (as a Lorentz metric on

$$C(M) = \left(\Lambda^{n+1,0}(M) \setminus \{0\}\right) \Big/ \mathbf{R}_+,$$

where n is the CR dimension) is available. Also, the work of G. Sparling [377], C.R. Graham [182], L. Koch [242]–[244], helped clarify a number of geometric facts (e.g., how the Fefferman metric may be singled out, in terms of curvature properties, from the set of all Lorentz metrics on $C(M)$ (cf. [182]), or providing a simple proof (cf. [242]) to H. Jacobowitz's theorem (cf. [220]) that nearby points on a strictly pseudoconvex CR manifold may be joined by a chain). Other properties are known, e.g., that certain Pontryagin forms of the Fefferman metric are obstructions to global CR equivalence to a sphere (and perhaps to global embeddability); cf. E. Barletta et al. [38]. The Fefferman metric remains however an insufficiently understood object and worth of further investigation.

One of the most spectacular results in this book is D. Jerison and J.M. Lee's solution (cf. [226]–[228] and our Chapter 3) to the *CR Yamabe problem*, which is the Yamabe problem for the Fefferman metric. As the Yamabe problem in Riemannian geometry (find a conformal transformation $\tilde{g} = fg,\ f > 0$, such that $\tilde{g}$ is of constant scalar curvature) the Yamabe problem for the Fefferman metric may be reformulated as a nonlinear PDE on $C(M)$ whose principal part is the Laplace–Beltrami operator of the metric; here the wave operator as the metric is Lorentzian, and hence nonelliptic. However, this equation may be shown (cf. [227]) to project on

$$b_n \Delta_b u + \rho u = \lambda u^{p-1},$$

the *CR Yamabe equation*, a nonlinear PDE on M, whose principal part is the *sub-Laplacian* Δ_b. The book presents the solution to the CR Yamabe problem only when $\lambda(M) \leq \lambda(S^{2n+1})$ (cf. Theorem 3.4 in Chapter 4), where $\lambda(M)$ is the CR analogue to the Yamabe invariant in Riemannian geometry; i.e.,

$$\lambda(M) = \inf\{\int_M \{b_n \|\pi_H \nabla u\|^2 + \rho u^2\}\theta \wedge (d\theta)^n : \int_M |u|^p \theta \wedge (d\theta)^n = 1\}.$$

The remaining case was dealt with by N. Gamara and R. Yacoub [164], who completed the solution to the CR Yamabe problem (see the comments at the end of Section 4.7). Δ_b is degenerate elliptic and subelliptic of order $1/2$ (and hence hypoelliptic). The authors of this book believe that subelliptic PDEs are bound to play within CR geometry the strong role played by elliptic theory in Riemannian geometry. A similar application is to use the Fefferman metric in the study of *pseudoharmonic maps*; cf. Chapter 4 (these are, locally, J. Jost and C.J. Xu's subelliptic harmonic maps; cf. [234]).

Another main theme of the book is represented by *pseudo-Einsteinian structures* (i.e., contact forms such that the pseudo-Hermitian Ricci tensor of their Tanaka–Webster connection is proportional to the Levi form) and the problem of local and global existence of pseudo-Einsteinian structures on CR manifolds. We present the achievements in the field, together with the *Lee conjecture* [that each compact strictly pseudoconvex CR manifold whose CR structure has a vanishing first Chern class ($c_1(T_{1,0}(M)) = 0$) must possess some global pseudo-Einsteinian structure]. The global problem turns out to be related to the theory of *CR immersions*, certain aspects of which are discussed in Chapter 6. The source mainly used for discussing pseudo-Einsteinian structures is, of course, the original paper [270]. However, our works [121] (solving the Lee conjecture on a compact strictly pseudoconvex CR manifold admitting a contact form whose corresponding characteristic direction is *regular* in the sense of R. Palais) [37] (demonstrating pseudo-Einsteinian contact forms on (total spaces of) tangent sphere bundles over real space forms $M^n(1)$) [68] (taking into account the relationship between the pseudo-Einsteinian condition and pseudo-Hermitian holonomy, i.e., the holonomy of the Tanaka–Webster connection), and the work by M.B. Stenzel [386] (producing pseudo-Einsteinian structures on boundaries of tubes $T^{*\epsilon}X$ over harmonic Riemannian manifolds (X, g)), extend the knowledge about pseudo-Einsteinian structures somewhat beyond the starting point of J.M. Lee [270]. As to the relationship between the global existence problem of pseudo-Einsteinian structures and the theory of CR, or rather pseudo-Hermitian, immersions (cf. [424] and [120]), let us mention that the Lee class may be interpreted as an obstruction to the existence of a pseudo-Hermitian immersion $f : M \to S^{2N+1}$, of a strictly pseudoconvex CR manifold M into an odd-dimensional sphere, such that f has flat normal Tanaka–Webster connection $\nabla^{\perp}$ (cf. [36] and the corollary to Theorem 6.1 in this book). The *Lee class* is a cohomology class $\gamma(M) \in H^1(M, \mathcal{P})$ with coefficients in the sheaf $\mathcal{P}$ of CR-pluriharmonic functions on M, as devised by J.M. Lee [270], such that $\gamma(M) = 0$ if and only if M admits a globally defined pseudo-Einsteinian contact form.

We deal with quasiconformal mappings of CR manifolds (a subject developed mainly by A. Korányi and H. Reimann [254]–[255]) in Chapter 7, with H. Urakawa's Yang–Mills connections (cf. [412]) on CR manifolds in Chapter 8, and with spectral geometry of CR manifolds (cf. A. Greenleaf [186]) in Chapter 9. A previous version of this text contained material devoted to the interplay between CR geometry and foliation theory, which in the meanwhile grew into an independent volume. While the presentation in Chapter 7 owes, as mentioned above, to A. Korányi and H. Reimann (cf. op. cit.), the observation that the ordinary Beltrami equations in several complex variables (cf. [419]) induce on $tial\Omega_n$ (the boundary of the Siegel domain Ω_n) the (tangential) Beltrami equations considered by A. Korányi and H. Reimann is new (cf. [41]). It is interesting to note that given a strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$, any biholomorphism F of Ω lifts to a C^∞ map

$$F^\sharp : \partial\Omega \times S^1 \to \partial\Omega \times S^1 \ , \quad F^\sharp(z, \gamma) := (F(z), \gamma - \arg(\det F'(z))),$$

preserving the "extrinsic" Fefferman metric (2.62) up to a conformal factor

$$\left(F^\sharp\right)^* g = \left|\det F'(z)\right|^{2/(n+1)} g \ \ ,$$

(cf. [138], p. 402, or by a simple calculation based on (2.62) in Chapter 2 of this book). When F is only a symplectomorphism of (Ω, ω), with $\omega := -i\partial\overline{\partial}\log K(z, z))$, extending smoothly to the boundary, a fundamental result of A. Korányi and H. Reimann, presented in Chapter 7, is that the boundary values f of such F constitute a contact transformation. Thus, in general, F is not a holomorphic map, nor are its boundary values f a CR map, both phenomena manifesting in the presence of a "dilatation" ($\mathrm{dil}(F)$ for F, and μ_f for f, themselves related in the limit as $z \to \partial\Omega$, cf. Theorem 7.7 in Chapter 7). Although $f^*G_\theta = \lambda_f G_\theta$ fails to hold (since f is not CR), one may "adjust" the complex structure J on $H(\partial\Omega)$ (cf. section 7.1) and get a new complex structure J_f such that $f^*G_\theta = \lambda_f G_f$, where $G_f(X, Y) := (d\theta)(X, J_f Y)$, $X, Y \in H(M)$. The problem of computing the Fefferman metric of (M, J_f, θ), or more generally of investigating the relationship (if any) between F_θ and the symplectomorphisms of (Ω, ω), remains unsolved.

As to Chapter 8, let us mention that while solving the inhomogeneous Yang–Mills equation

$$d_D^* R^D = 4ni\{d_M^c \rho - \rho\,\theta\} \otimes I \tag{0.1}$$

for a Hermitian connection $D \in \mathcal{C}(E, h)$, pseudo-Einsteinian structures come once again into the picture in a surprising way. The canonical line bundle $K(M) \to M$ over a pseudo-Einsteinian manifold M is a *quantum bundle* (in the sense of [259]): this is just the condition that the canonical S-connection of $K(M)$ has curvature of type $(1, 1)$, and one may use Theorem 8.2 to *explicitly* solve (0.1) (demonstrating among other things the strength of the purely differential-geometric approach to the study of the inhomogeneous Yang–Mills equations on CR manifolds).

This book also aims to explain how certain results in classical analysis apply to CR geometry (part of the needed material is taken from the fundamental paper by G.B. Folland and E.M. Stein [150]). This task, together with the authors' choice to give detailed proofs to a number of geometric facts, is expected to add to the clarity of exposition. It surely added in volume and prevented us from including certain modern, and still growing, subjects. A notable example is the theory of *homogeneous CR manifolds* (cf. H. Azad, A. Huckleberry, and W. Richthofer [26], A. Krüger [262], R. Lehmann and D. Feldmueller [277], and D.V. Alekseevski and A. Spiro [9]–[10]). See however our notes at the end of Chapter 5. Another absent protagonist is the theory of *deformation of CR structures* (cf. T. Akahori [3]–[6], T. Akahori and K. Miyajima [7], R.O. Buchweitz and J.J. Millson [78], J.J. Millson [302], and K. Miyajima [306]–[311]). The same holds for more recent work, such as H. Baum's (cf. [49]) on spinor calculus in the presence of the Fefferman metric, and F. Loose's (cf. [288]) initiating a study of the CR moment map, perhaps related to that of CR orbifolds (cf. [128]).

We may conclude that such objects as the Tanaka–Webster connection, the Fefferman metric, and pseudo-Einsteinian structures constitute the leitmotif of this book. More precisely, this book is an attempt to understand certain aspects of the relationship between Lorentzian geometry (on $(C(M), F_\theta)$) and pseudo-Hermitian geometry (on (M, θ)), a spectacular part of which is the relationship between hyperbolic and subelliptic PDEs (as demonstrated in Sections 2.5 and 4.4.3 of this monograph). The authors found a powerful source of techniques and ideas in the scientific creation of S.M. Webster and J.M. Lee, to whose papers they returned again and again over the years, and to whom they wish to express their gratitude.

Sorin Dragomir
Giuseppe Tomassini
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Differential Geometry and Analysis on CR Manifolds

1

CR Manifolds

Let Ω be a smooth domain in $\mathbf{C}^{n+1}$, i.e., there is an open neighborhood $U \supset \overline{\Omega}$ and a real-valued function $\rho \in C^2(U)$ such that $\Omega = \{z \in U : \rho(z) > 0\}$, $\mathbf{C}^{n+1} \setminus \overline{\Omega} = \{z \in U : \rho(z) < 0\}$, the boundary of Ω is given by $\partial\Omega = \{z \in U : \rho(z) = 0\}$, and $D\rho(z) \neq 0$ for any $z \in \partial\Omega$. Here $D\rho$ is the gradient

$$D\rho = \left(\frac{\partial \rho}{\partial x^1}, \dots, \frac{\partial \rho}{\partial x^{2n+2}}\right)$$

and $(x^1, \dots, x^{2n+2})$ are the Cartesian coordinates on $\mathbf{R}^{2n+2} \simeq \mathbf{C}^{n+1}$.

The Cauchy–Riemann equations in $\mathbf{C}^{n+1}$ induce on $\partial\Omega$ an overdetermined system of PDEs with smooth complex-valued coefficients

$$\overline{L}_\alpha u(z) \equiv \sum_{j=1}^{n+1} a_\alpha^j(z) \frac{\partial u}{\partial \overline{z}^j} = 0, \quad 1 \leq \alpha \leq n \tag{1.1}$$

(the *tangential Cauchy–Riemann equations*), $z \in V$, with $V \subseteq (\partial\Omega) \cap U$ open. Here L_α are linearly independent (at each point of V) and

$$\sum_{j=1}^{n+1} \overline{a}_\alpha^j(z) \frac{\partial \rho}{\partial z^j} = 0, \quad 1 \leq \alpha \leq n, \tag{1.2}$$

for any $z \in V$, i.e., L_α are purely tangential first-order differential operators (tangent vector fields on $\partial\Omega$). It then follows that

$$\left[L_\alpha, L_\beta\right] = C_{\alpha\beta}^\gamma(z) L_\gamma \tag{1.3}$$

for some complex-valued smooth functions $C_{\alpha\beta}^\gamma$ on V.

At each point $z \in V$ the $L_{\alpha,z}$'s span a complex n-dimensional subspace $T_{1,0}(\partial\Omega)_z$ of the complexified tangent space $T_z(\partial\Omega) \otimes_{\mathbf{R}} \mathbf{C}$. The bundle $T_{1,0}(\partial\Omega) \to \partial\Omega$ is the *CR structure* of $\partial\Omega$, and a bundle-theoretic recast of (1.1)–(1.3) consists in observing that

$$T_{1,0}(\partial\Omega) = [T(\partial\Omega) \otimes \mathbf{C}] \cap T^{1,0}(\mathbf{C}^{n+1}), \tag{1.4}$$

where $T^{1,0}(\mathbf{C}^{n+1})$ is the holomorphic tangent bundle over $\mathbf{C}^{n+1}$, and that $M = \partial\Omega$ satisfies the axioms (1.5)–(1.6) below. A C^1 function $u : \partial\Omega \to \mathbf{C}$ is a *CR function* if $\overline{Z}(u) = 0$ for any $Z \in T_{1,0}(\partial\Omega)$. Locally, a CR function is a solution of (1.1).

The pullback (via $j : \partial\Omega \subset U$) of the complex 1-form $\frac{i}{2}(\overline{\partial} - \partial)\rho$ is a *pseudo-Hermitian structure* θ on $\partial\Omega$. When $\partial\Omega$ is *nondegenerate* θ is a contact form. Everything stated above holds should one replace the boundary $\partial\Omega$ by some (open piece of a) smooth real hypersurface in $\mathbf{C}^{n+1}$.

As observed by N. Tanaka [398], and S. Webster [422], when θ is a contact form M may be described in terms of *pseudo-Hermitian geometry* (a term coined as a *seguito* of the—fundamental to this book—paper [422]), which complements the (better-known) *contact Riemannian geometry* (cf. [62]) and is well suited for capturing the convexity properties of M (as familiar in the analysis in several complex variables). M carries a semi-Riemannian metric g_θ (Riemannian, if M is strictly pseudoconvex) coinciding with the Levi form along the maximal complex distribution of M. This is the *Webster metric* (cf. Section 1.1.3). Of course, M carries also the Riemannian metric induced from the (flat Kähler) metric of $\mathbf{C}^{n+1}$, and the pseudo-Hermitian and contact Riemannian geometries do interact. However, that the two are quite different in character should be emphasized: for instance, *none* of the Webster metrics $g_{\lambda\theta}$, for every smooth $\lambda : M \to (0, +\infty)$, of the boundary of the Siegel domain $\Omega_{n+1} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C} : \operatorname{Im}(w) > \|z\|^2\}$ coincides with the metric induced from $\mathbf{C}^{n+1}$.

CR manifolds as in (1.3), or in (1.12) below, are *embedded*. The currently accepted concept of a CR manifold as a tool for studying the tangential Cauchy–Riemann equations by geometric methods is, however, more general. The manifold may be abstract, and not all CR manifolds embed, even locally (cf. Section 1.6). The CR codimension (cf. Section 1.1) may be > 1, and distinct from the codimension (when M is a CR submanifold). The Levi form may be vector, rather than scalar, valued (and then there is no natural notion of strict pseudoconvexity) or degenerate (and then the tools of pseudo-Hermitian geometry are not available).

According to our purposes in this book, that is, to describe (1.1) by means of pseudo-Hermitian geometry, we shall assume integrability, nondegeneracy, and CR codimension 1. The reader should nevertheless be aware of the existence of a large literature, with similar expectations, and not subject to our hypothesis.[1]

[1] For instance, H. Rossi and M. Vergne [356], deal with CR manifolds $\Sigma = \Sigma(V, N, E)$ of the form $\Sigma = \{(x + iy, u) : x, y \in \mathbf{R}^n,\ u \in E,\ y - N(u) \in V\}$, where E is a domain in $\mathbf{C}^m$, $N : E \to \mathbf{R}^n$ is a smooth function, and V is a submanifold in $\mathbf{R}^n$. These are in general degenerate (the Levi form has a nontrivial null space); yet this is not relevant to the purpose of analysis on Σ. Indeed, it may be shown (by a partial Fourier transform technique) that the CR functions on Σ satisfy a Paley–Wiener (type) theorem (cf. [356], p. 306). An application of their result (to constant-coefficient PDEs on the Heisenberg group) is given by D.E. Blair et al. [67]. To give one more example, we may quote the long series of papers by C.D. Hill and M. Nacinovich [202], in which the CR codimension is always > 1, and eventually only a small amount of pseudoconcavity is prescribed (cf. [203]).

Chapter 1 is organized as follows. In Section 1.1 we discuss the fundamentals (CR structures, the Levi form, characteristic directions, etc.) and examples (e.g., CR Lie groups). Sections 1.2 to 1.5 are devoted to the construction and principal properties of what appears to be the main geometric tool through this book, the *Tanaka–Webster connection*. Ample space is dedicated to curvature properties, details of which appear nowhere else in the mathematical literature, and to applications (due to S. Webster [422]) of the Chern–Moser theorem (CR manifolds with a vanishing Chern tensor are locally CR isomorphic to spheres) to several *pseudo-Hermitian space forms*. In Section 1.6 we discuss CR structures as G-structures and hint at some open problems.

1.1 CR manifolds

1.1.1 CR structures

Let M be a real m-dimensional C^∞ differentiable manifold. Let $n \in \mathbf{N}$ be an integer such that[2] $1 \leq n \leq [m/2]$. Let $T(M) \otimes \mathbf{C}$ be the complexified tangent bundle over M. Elements of $T(M) \otimes \mathbf{C}$ are of the form $u \otimes 1 + v \otimes i$, where $u, v \in T(M)$ are real tangent vectors (and $i = \sqrt{-1}$). For simplicity, we drop the tensor products and write merely $u + iv$ (a *complex* tangent vector on M). The following definition is central to this book.

Definition 1.1. Let us consider a complex subbundle $T_{1,0}(M)$ of the complexified tangent bundle $T(M) \otimes \mathbf{C}$, of complex rank n. If

$$T_{1,0}(M) \cap T_{0,1}(M) = (0) \tag{1.5}$$

then $T_{1,0}(M)$ is called an *almost CR structure* on M. Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and throughout an overbar denotes complex conjugation. The integers n and $k = m - 2n$ are respectively the *CR dimension* and *CR codimension* of the almost CR structure and (n, k) is its *type*. A pair $(M, T_{1,0}(M))$ consisting of an almost CR structure of type (n, k) is an *almost CR manifold* (of type (n, k)). $\square$

It is easy to see that an almost CR manifold of type $(n, 0)$ is an almost complex manifold (cf., e.g., [241], vol. II, p. 121).

Given a vector bundle $E \to M$ we denote by $\Gamma^\infty(U, E)$ the space of all C^∞ cross-sections in E defined on the open subset $U \subseteq M$. We often write $\Gamma^\infty(E)$ for $\Gamma^\infty(M, E)$ (the space of globally defined smooth sections). Also E_x is the fiber in E over $x \in M$.

Definition 1.2. An almost CR structure $T_{1,0}(M)$ on M is (*formally*) *integrable* if for any open set $U \subseteq M$,

$$\left[\Gamma^\infty(U, T_{1,0}(M)), \Gamma^\infty(U, T_{1,0}(M))\right] \subseteq \Gamma^\infty(U, T_{1,0}(M)). \tag{1.6}$$

[2] If $a \in \mathbf{R}$ then $[a] \in \mathbf{Z}$ denotes the integer part of a.

That is, for any two complex vector fields Z, W (defined on $U \subseteq M$) belonging to $T_{1,0}(M)$, their Lie bracket $[Z, W]$ belongs to $T_{1,0}(M)$, i.e., $[Z, W]_x \in T_{1,0}(M)_x$ for any $x \in U$. An integrable almost CR structure (of type (n, k)) is referred to as a *CR structure* (of type (n, k)), and a pair $(M, T_{1,0}(M))$ consisting of a C^∞ manifold and a CR structure (of type (n, k)) is a *CR manifold* (of type (n, k)). □

CR manifolds are the objects of a category whose arrows are smooth maps preserving CR structures. Precisely we have the following definition:

Definition 1.3. Let $(M, T_{1,0}(M))$ and $(N, T_{1,0}(N))$ be two CR manifolds (of arbitrary, but fixed type). A C^∞ map $f : M \to N$ is a *CR map* if

$$(d_x f)T_{1,0}(M)_x \subseteq T_{1,0}(N)_{f(x)}, \tag{1.7}$$

for any $x \in M$, where $d_x f$ is the ($\mathbf{C}$-linear extension to $T_x(M)\otimes_{\mathbf{R}}\mathbf{C}$ of the) differential of f at x. □

It is easy to see that the complex manifolds and holomorphic maps form a subcategory of the category of CR manifolds and CR maps.

Let $(M, T_{1,0}(M))$ be a CR manifold of type (n, k). Its *maximal complex*, or *Levi, distribution* is the real rank $2n$ subbundle $H(M) \subset T(M)$ given by

$$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

It carries the complex structure $J_b : H(M) \to H(M)$ given by

$$J_b(V + \overline{V}) = i(V - \overline{V}),$$

for any $V \in T_{1,0}(M)$. Here $i = \sqrt{-1}$. The (formal) integrability requirement (1.6) is equivalent to

$$[J_b X, Y] + [X, J_b Y] \in \Gamma^\infty(U, H(M)), \tag{1.8}$$

$$[J_b X, J_b Y] - [X, Y] = J_b\{[J_b X, Y] + [X, J_b Y]\}, \tag{1.9}$$

for any $X, Y \in \Gamma^\infty(U, H(M))$; cf. S. Greenfield [187]. This is formally similar to the notion of integrability of an almost complex structure (cf., e.g., [241], vol. II, p. 124). It should be noted, however, that in contrast to the case of an almost complex manifold, J_b is not defined on the whole of $T(M)$, and $[J_b X, Y] + [X, J_b Y]$ may not lie in $H(M)$ (thus requiring the axiom (1.8)). Proving the equivalence of (1.6) and (1.8)–(1.9) is an easy exercise and therefore left to the reader.

Let $f : M \to N$ be a CR mapping. Then (1.7) is equivalent to the prescriptions

$$(d_x f)H(M)_x \subseteq H(N)_{f(x)} \tag{1.10}$$

and

$$(d_x f) \circ J_{b,x} = J^N_{b,f(x)} \circ (d_x f), \tag{1.11}$$

for any $x \in M$. Here $J^N_b : H(N) \to H(N)$ denotes the complex structure in the Levi distribution $H(N)$ of N.

Definition 1.4. $f : M \to N$ is a *CR isomorphism* (or a *CR equivalence*) if f is a C^∞ diffeomorphism and a CR map. □

CR manifolds arise mainly as real submanifolds of complex manifolds. Let V be a complex manifold, of complex dimension N, and let $M \subset V$ be a real m-dimensional submanifold. Let us set

$$T_{1,0}(M) = T^{1,0}(V) \cap [T(M) \otimes \mathbf{C}], \tag{1.12}$$

where $T^{1,0}(V)$ is the holomorphic tangent bundle over V, i.e., locally the span of $\{\partial/\partial z^j : 1 \leq j \leq N\}$, where $(z^1, \dots, z^N)$ are local complex coordinates on V. The following result is immediate:

Proposition 1.1. *If M is a real hypersurface* $(m = 2N - 1)$ *then $T_{1,0}(M)$ is a CR structure of type* $(N - 1, 1)$.

In general the complex dimension of $T_{1,0}(M)_x$ may depend on $x \in M$. Nevertheless, if

$$\dim_{\mathbf{C}} T_{1,0}(M)_x = n \quad (= \ \text{const})$$

then $(M, T_{1,0}(M))$ is a CR manifold (of type (n, k)). The reader will meet no difficulty in checking both statements as a consequence of the properties $T^{1,0}(V) \cap T^{0,1}(V) = (0)$ (where $T^{0,1}(V) = \overline{T^{1,0}(V)}$) and $Z, W \in T^{1,0}(V) \Longrightarrow [Z, W] \in T^{1,0}(V)$.

Definition 1.5. If $k = 2N - m$ (i.e., the CR codimension of $(M, T_{1,0}(M))$ and the codimension of M as a real submanifold of V coincide) then $(M, T_{1,0}(M))$ is termed *generic*. □

1.1.2 The Levi form

Significant portions of the present text will be devoted to the study of CR manifolds of CR codimension $k = 1$ (referred to as well as CR manifolds of *hypersurface type*). Let M be a connected CR manifold of type $(n, 1)$. Assume M to be orientable. Let

$$E_x = \{\omega \in T_x^*(M) : \mathrm{Ker}(\omega) \supseteq H(M)_x\},$$

for any $x \in M$. Then $E \to M$ is a real line subbundle of the cotangent bundle $T^*(M) \to M$ and $E \simeq T(M)/H(M)$ (a vector bundle isomorphism). Since M is orientable and $H(M)$ is oriented by its complex structure J_b, it follows that E is orientable. Any orientable real line bundle over a connected manifold is trivial, so there exist globally defined nowhere vanishing sections $\theta \in \Gamma^\infty(E)$.

Definition 1.6. Any such section θ is referred to as a *pseudo-Hermitian structure* on M. Given a pseudo-Hermitian structure θ on M the *Levi form* L_θ is defined by

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \tag{1.13}$$

for any $Z, W \in T_{1,0}(M)$. □

Since $E \to M$ is a real line bundle, any two pseudo-Hermitian structures $\theta, \hat{\theta} \in \Gamma^\infty(E)$ are related by

$$\hat{\theta} = \lambda\, \theta, \tag{1.14}$$

for some nowhere-zero C^∞ function $\lambda : M \to \mathbf{R}$. Let us apply the exterior differentiation operator d to (1.14). We get

$$d\hat{\theta} = d\lambda \wedge \theta + \lambda d\theta.$$

Since $\mathrm{Ker}(\theta) = H(M)$ (the $\mathbf{C}$-linear extension of) θ vanishes on $T_{1,0}(M)$ and $T_{0,1}(M)$ as well. Consequently, the Levi form changes according to

$$L_{\hat{\theta}} = \lambda L_\theta \tag{1.15}$$

under any transformation (1.14) of the pseudo-Hermitian structure. This leads to a largely exploited analogy between CR and conformal geometry (cf., e.g., J.M. Lee [270, 271], D. Jerison and J.M. Lee [227, 228], C.R. Graham [182], etc.), a matter we will treat in detail in the subsequent chapters of this text.

Let $(M, T_{1,0}(M))$ be an orientable CR manifold of type $(n, 1)$ (of hypersurface type) and θ a fixed pseudo-Hermitian structure on M. Define the bilinear form G_θ by setting

$$G_\theta(X, Y) = (d\theta)(X, J_b Y), \tag{1.16}$$

for any $X, Y \in H(M)$. Note that L_θ and (the $\mathbf{C}$-bilinear extension to $H(M) \otimes \mathbf{C}$ of) G_θ coincide on $T_{1,0}(M) \otimes T_{0,1}(M)$. Then

$$G_\theta(J_b X, J_b Y) = G_\theta(X, Y), \tag{1.17}$$

for any $X, Y \in H(M)$. Indeed, we may (by (1.8)–(1.9)) perform the following calculation:

$$\begin{aligned} G_\theta(J_b X, J_b Y) - G_\theta(X, Y) &= -(d\theta)(J_b X, Y) - (d\theta)(X, J_b Y) \\ &= \frac{1}{2}\{\theta([J_b X, Y]) + \theta([X, J_b Y])\} \\ &= \frac{1}{2}\theta(J_b\{[X, Y] - [J_b X, J_b Y]\}) = 0, \end{aligned}$$

since $\theta \circ J_b = 0$. In particular, G_θ is symmetric.

Definition 1.7. We say that $(M, T_{1,0}(M))$ is *nondegenerate* if the Levi form L_θ is nondegenerate (i.e., if $Z \in T_{1,0}(M)$ and $L_\theta(Z, \overline{W}) = 0$ for any $W \in T_{1,0}(M)$ then $Z = 0$) for some choice of pseudo-Hermitian structure θ on M. If L_θ is positive definite (i.e., $L_\theta(Z, \overline{Z}) > 0$ for any $Z \in T_{1,0}(M)$, $Z \neq 0$) for some θ, then $(M, T_{1,0}(M))$ is said to be *strictly pseudoconvex*. □

If θ and $\hat{\theta} = \lambda\theta$ are two pseudo-Hermitian structures, then as a consequence of (1.15), L_θ is nondegenerate if and only if $L_{\hat{\theta}}$ is nondegenerate. Hence nondegeneracy is a *CR-invariant* property, i.e., it is invariant under a transformation (1.14). Of course, strict pseuoconvexity is not a CR-invariant property. Indeed, if L_θ is positive definite then $L_{-\theta}$ is negative definite.

Definition 1.8. Let M be a nondegenerate CR manifold and θ a fixed pseudo-Hermitian structure on M. The pair (M, θ) is referred to as a *pseudo-Hermitian manifold*. □

Let $f : M \to N$ be a CR map and θ, θ_N pseudo-Hermitian structures on M and N, respectively. Then $f^*\theta_N = \lambda\,\theta$, for some $\lambda \in C^\infty(M)$.

Definition 1.9. Let M and N be two CR manifolds and θ, θ_N pseudo-Hermitian structures on M and N, respectively. We say that a CR map $f : M \to N$ is a *pseudo-Hermitian map* if $f^*\theta_N = c\,\theta$, for some $c \in \mathbf{R}$. If $c = 1$ then f is referred to as an *isopseudo-Hermitian map*. □

Let us go back for a moment to the case of CR manifolds of arbitrary type. If $(M, T_{1,0}(M))$ is a CR manifold of type (n, k) then its *Levi form* is defined as follows. Let $x \in M$ and $v, w \in T_{1,0}(M)_x$. We set

$$L_x(v, w) = i\ \pi_x \left[V, \overline{W}\right]_x ,$$

where $\pi : T(M) \otimes \mathbf{C} \to (T(M) \otimes \mathbf{C})/(H(M) \otimes \mathbf{C})$ is the natural bundle map and $V, W \in \Gamma^\infty(T_{1,0}(M))$ are arbitrary C^∞ extensions of v, w (i.e., $V_x = v$, $W_x = w$). The definition of $L_x(v, w)$ does not depend on the choice of extensions of v, w because of

$$\pi_x \left[V, \overline{W}\right]_x = v^\alpha \overline{w^\beta} \pi_x \left[T_\alpha, T_{\overline{\beta}}\right]_x ,$$

where $v = v^\alpha T_{\alpha,x}$ and $w = w^\alpha T_{\alpha,x}$, for some local frame $\{T_1, \ldots, T_n\}$ of $T_{1,0}(M)$ defined on an open neighborhood of x (and $T_{\overline{\alpha}} = \overline{T_\alpha}$). Then $(M, T_{1,0}(M))$ is said to be *nondegenerate* if L is nondegenerate. However, since for $k \geq 2$ the Levi form L is vector valued, there isn't any obvious way to generalize the notion of strict pseudoconvexity (to the arbitrary CR codimension case).

Let us check that the new and old concepts of Levi form coincide (up to an isomorphism) when $k = 1$. Let $(M, T_{1,0}(M))$ be an oriented CR manifold (of hypersurface type) and θ a pseudo-Hermitian structure on M. Consider the bundle isomorphism

$$\Phi_\theta : \frac{T(M) \otimes \mathbf{C}}{H(M) \otimes \mathbf{C}} \to E$$

given by

$$(\Phi_\theta)_x\,(v + H(M)_x \otimes_{\mathbf{R}} \mathbf{C}) = \theta_x(v)\,\theta_x ,$$

for any $v \in T_x(M) \otimes \mathbf{C},\ x \in M$. Then

$$(\Phi_\theta)_x\, L_x(v, w) = 2\,(L_\theta)_x\,(v, w)\,\theta_x ,$$

for any $v, w \in T_{1,0}(M)_x$. □

1.1.3 Characteristic directions on nondegenerate CR manifolds

Let $(M, T_{1,0}(M))$ be a CR manifold and θ a fixed pseudo-Hermitian structure on M. As a consequence of the formal integrability property,

$$(d\theta)(Z, W) = 0, \quad Z, W \in T_{1,0}(M).$$

Of course, in the preceding identity $d\theta$ is thought of as extended by **C**-linearity (to $T(M) \otimes \mathbf{C}$). Moreover, we also have $(d\theta)(\overline{Z}, \overline{W}) = 0$ (by complex conjugation), for any $Z, W \in T_{1,0}(M)$. If $(M, T_{1,0}(M))$ is nondegenerate then $d\theta$ is nondegenerate on $H(M)$. Indeed, let us assume that $X = Z + \overline{Z} \in H(M)$ $(Z \in T_{1,0}(M))$ and

$$(d\theta)(X, Y) = 0, \tag{1.18}$$

for any $Y \in H(M)$. Let $X \rfloor$ denote the interior product with X. For instance, $(X \rfloor d\theta)(Y) = (d\theta)(X, Y)$ for any $Y \in \mathcal{X}(M)$. When $X \rfloor d\theta$ is extended by **C**-linearity, (1.18) continues to hold for any $Y \in H(M) \otimes \mathbf{C} = T_{1,0}(M) \oplus T_{0,1}(M)$; hence

$$0 = (d\theta)(X, \overline{W}) = i\, L_\theta(Z, \overline{W}),$$

for any $W \in T_{1,0}(M)$. It follows that $Z = 0$. □

Consequently we may establish the following proposition:

Proposition 1.2. *There is a unique globally defined nowhere zero tangent vector field T on M such that*

$$\theta(T) = 1, \quad T \rfloor d\theta = 0. \tag{1.19}$$

T is transverse to the Levi distribution $H(M)$.

To prove Proposition 1.2 one uses the following fact of linear algebra (together with the orientability assumption).

Proposition 1.3. *Let V be a real $(n+1)$-dimensional linear space and $H \subset V$ an n-dimensional subspace. Let ω be a skew-symmetric bilinear form on V. Assume that ω is nondegenerate on H. Then there is $v_0 \in V$, $v_0 \neq 0$, such that $\omega(v_0, v) = 0$ for any $v \in V$.*

Proof. Let us set

$$K = \{v \in V : \omega(v, u) = 0,\ \forall\, u \in H\}.$$

The proof of Proposition 1.3 is organized in two steps, as follows.
Step 1. *K is 1-dimensional.*
Indeed, given a linear basis $\{e_1, \dots, e_{n+1}\}$ of V with $\{e_1, \dots, e_n\} \subset H$, then for each $v = \sum_{j=1}^{n+1} \lambda_j e_j \in K$ we have $A\lambda = 0$ where $A = [a_{jk}]$, $a_{jk} = \omega(e_j, e_k)$, $1 \le j \le n+1$, $1 \le k \le n$, and $\lambda = (\lambda_1, \dots, \lambda_{n+1})^T$. Thus K is the solution

space to the homogeneous system $AX = 0$, so that $\dim_{\mathbf{R}} K = n + 1 - \text{rank}(A)$ and $\text{rank}(A) = n$ (by the nondegeneracy of ω on H). $\square$

Step 2. $H \cap K = (0)$.

If $v \in H \cap K$ then $v \in K$ yields $\omega(v, u) = 0$ for any $u \in H$ such that $v = 0$ (since $v \in H$ and ω is nondegenerate). $\square$

By **Steps 1** and **2**, $V = H \oplus K$. Let (by **Step 1**) $v_0 \in K$, $v_0 \neq 0$. This is the vector we are looking for. Indeed, given any $v \in V$ there are $w \in H$ and $\lambda \in \mathbf{R}$ with $v = w + \lambda v_0$ such that $\omega(v_0, v) = 0$ by the skew-symmetry of ω. $\square$

The tangent vector field T determined by (1.19) is referred to as the *characteristic direction* of (M, θ). Next, we may state the following result:

Proposition 1.4. *Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold, θ a pseudo-Hermitian structure on M, and T the corresponding characteristic direction. Then*

$$T(M) = H(M) \oplus \mathbf{R}T. \tag{1.20}$$

Indeed, let $X \in T(M)$ and set $Y = X - \theta(X)T$. Then $\theta(Y) = 0$, i.e., $Y \in \text{Ker}(\theta) = H(M)$. Proposition 1.4 is proved.

Using (1.20) we may extend G_θ to a semi-Riemannian metric g_θ on M, which will play a crucial role in the sequel.

Definition 1.10. Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold and θ a pseudo-Hermitian structure on M. Let g_θ be the semi-Riemannian metric given by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$. This is called the *Webster metric* of (M, θ). $\square$

Assume that $(M, T_{1,0}(M))$ is nondegenerate. It is not difficult to check that the signature (r, s) of $L_{\theta,x}$ doesn't depend on $x \in M$. Also, (r, s) is a CR-invariant. Moreover, the signature of the Webster metric g_θ is $(2r + 1, 2s)$. If $(M, T_{1,0}(M))$ is strictly pseudoconvex and θ is chosen in such a way that L_θ is positive definite, then g_θ is a Riemannian metric on M. Let $\pi_H : T(M) \to H(M)$ be the projection associated with the direct sum decomposition (1.20). If $\pi_H G_\theta$ denotes the $(0, 2)$-tensor field on M given by $(\pi_H G_\theta)(X, Y) = G_\theta(\pi_H X, \pi_H Y)$, for any $X, Y \in T(M)$, then the Webster metric may be written as

$$g_\theta = \pi_H G_\theta + \theta \otimes \theta.$$

g_θ is not a CR-invariant. To write the transformation law for g_θ (under a transformation $\hat{\theta} = \lambda\theta$ of the pseudo-Hermitian structure) is a rather tedious exercise. Of course $G_{\hat{\theta}} = \lambda G_\theta$ and $\hat{\theta} \otimes \hat{\theta} = \lambda^2\, \theta \otimes \theta$; yet π_H transforms as well. We will return to this matter in Chapter 2.

1.1.4 CR geometry and contact Riemannian geometry

We start be recalling a few notions of contact Riemannian geometry, following for instance D.E. Blair [62]. Let M be a $(2n+1)$-dimensional C^∞ manifold. Let (ϕ, ξ, η) be a synthetic object, consisting of a $(1, 1)$-tensor field $\phi : T(M) \to T(M)$, a tangent vector field $\xi \in \mathcal{X}(M)$, and a differential 1-form η on M. (ϕ, ξ, η) is an *almost contact structure* if

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \;\; \eta(\xi) = 1, \quad \eta \circ \phi = 0.$$

An almost contact structure (ϕ, ξ, η) is said to be *normal* if

$$[\phi, \phi] + 2(d\eta) \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A Riemannian metric g on M is said to be *compatible* with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in T(M)$. A synthetic object (ϕ, ξ, η, g) consisting of an almost contact structure (ϕ, ξ, η) and a compatible Riemannian metric g is said to be an almost contact metric structure. Given an almost contact metric structure (ϕ, ξ, η, g) one defines a 2-form Ω by setting $\Omega(X, Y) = g(X, \phi Y)$. (ϕ, ξ, η, g) is said to satisfy the *contact condition* if $\Omega = d\eta$, and if this is the case, (ϕ, ξ, η, g) is called a *contact metric* structure on M. A contact metric structure (ϕ, ξ, η, g) that is also normal is called a *Sasakian structure* (and g is a *Sasakian metric*).

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold and θ a pseudo-Hermitian structure on M. Let T be the characteristic direction of (M, θ). Let us extend J_b to a $(1, 1)$-tensor field on M by requiring that

$$J_b T = 0. \tag{1.21}$$

Then (by summarizing properties, old and new)

$$\begin{aligned} J_b^2 &= -I + \theta \otimes T, \\ J_b T &= 0, \quad \theta \circ J_b = 0, \quad g_\theta(X, T) = \theta(X), \\ g_\theta(J_b X, J_b Y) &= g_\theta(X, Y) - \theta(X)\theta(Y), \end{aligned}$$

for any $X, Y \in T(M)$. Therefore, if $(M, T_{1,0}(M))$ is a strictly pseudoconvex CR manifold and θ is a contact form such that L_θ is positive definite, then $(J_b, T, \theta, g_\theta)$ is an almost contact metric structure on M. Also

$$\Omega = -d\theta, \tag{1.22}$$

where Ω is defined by

$$\Omega(X, Y) = g_\theta(X, J_b Y),$$

for any $X, Y \in T(M)$. That is, if we set $\phi = J_b$, $\xi = -T$, $\eta = -\theta$, and $g = g_\theta$, then (ϕ, ξ, η, g) is a *contact metric structure* on M, provided that $(M, T_{1,0}(M))$ is strictly pseudoconvex. By (1.20) it suffices to check (1.22) on $H(M) \otimes H(M)$, respectively on $H(M) \otimes \mathbf{R}T$ and $\mathbf{R}T \otimes \mathbf{R}T$. Let $X, Y \in H(M)$. Then (by (1.17))

$$\Omega(X, Y) = g_\theta(X, J_b Y) = (d\theta)(X, J_b^2 Y) = -(d\theta)(X, Y).$$

Finally (by (1.19))

$$\Omega(T, X) = 0 = -(d\theta)(T, X),$$

for any $X \in T(M)$. Strictly pseudoconvex CR manifolds are therefore contact Riemannian manifolds, in a natural way. However, they might fail to be normal. As we shall see in the sequel, the almost contact structure (J_b, T, θ) is normal if and only if the Tanaka–Webster connection of (M, θ) (to be introduced in Section 1.2) has a vanishing pseudo-Hermitian torsion ($\tau = 0$). The converse, that is, which almost contact manifolds are CR manifolds, was taken up by S. Ianuş [214]. Indeed, an almost contact manifold $(M, (\phi, \xi, \eta))$ possesses a natural almost CR structure $T_{1,0}(M)$ defined as the eigenbundle $\mathrm{Eigen}(J_b^{\mathbf{C}}; i)$ of $J_b^{\mathbf{C}}$ corresponding to the eigenvalue i. Here J_b is the restriction of ϕ to $H(M) = \mathrm{Ker}(\eta)$ and $J_b^{\mathbf{C}}$ is the $\mathbf{C}$-linear extension of J_b to $H(M) \otimes \mathbf{C}$. In general, $T_{1,0}(M)$ may fail to be integrable. By a result of S. Ianuş (cf. op. cit.), if (ϕ, ξ, η) is normal then $T_{1,0}(M)$ is a CR structure. The converse is not true, in general. The question (of characterizing the almost contact manifolds whose natural almost CR structure is integrable) has been settled by S. Tanno [401], who built a tensor field Q (in terms of (ϕ, ξ, η)) such that $Q = 0$ if and only if $T_{1,0}(M)$ is integrable. This matter (together with the implication $[\phi, \phi] + 2(d\eta) \otimes \xi = 0 \Longrightarrow Q = 0$) will be examined in the sequel.

1.1.5 The Heisenberg group

Let us set $\mathbf{H}_n = \mathbf{C}^n \times \mathbf{R}$, thought of as endowed with the natural coordinates $(z, t) = (z^1, \dots, z^n, t)$. $\mathbf{H}_n$ may be organized as a group with the group law

$$(z, t) \cdot (w, s) = (z + w, t + s + 2\mathrm{Im}\langle z, w\rangle),$$

where $\langle z, w\rangle = \delta_{jk} z^j \overline{w^k}$. This actually makes $\mathbf{H}_n$ into a Lie group, referred to as the *Heisenberg group*. A good bibliographical reference is the paper by G.B. Folland and E.M. Stein [150], pp. 434–437, yet the mathematical literature (dealing with both geometric and analysis aspects) on the Heisenberg group occupies a huge (and still growing) volume. Let us consider the complex vector fields on $\mathbf{H}_n$,

$$T_j = \frac{\partial}{\partial z^j} + i\overline{z}^j \frac{\partial}{\partial t}, \tag{1.23}$$

where

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)$$

and $z^j = x^j + iy^j$, $1 \le j \le n$. Let us define $T_{1,0}(\mathbf{H}_n)_{(z,t)}$ as the space spanned by the $T_{j,(z,t)}$'s, i.e.,

$$T_{1,0}(\mathbf{H}_n)_{(z,t)} = \sum_{j=1}^{n} \mathbf{C} T_{j,(z,t)}, \tag{1.24}$$

for any $(z, t) \in \mathbf{H}_n$. Since

$$[T_j, T_k] = 0, \quad 1 \le j, k \le n,$$

it follows that $(\mathbf{H}_n, T_{1,0}(\mathbf{H}_n))$ is a CR manifold of type $(n, 1)$ (a CR manifold of hypersurface type). Next, let us consider the real 1-form θ_0 on $\mathbf{H}_n$ defined by

$$\theta_0 = dt + i \sum_{j=1}^{n} \left(z^j d\overline{z}^j - \overline{z}^j dz^j \right). \tag{1.25}$$

Then θ_0 is a pseudo-Hermitian structure on $(\mathbf{H}_n, T_{1,0}(\mathbf{H}_n))$. By differentiating (1.25) we obtain

$$d\theta_0 = 2i \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j.$$

Then, by taking into account (1.13), it follows that

$$L_{\theta_0}(T_j, T_{\overline{k}}) = \delta_{jk},$$

where $T_{\overline{j}} = \overline{T_j}$, $1 \le j \le n$.

Our choice of θ_0 shows that $(\mathbf{H}_n, T_{1,0}(\mathbf{H}_n))$ is a strictly pseudoconvex CR manifold. Its Levi distribution $H(\mathbf{H}_n)$ is spanned by the (left-invariant) tangent vector fields $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$, where

$$X_j = \frac{\partial}{\partial x^j} + 2y^j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y^j} - 2x^j \frac{\partial}{\partial t}, \quad 1 \le j \le n.$$

The reader may easily check that $T = \partial/\partial t$ is the characteristic direction of $(\mathbf{H}_n, \theta_0)$.

For $n = 1$, $T_{\overline{1}} = \partial/\partial\overline{z} - iz\partial/\partial t$ is the *Lewy operator* (discovered by H. Lewy [284], in connection with the boundary behavior of holomorphic functions on $\Omega_2 = \{(z, w) \in \mathbf{C}^2 : \mathrm{Im}(w) > |z|^2\}$). The Lewy operator exhibits interesting unsolvability features described, for instance, in [232], pp. 235–239. See also L. Ehrenpreis [134], for two new approaches to the Lewy unsolvability phenomenon (one based on the existence of peak points in the kernel of $T_{\overline{1}}$, and the second on a Hartogs-type extension property), both with ramifications in the area of topological algebra.

Definition 1.11. The map $D_\delta : \mathbf{H}_n \to \mathbf{H}_n$ given by $D_\delta(z, t) = (\delta z, \delta^2 t)$, for any $(z, t) \in \mathbf{H}_n$, is called the *dilation* by the factor $\delta > 0$. □

It is an easy exercise to prove the following result:

Proposition 1.5. *Each dilation is a group homomorphism and a CR isomorphism.*

The Euclidean norm of $x = (z, t) \in \underline{\mathrm{H}}_n$ is denoted by $\|x\|$ (i.e., $\|x\|^2 = \|z\|^2 + t^2$). The Euclidean norm is not homogeneous with respect to dilations. However, $\mathbf{H}_n$ carries another significant function, the Heisenberg norm, which enjoys the required homogeneity property.

Definition 1.12. The *Heisenberg norm* is

$$|x| = (\|z\|^4 + t^2)^{1/4},$$

for any $x \in \mathbf{H}_n$. □

The Heisenberg norm is homogeneous with respect to dilations, i.e.,

$$|D_\delta x| = \delta |x|, \quad x \in \mathbf{H}_n.$$

Let us consider the transformation $T^\delta = D_{1/\delta}$. Then

$$(d_x T^\delta) T_{j,x} = \delta^{-1} T_{j, T^\delta(x)},$$

i.e., the T_j are homogeneous of degree -1 with respect to dilations. As to the form θ_0 given by (1.25), it satisfies

$$(D_\delta^* \theta)_x = \delta^2 \theta_{0, D_\delta x}.$$

The following inequalities hold on the Heisenberg group:

Proposition 1.6.

$$\|x\| \leq |x| \leq \|x\|^{1/2}, \tag{1.26}$$

for any $x \in \mathbf{H}_n$ *such that* $|x| \leq 1$.

We shall occasionally need the following:

Proposition 1.7. *There is a constant* $\gamma \geq 1$ *such that*

$$|x + y| \leq \gamma(|x| + |y|), \tag{1.27}$$

$$|xy| \leq \gamma(|x| + |y|), \tag{1.28}$$

for any $x, y \in \mathbf{H}_n$.

Definition 1.13. The inequalities (1.27)–(1.28) are called the *triangle inequalities*. □

The proof of (1.27)–(1.28) is elementary. Indeed, by homogeneity we may assume that $|x| + |y| = 1$. The set of all $(x, y) \in \mathbf{H}_n \times \mathbf{H}_n$ satisfying this equation is compact; hence we may take γ to be the larger of the maximum values of $|x + y|$ and $|xy|$ on this set. □

The Heisenberg group may be identified with the boundary of a domain in $\mathbf{C}^{n+1}$. Indeed, let Ω_{n+1} be the *Siegel domain*, i.e.,

$$\Omega_{n+1} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C} : v > \|z\|^2\},$$

where $z = (z^1, \dots, z^n)$ and $w = u + iv$ $(u, v \in \mathbf{R})$. Also $\|z\|^2 = \sum_{j=1}^n |z^j|^2$. Let then

$$f : \mathbf{H}_n \to \partial\Omega_{n+1}, \quad f(z, t) = (z, t + i\|z\|^2),$$

for any $(z, t) \in \mathbf{H}_n$. f is a CR isomorphism, where the boundary $\partial\Omega_{n+1} = \{(z, w) : \|z\|^2 = v\}$ of the Siegel domain is thought of as a CR manifold (of hypersurface type) with the CR structure induced from $\mathbf{C}^{n+1}$. Computing the differential df (on the generators T_j, $T_{\overline{j}}$, and T) is a tedious but useful exercise (left to the reader).

Another useful identification is that of the Heisenberg group and the sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$ minus a point. Let $\Omega \subset \mathbf{C}^N$ be a domain with smooth boundary $\partial\Omega$, i.e., there is an open neighborhood U of the closure $\overline{\Omega}$ in $\mathbf{C}^N$ and a C^∞ function $\rho : U \to \mathbf{R}$ such that $\Omega = \{x \in U : \rho(x) > 0\}$ (and $\partial\Omega = \{x \in U : \rho(x) = 0\}$) and $(D\rho)_x \neq 0$ at any $x \in \partial\Omega$. Let $T_{1,0}(\partial\Omega)$ be the induced CR structure on $\partial\Omega$, as a real hypersurface in $\mathbf{C}^N$. Let θ be the pullback to $\partial\Omega$ of the real 1-form $i(\overline{\partial} - \partial)\rho$ on U. Then θ is a pseudo-Hermitian structure on $(\partial\Omega, T_{1,0}(\partial\Omega))$. As we just saw, the boundary of the Siegel domain is a strictly pseudoconvex CR manifold. Also, the sphere $S^{2n+1} \subset \mathbf{C}^{2n+1}$ is a strictly pseudoconvex manifold, since the boundary of the unit ball $B_{n+1} = \{z \in \mathbf{C}^{n+1} : |z| < 1\}$, and the (restriction to $S^{2n+1} \setminus \{e_1\}$ of the) Cayley transform

$$\Phi : \mathbf{C}^{n+1} \setminus \{z_1 = 1\} \to \mathbf{C}^{n+1},$$
$$\Phi(z) = i\,\frac{e_1 + z}{1 - z_1}, \;\; z_1 \neq 1, \;\; e_1 = (1, 0, \dots, 0),$$

gives a CR isomorphism $S^{2n+1} \setminus \{e_1\} \simeq \partial\Omega_{n+1}$ (and thus a CR isomorphism $S^{2n+1} \setminus \{e_1\} \simeq \mathbf{H}_n$).

1.1.6 Embeddable CR manifolds

Let $M \subset \mathbf{C}^N$ be a real m-dimensional submanifold. If M is a CR manifold (of type (n, k)) whose CR structure is given by (1.12) with $V = \mathbf{C}^N$, then M is referred to as an *embedded* (or *realized*) *CR manifold*.

Let $(M, T_{1,0}(M))$ be a CR manifold. If in some neighborhood of each point $x \in M$, $(M, T_{1,0}(M))$ is CR isomorphic to an embedded CR manifold, then $(M, T_{1,0}(M))$ is termed *locally embeddable*. If a global isomorphism with an embedded CR manifold exists, then $(M, T_{1,0}(M))$ is called *embeddable* (or *realizable*).

Let $(M, T_{1,0}(M))$ be a CR manifold. We say that $(M, T_{1,0}(M))$ is *real analytic* if M is a real analytic manifold and $T_{1,0}(M)$ is a real analytic subbundle of $T(M) \otimes \mathbf{C}$, i.e., $T_{1,0}(M)$ is locally generated by real analytic vector fields. By a (classical) result of A. Andreotti and C.D. Hill [13], any real analytic CR manifold $(M, T_{1,0}(M))$

of type (n,k), $k \geq 1$, is locally embeddable. Precisely, for any $x \in M$, there is a neighborhood U of x in M such that $(U, T_{1,0}(M)_{|U})$ is CR isomorphic via a real analytic CR map to a real analytic generic embedded CR manifold in $\mathbf{C}^{n+k}$. Here $T_{1,0}(M)_{|U}$ is the pullback of $T_{1,0}(M)$ by $\iota : U \subseteq M$. A proof of the embeddability result of A. Andreotti and C.D. Hill (cf. op. cit.) is already available in book form, cf. A. Boggess [70], pp. 169–172, and will not be reproduced here. A discussion of *characteristic coordinates* (the relevant ingredient in the proof) is also available in the book by D.E. Blair [62], pp. 57–60, in the context of the geometric interpretation of normal almost contact structures ([62], pp. 61–63). By a result of A. Andreotti and G.A. Fredricks [15], real analytic CR manifolds are also *globally embeddable*, i.e., globally isomorphic to a generic CR submanifold of some complex manifold. This generalizes a result by H.B. Shutrick [373], on the existence of complexifications.

The *embedding problem* (i.e., decide whether a given abstract CR manifold is (locally) embeddable) was first posed by J.J. Kohn [246], and subsequently solved to a large extent by M. Kuranishi [263]. By M. Kuranishi's result, each strictly pseudoconvex CR manifold of real dimension $2n+1 \geq 9$ is locally embeddable in $\mathbf{C}^{n+1}$. T. Akahori [2], settled the question in dimension 7. A much simpler proof was given by S. Webster [429]. The embedding problem is open in dimension 5, while L. Nirenberg [326], built a counterexample in dimension 3 (as a perturbation of the CR structure of $\mathbf{H}_1$). More generally, by a result of H. Jacobowitz and F. Trèves [224], analytically small perturbations of 3-dimensional embeddable strictly pseudoconvex CR manifolds are known to be nonembeddable. M.S. Baouendi, L.P. Rothschild, and F. Treves [30], showed that the existence of a local transverse CR action implies local embeddability. As a global version of this result, though confined to the 3-dimensional case, we may quote a result by L. Lempert [278]: Let M be a 3-dimensional ($n=1$) CR manifold admitting a smooth CR action of $\mathbf{R}$ that is transverse. Then M is the boundary of a strictly pseudoconvex complex surface, i.e., it is embeddable. More recently, by a result of Z. Balogh and C. Leuenberger [29], if a CR manifold M of hypersurface type admits a local semi-extendable $\mathbf{R}$-action then M is locally realizable as the boundary of a complex manifold.

Embeddability is related to solvability of certain PDEs, as shown by the following example, due to C.D. Hill [200]. Let $t = (t_1, t_2, t_3)$ be the Cartesian coordinates on $\mathbf{R}^3$. The natural coordinates on $\mathbf{R}^5 = \mathbf{R}^3 \times \mathbf{C}$ are denoted by (t, z). Let

$$L = T_{\bar{1}} = \frac{1}{2}\left(\frac{\partial}{\partial t_1} + i\,\frac{\partial}{\partial t_2}\right) - i(t_1 + i t_2)\frac{\partial}{\partial t_3}$$

be the Lewy operator on $\mathbf{H}_1 = \mathbf{R}^3$. Given a C^∞ function $\omega : \mathbf{H}_1 \to \mathbf{C}$ we consider the first-order PDE

$$L\chi = \omega. \tag{1.29}$$

Definition 1.14. We say that (1.29) is *solvable* at a point $t_0 \in \mathbf{R}^3$ if there is an open set $U \subseteq \mathbf{R}^3$ such that $t_0 \in U$ and there is a C^∞ function $\chi : U \to \mathbf{C}$ such that $L\chi = \omega$ on U. □

Moreover, let us consider the complex vector fields $P, Q \in T(\mathbf{H}_1 \times \mathbf{C}) \otimes \mathbf{C}$ given by

$$P = \frac{\partial}{\partial \overline{z}}, \quad Q = L + \omega(t) \frac{\partial}{\partial z}. \tag{1.30}$$

Clearly $\{P, Q, \overline{P}, \overline{Q}\}$ are linearly independent at each point of $\mathbf{H}_1 \times \mathbf{C}$ and $[P, Q] = 0$. Consequently

$$T_{0,1}(\mathbf{H}_1 \times \mathbf{C})_{(t,z)} = \mathbf{C}P_{(t,z)} + \mathbf{C}Q_{(t,z)}, \quad (t, z) \in \mathbf{H}_1 \times \mathbf{C},$$

gives a CR structure on $\mathbf{H}_1 \times \mathbf{C}$. We shall prove the following theorem:

Theorem 1.1. (C.D. Hill [200])
The CR structure (1.30) *is locally embeddable at* $(t_0, z_0) \in \mathbf{H}_1 \times \mathbf{C}$ *if and only if* (1.29) *is solvable at* t_0.

On the other hand, by a result of H. Lewy [284], there is $\omega \in C^\infty(\mathbf{R}^3)$ such that for any open set $U \subseteq \mathbf{R}^3$ the equation (1.29) has no solution $\chi \in C^1(U)$. Hence the CR structure (1.30) is not locally embeddable in general.

Proof of Theorem 1.1. As shown in Section 1.1.5, the map

$$\mathbf{R}^3 \to \mathbf{C}^2, \quad t \mapsto (v_1(t), v_2(t)),$$
$$v_1(t) = t_1 + it_2, \quad v_2(t) = t_3 + i(t_1^2 + t_2^2),$$

embeds $\mathbf{H}_1$ globally into $\mathbf{C}^2$. The functions $v_j : \mathbf{H}_1 \to \mathbf{C}$ form a maximal set of functionally independent *characteristic coordinates* (in the sense of A. Andreotti and C.D. Hill [13]), i.e., $Lv_j = 0$ and $dv_1 \wedge dv_2 \neq 0$. If we adopt the terminology in Chapter 6, the map $v = (v_1, v_2) : \mathbf{H}_1 \to \mathbf{C}^2$ is a *CR immersion*, i.e., an immersion and a CR map (and it determines a CR isomorphism $\mathbf{H}_1 \simeq \partial\Omega_2$).

Let us prove first the sufficiency. Assume that (1.29) is solvable at t_0, i.e., there is $\chi \in C^\infty(U)$ with $t_0 \in U \subseteq \mathbf{H}_1$ and $L\chi = \omega$ on U. Let us consider the function

$$v_3 : U \times \mathbf{C} \to \mathbf{C}, \quad v_3(t, z) = z - \chi(t).$$

A calculation shows that

$$Pv_j = 0, \quad Qv_j = 0, \quad j \in \{1, 2, 3\}, \quad dv_1 \wedge dv_2 \wedge dv_3 \neq 0;$$

that is,

$$\varphi : U \times \mathbf{C} \to \mathbf{C}^3, \quad \varphi(t, z) = (v(t), v_3(t, z)), \quad t \in U, \quad z \in \mathbf{C},$$

is a CR immersion (of a neighborhood of (t_0, z_0)).

The proof of necessity is more involved. Let us assume that there is a CR immersion $u = (u_1, u_2, u_3) : V \to \mathbf{C}^3$ of an open set $V \subseteq \mathbf{H}_1 \times \mathbf{C}$ with $(t_0, z_0) \in V$, that is, $Pu_j = 0$, $Qu_j = 0$, and $du_1 \wedge du_2 \wedge du_3 \neq 0$ on V. In particular, each u_j is holomorphic with respect to the z-variable. Consequently the Jacobian matrix of u has the form