

Differential geometry of complex hypersurfaces II*

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In this paper we continue the study of complex hypersurfaces of complex space forms (i. e. Kählerian manifolds of constant holomorphic sectional curvature) begun in [8]. The main results are: the determination of the holonomy groups of such hypersurfaces, a generalization of the main theorem of [8] on Einstein hypersurfaces, the non-existence of a certain type of hypersurface in the complex projective space, and some results concerning the curvature of complex curves.

Let \tilde{M} be a complex space form (which in general will not be complete) of complex dimension $n+1$ and let M be an immersed complex hypersurface in \tilde{M} . In §1 we show that the rank of the second fundamental form of M is intrinsic and that M is rigid in \tilde{M} , if the latter is simply connected and complete. The local version of rigidity is contained as a special case in the work of Calabi [1], but our method is more direct and more in the line of classical differential geometry.

The holonomy group of M (with respect to the induced Kähler metric) is studied in §2. If the holomorphic sectional curvature \tilde{c} of \tilde{M} is negative, the holonomy group is always $U(n)$. In the case where $\tilde{c} > 0$ (e. g. $\tilde{M} = P^{n+1}(C)$), the holonomy group of M is either $U(n)$ or $SO(n) \times S^1$ (S^1 denotes the circle group), the latter case arising only when M is locally holomorphically isometric to the complex quadric Q^n in $P^{n+1}(C)$. When $\tilde{c} = 0$ (i. e. when \tilde{M} is flat), the holonomy group of M depends on the rank of the second fundamental form and we obtain a result of Kerbrat [3] more directly.

In §3 we first obtain the following generalized local version of the classification theorem of [8]. If the Ricci tensor S of M is parallel (i. e. $\nabla S = 0$), then M is totally geodesic in \tilde{M} or else $\tilde{c} > 0$ and M is locally a complex quadric. To prove this we modify Theorem 2 [8] to show that M is locally symmetric when its Ricci tensor is parallel, and obtain the local classification without using the list of irreducible Hermitian symmetric spaces. This local version was proved by Chern [2] with the original assumption that M is Einstein, and Takahashi [9] has shown that M is Einstein if its Ricci tensor

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is parallel. It is worth noting that when $\tilde{c} \neq 0$ this latter result follows immediately from Theorem 2 of §2. We conclude this section with a better global version of the classification theorem of [8]—here the proof is made considerably more elementary than the original one and simple-connectedness of the hypersurfaces is no longer assumed in the case $\tilde{c} \leq 0$.

We show, in §4, that the rank of the second fundamental form cannot be identically equal to 2 on a compact complex hypersurface in $P^{n+1}(C)$, $n \geq 3$. In §5 we discuss the Gaussian mapping of a complex hypersurface M in C^{n+1} into $P^n(C)$; we find that its Jacobian is essentially the second fundamental form and we show how the Gaussian mapping relates the Kählerian connections of M and $P^n(C)$.

The study of complex curves in a 2-dimensional complex space form is taken up in §6. First we take care of the case $n = 1$ in Theorems 4 and 5. We then obtain some characterizations of P^1 and Q^1 among closed nonsingular complex curves in $P^2(C)$ by curvature conditions.

We shall use the same notation as in [8].

§1. Rigidity.

Let M be a Kähler manifold of complex dimension n and let f be a Kählerian immersion (i. e. a complex isometric immersion) of M as a complex hypersurface in a space \tilde{M} of constant holomorphic curvature \tilde{c} . For each point $x_0 \in M$ there is a neighborhood $U(x_0)$ of x_0 in M on which Gauss' equation for the immersion f may be written as

$$R(X, Y) = \tilde{R}(X, Y) + D(X, Y)$$

with

$$\tilde{R}(X, Y) = \frac{\tilde{c}}{4} \{X \wedge Y + JX \wedge JY + 2g(X, JY)J\}$$

and

$$D(X, Y) = AX \wedge AY + JAX \wedge JAY,$$

where $X \wedge Y$ denotes the skew-symmetric endomorphism which maps Z upon $g(Y, Z)X - g(X, Z)Y$, and X, Y, Z are tangent vectors to M (see Proposition 3 [8]). Whereas A depends on the immersion f and on a local choice of unit vector field normal to M , the following lemma shows that its kernel does not.

LEMMA 1. *At each point $x \in U(x_0)$ we have*

$$\begin{aligned} \text{Ker } A &= \{X \in T_x(M) \mid D(X, Y) = 0 \quad \text{for all } Y \in T_x(M)\} \\ &= \{X \in T_x(M) \mid (R - \tilde{R})(X, Y) = 0 \quad \text{for all } Y \in T_x(M)\}. \end{aligned}$$

PROOF. Clearly $\text{Ker } A$ is contained in the subspace defined by D . On the other hand, if $X \in \text{Ker } A$ then $D(X, JX) = -2AX \wedge JAX \neq 0$, and the first

Now

$$(R - \check{R})(e_i, Je_i) = -2Ae_i \wedge JAe_i = -2\bar{A}e_i \wedge J\bar{A}e_i$$

and the middle form of this identity being nonzero when $i \leq k$, we see that $\bar{A}e_i$ is a linear combination of Ae_i and JAe_i , say

$$\bar{A}e_i = \alpha_i Ae_i + \beta_i JAe_i.$$

It is then clear that $\alpha_i^2 + \beta_i^2 = 1$. From

$$\begin{aligned} R(e_i, e_j) - \check{R}(e_i, e_j) &= Ae_i \wedge Ae_j + JAe_i \wedge JAe_j \\ &= \bar{A}e_i \wedge \bar{A}e_j + J\bar{A}e_i \wedge J\bar{A}e_j \end{aligned}$$

we can easily deduce that $\alpha_i = \alpha_j = \alpha$, say, and $\beta_i = \beta_j = \beta$, say, for $1 \leq i, j \leq k$. However $\text{Ker } A = \text{Ker } \bar{A}$, by virtue of Lemma 1, and therefore $\bar{A} = \alpha A + \beta JA$ with $\alpha^2 + \beta^2 = 1$ at each point of a neighborhood of x_0 . By virtue of the assumption on the rank of M at x_0 we can find a differentiable vector field X on a neighborhood of x_0 such that $AX \neq 0$; and, since $\alpha = \frac{g(\bar{A}X, AX)}{g(AX, AX)}$, it follows that α (and similarly β) is a differentiable function on a neighborhood of x_0 . We may then define a differentiable function θ on a neighborhood $U(x_0)$ of x_0 such that $\alpha = \cos \theta$ and $\beta = \sin \theta$. Then $\xi' = \cos \theta \xi + \sin \theta J\xi$ is a unit normal vector field on $U(x_0)$ with respect to the immersion f and clearly $A' = \bar{A}$. By Lemma 2, it follows that $s' = \bar{s}$ also.

THEOREM 1. *A connected Kählerian hypersurface M of complex dimension $n \geq 1$ of a simply connected complete complex space form \tilde{M} is rigid in \tilde{M} .*

PROOF. If $R = \check{R}$ at every point of M , then M has constant holomorphic sectional curvature \check{c} . Therefore, by Corollary 2 of [8, §3], M is totally geodesic in \tilde{M} and thus is rigid. If $R \neq \check{R}$ at some point of M , let x_0 be a point where the rank of M is maximal. Let $f, \bar{f}: M \rightarrow \tilde{M}$ be two Kählerian immersions. By virtue of Lemma 3, there exists a neighborhood $U(x_0)$ of x_0 and suitably chosen unit normal vector fields ξ and $\bar{\xi}$ on $U(x_0)$ with respect to the immersions f and \bar{f} respectively such that $A = \bar{A}$ and $s = \bar{s}$ on $U(x_0)$. We now resort to local coordinates to show that f and \bar{f} differ by a holomorphic motion ϕ of \tilde{M} on $U(x_0)$, that is, $\bar{f} = \phi \circ f$ on $U(x_0)$; and, by analyticity, this will then hold on all of M . In fact, since the group of holomorphic isometries of \tilde{M} is transitive on the set of unitary frames, we may assume without loss of generality that

$$f(x_0) = \bar{f}(x_0), \quad f_*(x_0) = \bar{f}_*(x_0), \quad \xi(x_0) = \bar{\xi}(x_0),$$

where f_* and \bar{f}_* denote the differentials of f and \bar{f} , respectively, and prove that $f = \bar{f}$ in a neighborhood of x_0 . Let (x^1, \dots, x^{2n}) be a system of local coordinates on $U(x_0)$ and let (u^1, \dots, u^{2n+2}) be a system of local coordinates on a neighborhood of $f(x_0)$ in \tilde{M} derived from a system of complex coordinates.

We agree on the following ranges for the indices :

$$1 \leq i, j, k, l \leq 2n, \quad 1 \leq p, q, r, s \leq 2n+2.$$

Our notation (in the summation convention) will be

$$f^p(x) = u^p(f(x)), \quad f_i^p = \frac{\partial f^p}{\partial x^i}, \quad f_{ij}^p = \frac{\partial^2 f^p}{\partial x^i \partial x^j}, \quad \text{etc.}, \quad f_* \left(\frac{\partial}{\partial x^i} \right) = f_i^p \left(\frac{\partial}{\partial u^p} \right),$$

$$\xi = \xi^r \frac{\partial}{\partial u^r}, \quad J\xi = (J\xi)^r \frac{\partial}{\partial u^r}, \quad \xi_i^r = \frac{\partial \xi^r}{\partial x^i}, \quad \xi_{ij}^r = \frac{\partial^2 \xi^r}{\partial x^i \partial x^j}, \quad \text{etc.}$$

The corresponding notation for \bar{f} is then self-explanatory. We also use

$$h_{ij} = h \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right), \quad k_{ij} = k \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right),$$

$$A \frac{\partial}{\partial x^i} = a_i^j \frac{\partial}{\partial x^j}, \quad s \left(\frac{\partial}{\partial x^i} \right) = s_i.$$

(Note that we have $A = \bar{A}$ and $s = \bar{s}$ so that we do not need the corresponding notation for \bar{f} here). The Christoffel symbols are denoted by Γ_{jk}^i for (x^1, \dots, x^{2n}) and by Γ_{pq}^r for (u^1, \dots, u^{2n+2}) . We note that $(J\xi)^r = -\xi^{r+n+1}$ and $(J\xi)^{r+n+1} = \xi^r$ (indices are understood here modulo $2n+2$) because of the nature of the coordinate system (u^1, \dots, u^{2n+2}) . The equations

$$\bar{V}_{f_* \left(\frac{\partial}{\partial x^i} \right)} f_* \left(\frac{\partial}{\partial x^j} \right) = f_* \left[V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right] + h \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] \xi + k \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] J\xi,$$

$$\bar{V}_{f_* \left(\frac{\partial}{\partial x^i} \right)} \xi = -f_* \left[A \frac{\partial}{\partial x^i} \right] + s \left[\frac{\partial}{\partial x^i} \right] J\xi$$

for the immersion f then yield

$$(I) \quad f_{ij}^r = -f_i^p f_j^q \Gamma_{pq}^r + f_k^r \Gamma_{ij}^k + h_{ij} \xi^r + k_{ij} (J\xi)^r,$$

$$(II) \quad \xi_i^r = -f_i^p \xi^q \Gamma_{pq}^r - a_i^j f_j^r + s_i (J\xi)^r.$$

We denote the corresponding equations for the immersion \bar{f} by (\bar{I}) and (\bar{II}) . At x_0 we have

$$(1) \quad f^p(x_0) = \bar{f}^p(x_0), \quad f_i^p(x_0) = \bar{f}_i^p(x_0), \quad \xi^r(x_0) = \bar{\xi}^r(x_0), \quad (J\xi)^r(x_0) = (J\bar{\xi})^r(x_0).$$

We wish to show that $f = \bar{f}$ in a neighborhood of x_0 ; since f^p and \bar{f}^p are real analytic it suffices to prove

$$(2) \quad f_{ij}^p(x_0) = \bar{f}_{ij}^p(x_0),$$

$$(4) \quad f_{ijk}^p(x_0) = \bar{f}_{ijk}^p(x_0),$$

and so on for all higher-order derivatives at x_0 . (2) follows from (I), (\bar{I}) , (1) and the equation $A = \bar{A}$ on $U(x_0)$, while

$$(3) \quad \xi_i^r(x_0) = \bar{\xi}_i^r(x_0)$$

follows from (II), (\bar{II}) , (1) and the equations $A = \bar{A}$ and $s = \bar{s}$ on $U(x_0)$. Now

f_{ijk}^r and \bar{f}_{ijk}^r are obtained by differentiating (I) and (\bar{I}) and we deduce (4) from the equations (1), (2), (3) and the equation $A = \bar{A}$ on $U(x_0)$. In the same manner ξ_{ij}^r and $\bar{\xi}_{ij}^r$ are obtained by differentiating (II) and (\bar{II}) . Using the previous equations together with the equations $A = \bar{A}$ and $s = \bar{s}$ on $U(x_0)$, we infer

$$(5) \quad \xi_{ij}^r(x_0) = \bar{\xi}_{ij}^r(x_0).$$

We can then easily obtain

$$(6) \quad f_{ijkl}^p(x_0) = \bar{f}_{ijkl}^p(x_0).$$

The equalities for higher-order derivatives are obtained in the same fashion. Thus $f = \bar{f}$ in a neighborhood of x_0 and this completes the proof.

§ 2. Holonomy.

In this section we study the restricted holonomy group H of a complex hypersurface M in a space \tilde{M} of constant holomorphic sectional curvature \tilde{c} . When the complex dimension n of M equals 1 it is clear that either $H = U(1)$ or M is flat. In this latter case the results of § 6 will show that $\tilde{c} = 0$. It will then be clear that Theorems 2 and 3 in this section are valid for $n = 1$. We therefore assume $n \geq 2$ in the following.

On a neighborhood $U(x_0)$ of any point $x_0 \in M$, the Riemannian and Ricci curvature tensors of M may be written as

$$(7) \quad R(X, Y) = \frac{\tilde{c}}{4} \{X \wedge Y + JX \wedge JY + 2g(X, JY)J\} + AX \wedge AY + JAX \wedge JAY,$$

$$(8) \quad S(X, Y) = \frac{(n+1)\tilde{c}}{2} g(X, Y) - 2g(A^2 X, Y),$$

where $X, Y \in T_x(M)$ and $x \in U(x_0)$ [8]. We pick an orthonormal basis $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ of $T_{x_0}(M)$ with respect to which the matrix of A is of the form

$$\left(\begin{array}{cccc} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_n & \\ & & & -\lambda_1 \\ & & & & \ddots \\ & & & & & -\lambda_n \end{array} \right),$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. With respect to this basis the Lie algebra of the group of unitary transformations of the tangent space $T_{x_0}(M)$ may be identified with the Lie algebra of all block matrices of the form $\begin{bmatrix} C & -D \\ D & C \end{bmatrix}$, where C and D are respectively skew-symmetric and symmetric $n \times n$ real matrices. The holonomy algebra \mathfrak{h} is thereby identified with a Lie subalgebra (also denoted

\mathfrak{h}) of this matrix algebra. For the sake of brevity we frequently use the same symbol to denote an endomorphism of $T_{x_0}(M)$ and its matrix with respect to the above basis. We shall say that M is *nondegenerate* when $J \in \mathfrak{h}$ and this definition is independent of the point x_0 (see [4], where the notion of non-degeneracy was defined to mean $J \in H$).

In this section all indices range from 1 to n and we agree that $i \neq j$. Let E_j^i denote the $n \times n$ matrix whose (i, j) entry (i -th row, j -th column) is 1 and whose (j, i) entry is -1 , all other entries being zero. For $p \neq q$ as well as $p = q$, let F_q^p denote the $n \times n$ matrix whose (p, q) and (q, p) entries equal 1, all other entries being zero. Setting $K_j^i = \begin{bmatrix} E_j^i & 0 \\ 0 & E_j^i \end{bmatrix}$ and $S_q^p = \begin{bmatrix} 0 & -F_q^p \\ F_q^p & 0 \end{bmatrix}$ (including $p = q$), the following identities are readily verified (assuming $i \neq j$ as agreed):

$$(9) \quad \begin{cases} [K_j^i, S_k^i] = -S_k^i & (k \neq j), \\ [K_j^i, S_j^i] = 2(S_i^i - S_j^i), \\ [S_j^i, S_i^i] = K_j^i, \end{cases}$$

where $[,]$ denotes the usual bracket operation.

The holonomy algebra \mathfrak{h} contains all curvature transformations of $T_{x_0}(M)$ and in particular the endomorphisms $R(e_i, e_j)$, $R(e_i, J e_j)$ and $R(e_i, J e_i)$ for all i, j . Their matrices with respect to the above basis are respectively

$$\left(\lambda_i \lambda_j + \frac{\tilde{c}}{4}\right) K_j^i, \quad -\left(\lambda_i \lambda_j - \frac{\tilde{c}}{4}\right) S_j^i \quad \text{and} \quad -\frac{\tilde{c}}{2} J + 2\left(\lambda_i^2 - \frac{\tilde{c}}{4}\right) S_i^i,$$

as may be verified by using (7). In the proofs which follow we make repeated use of the fact that these are elements of \mathfrak{h} .

LEMMA 4. Let $\tilde{c} > 0$.

- i) $K_l^k \in \mathfrak{h}$ for all k, l ($k \neq l$).
- ii) If $S_j^i \in \mathfrak{h}$ for some j , then $\mathfrak{h} = \mathfrak{u}(n)$.
- iii) If $S_j^i \in \mathfrak{h}$ and $\lambda_i \neq \lambda_j$ for some pair (i, j) , then $\mathfrak{h} = \mathfrak{u}(n)$.

PROOF. i) Since $\lambda_k \geq 0$ for all k and $\tilde{c} > 0$, $R(e_k, e_l) \in \mathfrak{h}$ implies $K_l^k \in \mathfrak{h}$ for every pair (k, l) .

ii) For $k \neq j$, we have $[K_k^j, S_j^i] = -S_k^i \in \mathfrak{h}$ using (i) and the assumption. Thus $[K_k^j, S_k^i] = 2(S_j^i - S_k^i) \in \mathfrak{h}$ and hence $S_k^i \in \mathfrak{h}$ for all k . In addition, $[K_l^k, S_k^k] = -S_l^k \in \mathfrak{h}$ when $k \neq l$. Since K_j^i for all $i \neq j$ and S_q^p for all p, q together span $\mathfrak{u}(n)$, we have $\mathfrak{h} = \mathfrak{u}(n)$.

iii) By (i) and by the assumption, we have $[K_j^i, S_j^i] = 2(S_i^i - S_j^i) \in \mathfrak{h}$. Since

$$\begin{aligned} R(e_i, J e_i) - R(e_j, J e_j) &= -\frac{\tilde{c}}{2}(S_i^i - S_j^i) + 2(\lambda_i^2 S_i^i - \lambda_j^2 S_j^i) \\ &= \left(2\lambda_i^2 - \frac{\tilde{c}}{2}\right)(S_i^i - S_j^i) + 2(\lambda_i^2 - \lambda_j^2)S_j^i \end{aligned}$$

belongs to \mathfrak{h} , we infer that $(\lambda_i^2 - \lambda_j^2)S_j^i \in \mathfrak{h}$ and hence $S_j^i \in \mathfrak{h}$ since $\lambda_i \neq \lambda_j$. By

(ii), we have $\mathfrak{h} = \mathfrak{u}(n)$.

THEOREM 2. *Let M be a complex hypersurface of complex dimension $n \geq 1$ in a space \tilde{M} of constant holomorphic curvature \tilde{c} ($\neq 0$) and let H be the restricted holonomy group of M (with respect to the induced Kählerian structure). Then*

i) *if $\tilde{c} < 0$, H is always isomorphic to $U(n)$.*

ii) *if $\tilde{c} > 0$, H is isomorphic either to $U(n)$ or to $SO(n) \times S^1$,*

where S^1 denotes the circle group, the second case arising only when M is locally holomorphically isometric to the complex quadric Q^n in $P^{n+1}(C)$.

PROOF. i) Since $\tilde{c} < 0$, the Ricci tensor is negative definite according to (8) and M is therefore nondegenerate (see [4]; actually it was proved there that $J \in H$ but the proof shows that $J \in \mathfrak{h}$). Since $R(e_i, J e_j) = \left(\frac{\tilde{c}}{4} - \lambda_i \lambda_j\right) S_j^i \in \mathfrak{h}$ and since $\lambda_k \geq 0$ for all k and $\tilde{c} < 0$, we have $S_j^i \in \mathfrak{h}$ for every pair (i, j) . Since $R(e_i, J e_i) \in \mathfrak{h}$ and $J \in \mathfrak{h}$, we have $S_i^i \in \mathfrak{h}$. Thus $K_j^i = [S_j^i, S_i^i] \in \mathfrak{h}$ for all i, j . Hence $\mathfrak{h} = \mathfrak{u}(n)$.

ii) We first dispense with the case where M is an Einstein manifold, in which case $A^2 = \lambda^2 I$. Since $\sum_{r=1}^n R(e_r, J e_r) = -\rho J \in \mathfrak{h}$, where ρ is the Ricci curvature of M , and since ρ is nonzero in view of Proposition 9 [8], we deduce that $J \in \mathfrak{h}$. From the curvature transformations $R(e_i, e_j)$, $R(e_i, J e_j)$ and $R(e_i, J e_i)$ we conclude that all K_j^i ($i \neq j$) and S_j^i ($i = j$ included) are contained in \mathfrak{h} , that is, $H = U(n)$, unless $\lambda^2 = \tilde{c}/4$ (i. e. $\rho = n\tilde{c}/2$). At any rate we know that M is locally symmetric so that the curvature transformations at any point x_0 generate the holonomy algebra \mathfrak{h} . If $\lambda^2 = \tilde{c}/4$, we readily see that \mathfrak{h} is generated by J and by all K_j^i , that is $H = SO(n) \times S^1$. On the other hand, the complex quadric $Q^n = SO(n+2)/SO(n) \times SO(2)$ imbedded in $P^{n+1}(C)$ with holomorphic curvature \tilde{c} is Einstein and has holonomy group isomorphic to $SO(n) \times SO(2)$ (i. e. $SO(n) \times S^1$). Thus $\lambda^2 = \tilde{c}/4$ for Q^n . Now if $\lambda^2 = \tilde{c}/4$ for M , the same argument as was used in Proposition 11 of [8] can be applied locally to show that M is locally holomorphically isometric to Q^n . We have thus taken care of Theorem 2 in the case where M is Einstein (getting a more precise result than Proposition 10 of [8]).

If M is not an Einstein manifold we may assume that the characteristic roots of A^2 at x_0 are not all equal. By (i) of Lemma 4 we know that $K_l^k \in \mathfrak{h}$ for all k, l . If $4\lambda_i^2 = \tilde{c}$ for some i , then $R(e_i, J e_i) = -\frac{\tilde{c}}{2} J \in \mathfrak{h}$. By the assumption on A^2 at x_0 , we have $4\lambda_j^2 \neq \tilde{c}$ for some j and consequently $S_j^j \in \mathfrak{h}$ from $R(e_j, J e_j) = -\frac{\tilde{c}}{2} J + 2\left(\lambda_j^2 - \frac{\tilde{c}}{4}\right) S_j^j \in \mathfrak{h}$. By (ii) of Lemma 4 we conclude that $\mathfrak{h} = \mathfrak{u}(n)$, that is, $H = U(n)$. We may therefore suppose $4\lambda_i^2 \neq \tilde{c}$ for every i . If $4\lambda_1^2 < \tilde{c}$, then $4\lambda_1 \lambda_n < \tilde{c}$, since $\lambda_1 > \lambda_n$; therefore $R(e_1, J e_n) \in \mathfrak{h}$ implies $S_n^1 \in \mathfrak{h}$. By (iii) of Lemma

4 we have $\mathfrak{h} = \mathfrak{u}(n)$. Similarly, if $4\lambda_n^2 > \tilde{c}$, we find $\mathfrak{h} = \mathfrak{u}(n)$ again. Thus we are led to suppose

$$\lambda_1^2 \geq \lambda_2^2 \geq \dots \geq \lambda_m^2 > \frac{\tilde{c}}{4} > \lambda_{m+1}^2 \geq \dots \geq \lambda_n^2, \quad 1 \leq m < n.$$

Taking the case $n \geq 3$, we see that if $m \geq 2$ then $\lambda_1 \lambda_m \geq \lambda_m^2 > \frac{\tilde{c}}{4}$, so that $S_m^1 \in \mathfrak{h}$; however $[K_n^1, S_m^1] = -S_m^2 \in \mathfrak{h}$ and $\lambda_m \neq \lambda_n$. Thus $\mathfrak{h} = \mathfrak{u}(n)$ again, by (iii) of Lemma 4. If $n \geq 3$ and $m = 1$, then $\lambda_2 \lambda_n \leq \lambda_2^2 < \frac{\tilde{c}}{4}$ and $S_n^2 \in \mathfrak{h}$ so that $[K_1^2, S_n^2] = -S_n^1 \in \mathfrak{h}$. Thus $\mathfrak{h} = \mathfrak{u}(n)$ again. Finally, we suppose $n = 2$, in which case $m = 1$. If $J \in \mathfrak{h}$, then the Ricci tensor is singular everywhere [4], or what amounts to the same thing, $A^2 - \frac{3\tilde{c}}{4}I$ is singular everywhere. Thus $(\lambda_1^2 - \frac{3\tilde{c}}{4})(\lambda_2^2 - \frac{3\tilde{c}}{4}) = 0$. Since $\lambda_2^2 < \frac{\tilde{c}}{4}$, we must have $\lambda_1^2 = \frac{3\tilde{c}}{4}$. Since $\lambda_1 \lambda_2 = \frac{\tilde{c}}{4}$, we have $\lambda_2^2 = \frac{\tilde{c}}{12}$. We see then that $R(e_1, J e_1)$ and $R(e_2, J e_2)$ are linear combinations of S_1^1 and S_2^2 , from which we can solve for S_1^1 and S_2^2 . Thus $S_1^1, S_2^2 \in \mathfrak{h}$ and hence $J = S_1^1 + S_2^2 \in \mathfrak{h}$. We have thus shown $J \in \mathfrak{h}$. Now $\lambda_1^2 > \frac{\tilde{c}}{4}$ and $R(e_1, J e_1) \in \mathfrak{h}$ imply $S_1^1 \in \mathfrak{h}$. By (ii) of Lemma 4 we have $\mathfrak{h} = \mathfrak{u}(2)$. This completes the proof of Theorem 2.

COROLLARY. *Let M be a complete complex hypersurface in $P^{n+1}(C)$ or in D^{n+1} . Then the largest connected group of affine transformations of M (with respect to the induced Kählerian connection) preserves the complex structure.*

PROOF. This follows from Theorem 2 and from Theorem 3 of [4].

The following theorem has been obtained by Kerbrat [3] using a different method.

THEOREM 3. *Let M be a complex n -dimensional hypersurface in a flat Kähler manifold \tilde{M} . If at some point the rank of M equals $2n$, then the restricted holonomy group of M is isomorphic to $U(n)$.*

PROOF. We may suppose that $\text{rank } A = 2n$ at x_0 . An examination of the curvature transformations reveals that $K_j^i, S_j^i, S_i^j \in \mathfrak{h}$ for all i, j . Thus H is isomorphic to $U(n)$.

§ 3. Hypersurfaces with parallel Ricci tensor.

On a neighborhood $U(x_0)$ of each point x_0 of a complex hypersurface M in a space \tilde{M} of constant holomorphic curvature \tilde{c} , Codazzi's equation

$$(\nabla_x A)Y - (\nabla_Y A)X - s(X)JAY + s(Y)JAX = 0$$

holds, where $X, Y \in T_x(M)$ and x is any point of $U(x_0)$. When the simpler equation $(\nabla_x A)Y = s(X)JAY$ is valid on a neighborhood of every point in M

we say that Codazzi's equation reduces. We have

LEMMA 5. *The following conditions are equivalent on M :*

- i) *Codazzi's equation reduces.*
- ii) *The Ricci tensor of M is parallel, that is $\nabla S = 0$.*
- iii) *M is locally symmetric.*

REMARK. This result has been obtained independently by T. Takahashi [9] using another method. In the case $\tilde{c} \neq 0$ we know by Theorem 2 in § 2 that either M is locally Q^n , which is Einstein, or the holonomy group of M is $U(n)$. In the second case, $\nabla S = 0$ implies that M is Einstein because M is irreducible. Thus Lemma 5 generalizes Theorem 2 of [8] only in the case $\tilde{c} = 0$. We shall, however, give a direct proof of (ii) \rightarrow (i).

PROOF. The proof of Theorem 2 [8] shows that (i) implies (iii). (iii) implies (ii) trivially. We now show that (ii) implies (i). $\nabla S = 0$ is equivalent to $\nabla A^2 = 0$ and this in turn implies that the characteristic roots of A^2 together with their multiplicities are constant on M . Consequently, if $A^2 = 0$ at one point then A^2 vanishes identically and Codazzi's equation reduces trivially. Assuming that this is not the case, let $\lambda_1, \dots, \lambda_r$ be the distinct positive characteristic roots of A on $U(x_0)$. Consider the distributions on $U(x_0)$ defined by

$$\begin{aligned} T_i^+(x) &= \{X \in T_x(M) \mid AX = \lambda_i X\}, \\ T_i^-(x) &= \{X \in T_x(M) \mid AX = -\lambda_i X\}, \\ T_i(x) &= T_i^+(x) \oplus T_i^-(x), \\ T^0(x) &= \{X \in T_x(M) \mid AX = 0\}. \end{aligned}$$

Clearly J interchanges $T_i^+(x)$ and $T_i^-(x)$. When X is an arbitrary vector field and Y is a vector field in T^0 we deduce from

$$0 = (\nabla_X A^2)(Y) = \nabla_X(A^2 Y) - A^2(\nabla_X Y) = -A^2(\nabla_X Y)$$

that $\nabla_X Y \in T^0$. Hence T^0 is parallel. (A similar argument shows that each T_i is parallel.)

If $Y \in T^0$, we have $(\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y = 0$. On the other hand, we have $s(X)JAY = 0$ so that $(\nabla_X A)Y = s(X)JAY$. By Codazzi's equation we also obtain $(\nabla_Y A)X = s(Y)JAX$. In other words, the reduced Codazzi equation holds when X or Y is in T^0 . Now $\nabla A^2 = 0$ being equivalent to $(\nabla_X A)A + A(\nabla_X A) = 0$ (for all X), we see that $(\nabla_X A)T_i^+ \subset T_i^-$ and $(\nabla_X A)T_i^- \subset T_i^+$. By virtue of Codazzi's equation the reduced Codazzi equation holds for vector fields $X \in T_i$ and $Y \in T_j$ ($i \neq j$). We draw the same conclusion when $X \in T_i^+$ and $Y \in T_i^-$, or vice versa. Finally, if $X, Y \in T_i^+$ (or T_i^-), then using $J(\nabla_X A) = -(\nabla_X A)J$ and $JY \in T_i^-$ we get

$$(\nabla_X A)Y = -JJ(\nabla_X A)Y = J(\nabla_X A)JY = Js(X)JA(JY) = s(X)JAY.$$

In short, we have shown that the equation $(\nabla_X A)Y = s(X)JAY$ holds for all

X, Y .

THEOREM 4. *Let M be a complex hypersurface of complex dimension $n \geq 1$ in a space \tilde{M} of constant holomorphic curvature \tilde{c} . If the Ricci tensor of M is parallel, then either M is of constant holomorphic curvature \tilde{c} and totally geodesic in \tilde{M} or M is locally holomorphically isometric to the complex quadric Q^n in $P^{n+1}(C)$, the latter case arising only when $\tilde{c} > 0$.*

PROOF. When $n = 1$ the condition $\nabla S = 0$ simply means that M is of constant curvature and the classification obtained in § 6 will show that Theorem 4 is valid.

Assume $n \geq 2$. Let $\tilde{c} \neq 0$. In view of Lemma 5, M is locally symmetric. Consequently, each $\tau \in H$, considered as parallel displacement of $T_{x_0}(M)$ along a closed curve through x_0 , maps the curvature tensor R_{x_0} at x_0 into R_{x_0} . Thus if M has restricted holonomy group $U(n)$ then, since $U(n)$ acts transitively on the set of holomorphic planes at x_0 , we conclude that all holomorphic planes at x_0 have the same sectional curvature; since x_0 is an arbitrary point, M has constant holomorphic sectional curvature and immerses totally geodesically in \tilde{M} (see Theorem 1 [8]). If the restricted holonomy group of M is not $U(n)$, M is locally holomorphically isometric to Q^n and $\tilde{c} > 0$, by virtue of Theorem 2.

Let $\tilde{c} = 0$. The roots of A^2 are constant in value and multiplicity on M , since $\nabla A^2 = 0$. Let us now suppose that $A^2 \neq 0$ and choose a basis $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ of $T_{x_0}(M)$ diagonalizing A in the manner described in the previous section. Using the computations of § 2 and the fact that $\nabla R = 0$ and $\tilde{c} = 0$, we find

$$\begin{aligned} 0 &= (R(e_i, e_j)R)(e_i, Je_j) = [R(e_i, e_j), R(e_i, Je_j)] - R(R(e_i, e_j)e_i, Je_j) - R(e_i, R(e_i, e_j)Je_j) \\ &= -\lambda_i^2\lambda_j^2[K_j^i, S_j^i] + \lambda_i\lambda_jR(e_j, Je_j) - \lambda_i\lambda_jR(e_i, Je_j) \\ &= -2\lambda_i^2\lambda_j^2(S_i^i - S_j^j) + 2\lambda_i\lambda_j^3S_j^j - 2\lambda_i^3\lambda_jS_i^i \\ &= -2\lambda_i^2\lambda_j(\lambda_i + \lambda_j)S_i^i + 2\lambda_i\lambda_j^2(\lambda_i + \lambda_j)S_j^j. \end{aligned}$$

Thus $\lambda_i\lambda_j = 0$ or $\lambda_i + \lambda_j = 0$. Since $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ and $\lambda_1 > 0$, A^2 has precisely one nonzero characteristic root λ_1^2 and its multiplicity is 2. We confine our attention to the distributions T_1^+, T_1^-, T_1 and T^0 on $U(x_0)$, as defined in Lemma 5. We have already seen that T_1 and T^0 are parallel on M and that the reduced Codazzi equation holds by virtue of Lemma 5. Thus if Z is an arbitrary vector and W is a unit vector field in T_1^+ , then

$$s(Z)JAW = (\nabla_Z A)W = \nabla_Z(AW) - A\nabla_Z W = \lambda_1\nabla_Z W - A\nabla_Z W.$$

But since T_1 is parallel and (real) 2-dimensional and W is a unit vector in T_1^+ , we see that $\nabla_Z W \in T_1^-$ and $\lambda_1\nabla_Z W - A\nabla_Z W = 2\lambda_1\nabla_Z W$. Therefore, the equation above reduces to $\lambda_1 s(Z)JW = 2\lambda_1\nabla_Z W$, that is, $\nabla_Z W = \frac{1}{2}s(Z)JW$. It is an easy matter to verify that $R(X, Y)W = ds(X, Y)JW$, for arbitrary vectors

X, Y , so that $S(W, W) = R(W, JW, W, JW) = ds(JW, W)$. By virtue of proposition 4 [8], $S(W, W) = -2ds(JW, W)$. Hence $0 = S(W, W) = -2\lambda_1^2$ and this is a contradiction. Therefore $A^2 = 0$ and M is flat and totally geodesic in \tilde{M} . This completes the proof of Theorem 4.

With a view to obtaining a global version of this theorem, we suppose that M is a complete complex hypersurface in \tilde{M} with parallel Ricci tensor. f will denote the Kählerian immersion of M in \tilde{M} . Let \hat{M} be the universal covering manifold of M and let π be the covering map. On \hat{M} we take the Kählerian structure which makes π a holomorphic isometric immersion; \hat{M} is then simply-connected and complete and its Ricci tensor is parallel. Moreover $f \circ \pi$ is a holomorphic isometric immersion of \hat{M} in \tilde{M} .

If $\tilde{M} = P^{n+1}(C)$ then, in view of Theorem 4, \hat{M} is holomorphically isometric either to $P^n(C)$ or to Q^n and, by rigidity (Theorem 1), \hat{M} immerses either onto a projective hyperplane or onto a complex quadric in $P^{n+1}(C)$. In either case $f \circ \pi(\hat{M})$ is a simply-connected manifold and since $f \circ \pi$ is a covering map (see Theorem 4.6 in [5, p. 176]), it is one-to-one. Hence π is one-to-one and therefore M is holomorphically isometric either to P^n or to Q^n . The same type of argument can be applied when $\tilde{M} = D^{n+1}$ or C^{n+1} (without assuming that M is simply connected). We thus obtain the following improved form of Theorem 3 of [8]:

THEOREM 5.

- i) $P^n(C)$ and the complex quadric Q^n are the only complete complex hypersurfaces in $P^{n+1}(C)$ which have parallel Ricci tensors¹⁾.
- ii) D^n (resp. C^n) is the only complete complex hypersurface in D^{n+1} (resp. C^{n+1}) which has parallel Ricci tensor.

§ 4. Hypersurfaces of rank 2 in $P^{n+1}(C)$.

The main purpose of this section is to prove that in $P^{n+1}(C)$, $n \geq 3$, there is no compact complex hypersurface M which has rank 2 everywhere. We must, however, develop a few preliminary results on the nullity space of a curvature-type tensor field, which are generalized adaptations of some results of Maltz [6].

In general, let M be a Riemannian manifold with metric g and let D be a tensor field of type (1, 3) on M . We shall say that D is curvature-type if it satisfies the following conditions:

- i) $D(X, Y)$ is a skew-symmetric transformation for any pair of vectors X and Y ,

1) After the completion of our work we learned of a further generalization of (i) by S. Kobayashi (Hypersurfaces of complex projective space with constant scalar curvature, to appear).

$$\text{ii) } \quad D(Y, X) = -D(X, Y),$$

$$\text{iii) } \quad \mathfrak{S}\{D(X, Y)Z\} = 0,$$

where \mathfrak{S} is the cyclic sum taken over X, Y and Z ,

$$\text{iv) } \quad \mathfrak{S}\{(\nabla_x D)(Y, Z)\} = 0.$$

It is well known that the Riemannian curvature tensor field R of M satisfies these conditions. We also note that (i), (ii) and (iii) imply

$$\text{v) } \quad g(D(X, Y)Z, W) = g(D(Z, W)X, Y),$$

as is the case for R (see [5], p. 198).

We define the *nullity space* T_x^0 of D at each point $x \in M$ to be the subspace $\{X \mid D(X, Y) = 0 \text{ for all } Y \in T_x(M)\}$ of $T_x(M)$; its dimension is called the *index of nullity* of D . Let T_x^1 be the orthogonal complement of T_x^0 . The following lemmas can be proved in exactly the same way as those in [6].

LEMMA 6.

i) If $X \in T_x^0$, then $D(Y, Z)X = 0$ for all $Y, Z \in T_x(M)$.

ii) T_x^1 coincides with the subspace spanned by all $D(X, Y)Z$, where $X, Y, Z \in T_x(M)$.

LEMMA 7. Assume that the index of nullity of a curvature-type tensor field D is constant on M . Then the distribution $T^0: x \rightarrow T_x^0$ is involutive and totally geodesic (that is, $\nabla_x T^0 \subset T^0$ for any vector $X \in T^0$ so that any integral manifold of T^0 is a totally geodesic submanifold of M).

We shall apply the foregoing lemma to the situation where M is a complex hypersurface in a space \tilde{M} of constant holomorphic curvature \tilde{c} . The curvature tensor R of M is given by Gauss' equation

$$R(X, Y) = \tilde{R}(X, Y) + D(X, Y),$$

the expressions for $\tilde{R}(X, Y)$ and $D(X, Y)$ being as in §1. Since both R and \tilde{R} are curvature-type tensor fields on M , so is their difference D . We know (Lemma 1, §1) that the nullity space T_x^0 coincides with the kernel of A at x . Hence $\dim T_x^0$ equals the rank of M at x . Assume now that this is constant on M . The distribution T^0 is integrable and totally geodesic by Lemma 7; it is also invariant by the complex structure J , because $JA = -AJ$. If M^0 is a maximal integral manifold of T^0 , we conclude that M^0 is a complex submanifold of M which is totally geodesic in M . The curvature tensor R^0 of M^0 (with respect to the metric induced from that of M) is given by $R^0(X, Y) = R(X, Y)$, where $X, Y \in T_x(M^0)$, which is equal to $\tilde{R}(X, Y)$, since $D(X, Y) = 0$ for $X, Y \in T_x(M^0) = T_x^0$. Thus

$$R^0(X, Y) = \frac{\tilde{c}}{4} \{X \wedge Y + JX \wedge JY + 2g(X, JY)J\},$$

which means that M^0 has constant holomorphic curvature \tilde{c} .

Considering M^0 as a complex submanifold of \tilde{M} , we may establish the formula

$$\tilde{K}(X) = K^0(X) + 2 \sum_{i=1}^k \{g(A_i X, X)^2 + g(JA_i X, X)^2\}$$

(for a unit vector X tangent to M^0) relating the sectional curvatures $\tilde{K}(X)$ and $K^0(X)$ in \tilde{M} and M^0 , respectively, of the holomorphic plane generated by X . In this formula A_1, \dots, A_k are the second fundamental forms corresponding to a choice of an orthonormal family of vector fields ξ_1, \dots, ξ_k normal to M^0 , and k is the complex codimension of M^0 in \tilde{M} . This formula generalizes that of Corollary 2 [8]. Since $\tilde{K}(X) = K^0(X) = \tilde{c}$ in our case, it follows that each A_i is identically zero, which means that M^0 is totally geodesic in \tilde{M} .

Let us now assume that M is a complete complex hypersurface in $P^{n+1}(C)$, C^{n+1} or D^{n+1} such that the rank of M is everywhere equal to $2r$. We show that M^0 is then complete. Let $\gamma(s)$ be a geodesic in M^0 defined on $a < s < b$. Since M is complete and M^0 is totally geodesic in M , $\gamma(s)$ can be extended as a geodesic $\gamma^*(s)$ in M , defined for all values of s . Let $(x^1, \dots, x^{2m}, x^{2m+1}, \dots, x^{2n})$, where $m = n - r$, be a system of local coordinates on M with origin $\gamma^*(b)$, such that $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{2m}} \right\}$ is a local basis for T^0 . When s is in a certain neighborhood of b , say $(b - \epsilon, b + \epsilon)$, we may express $\gamma^*(s)$ by the set of equations $x^i(\gamma^*(s)) = f^i(s)$, $1 \leq i \leq 2n$. Since $\gamma^*(s) \in M^0$ when $a < s < b$ we must then have $f^i(s) = c^i$ (a constant) for $2m + 1 \leq i \leq 2n$. Letting s approach b from below we find that $0 = f^i(b) = c^i$, $2m + 1 \leq i \leq 2n$. Thus $\gamma^*(b)$ is in the maximal integral manifold which contains $\gamma(s)$, $a < s < b$. In other words $\gamma^*(b) \in M^0$ and it is possible to extend $\gamma(s)$ as a geodesic in M^0 for parameter values larger than b . Thus M^0 is complete.

Since we know that any complete totally geodesic complex n -dimensional submanifold of $P^{n+1}(C)$, C^{n+1} , or D^{n+1} is of the form $P^m(C)$, C^m , D^m , respectively, we obtain

PROPOSITION 1. *Let M be a complex hypersurface of $\tilde{M} = P^{n+1}(C)$, C^{n+1} , or D^{n+1} . If the rank (of the second fundamental form) of M is everywhere equal to a constant, $2r$, then M contains a complete totally geodesic complex $(n - r)$ -dimensional submanifold of \tilde{M} , namely $P^{n-r}(C)$, C^{n-r} , D^{n-r} , respectively.*

We now prove the main theorem of this section

THEOREM 6. *Let M be a compact complex hypersurface of $P^{n+1}(C)$, $n \geq 3$. The rank (of the second fundamental form) of M cannot be identically equal to 2.*

REMARK. For $n = 1$, the quadrics are the only closed complex curves in $P^2(C)$ of rank identically equal to 2 (see (i) of Theorem 9 in § 6). The case $n = 2$ remains unsettled.

PROOF. By virtue of Proposition 1, M contains a projective subspace P^{n-1} . Choose a system of homogeneous coordinates $(z_0, z_1, \dots, z_{n+1})$ in $P^{n+1}(C)$ such that P^{n-1} is given by $z_0 = z_1 = 0$. By a theorem of Chow the compact complex hypersurface M can be defined by $f = 0$, where f is a homogeneous polynomial in z_0, z_1, \dots, z_{n+1} such that the partial derivatives $\frac{\partial f}{\partial z_k}$ ($0 \leq k \leq n+1$) are not all zero at any point of M . We write f in the form

$$f(z_0, \dots, z_{n+1}) = F(z_2, \dots, z_{n+1}) + z_0 f_0(z_2, \dots, z_{n+1}) + z_1 f_1(z_2, \dots, z_{n+1}) + \sum_{k+l \geq 2} z_0^k z_1^l f_{kl}(z_2, \dots, z_{n+1}),$$

where F, f_0, f_1 and f_{kl} are homogeneous polynomials in the variables z_2, \dots, z_{n+1} . Since $P^{n-1} \subset M$, we have $f(0, 0, z_2, \dots, z_{n+1}) = 0$ for all z_2, \dots, z_{n+1} . Thus F is identically zero and

$$f = z_0 f_0 + z_1 f_1 + \sum_{k+l \geq 2} z_0^k z_1^l f_{kl}.$$

Consequently

$$\frac{\partial f}{\partial z_0} = f_0 + \sum_{k+l \geq 2} k z_0^{k-1} z_1^l f_{kl},$$

$$\frac{\partial f}{\partial z_1} = f_1 + \sum_{k+l \geq 2} l z_0^k z_1^{l-1} f_{kl}$$

and

$$\frac{\partial f}{\partial z_j} = z_0 \frac{\partial f_0}{\partial z_j} + z_1 \frac{\partial f_1}{\partial z_j} + \sum_{k+l \geq 2} z_0^k z_1^l \frac{\partial f_{kl}}{\partial z_j}$$

for $j \geq 2$. At $(0, 0, z_2, \dots, z_{n+1}) \in P^{n-1} \subset M$, we have $\frac{\partial f}{\partial z_j} = 0$ for $j \geq 2$, $\frac{\partial f}{\partial z_0} = f_0$ and $\frac{\partial f}{\partial z_1} = f_1$. Lemma 8 will show, however, that unless f_0 and f_1 are constants there exist z_2, \dots, z_{n+1} (not all zero) for which $f_0 = f_1 = 0$. This would mean that there is a point $(0, 0, z_2, \dots, z_{n+1}) \in M$ where all the partial derivatives $\frac{\partial f}{\partial z_k}$ are zero. Thus f_0 and f_1 are constants, so that f is of degree 1 and is given by $f = c_0 z_0 + c_1 z_1$, where c_0, c_1 are constants; therefore M is a projective hyperplane in P^{n+1} and thus M is of rank zero everywhere. This is a contradiction.

The following lemma occurs as a particular case of the main theorem of § 5 in Samuel's book [7], although it is easy to give a direct proof using the theory of resultants.

LEMMA 8. *For any two non-constant homogeneous polynomials $g, h \in C[x_1, \dots, x_n]$, $n \geq 3$, there is a non-trivial solution of $g = h = 0$.*

§ 5. Hypersurfaces in C^{n+1} .

To begin with, we suppose that M is a complex hypersurface in an arbitrary Kählerian manifold \tilde{M} . For any vector field X on M and for any field of vectors ξ normal to M in \tilde{M} , we define $\hat{V}_X\xi$ to be the normal component of $\tilde{V}_X\xi$, where \tilde{V} refers, as in [8], to covariant differentiation in \tilde{M} . We may easily verify that \hat{V} is a linear connection in the normal bundle over M , which we call the normal connection for the hypersurface M . The relative curvature tensor \hat{R} of M (that is, the curvature tensor of the normal connection of M) is given by

$$\hat{R}(X, Y)\xi = [\hat{V}_X, \hat{V}_Y]\xi - \hat{V}_{[X, Y]}\xi,$$

where X and Y are vector fields tangent to M . If ξ is a field of unit normals, $\hat{V}_X\xi$ is equal to $s(X)J\xi$ and, by an easy computation, we find

PROPOSITION 2. *The relative curvature tensor \hat{R} of M is expressed by*

$$\hat{R}(X, Y)\xi = 2ds(X, Y)J\xi,$$

where ξ is a field of unit normals to M .

Now assume that \tilde{M} has constant holomorphic sectional curvature \tilde{c} . According to Proposition 4 of [8], we have

$$\tilde{S}(X, JY) = S(X, JY) + 2ds(X, Y),$$

where \tilde{S} and S denote the Ricci tensors of \tilde{M} and M , respectively. We shall prove

THEOREM 7. *Let M be a complex hypersurface of complex dimension $n \geq 1$ in a space \tilde{M} of constant holomorphic curvature \tilde{c} . The following conditions are equivalent:*

- i) *The normal connection of M is trivial, that is, $\hat{R} = 0$.*
- ii) *$S = \tilde{S}$ on M .*
- iii) *$S = 0$ on M .*
- iv) *$\tilde{c} = 0$ and M is totally geodesic in \tilde{M} .*

PROOF. It is clear that iv) implies each of the other conditions, while the equivalence of i) and ii) follows from Proposition 2 above. Assuming ii) we see that M is Einstein. By Theorem 4, M is then totally geodesic in \tilde{M} or else $\tilde{c} > 0$ and M is locally holomorphically isometric to Q^n in $P^{n+1}(C)$. Thus $S = (n+1)\frac{\tilde{c}}{2}g$ or else $\tilde{c} > 0$ and $S = \frac{n\tilde{c}}{2}g$. However, $\tilde{S} = (n+2)\frac{\tilde{c}}{2}g$. Therefore $\tilde{c} = 0$ and $S = 0$ and consequently M is totally geodesic in \tilde{M} . In other words, ii) implies both iii) and iv). If $S = 0$, then M is Einstein and it is clear from the above that $\tilde{c} = 0$ and M is totally geodesic in \tilde{M} . Thus iii) implies iv) and the equivalence of all four conditions is proved.

The general object of the remainder of this section is to define the Gaussian

mapping of a complex hypersurface in complex Euclidean space C^{n+1} into the complex projective space $P^n(C)$, and to give a geometric interpretation thereof. It is convenient to begin by establishing a relationship between the Riemannian connection on the sphere S^{2n+1} and the Kählerian connection on $P^n(C)$ (for the Fubini-Study metric, of course).

$P^n(C)$ can be regarded as the base of a principal fibre bundle S^{2n+1} (unit sphere in C^{n+1}) on which the structure group $S^1 = \{e^{i\theta} | \theta \in R\}$ acts as follows: $S^{2n+1} \times S^1 \ni (z, e^{i\theta}) \rightarrow ze^{i\theta} \in S^{2n+1}$. π denotes the canonical projection of S^{2n+1} onto $P^n(C)$ and $g(z, w) = \text{Re} \left(\sum_{k=0}^n z^k w^{-k} \right)$ for $z = (z^0, z^1, \dots, z^n)$, $w = (w^0, w^1, \dots, w^n)$ defines the Euclidean metric on C^{n+1} . With the natural identification between vectors tangent to S^{2n+1} and vectors C^{n+1} , we have

$$T_z(S^{2n+1}) = \{w \in C^{n+1} | g(z, w) = 0\}$$

for each $z \in S^{2n+1}$. The orthogonal complement of

$$T'_z = \{w \in C^{n+1} | g(z, w) = g(iz, w) = 0\}$$

in $T_z(S^{2n+1})$ is the 1-dimensional subspace $\{iz\}$ which is spanned by the vector iz (in the sense of the above identification). The distribution T' defines a connection in the principal fibre bundle $S^{2n+1}(P^n(C), S^1)$, that is, T'_z is complementary to the subspace $\{iz\}$ tangent to the fibre through z , and T' is invariant by the action of S^1 . Thus the projection π induces a linear isomorphism of T'_z onto $T_{\pi(z)}(P^n(C))$ and π maps $\{iz\}$ into zero for each $z \in S^{2n+1}$.

The classical Fubini-Study metric of holomorphic sectional curvature 1 is nothing but the metric \tilde{g} defined by $\tilde{g}(\tilde{X}, \tilde{Y}) = 4g(X', Y')$, where $\tilde{X}, \tilde{Y} \in T_p(P^n(C))$ and X', Y' are their respective horizontal lifts at z ($\pi(z) = p$). Since g is invariant by S^1 , the definition of $\tilde{g}(\tilde{X}, \tilde{Y})$ is independent of the choice of z . We might also observe that the complex structure in T'_z (defined by multiplication of vectors by i) induces the canonical complex structure J on $P^n(C)$, when transferred by means of π . (What we have said so far is more or less well known.)

The horizontal lift of a vector field \tilde{X} on $P^n(C)$ will be denoted by X' . If \tilde{X} and \tilde{Y} are vector fields on $P^n(C)$, then the vector fields X' and Y' are invariant by S^1 ; since the Riemannian connection on S^{2n+1} is invariant by S^1 , it follows that $\nabla'_{X'} Y'$ (where ∇' denotes covariant differentiation on S^{2n+1}) is also invariant by S^1 and hence projectable, that is, there exists a vector field \tilde{Z} on $P^n(C)$ such that $\pi_*(\nabla'_{X'} Y')_z = \tilde{Z}_{\pi(z)}$ for all $z \in S^{2n+1}$.

PROPOSITION 3. *For every pair of vector fields \tilde{X}, \tilde{Y} on $P^n(C)$ the vector field $\nabla'_{X'} Y'$ on S^{2n+1} is projectable and $\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \pi_*(\nabla'_{X'} Y')$ defines the Kählerian connection on $P^n(C)$.*

PROOF. To prove this we verify the following:

i) $\tilde{\nabla}$ is a linear connection. Obviously $\tilde{\nabla}_{\tilde{X}} \tilde{Y}$ is bi-additive in X and Y .

For any differentiable function \tilde{f} on $P^n(C)$ we let $f' = \tilde{f} \circ \pi$ be its lift to S^{2n+1} . Then $f'X'$ is the horizontal lift of $\tilde{f}\tilde{X}$ and $\nabla'_{f'X'}Y' = f'\nabla'_{X'}Y'$ is projectable. Thus

$$\tilde{\nabla}'_{\tilde{f}\tilde{X}}\tilde{Y}' = \pi_*(\nabla'_{f'X'}Y') = \pi_*(f'\nabla'_{X'}Y') = \tilde{f}\tilde{\nabla}'_{\tilde{X}}\tilde{Y}'.$$

Similarly, we can prove

$$\tilde{\nabla}'_{\tilde{X}}(\tilde{f}\tilde{Y}') = (\tilde{X}\tilde{f})\tilde{Y}' + \tilde{f}\tilde{\nabla}'_{\tilde{X}}\tilde{Y}'.$$

ii) *The torsion tensor of $\tilde{\nabla}'$ is zero.* If \tilde{X} and \tilde{Y} are vector fields on $P^n(C)$ then $[X', Y']$ is projectable and $\pi_*[X', Y'] = [\tilde{X}, \tilde{Y}]$. Consequently

$$\tilde{\nabla}'_{\tilde{X}}\tilde{Y}' - \tilde{\nabla}'_{\tilde{Y}}\tilde{X}' - [\tilde{X}, \tilde{Y}] = \pi_*(\nabla'_{X'}Y' - \nabla'_{Y'}X' - [X', Y']) = 0.$$

iii) *∇' is a metric connection for \tilde{g} .* Let \tilde{X}, \tilde{Y} and \tilde{Z} be vector fields on $P^n(C)$. On S^{2n+1} we have

$$X'g(Y', Z') = g(\nabla'_{X'}Y', Z') + g(Y', \nabla'_{X'}Z').$$

Denoting by h the horizontal component of vector fields on S^{2n+1} , we see that

$$\begin{aligned} g_z(\nabla'_{X'}Y', Z') &= g_z(h(\nabla'_{X'}Y'), Z') \\ &= \frac{1}{4} \tilde{g}_p(\pi_*h(\nabla'_{X'}Y'), \pi_*Z') = \frac{1}{4} \tilde{g}_p(\tilde{\nabla}'_{\tilde{X}}\tilde{Y}', \tilde{Z}'), \end{aligned}$$

where $\pi(z) = p$. Similarly, we have $g_z(Y', \nabla'_{X'}Z') = \frac{1}{4} \tilde{g}_p(\tilde{Y}', \tilde{\nabla}'_{\tilde{X}}\tilde{Z}')$. On the other hand we have $g(Y', Z') = \tilde{f} \circ \pi$, where $\tilde{f} = \frac{1}{4} \tilde{g}(\tilde{Y}', \tilde{Z}')$, so that

$$X'_z g(Y', Z') = X'_z(\tilde{f} \circ \pi) = (\pi_*X'_z)\tilde{f} = \tilde{X}_p\tilde{f} = \frac{1}{4} \tilde{X}_p\tilde{g}(\tilde{Y}', \tilde{Z}').$$

The metric condition for ∇' therefore gives rise to the same condition for $\tilde{\nabla}'$, that is,

$$\tilde{X}\tilde{g}(\tilde{Y}', \tilde{Z}') = \tilde{g}(\tilde{\nabla}'_{\tilde{X}}\tilde{Y}', \tilde{Z}') + \tilde{g}(\tilde{Y}', \tilde{\nabla}'_{\tilde{X}}\tilde{Z}').$$

REMARK. If z_t is a horizontal curve on S^{2n+1} and Y'_t is a family of horizontal vectors defined along z_t , then $\pi_*(\nabla'_{z'_t}Y'_t) = \tilde{\nabla}'_{\pi_*(z'_t)}\pi_*(Y'_t)$ along z_t , where \dot{z}_t is the velocity vector of the curve z_t at time t .

To verify this for each t_0 we extend \dot{z}_t and Y'_t , respectively, to horizontal vector fields Z' and Y' in a neighborhood of z_{t_0} , as follows: extend $\pi_*(\dot{z}_t)$ (resp. $\pi_*(Y'_t)$) to a neighborhood of $\pi(z_{t_0})$ and let Z' (resp. Y') be its horizontal lift. We then have $\pi_*(\nabla'_{Z'}Y') = \tilde{\nabla}'_{Z'}\tilde{Y}'$, which implies $\pi_*(\nabla'_{z'_t}Y'_t) = \tilde{\nabla}'_{\pi_*(z'_t)}\pi_*(Y'_t)$ at z_{t_0} .

Turning our attention now to a complex hypersurface M in C^{n+1} , we shall first define a generalized Gaussian mapping of M into $P^n(C)$.

For each point $x \in M$ we can choose a unit vector ξ normal to M at x . As a vector in C^{n+1} , it is determined to within a multiple of the form $e^{i\theta}$.

Thus $\phi(x) = \pi(\xi) \in P^n(C)$ is well defined and the mapping $\phi: M \rightarrow P^n(C)$ is called the *Gaussian mapping* of M . We can relate ϕ to the second fundamental form A of M (in the formalism of [8]) as follows:

Let $X \in T_x(M)$ and take a curve x_t on M such that $x_0 = x$ and $(\vec{x}_t)_{t=0} = X$. Choose a (differentiable) family of unit normals ξ_t along x_t . The differential ϕ_* of ϕ maps X upon

$$\left(\frac{d\pi(\xi_t)}{dt}\right)_{t=0} = \pi_*\left(\frac{d\xi_t}{dt}\right)_{t=0} \in T_{\phi(x)}(P^n(C)),$$

where $\left(\frac{d\xi_t}{dt}\right)_{t=0}$ is the tangent vector of the curve ξ_t on S^{2n+1} at ξ_0 . On the other hand, the Weingarten formula for M as a complex hypersurface in C^{n+1} (with the flat connection D) gives

$$\left(\frac{d\xi_t}{dt}\right)_{t=0} = D_x \xi = -AX + s(X)J\xi, \quad \text{where } J\xi = i\xi.$$

Since $J\xi$ is the initial tangent vector of the curve $e^{i\theta}\xi$ on S^{2n+1} , we have $\pi_*(J\xi) = 0$. Hence

$$\phi_*(X) = -\pi_*(AX).$$

The vector AX , considered by translation as a tangent vector to S^{2n+1} at ξ , belongs to T'_ξ because it is perpendicular to $J\xi$. Since $\pi_*: T'_\xi \rightarrow T_{\pi(\xi)}(P^n(C))$ is one-to-one, we conclude that

- i) $\phi_*(X) = 0$ if and only if $AX = 0$.
- ii) The rank of ϕ_* is equal to the rank of A .

Since $\phi_*(JX) = -\pi_*(AJX) = \pi_*(JAX)$ and since the complex structure J on T'_ξ corresponds to the complex structure \check{J} on $T_{\pi(\xi)}(P^n(C))$, by means of π , we have

$$\phi_*(JX) = \check{J}\pi_*(AX) = -\check{J}\phi_*(X),$$

namely,

- iii) the Gaussian mapping ϕ is anti-holomorphic.

EXAMPLES.

- i) If M is a hyperplane C^n in C^{n+1} we have a constant unit normal ξ over M , so that $\phi(M)$ is a single point in $P^n(C)$.
- ii) If M is of the form $K \times C^{n-1}$, where K is a complex curve in a plane C^2 perpendicular to C^{n-1} , then the rank of ϕ is ≤ 2 everywhere and $\phi(M)$ lies in a projective line $P^1(C)$ in $P^n(C)$. It will be interesting to find an appropriate converse of this proposition.

In relating the Kählerian connection on M to that on $P^n(C)$, the following lemma will be useful.

LEMMA 9. *Let x_t be a differentiable curve on M . Then there is a family of unit normals ξ_t along x_t which, as a curve in S^{2n+1} , is horizontal.*

PROOF. For an arbitrary family of unit normals η_t along x_t we consider

a family of unit normals given by $\xi_t = a\eta_t + bJ\eta_t$, where $a = a(t)$ and $b = b(t)$ are differentiable functions such that $a^2 + b^2 = 1$. We show that by choosing a and b suitably we can make ξ_t horizontal, that is $g\left(\frac{d\xi_t}{dt}, J\xi_t\right) = 0$ for all t . It is readily verified that

$$g\left(\frac{d\xi_t}{dt}, J\xi_t\right) = g\left(\frac{d\eta_t}{dt}, J\eta_t\right) + a\frac{db}{dt} - b\frac{da}{dt}.$$

Thus our purpose will be achieved if we can choose a and b such that

$$a\frac{db}{dt} - b\frac{da}{dt} = k(t) \quad \text{and} \quad a^2 + b^2 = 1,$$

where $k(t) = -g\left(\frac{d\eta_t}{dt}, J\eta_t\right)$. Since $a^2 + b^2 = 1$ implies $a\frac{da}{dt} + b\frac{db}{dt} = 0$, we have

$$\frac{da}{dt} = -bk(t) \quad \text{and} \quad \frac{db}{dt} = ak(t).$$

Thus we may take

$$a(t) = \cos l(t) - \sin l(t), \quad b(t) = \sin l(t) + \cos l(t),$$

where $l(t) = \int k(t)dt$.

THEOREM 8. *Let M be a complex hypersurface in C^{n+1} and let Y_t be a family of vectors tangent to M along a curve x_t . Choose a family of unit normals ξ_t along x_t as in Lemma 9 and let Y'_t be the vector tangent to S^{2n+1} at ξ_t which is parallel to Y_t in C^{n+1} . Let $\tilde{Y}_t = \pi_*(Y'_t)$. Then Y_t is parallel along x_t on M if and only if \tilde{Y}_t is parallel along $\phi(x_t)$ on $P^n(C)$.*

PROOF. For M we have

$$(10) \quad \frac{dY_t}{dt} = D_{\vec{x}_t} Y_t = \nabla_{\vec{x}_t} Y_t + h(\vec{x}_t, Y_t)\xi_t + k(\vec{x}_t, Y_t)J\xi_t,$$

where D is the flat connection in C^{n+1} and ∇ is the Kählerian connection on M . On the other hand, for S^{2n+1} (with the Riemannian connection ∇') we get

$$(11) \quad \frac{dY'_t}{dt} = \frac{dY'_t}{dt} = D_{\vec{\xi}_t} Y'_t = \nabla'_{\vec{\xi}_t} Y'_t + h'(\vec{\xi}_t, Y'_t)\xi_t,$$

where h' is the second fundamental form of S^{2n+1} with respect to the unit normals ξ_t . Equations (10) and (11) yield

$$\nabla_{\vec{x}_t} Y_t + \{h(\vec{x}_t, Y_t) - h'(\vec{\xi}_t, Y'_t)\}\xi_t + k(\vec{x}_t, Y_t)J\xi_t = \nabla'_{\vec{\xi}_t} Y'_t$$

(considered as an identity among vectors in C^{n+1}). Therefore

$$\nabla_{\vec{x}_t} Y_t = \nabla'_{\vec{\xi}_t} Y'_t - k(\vec{x}_t, Y_t)J\xi_t.$$

Thus, if $\nabla_{\vec{x}_t} Y_t = 0$, the fact that both $\vec{\xi}_t$ and Y'_t are horizontal and that $J\xi_t$ is vertical in $T_{\vec{\xi}_t}(S^{2n+1})$ implies

$$0 = \pi_* (\nabla'_{\xi_t} Y'_t - k(\hat{x}_t, Y_t) J\xi_t) = \pi_* (\nabla'_{\xi_t} Y'_t) = \tilde{\nabla}_{\pi(\xi_t)} \pi_*(Y'_t) = \tilde{\nabla}_{\phi(x_t)} \tilde{Y}'_t,$$

in view of the remark following Proposition 3. Conversely, suppose $\tilde{\nabla}_{\phi(x_t)} \tilde{Y}'_t = 0$. Then $\nabla'_{\xi_t} Y'_t$ must be vertical, that is, in the direction of $J\xi_t$. From (11) we see that $\frac{dY'_t}{dt}$ is a linear combination of ξ_t and $J\xi_t$. Therefore $\nabla_{x_t} Y_t = 0$, by virtue of (10).

REMARK. For a complex n -dimensional submanifold M of C^{n+k} there is a naturally defined mapping $\phi: M \rightarrow U(n+k)/U(n) \times U(k)$ and an associated mapping of the bundle of unitary frames over M into $U(n+k)/U(k)$. This bundle mapping was studied by Kerbrat [3]. For an n -dimensional (real) orientable submanifold in real Euclidean space R^{n+k} , there is a naturally defined mapping $\phi: M \rightarrow SO(n+k)/SO(n) \times SO(k)$. If $k=2$, the latter space can be identified with the quadric Q^n in $P^{n+1}(C)$ and we may relate the Riemannian connection on M to the Kählerian connection on Q^n by means of ϕ in a geometric manner similar to that of Theorem 8.

§ 6. Complex curves.

We now turn to (nonsingular) complex curves in a complex 2-dimensional space \tilde{M} of constant holomorphic sectional curvature \tilde{c} and we derive a very convenient formula for their curvature. If M is such a curve, then $A^2 = \lambda_1^2 I$ on M . Since the curvature K of M is given by $K = -2\lambda_1^2 + \tilde{c}$ (Corollary 3, [8]), we have $K \leq \tilde{c}$ on M . We now suppose that $K(x_0) \neq \tilde{c}$, so that $\lambda_1^2 \neq 0$ in a neighborhood $U(x_0)$ of x_0 ; let λ_1 denote its positive square root. Consider the distributions T_1^+, T_1^- on $U(x_0)$ as defined in § 3. Codazzi's equation may be written

$$\nabla_Y(AZ) - \nabla_Z(AY) - A\nabla_Y Z + A\nabla_Z Y - s(Y)JAZ + s(Z)JAY = 0$$

and, supposing that the vector fields Y and Z belong to T_1^- and T_1^+ respectively, we obtain

$$Y(\lambda_1)Z + \lambda_1 \nabla_Y Z + Z(\lambda_1)Y + \lambda_1 \nabla_Z Y - A\nabla_Y Z + A\nabla_Z Y - \lambda_1 s(Y)JZ - \lambda_1 s(Z)JY = 0.$$

If, in addition, Y and Z are unit vector fields, a consideration of the T_1^- -component of this equation yields

$$-A\nabla_Y Z + \lambda_1 \nabla_Y Z + Z(\lambda_1)Y - \lambda_1 s(Y)JZ = 0,$$

i. e. $2\lambda_1 \nabla_Y Z + Z(\lambda_1)Y - \lambda_1 s(Y)JZ = 0,$

i. e. $\nabla_Y Z = -\frac{1}{2} \left\{ Z \left(\frac{1}{2} \ln \lambda_1^2 \right) Y - s(Y)JZ \right\}$

$$= -\frac{1}{2} \{ Z(\mu)Y - s(Y)JZ \},$$

where $\mu = -\frac{1}{2} \ln \lambda_1^2$. However, $Z(\mu)Y = -(JY)(\mu)JZ$ since $JY = \pm Z$. Hence

$$\nabla_Y Z = \frac{1}{2} [(JY)\mu + s(Y)] JZ.$$

Note that this still holds if Y is an arbitrary vector field in T_1^- . Also, if Z is a unit vector field in T_1^- (instead of T_1^+), then JZ is a unit vector field in T_1^+ so that the formula above is valid when we replace Z by JZ . Using $\nabla_Y(JZ) = J(\nabla_Y Z)$ we obtain again

$$\nabla_Y Z = \frac{1}{2} \{(JY)\mu + s(Y)\} JZ,$$

where Z is a unit vector field in T_1^- .

Similarly, we obtain

$$\nabla_Z Y = \frac{1}{2} \{(JZ)\mu + s(Z)\} JY$$

on considering the T_1^+ -component of Codazzi's equation. Note that this still holds if Z is an arbitrary vector field in T_1^+ and if Y is a unit vector field in T_1^+ . Combining all cases we conclude that

$$\nabla_Y Z = \frac{1}{2} \{(JY)\mu + s(Y)\} JZ$$

when Z is a unit vector field in either T_1^+ or T_1^- and Y is an arbitrary vector.

It may be readily verified that

$$\nabla_X \nabla_Y Z = \frac{1}{2} \{X(JY)\mu + Xs(Y)\} JZ - \frac{1}{4} \{(JY)\mu + s(Y)\} \{(JX)\mu + s(X)\} Z,$$

where Z is a unit vector field in T_1^+ . Therefore

$$\begin{aligned} R(JZ, Z)Z &= \frac{1}{2} \{(JZ)(JZ)\mu + ZZ\mu - J([JZ, Z])\mu\} JZ + ds(JZ, Z)JZ \\ &= \frac{1}{2} \{ZZ\mu + (JZ)(JZ)\mu - (\nabla_Z Z + \nabla_{JZ} JZ)\mu\} JZ + ds(JZ, Z)JZ \\ &= \left\{ \frac{1}{2} \Delta\mu + ds(JZ, Z) \right\} JZ, \end{aligned}$$

where $\Delta\mu$ denotes the Laplacian of μ . Since $K = g(R(JZ, Z)Z, JZ)$, we obtain $K = \frac{1}{2} \Delta\mu + ds(JZ, Z)$ and, using (12), we have

PROPOSITION 4. *Let M be a complex curve in a complex 2-dimensional space \tilde{M} of constant holomorphic curvature \tilde{c} . The curvature of M is given by*

$$K = \frac{1}{3} \Delta\mu + \frac{\tilde{c}}{2}$$

on the open set $\{x \in M \mid K(x) \neq \tilde{c}\}$, where $\mu = \frac{1}{2} \ln \lambda_1^2$ and λ_1^2 is defined by

$$A^2 = \lambda_1^2 I.$$

It is now easy to prove Theorem 4 for the case $n=1$. Let M be of constant curvature but not totally geodesic in \tilde{M} , then λ_1^2 is a nonzero constant, so that $\Delta\mu=0$ on M . Thus $K=\tilde{c}/2$ by Proposition 4, and since $K=-2\lambda_1^2+\tilde{c}$, it follows that $\lambda_1^2=\tilde{c}/4$. In particular $\tilde{c}>0$. However the complex quadric Q^1 in $P^2(C)$ is of constant curvature and is not totally geodesic in $P^2(C)$; consequently $\lambda_1^2=\tilde{c}/4$ on Q^1 also. Thus if M is of constant curvature but not totally geodesic in \tilde{M} then $\tilde{c}>0$ and M is locally holomorphically isometric to Q^1 in $P^2(C)$.

Consider $P^2(C)$ with the Fubini-Study metric of holomorphic curvature 1. Let M be a (nonsingular) closed complex curve in $P^2(C)$ and suppose $K<1$ at every point of M . Then $K=\frac{1}{3}\Delta\mu+\frac{1}{2}$ is a global formula for the curvature of M . Let dv denote the area element of the Riemannian manifold M . By virtue of the Gauss-Bonnet theorem and Green's theorem we obtain

$$2\pi\chi = 4\pi(1-p) = \int_M K dv = \int_M \left(\frac{1}{3}\Delta\mu + \frac{1}{2}\right) dv = \frac{1}{2} \int_M dv > 0,$$

where χ and p are the Euler characteristic and genus of M , respectively. The genus of M is therefore zero. However M , being a closed complex curve in $P^2(C)$, is algebraic and its genus is given by $p = \frac{(n-1)(n-2)}{2}$, where n is the order of the curve M (see p. 179, [10]). Thus M is of order 1 or 2, that is, M is either a projective line or a quadric. However, K is identically equal to 1 on a projective line. Thus M is a quadric, which is congruent to $Q^1: z_0^2+z_1^2+z_2^2=0$ by a projective transformation of $P^2(C)$ but not necessarily by a holomorphic motion of $P^2(C)$.

If further we assume either $K \leq 1/2$ everywhere or $1/2 \leq K < 1$ everywhere, then $\Delta\mu = 3(K-1/2)$ is of constant sign on M and, by Green's Theorem, we must have $\Delta\mu=0$ on M , that is, $K \equiv 1/2$. In either case M is congruent to Q^1 by a holomorphic motion of $P^2(C)$.

We now show that M is a projective line if $K > 1/2$ everywhere on M . If not there would be a point $x_0 \in M$ of minimum curvature $K(x_0) < 1$. Then $\mu(x_0)$ is a maximum for μ so that $\Delta\mu = 3(K-1/2) \leq 0$ at x_0 . In other words $K(x_0) \leq 1/2$ and this is a contradiction.

We summarize these results in the following theorem.

THEOREM 9. *On an arbitrary nonsingular complex curve M in the projective plane $P^2(C)$, the curvature K of M (considered with the induced Kähler structure) satisfies $K \leq 1$ everywhere. If M is closed, the following results hold:*

i) *If $K < 1$ everywhere, then M is (complex analytically) a quadric²⁾.*

2) Blaine Lawson has given us an example which shows that M need not be holomorphically isometric to the quadric Q^1 .

ii) If $K \leq 1/2$ everywhere or if $1/2 \leq K < 1$ everywhere, then M is congruent to the quadric Q^1 by a holomorphic motion of $P^2(C)$, and of course $K=1/2$ everywhere.

iii) If $K > 1/2$ everywhere, then M is a projective line and $K=1$ everywhere.
From ii) and iii) we also obtain

COROLLARY. Any closed nonsingular complex curve in $P^2(C)$ has a point where $K \geq 1/2$. If M is not a projective line, it has a point where $K=1/2$.

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