

# DIFFERENTIAL GEOMETRY OF SUBMANIFOLDS OF PROJECTIVE SPACE\*

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**Abstract.** These are lecture notes on the rigidity of submanifolds of projective space “resembling” compact Hermitian symmetric spaces in their homogeneous embeddings. The results of [16, 20, 29, 18, 19, 10, 31] are surveyed, along with their classical predecessors. The notes include an introduction to moving frames in projective geometry, an exposition of the Hwang-Yamaguchi rigidity theorem and a new variant of the Hwang-Yamaguchi theorem.

## 1. Overview.

- Introduction to the local differential geometry of submanifolds of projective space.
- Introduction to moving frames for projective geometry.
- How much must a submanifold  $X \subset \mathbb{P}^N$  resemble a given submanifold  $Z \subset \mathbb{P}^M$  infinitesimally before we can conclude  $X \simeq Z$ ?
- To what order must a line field on a submanifold  $X \subset \mathbb{P}^N$  have contact with  $X$  before we can conclude the lines are contained in  $X$ ?
- Applications to algebraic geometry.
- A new variant of the Hwang-Yamaguchi rigidity theorem.
- An exposition of the Hwang-Yamaguchi rigidity theorem in the language of moving frames.

Representation theory and algebraic geometry are natural tools for studying submanifolds of projective space. Recently there has also been progress the other way, using projective differential geometry to prove results in algebraic geometry and representation theory. These talks will focus on the basics of submanifolds of projective space, and give a few applications to algebraic geometry. For further applications to algebraic geometry the reader is invited to consult chapter 3 of [11] and the references therein.

Due to constraints of time and space, applications to representation theory will not be given here, but the interested reader can consult [23] for an overview. Entertaining applications include new proofs of the classification of compact Hermitian symmetric spaces, and of complex simple Lie algebras, based on the geometry of rational homogeneous varieties (instead of root systems), see [22]. The applications are not limited to classical representation theory. There are applications to Deligne’s conjectured categorical generalization of the exceptional series [25], to Vogel’s proposed

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*Universal Lie algebra* [27], and to the study of the intermediate Lie algebra  $\mathfrak{e}_{7\frac{1}{2}}$  [26].

**Notations, conventions.** I mostly work over the complex numbers in the complex analytic category, although most of the results are valid in the  $C^\infty$  category and over other fields, even characteristic  $p$ , as long as the usual precautions are taken. When working over  $\mathbb{R}$ , some results become more complicated as there are more possible normal forms. I use notations and the ordering of roots as in [2] and label maximal parabolic subgroups accordingly, e.g.,  $P_k$  refers to the maximal parabolic obtained by omitting the spaces corresponding to the simple root  $\alpha_k$ .  $\langle v_1, \dots, v_k \rangle$  denotes the linear span of the vectors  $v_1, \dots, v_k$ . If  $X \subset \mathbb{P}V$  is a subset,  $\bar{X} \subset V$  denotes the corresponding cone in  $V \setminus 0$ , the inverse image of  $X$  under the projection  $V \setminus 0 \rightarrow \mathbb{P}V$ . and  $\overline{X} \subset \mathbb{P}V$  denotes the Zariski closure of  $X$ , the zero set of all the homogeneous polynomials vanishing on  $X$ . When we write  $X^n$ , we mean  $\dim(X) = n$ . We often use  $Id$  to denote the identity matrix or identity map. Repeated indicies are to be summed over.

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## 2. Submanifolds of projective space.

**2.1. Projective geometry.** Let  $V$  be a vector space and let  $\mathbb{P}V$  denote the associated projective space. We think of  $\mathbb{P}V$  as the quotient of  $GL(V)$ , the general linear group of invertible endomorphisms of  $V$ , by the subgroup  $P_1$  preserving a line. For example if we take the line

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle$$

then

$$P_1 = \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & * & \dots & * \\ 0 & * & \dots & * \end{pmatrix}$$

where, if  $\dim V = N + 1$  the blocking is  $(1, N) \times (1, N)$ , so  $P_1$  is the group of invertible matrices with zeros in the lower left hand block.

In the spirit of Klein, we consider two submanifolds  $M_1, M_2 \subset \mathbb{P}V$  to be *equivalent* if there exists some  $g \in GL(V)$  such that  $g.M_1 = M_2$  and define the corresponding notion of local equivalence.

Just as in the geometry of submanifolds of Euclidean space, we will look for *differential invariants* that will enable us to determine if a neighborhood (germ) of a point of  $M_1$  is equivalent to a neighborhood (germ) of a point of  $M_2$ . These invariants will be obtained by taking derivatives at a point in a geometrically meaningful way. Recall that second derivatives furnish a complete set of differential invariants for surfaces in Euclidean three space- the vector-bundle valued Euclidean first and second fundamental forms, and two surfaces are locally equivalent iff there exists a local diffeomorphism  $f : M_1 \rightarrow M_2$  preserving the first and second fundamental forms.

The group of admissible motions in projective space is larger than the corresponding Euclidean group so we expect to have to take more derivatives to determine equivalence in the projective case than the Euclidean. For example, it was long known that for hypersurfaces  $X^n \subset \mathbb{P}^{n+1}$  that one needs at least three derivatives, and Jensen and Musso [12] showed that for most hypersurfaces, when  $n > 1$ , three derivatives are sufficient. For curves in the plane, by a classical result of Monge, one needs six derivatives!

In order to take derivatives in a way that will facilitate extracting geometric information from them, we will use the *moving frame*. Before developing the moving frame in §3, we discuss a few coarse invariants without machinery and state several rigidity results.

**2.2. Asymptotic directions.** Fix  $x \in X^n \subset \mathbb{P}V$ . After taking one derivative, we have the tangent space  $T_x X \subset T_x \mathbb{P}V$ , which is the set of tangent directions to lines in  $\mathbb{P}V$  having contact with  $X$  at  $x$  to order at least one. Since we are discussing directions, it is better to consider  $\mathbb{P}T_x X \subset \mathbb{P}T_x \mathbb{P}V$ . Inside  $\mathbb{P}T_x X$  is  $\mathcal{C}_{2,X,x} \subset \mathbb{P}T_x X$ , the set of tangent directions to lines having contact at least two with  $X$  at  $x$ , these are called the *asymptotic directions* in Euclidean geometry, and we continue to use the same terminology in the projective setting. Continuing, we define  $\mathcal{C}_{k,X,x}$  for all  $k$ , and finally,  $\mathcal{C}_{\infty,X,x}$ , which, in the analytic category, equals  $\mathcal{C}_{X,x}$ , the lines on (the completion of)  $X$  through  $x$ . When  $X$  is understood we sometimes write  $\mathcal{C}_{k,x}$  for  $\mathcal{C}_{k,X,x}$ .

*What does  $\mathcal{C}_{2,X,x}$ , or more generally  $\mathcal{C}_{k,X,x}$  tell us about the geometry of  $X$ ?*

That is, what can we learn of the macroscopic geometry of  $X$  from the microscopic geometry at a point? To increase the chances of getting meaningful information, from now on, when we are in the analytic or algebraic category, we will work at a *general point*. Loosely speaking, after taking  $k$

derivatives there will be both discrete and continuous invariants. A general point is one where all the discrete invariants are locally constant.

(To be more precise, if one is in the analytic category, one should really speak of  $k$ -general points (those that are general to order  $k$ ), to insure there is just a finite number of discrete invariants. In everything that follows we will be taking just a finite number of derivatives and we should say we are working at a  $k$ -general point where  $k$  is larger than the number of derivatives we are taking.)

When we are in the  $C^\infty$  category, we will work in open subsets and require whatever property we are studying at a point holds at all points in the open subset.

For example, if  $X^n$  is a hypersurface, then  $\mathcal{C}_{2,X,x}$  is a degree two hypersurface in  $\mathbb{P}T_xX$  (we will prove this below), and thus its only invariant is its rank  $r$ . In particular, if  $X$  is a smooth algebraic variety and  $x \in X_{\text{general}}$  the rank is  $n$  (see e.g., [6, 11]) and thus we do not get much information. (In contrast, if  $r < n$ , then the Gauss map of  $X$  is degenerate and  $X$  is (locally) ruled by  $\mathbb{P}^{n-r}$ 's.)

More generally, if  $X^n \subset \mathbb{P}^{n+a}$ , then  $\mathcal{C}_{2,X,x}$  is the intersection of at most  $\min(a, \binom{n+1}{2})$  quadric hypersurfaces, and one generally expects that equality holds. In particular, if the codimension is sufficiently large we expect  $\mathcal{C}_{2,x}$  to be empty and otherwise it should have codimension  $a$ . When this fails to happen, there are often interesting consequences for the macroscopic geometry of  $X$ .

**2.3. The Segre variety and Griffiths-Harris conjecture.** Let  $A, B$  be vector spaces and let  $V = A \otimes B$ . Let

$$X = \mathbb{P}(\text{rank one tensors}) \subset \mathbb{P}V.$$

Recall that every rank one matrix (i.e., rank one tensor expressed in terms of bases) is the matrix product of a column vector with a row vector, and that this representation is unique up to a choice of scale, so when we projectivize (and thus introduce another choice of scale) we obtain

$$X \simeq \mathbb{P}A \times \mathbb{P}B.$$

$X$  is called the *Segre variety* and is often written  $X = \text{Seg}(\mathbb{P}A \times \mathbb{P}B) \subset \mathbb{P}(A \otimes B)$ .

We calculate  $\mathcal{C}_{2,x}$  for the Segre. We first must calculate  $T_xX \subset T_x\mathbb{P}V$ . We identify  $T_x\mathbb{P}V$  with  $V \bmod \hat{x}$  and locate  $T_xX$  as a subspace of  $V \bmod \hat{x}$ .

Let  $x = [a_0 \otimes b_0] \in \text{Seg}(\mathbb{P}A \times \mathbb{P}B)$ . A curve  $x(t)$  in  $X$  with  $x(0) = x$  is given by curves  $a(t) \subset A$ ,  $b(t) \subset B$ , with  $a(0) = a_0$ ,  $b(0) = b_0$  by taking  $x(t) = [a(t) \otimes b(t)]$ .

$$\frac{d}{dt}\Big|_{t=0} a_t \otimes b_t = a'_0 \otimes b_0 + a_0 \otimes b'_0$$

and thus

$$T_x X = (A/a_0) \otimes b_0 \oplus a_0 \otimes B/b_0 \text{ mod } a_0 \otimes b_0.$$

Write  $A' = (A/a_0) \otimes b_0$ ,  $B' = a_0 \otimes (B/b_0)$  so

$$T_x X \simeq A' \oplus B'.$$

We now take second derivatives modulo the tangent space to see which tangent directions have lines osculating to order two (these will be the derivatives that are zero modulo the tangent space).

$$\frac{d^2}{(dt)^2} \Big|_{t=0} a_t \otimes b_t = a_0'' \otimes b_0 + a_0' \otimes b_0' + a_0 \otimes b_0'' \text{ mod } \hat{x} \quad (2.1)$$

$$\equiv a_0' \otimes b_0' \text{ mod } \hat{T}_x X. \quad (2.2)$$

Thus we get zero iff either  $a_0' = 0$  or  $b_0' = 0$ , i.e.,

$$\mathcal{C}_{2,X,x} = \mathbb{P}A' \sqcup \mathbb{P}B' \subset \mathbb{P}(A' \oplus B')$$

i.e.,  $\mathcal{C}_{2,X,x}$  is the disjoint union of two linear spaces, of dimensions  $\dim A - 2$ ,  $\dim B - 2$ . Note that  $\dim \mathcal{C}_{2,x}$  is much larger than expected.

For example, consider the case  $Seg(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$ . Here  $\mathcal{C}_{2,x}$  is defined by four quadratic polynomials on  $\mathbb{P}^3 = \mathbb{P}(T_x X)$ , so one would have expected  $\mathcal{C}_{2,x}$  to be empty. This rather extreme pathology led Griffiths and Harris to conjecture:

**CONJECTURE 2.1** (Griffiths-Harris, 1979 [6]). *Let  $Y^4 \subset \mathbb{P}^8$  be a variety not contained in a hyperplane and let  $y \in Y_{general}$ . If  $\mathcal{C}_{2,Y,y} = \mathbb{P}^1 \sqcup \mathbb{P}^1 \subset \mathbb{P}^3 = \mathbb{P}(T_y Y)$ , then  $Y$  is isomorphic to  $Seg(\mathbb{P}^2 \times \mathbb{P}^2)$ .*

(The original statement of the conjecture was in terms of the projective second fundamental form defined below.) Twenty years later, in [16] I showed the conjecture was true, and moreover in [16, 20] I showed:

**THEOREM 2.1.** *Let  $X^n = G/P \subset \mathbb{P}V$  be a rank two compact Hermitian symmetric space (CHSS) in its minimal homogeneous embedding, other than a quadric hypersurface. Let  $Y^n \subset \mathbb{P}V$  be a variety not contained in a hyperplane and let  $y \in Y_{general}$ . If  $\mathcal{C}_{2,Y,y} \simeq \mathcal{C}_{2,X,x}$  then  $Y$  is projectively isomorphic to  $X$ .*

An analogous result is true in the  $C^\infty$  category, namely

**THEOREM 2.2.** *Let  $X^n = G/P \subset \mathbb{P}V$  be a rank two compact Hermitian symmetric space (CHSS) in its minimal homogeneous embedding, other than a quadric hypersurface. Let  $Y^n \subset \mathbb{P}W$  be a smooth submanifold not contained in a hyperplane. If  $\mathcal{C}_{2,Y,y} \simeq \mathcal{C}_{2,X,x}$  for all  $y \in Y$ , then  $Y$  is projectively isomorphic to an open subset of  $X$ .*

The situation of the quadric hypersurface is explained below (Fubini's theorem) - to characterize it, one must have  $\mathcal{C}_{3,Y,y} = \mathcal{C}_{3,Q,x}$ .

The rank two CHSS are  $Seg(\mathbb{P}A \times \mathbb{P}B)$ , the Grassmanians of two-planes  $G(2, V)$ , the quadric hypersurfaces, the complexified Cayley plane

$\mathbb{O}\mathbb{P}^2 = E_6/P_6$ , and the spinor variety  $D_5/P_5$  (essentially the isotropic 5-planes through the origin in  $\mathbb{C}^{10}$  equipped with a quadratic form - the set of such planes is disconnected and the spinor variety is one (of the two) isomorphic components. The minimal homogenous embedding is also in a smaller linear space than the Plucker embedding of the Grassmannian.) The only rank one CHSS is projective space  $\mathbb{P}^n$ . The rank two CHSS and projective space are examples of *rational homogeneous varieties*.

**2.4. Homogeneous varieties.** Let  $G \subset GL(V)$  be a reductive group acting irreducibly on a vector space  $V$ . Then there exists a unique closed orbit  $X = G/P \subset \mathbb{P}V$ , which is called a *rational homogeneous variety*. (Equivalently,  $X$  may be characterized as the orbit of a highest weight line, or as the minimal orbit.)

Note that if  $X = G/P \subset \mathbb{P}V$  is homogenous,  $T_x X$  inherits additional structure beyond that of a vector space. Namely, consider  $x = [Id]$  as the class of the identity element for the projection  $G \rightarrow G/P$ . Then  $P$  acts on  $T_x X$  and, as a  $P$ -module  $T_x X \simeq \mathfrak{g}/\mathfrak{p}$ . For example, in the case of the Segre,  $T_x X$  was the direct sum of two vector spaces.

A homogeneous variety  $X = G/P$  is a *compact Hermitian symmetric space*, or CHSS for short, if  $P$  acts irreducibly on  $T_x X$ . The *rank* of a CHSS is the number of its last nonzero fundamental form in its minimal homogeneous embedding. (This definition agrees with the standard one.) Fundamental forms are defined in §3.4.

EXERCISE 2.1. *The Grassmannian of  $k$ -planes through the origin in  $V$ , which we denote  $G(k, V)$ , is homogenous for  $GL(V)$  (we have already seen the special case  $G(1, V) = \mathbb{P}V$ ).*

*Determine the group  $P_k \subset GL(V)$  that stabilizes a point. Show that  $T_E G(k, V) \simeq E^* \otimes V/E$  in two different ways - by an argument as in the Segre case above and by determining the structure of  $\mathfrak{g}/\mathfrak{p}$ .*

While all homogeneous varieties have many special properties, the rank at most two CHSS (other than the quadric hypersurface) are distinguished by the following property:

PROPOSITION 2.1. [21] *Theorem 2.1 is sharp in the sense that no other homogeneous variety is completely determined by its asymptotic directions at a general point other than a linearly embedded projective space.*

Nevertheless, there are significant generalizations of Theorem 2.1 due to Hwang-Yamaguchi and Robles discussed below in §3.5. To state these results we will need definitions of the fundamental forms and Fubini cubic forms, which are given in the next section.

However, with an additional hypothesis - namely that the unknown variety has the correct codimension, we obtain the following result (which appears here for the first time):

THEOREM 2.3. *Let  $X^n \subset \mathbb{C}\mathbb{P}^{n+a}$  be a complex submanifold not contained in a hyperplane. Let  $x \in X$  be a general point. Let  $Z^n \subset \mathbb{P}^{n+a}$  be an irreducible compact Hermitian symmetric space in its minimal homo-*

geneous embedding, other than a quadric hypersurface. If  $\mathcal{C}_{2,X,x} = \mathcal{C}_{2,Z,z}$  then  $\overline{X} = Z$ .

REMARK 2.1. The Segre variety has  $\mathcal{C}_{2,x} = \mathcal{C}_x = \mathbb{P}A' \sqcup \mathbb{P}B' \subset \mathbb{P}(A' \oplus B') = \mathbb{P}T_x X$ . To see this, note that a matrix has rank one iff all its  $2 \times 2$  minors are zero, and these minors provide defining equations for the Segre. In general, if a variety is defined by equations of degree at most  $d$ , then any line having contact to order  $d$  at any point must be contained in the variety. In fact, by an unpublished result of Kostant, all homogeneously embedded rational homogeneous varieties  $G/P$  are cut out by quadratic equations so  $\mathcal{C}_{2,G/P,x} = \mathcal{C}_{G/P,x}$ .

**3. Moving frames and differential invariants.** For more details regarding this section, see chapter 3 of [11].

Once and for all fix index ranges  $1 \leq \alpha, \beta, \gamma \leq n$ ,  $n+1 \leq \mu, \nu \leq n+a$ ,  $0 \leq A, B, C \leq n+a = N$ .

**3.1. The Maurer-Cartan form of  $GL(V)$ .** Let  $\dim V = N+1$ , denote an element  $f \in GL(V)$  by  $f = (e_0, \dots, e_N)$  where we may think of the  $e_A$  as column vectors providing a basis of  $V$ . (Once a reference basis of  $V$  is fixed,  $GL(V)$  is isomorphic to the space of all bases of  $V$ .) Each  $e_A$  is a  $V$ -valued function on  $GL(V)$ ,  $e_A : GL(V) \rightarrow V$ . For any differentiable map between manifolds we can compute the induced differential

$$de_A|_f : T_f GL(V) \rightarrow T_{e_A} V$$

but now since  $V$  is a vector space, we may identify  $T_{e_A} V \simeq V$  and consider

$$de_A : T_f GL(V) \rightarrow V$$

i.e.,  $de_A$  is a  $V$ -valued one-form on  $GL(V)$ . As such, we may express it as

$$de_A = e_0 \omega_A^0 + e_1 \omega_A^1 + \dots + e_N \omega_A^N$$

where  $\omega_B^A \in \Omega^1(GL(V))$  are ordinary one-forms (This is because  $e_0, \dots, e_N$  is a basis of  $V$  so any  $V$ -valued one form is a linear combination of these with scalar valued one forms as coefficients.) Collect the forms  $\omega_B^A$  into a matrix  $\Omega = (\omega_B^A)$ . Write  $df = (de_0, \dots, de_N)$ , so  $df = f\Omega$  or

$$\Omega = f^{-1}df$$

$\Omega$  is called the *Maurer-Cartan form* for  $GL(V)$ . Note that  $\omega_B^A$  measures the infinitesimal motion of  $e_B$  towards  $e_A$ .

**Amazing fact:** we can compute the exterior derivative of  $\Omega$  algebraically! We have  $d\Omega = d(f^{-1}) \wedge df$  so we need to calculate  $d(f^{-1})$ . Here is where an extremely useful fact comes in:

*The derivative of a constant function is zero.*

We calculate  $0 = d(Id) = d(f^{-1}f) = d(f^{-1})f + f^{-1}df$ , and thus  $d\Omega = -f^{-1}df f^{-1} \wedge df$  but we can move the scalar valued-matrix  $f^{-1}$  across the wedge product to conclude

$$d\Omega = -\Omega \wedge \Omega$$

which is called the *Maurer-Cartan equation*. The notation is such that  $(\Omega \wedge \Omega)_B^A = \omega_C^A \wedge \omega_B^C$ .

**3.2. Moving frames for  $X \subset \mathbb{P}V$ .** Now let  $X^n \subset \mathbb{P}^{n+a} = \mathbb{P}V$  be a submanifold. We are ready to take derivatives. Were we working in coordinates, to take derivatives at  $x \in X$ , we might want to choose coordinates such that  $x$  is the origin. We will make the analogous adaptation using moving frames, but the advantage of moving frames is that all points will be as if they were the origin of a coordinate system. To do this, let  $\pi : \mathcal{F}_X^0 := GL(V)|_X \rightarrow X$  be the restriction of  $\pi : GL(V) \rightarrow \mathbb{P}V$ .

Similarly, we might want to choose local coordinates  $(x^1, \dots, x^{n+a})$  about  $x = (0, \dots, 0)$  such that  $T_x X$  is spanned by  $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ . Again, using moving frames the effect will be as if we had chosen such coordinates about each point simultaneously. To do this, let  $\pi : \mathcal{F}^1 \rightarrow X$  denote the sub-bundle of  $\mathcal{F}_X^0$  preserving the flag

$$\hat{x} \subset \hat{T}_x X \subset V.$$

Recall  $\hat{x} \subset V$  denotes the line corresponding to  $x$  and  $\hat{T}_x X$  denotes the affine tangent space  $T_v \hat{X} \subset V$ , where  $[v] = x$ . Let  $(e_0, \dots, e_{n+a})$  be a basis of  $V$  with dual basis  $(e^0, \dots, e^{n+a})$  adapted such that  $e_0 \in \hat{x}$  and  $\{e_0, e_\alpha\}$  span  $\hat{T}_x X$ . Write  $T = T_x X$  and  $N = N_x X = T_x \mathbb{P}V / T_x X$ .

REMARK 3.1. (*Aside for the experts*) I am slightly abusing notation in this section by identifying  $\hat{T}_x X / \hat{x}$  with  $T_x X := (\hat{T}_x X / \hat{x}) \otimes \hat{x}^*$  and similarly for  $N_x X$ .

The fiber of  $\pi : \mathcal{F}^1 \rightarrow X$  over a point is isomorphic to the group

$$G_1 = \left\{ g = \begin{pmatrix} g_0^0 & g_\beta^0 & g_\nu^0 \\ 0 & g_\beta^\alpha & g_\nu^\alpha \\ 0 & 0 & g_\nu^\mu \end{pmatrix} \mid g \in GL(V) \right\}.$$

While  $\mathcal{F}^1$  is not in general a Lie group, since  $\mathcal{F}^1 \subset GL(V)$ , we may pull back the Maurer-Cartan form on  $GL(V)$  to  $\mathcal{F}^1$ . Write the pullback of the Maurer-Cartan form to  $\mathcal{F}^1$  as

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & \omega_\nu^0 \\ \omega_0^\alpha & \omega_\beta^\alpha & \omega_\nu^\alpha \\ \omega_0^\mu & \omega_\beta^\mu & \omega_\nu^\mu \end{pmatrix}.$$

The definition of  $\mathcal{F}^1$  implies that  $\omega_0^\mu = 0$  because

$$de_0 = \omega_0^0 e_0 + \dots + \omega_0^n e_n + \omega_0^{n+1} e_{n+1} + \dots + \omega_0^{n+a} e_{n+a}$$



but we have required that  $e_0$  only move towards  $e_1, \dots, e_n$  to first order. Similarly, because  $\dim X = n$ , the adaptation implies that the forms  $\omega_0^\alpha$  are all linearly independent.

At this point you should know what to do - seeing something equal to zero, we differentiate it. Thanks to the Maurer-Cartan equation, we may calculate the derivative algebraically. We obtain

$$0 = d(\omega_0^\mu) = -\omega_\alpha^\mu \wedge \omega_0^\alpha \quad \forall \mu.$$

Since the one-forms  $\omega_0^\alpha$  are all linearly independent, it is clear that the  $\omega_\alpha^\mu$  must be linear combinations of the  $\omega_0^\beta$ , and in fact the *Cartan lemma* (see e.g., [11], p 314) implies that the dependence is symmetric. More precisely (exercise!) there exist functions

$$q_{\alpha\beta}^\mu : \mathcal{F}^1 \rightarrow \mathbb{C}$$

with  $\omega_\alpha^\mu = q_{\alpha\beta}^\mu \omega_0^\beta$  and moreover  $q_{\alpha\beta}^\mu = q_{\beta\alpha}^\mu$ . One way to understand the equation  $\omega_\alpha^\mu = q_{\alpha\beta}^\mu \omega_0^\beta$  is that the infinitesimal motion of the embedded tangent space (the infinitesimal motion of the  $e_\alpha$ 's in the direction of the  $e_\mu$ 's) is determined by the motion of  $e_0$  towards the  $e_\alpha$ 's and the coefficients  $q_{\alpha\beta}^\mu$  encode this dependence.

Now  $\pi : \mathcal{F}^1 \rightarrow X$  was defined geometrically (i.e., without making any arbitrary choices) so any function on  $\mathcal{F}^1$  invariant under the action of  $G_1$  descends to be a well defined function on  $X$ , and will be a *differential invariant*. Our functions  $q_{\alpha\beta}^\mu$  are not invariant under the action of  $G_1$ , but we can form a tensor from them that is invariant, which will lead to a *vector-bundle valued differential invariant* for  $X$  (the same phenomenon happens in the Euclidean geometry of submanifolds).

Consider

$$\tilde{I}I_f = F_{2,f} := \omega_0^\alpha \omega_\alpha^\mu \otimes (e_\mu \bmod \hat{T}_x X) = q_{\alpha\beta}^\mu \omega_0^\alpha \omega_0^\beta \otimes (e_\mu \bmod \hat{T}_x X)$$

$\tilde{I}I \in \Gamma(\mathcal{F}^1, \pi^*(S^2 T^* X \otimes NX))$  is constant on the fiber and induces a tensor  $II \in \Gamma(X, S^2 T^* X \otimes NX)$ . called the *projective second fundamental form*.

Thinking of  $II_x : N_x^* X \rightarrow S^2 T_x^* X$ , we may now properly define the asymptotic directions by

$$\mathcal{C}_{2,x} := \mathbb{P}(\text{Zeros}(II_x(N_x^* X)) \subset \mathbb{P}T_x X).$$

### 3.3. Higher order differential invariants: the Fubini forms.

We continue differentiating constant functions:

$$0 = d(\omega_\alpha^\mu - q_{\alpha\beta}^\mu \omega_0^\beta)$$

yields functions  $r_{\alpha\beta\gamma}^\mu : \mathcal{F}^1 \rightarrow \mathbb{C}$ , symmetric in their lower indices, that induce a tensor  $F_3 \in \Gamma(\mathcal{F}^1, \pi^*(S^3 T^* X \otimes NX))$  called the *Fubini cubic form*.

Unlike the second fundamental form, it does *not* descend to be a tensor over  $X$  because it varies in the fiber. We discuss this variation in §6. Such tensors provide *relative differential invariants* and by successive differentiations, one obtains a series of invariants  $F_k \in \Gamma(\mathcal{F}^1, \pi^*(S^k T^* \otimes N))$ . For example,

$$\begin{aligned} F_3 &= r_{\alpha\beta\gamma}^\mu \omega_0^\alpha \omega_0^\beta \omega_0^\gamma \otimes e_\mu \\ F_4 &= r_{\alpha\beta\gamma\delta}^\mu \omega_0^\alpha \omega_0^\beta \omega_0^\gamma \omega_0^\delta \otimes e_\mu \end{aligned}$$

where the functions  $r_{\alpha\beta\gamma}^\mu, r_{\alpha\beta\gamma\delta}^\mu$  are given by

$$r_{\alpha\beta\gamma}^\mu \omega_0^\gamma = -dq_{\alpha\beta}^\mu - q_{\alpha\beta}^\mu \omega_0^0 - q_{\alpha\beta}^\nu \omega_\nu^\mu + q_{\alpha\delta}^\mu \omega_\beta^\delta + q_{\beta\delta}^\mu \omega_\alpha^\delta \quad (3.1)$$

$$\begin{aligned} r_{\alpha\beta\gamma\delta}^\mu \omega_0^\delta &= -dr_{\alpha\beta\gamma}^\mu - 2r_{\alpha\beta\gamma}^\mu \omega_0^0 - r_{\alpha\beta\gamma}^\nu \omega_\nu^\mu \\ &\quad + \mathfrak{S}_{\alpha\beta\gamma}(r_{\alpha\beta\epsilon}^\mu \omega_\gamma^\epsilon + q_{\alpha\beta}^\mu \omega_\gamma^0 - q_{\alpha\epsilon}^\mu q_{\beta\gamma}^\nu \omega_\nu^\epsilon). \end{aligned} \quad (3.2)$$

We define  $\mathcal{C}_{k,x} := \text{Zeros}(F_{2,f}, \dots, F_{k,f}) \subset \mathbb{P}T_x X$ , which is independent of our choice of  $f \in \pi^{-1}(x)$ .

If one chooses local affine coordinates  $(x^1, \dots, x^{n+a})$  such that  $x = (0, \dots, 0)$  and  $T_x X = \langle \frac{\partial}{\partial x^\alpha} \rangle$ , and writes  $X$  as a graph

$$x^\mu = q_{\alpha\beta}^\mu x^\alpha x^\beta - r_{\alpha\beta\gamma}^\mu x^\alpha x^\beta x^\gamma + r_{\alpha\beta\gamma\delta}^\mu x^\alpha x^\beta x^\gamma x^\delta + \dots$$

then there exists a local section of  $\mathcal{F}^1$  such that

$$\begin{aligned} F_2|_x &= q_{\alpha\beta}^\mu dx^\alpha dx^\beta \otimes \frac{\partial}{\partial x^\mu} \\ F_3|_x &= r_{\alpha\beta\gamma}^\mu dx^\alpha dx^\beta dx^\gamma \otimes \frac{\partial}{\partial x^\mu} \\ F_4|_x &= r_{\alpha\beta\gamma\delta}^\mu dx^\alpha dx^\beta dx^\gamma dx^\delta \otimes \frac{\partial}{\partial x^\mu} \end{aligned}$$

and similarly for higher orders.

(3.1) is a system of  $a \binom{n+1}{2}$  equations with one-forms as coefficients for the  $a \binom{n+2}{3}$  coefficients of  $F_3$  and is overdetermined if we assume  $dq_{\alpha\beta}^\mu = 0$ , as we do in the rigidity problems. One can calculate directly in the Segre  $\text{Seg}(\mathbb{P}^m \times \mathbb{P}^r)$ ,  $m, r > 1$  case that the only possible solutions are normalizable to zero by a fiber motion as described in (6.4). The situation is the same for  $F_4, F_5$  in this case.

In general, once  $F_k, \dots, F_{2k-1}$  are normalized to zero at a general point, it is automatic that all higher  $F_j$  are zero, see [15]. Thus one has the entire Taylor series and has completely identified the variety. This was the method of proof used in [16], as the rank two CHSS have all  $F_k$  normalizable to zero when  $k > 2$ .

Another perspective, for those familiar with  $G$ -structures, is that one obtains rigidity by reducing  $\mathcal{F}^1$  to a smaller bundle which is isomorphic to  $G$ , where the homogeneous model is  $G/P$ .

Yet another perspective, for those familiar with exterior differential systems, is that after three prolongations, the EDS defined by  $I = \{\omega_0^\mu, \omega_\alpha^\mu - q_{\alpha\beta}^\mu \omega_0^\beta\}$  on  $GL(V)$  becomes involutive, in fact Frobenius.

**3.4. The higher fundamental forms.** A component of  $F_3$  does descend to a well defined tensor on  $X$ . Namely, considering  $F_3 : N^* \rightarrow S^3 T^*$ , if we restrict  $F_3|_{\ker F_2}$ , we obtain a tensor  $\mathbb{F}_3 = III \in S^3 T^* \otimes N_3$  where  $N_3 = T_x \mathbb{P}V / \{T_x X + II(S^2 T_x X)\}$ . One continues in this manner to get a series of tensors  $\mathbb{F}_k$  called the *fundamental forms*.

Geometrically,  $II$  measures how  $X$  is leaving its embedded tangent space at  $x$  to first order,  $III$  measures how  $X$  is leaving its second osculating space at  $x$  to first order while  $F_3 \bmod III$  measures how  $X$  is moving away from its embedded tangent space to second order.

**3.5. More rigidity theorems.** Now that we have defined fundamental forms, we may state:

**THEOREM 3.1** (Hwang-Yamaguchi). [10] *Let  $X^n \subset \mathbb{C}\mathbb{P}^{n+a}$  be a complex submanifold. Let  $x \in X$  be a general point. Let  $Z$  be an irreducible rank  $r$  compact Hermitian symmetric space in its natural embedding, other than a quadric hypersurface. If there exists linear maps  $f : T_x X \rightarrow T_z Z$ ,  $g_k : N_{k,x} X \rightarrow N_{k,z} Z$  such that the induced maps  $S^k T_x^* X \otimes N_x X \rightarrow S^k T_z^* Z \otimes N_z Z$  take  $\mathbb{F}_{k,X,x}$  to  $\mathbb{F}_{k,Z,z}$  for  $2 \leq k \leq r$ , then  $\overline{X} = Z$ .*

In [22] we calculated the differential invariants of the *adjoint varieties*, the closed orbits in the projectivization of the adjoint representation of a simple Lie algebra. (These are the homogeneous complex contact manifolds in their natural homogeneous embedding.) The adjoint varieties have  $III = 0$ , but, in all cases but  $v_2(\mathbb{P}^{2n-1}) = C_n/P_1 \subset \mathbb{P}(\mathfrak{c}_n) = \mathbb{P}(S^2 \mathbb{C}^{2n})$  which we exclude from discussion in the remainder of this paragraph, the invariants  $F_3, F_4$  are not normalizable to zero, even though  $\mathcal{C}_{3,x} = \mathcal{C}_{4,x} = \mathcal{C}_{2,x} = \mathcal{C}_x$ . In a normalized frame  $\mathcal{C}_x$  is contained in a hyperplane  $H$  and  $F_4$  is the equation of the *tangential variety* of  $\mathcal{C}_x$  in  $H$ , where the tangential variety  $\tau(X) \subset \mathbb{P}V$  of an algebraic manifold  $X \subset \mathbb{P}V$  is the union of the points on the embedded tangent lines ( $\mathbb{P}^1$ 's) to the manifold. In this case the tangential variety is a hypersurface in  $H$ , except for  $\mathfrak{g} = \mathfrak{a}_n = \mathfrak{sl}_{n+1}$  which is discussed below. Moreover,  $F_3$  consists of the defining equations for the singular locus of  $\tau(\mathcal{C}_x)$ . In [22] we speculated that the varieties  $X_{ad}$ , with the exception of  $v_2(\mathbb{P}^{2n-1})$  (which is rigid to order three, see [16]) would be rigid to order four, but not three, due to the nonvanishing of  $F_4$ . Thus the following result came as a suprise to us:

**THEOREM 3.2** (Robles, [31]). *Let  $X^{2(m-2)} \subset \mathbb{C}\mathbb{P}^{m^2-2}$  be a complex submanifold. Let  $x \in X$  be a general point. Let  $Z \subset \mathbb{P}\mathfrak{sl}_m$  be the adjoint variety. If there exist linear maps  $f : T_x X \rightarrow T_z Z$ ,  $g : N_x X \rightarrow N_z Z$  such that the induced maps  $S^k T_x^* X \otimes N_x X \rightarrow S^k T_z^* Z \otimes N_z Z$  take  $\mathbb{F}_{k,X,x}$  to  $\mathbb{F}_{k,Z,z}$  for  $k = 2, 3$ , then  $\overline{X} = Z$ .*

Again, the corresponding result holds in the  $C^\infty$  category.

The adjoint variety of  $\mathfrak{sl}_m = \mathfrak{sl}(W)$  has the geometric interpretation of the variety of flags of lines inside hyperplanes inside  $W$ , or equivalently as the traceless, rank one matrices. It has  $C_{2,z}$  the union of two disjoint linear spaces in a hyperplane in  $\mathbb{P}T_z Z$ . The quartic  $F_4$  is the square of a quadratic equation (whose zero set contains the two linear spaces), and the cubics in  $F_3$  are the derivatives of this quartic, see [22], §6.

### 3.6. The prolongation property and proof of Theorem 2.3.

The precise restrictions  $II$  places on the  $F_k$  in general is not known at this time. However, there is a strong restriction  $II$  places on the higher fundamental forms that dates back to Cartan. We recall a definition from exterior differential systems:

Let  $U, W$  be vector spaces. Given a linear subspace  $A \subset S^k U^* \otimes W$ , define the  $j$ -th prolongation of  $A$  to be  $A^{(j)} := (A \otimes S^j U^*) \cap (S^{k+j} U^* \otimes W)$ . Thinking of  $A$  as a collection of  $W$ -valued homogeneous polynomials on  $U$ , the  $j$ -th prolongation of  $A$  is the set of all homogeneous  $W$ -valued polynomials of degree  $k+j$  on  $U$  with the property that all their  $j$ -th order partial derivatives lie in  $A$ .

PROPOSITION 3.1 (Cartan [3] p 377). *Let  $X^n \subset \mathbb{P}^{n+a}$ , and let  $x \in X$  be a general point. Then  $\mathbb{F}_{k,x}(N_k^*) \subseteq \mathbb{F}_{2,x}(N_2^*)^{(k-2)}$ . (Here  $W$  is taken to be the trivial vector space  $\mathbb{C}$  and  $U = T_x X$ .)*

PROPOSITION 3.2. [21] *Let  $X = G/P \subset \mathbb{P}V$  be a CHSS in its minimal homogeneous embedding. Then  $\mathbb{F}_{k,x}(N_k^*) = \mathbb{F}_{2,x}(N_2^*)^{(k-2)}$ . Moreover, the only nonzero components of the  $F_k$  are the fundamental forms.*

*The only homogeneous varieties having the property that the only nonzero components of the  $F_k$  are the fundamental forms are the CHSS.*

*Proof.* [Proof of Theorem 2.3] The strict prolongation property for CHSS in their minimal homogeneous embedding implies that any variety with the same second fundamental form at a general point as a CHSS in its minimal homogeneous embedding can have codimension at most that of the corresponding CHSS, and equality holds iff all the other fundamental forms are the prolongations of the second.  $\square$

**4. Bertini type theorems and applications.** The results discussed so far dealt with homogeneous varieties. We now broaden our study to various pathologies of the  $C_{k,x}$ .

Let  $T$  be a vector space. The classical *Bertini theorem* implies that for a linear subspace  $A \subset S^2 T^*$ , if  $q \in A$  is such that  $\text{rank}(q) \geq \text{rank}(q')$  for all  $q' \in A$ , then  $u \in q_{\text{sing}} := \{v \in T \mid q(v, w) = 0 \forall w \in T\}$  implies  $u \in \text{Zeros}(A) := \{v \in T \mid Q(v, v) = 0 \forall Q \in A\}$ .

THEOREM 4.1 (Mobile Bertini). *Let  $X^n \subset \mathbb{P}V$  be a complex manifold and let  $x \in X$  be a general point. Let  $q \in II(N_x^* X)$  be a generic quadric. Then  $q_{\text{sing}}$  is tangent to a linear space on  $\overline{X}$ .*

For generalizations and variations, see [20].

The result holds in the  $C^\infty$  category if one replaces a general point by all points and that the linear space is contained in  $X$  as long as  $X$  continues

(e.g., it is contained in  $X$  if  $X$  is complete).

*Proof.* Assume  $v = e_1 \in q_{sing}$  and  $q = q_{\alpha\beta}^{n+1} \omega_0^\alpha \omega_0^\beta$ . Our hypotheses imply  $q_{1\beta}^{n+1} = 0$  for all  $\beta$ . Formula (3.1) reduces to

$$r_{11\beta}^{n+1} \omega_0^\beta = -q_{11}^\mu \omega_\mu^{n+1}.$$

If  $q$  is generic we are working on a reduction of  $\mathcal{F}^1$  where the  $\omega_\nu^\mu$  are independent of each other and independent of the semi-basic forms (although the  $\omega_\beta^\alpha$  will no longer be independent of the  $\omega_0^\alpha, \omega_\nu^\mu$ ); thus the coefficients on both sides of the equality are zero, proving both the classical Bertini theorem and  $v \in r_{sing}$  where  $r$  is a generic cubic in  $F_3(N^*)$ . Then using the formula for  $F_4$  one obtains  $v \in Zeros(F_3)$  and  $v \in s_{sing}$  where  $s$  is a generic element of  $F_4(N^*)$ . One then concludes by induction.  $\square$

REMARK 4.1. *The mobile Bertini theorem essentially dates back to B. Segre [35], and was rediscovered in various forms in [6, 4]. Its primary use is in the study of varieties  $X^n \subset \mathbb{P}V$  with defective dual varieties  $X^* \subset \mathbb{P}V^*$ , where the dual variety of a smooth variety is the set of tangent hyperplanes to  $X$ , which is usually a hypersurface. The point is that a generic quadric in  $II(N_{x,X}^*)$  (with  $x \in X_{general}$ ) is singular of rank  $r$  iff  $\text{codim } X^* = n - r + 1$ .*

EXAMPLE 1. *Taking  $X = \text{Seg}(\mathbb{P}A \times \mathbb{P}B)$  and keeping the notations of above, let  $Y$  have the same second fundamental form of  $X$  at a general point  $y \in Y$  so we inherit an identification  $T_y Y \simeq A' \oplus B'$ . If  $\dim B = b > a = \dim A$ , then mobile Bertini implies that  $\mathbb{P}B'$  is actually tangent to a linear space on  $Y$ , because the maximum rank of a quadric is  $a - 1$ . So at any point  $[b] \in \mathbb{P}B'$  there is even a generic quadric singular at  $[b]$ . (Of course the directions of  $\mathbb{P}A'$  are also tangent to lines on  $Y$  because the Segre is rigid.)*

*While in [16] I did not calculate the rigidity of  $\text{Seg}(\mathbb{P}^1 \times \mathbb{P}^n) \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^{n+1})$ , the rigidity follows from the same calculations, however one must take additional derivatives to get the appropriate vanishing of the Fubini forms. However there is also an elementary proof of rigidity in this case using the mobile Bertini theorem [20]. Given a variety  $Y^{n+1} \subset \mathbb{P}^N$  such that at a general point  $y \in Y$ ,  $\mathcal{C}_{2,y}$  contains a  $\mathbb{P}^{n-1}$ , by 4.1 the resulting  $n$ -plane field on  $TY$  is integrable and thus  $Y$  is ruled by  $\mathbb{P}^n$ 's. Such a variety arises necessarily from a curve in the Grassmannian  $G(n+1, N+1)$  (as the union of the points on the  $\mathbb{P}^n$ 's in the curve). But in order to also have the  $\mathbb{P}^0$  factor in  $\mathcal{C}_{2,y}$ , such a curve must be a line and thus  $Y$  must be the Segre.*

The mobile Bertini theorem describes consequences of  $\mathcal{C}_{2,x}$  being pathological. Here are some results when  $\mathcal{C}_{k,x}$  is pathological for  $k > 2$ .

THEOREM 4.2 (Darboux). *Let  $X^2 \subset \mathbb{P}^{2+a}$  be an analytic submanifold and let  $x \in X_{general}$ . If there exists a line  $l$  having contact to order three with  $X$  at  $x$ , then  $l \subset \bar{X}$ . In other words, for surfaces in projective space,*

$$\mathcal{C}_{3,x} = \mathcal{C}_x \quad \forall x \in X_{general}.$$

The  $C^\infty$  analogue holds replacing general points by all points and lines by line segments contained in  $X$ .

There are several generalizations of this result in [18, 19]. Here is one of them:

**THEOREM 4.3.** [19] *Let  $X^n \subset \mathbb{P}^{n+1}$  be an analytic submanifold and let  $x \in X_{\text{general}}$ . If  $\Sigma \subseteq \mathcal{C}_{k,x}$  is an irreducible component with  $\dim \Sigma > n - k$ , then  $\Sigma \subset \mathcal{C}_x$ .*

The  $C^\infty$  analog holds with the by now obvious modifications.

The proof is similar to that of the mobile Bertini theorem.

**EXERCISE 4.1.** *One of my favorite problems to put on an undergraduate differential geometry exam is: Prove that a surface in Euclidean three space that has more than two lines passing through each point is a plane (i.e., has an infinite number of lines passing through). In [30], Mezzetti and Portelli showed that a 3-fold having more than six lines passing through a general point must have an infinite number. Show that an  $n$ -fold having more than  $n!$  lines passing through a general point must have an infinite number passing through each point. (See [19] if you need help.)*

The rigidity of the quadric hypersurface is a classical result:

**THEOREM 4.4 (Fubini).** *Let  $X^n \subset \mathbb{P}^{n+1}$  be an analytic submanifold or algebraic variety and let  $x \in X_{\text{general}}$ . Say  $\mathcal{C}_{3,x} = \mathcal{C}_{2,x}$ . Let  $r = \text{rank} \mathcal{C}_{2,x}$ . Then*

- *If  $r > 1$ , then  $X$  is a quadric hypersurface of rank  $r$ .*
- *If  $r = 1$ , then  $X$  has a one-dimensional Gauss image. In particular, it is ruled by  $\mathbb{P}^{n-1}$ 's.*

In all situations, the dimension of the Gauss image of  $X$  is  $r$ , see [11], §3.4.

One way to prove Fubini's theorem (assuming  $r > 1$ ) is to first use mobile Bertini to see that  $X$  contains large linear spaces, then to note that degree is invariant under linear section, so one can reduce to the case of a surface. But then it is elementary to show that the only analytic surface that is doubly ruled by lines is the quadric surface.

Another way to prove Fubini's theorem (assuming  $r > 1$ ) is to use moving frames to reduce the frame bundle to  $O(n+2)$ .

*What can we say in higher codimension? Consider codimension two. What are the varieties  $X^n \subset \mathbb{P}^{n+2}$  such that for general  $x \in X$  we have  $\mathcal{C}_{3,x} = \mathcal{C}_{2,x}$ ?*

Note that there are two principal difficulties in codimension two. First, in codimension one, having  $\mathcal{C}_{3,x} = \mathcal{C}_{2,x}$  implies that  $F_3$  is normalizable to zero - this is no longer true in codimension greater than one. Second, in codimension one, there is just one quadratic form in  $II$ , so its only invariant is its rank. In larger codimension there are moduli spaces, although for pencils at least there are normal forms, conveniently given in [9].

For examples, we have inherited from Fubini's theorem:

- 0.  $\mathbb{P}^n \subset \mathbb{P}^{n+2}$  as a linear subspace
- 0'.  $Q^n \subset \mathbb{P}^{n+1} \subset \mathbb{P}^{n+2}$  a quadric
- 0''. A variety with a one-dimensional Gauss image.

To these it is easy to see the following are also possible:

- 1. The (local) product of a curve with a variety with a one dimensional Gauss image.
- 2. The intersection of two quadric hypersurfaces.
- 3. A (local) product of a curve with a quadric hypersurface.

There is one more example we have already seen several times in these lectures:

- 4. The Segre  $Seg(\mathbb{P}^1 \times \mathbb{P}^2) \subset \mathbb{P}^5$  or a cone over it.

**THEOREM 4.5** (Codimension two Fubini). [29] *Let  $X^n \subset \mathbb{P}^{n+2}$  be an analytic submanifold and let  $x \in X_{general}$ . If  $\mathcal{C}_{3,x} = \mathcal{C}_{2,x}$  then  $X$  is (an open subset of) one of 0, 0', 0'', 1 – 4 above.*

Here if one were to work over  $\mathbb{R}$ , the corresponding result would be more complicated as there are more normal forms for pencils of quadrics.

**5. Applications to algebraic geometry.** One nice aspect of algebraic geometry is that spaces parametrizing algebraic varieties tend to also be algebraic varieties (or at least *stacks*, which is the algebraic geometer’s version of an orbifold). For example, let  $Z^n \subset \mathbb{P}^{n+1}$  be a hypersurface. The study of the images of holomorphic maps  $f : \mathbb{P}^1 \rightarrow Z$ , called *rational curves on  $Z$*  is of interest to algebraic geometers and physicists. One can break this into a series of problems based on the degree of  $f(\mathbb{P}^1)$  (that is, the number of points in the intersection  $f(\mathbb{P}^1) \cap H$  where  $H = \mathbb{P}^n$  is a general hyperplane). When the degree is one, these are just the lines on  $Z$ , and already here there are many open questions. Let  $\mathbb{F}(Z) \subset \mathbb{G}(\mathbb{P}^1, \mathbb{P}^{n+1}) = G(2, \mathbb{C}^{n+2})$  denote the variety of lines (i.e. linear  $\mathbb{P}^1$ ’s) on  $Z$ .

Note that if the lines are distributed evenly on  $Z$  and  $z \in Z_{general}$ , then  $\dim \mathbb{F}(Z) = \dim \mathcal{C}_{z,Z} + n - 1$ . We always have  $\dim \mathbb{F}(Z) \geq \dim \mathcal{C}_{z,Z} + n - 1$  (exercise!), so the microscopic geometry bounds the macroscopic geometry.

**EXAMPLE 2.** *Let  $Z = \mathbb{P}^n$ , then  $\mathbb{F}(Z) = G(2, n + 1)$ . In particular,  $\dim \mathbb{F}(Z) = 2n - 2$  and this is the largest possible dimension, and  $\mathbb{P}^n$  is the only variety with  $\dim \mathbb{F}(Z) = 2n - 2$ .*

Which are the varieties  $X^n \subset \mathbb{P}^V$  with  $\dim \mathbb{F}(X) = 2n - 3$ ? A classical theorem states that in this case  $X$  must be a quadric hypersurface.

Rogora, in [32], classified all  $X^n \subset \mathbb{P}^{n+a}$  with  $\dim \mathbb{F}(X) = 2n - 4$ , with the extra hypothesis  $\text{codim } X > 2$ . The only “new” example is the Grassmannian  $G(2, 5)$ . (A classification in codimension one would be quite difficult.) A corollary of the codimension two Fubini theorem is that Rogora’s theorem in codimension two is nearly proved - nearly and not completely because one needs to add the extra hypothesis that  $\mathcal{C}_{2,x}$  has only one component or that  $\mathcal{C}_{3,x} = \mathcal{C}_{2,x}$ .

If one has extra information about  $X$ , one can say more. Say  $X$  is a hypersurface of degree  $d$ . Then it is easy to show that  $\dim \mathbb{F}(X) \geq 2n - 1 - d$ .

CONJECTURE 5.1 (Debarre, de Jong). *If  $X^n \subset \mathbb{P}^{n+1}$  is smooth and  $n > d = \deg(X)$ , then  $\dim \mathbb{F}(X) = 2n - 1 - d$ .*

Without loss of generality it would be sufficient to prove the conjecture when  $n = d$  (slice by linear sections to reduce the dimension). The conjecture is easy to show when  $n = 2$ , it was proven by Collino when  $n = 3$ , by Debarre when  $n = 4$ , and the proof of the  $n = 5$  case was the PhD thesis of R. Beheshti [1]. Beheshti's thesis had three ingredients, a general lemma (that  $\mathbb{F}(X)$  could not be uniruled by rational curves), Theorem 4.3 above, and a case by case argument. As a corollary of the codimension two Fubini theorem one obtains a new proof of Beheshti's theorem eliminating the case by case argument (but there is a different case by case argument buried in the proof of the codimension two Fubini theorem). More importantly, the techniques should be useful in either proving the theorem, or pointing to where one should look for potential counter-examples for  $n > 5$ .

### 6. Moving frames proof of the Hwang-Yamaguchi theorem.

The principle of calculation in [20] was to use mobile Bertini theorems and the decomposition of the spaces  $S^d T^* \otimes N$  into irreducible  $R$ -modules, where  $R \subset GL(T) \times GL(N)$  is the subgroup preserving  $II \in S^2 T^* \otimes N$ . One can isolate where each  $F_k$  can "live" as the intersection of two vector spaces (one of which is  $S^k T^* \otimes N$ , the other is  $(\mathfrak{g}^\perp)_{k-3}$  defined below). Then, since  $R$  acts on fibers, we can decompose  $S^k T^* \otimes N$  and  $(\mathfrak{g}^\perp)_{k-3}$  into  $R$ -modules and in order for a module to appear, it must be in both the vector spaces. This combined with mobile Bertini theorems reduces the calculations to almost nothing.

Hwang and Yamaguchi use representation theory in a more sophisticated way via a theory developed by Se-ashi [34]. What follows is a proof of their result in the language of moving frames.

Let  $Z = G/P \subset \mathbb{P}W$  be a CHSS in its minimal homogeneous embedding. For the moment we restrict to the case where  $III_Z = 0$  (i.e.,  $\text{rank } Z = 2$ ). Let  $X \subset \mathbb{P}V$  be an analytic submanifold, let  $x \in X$  be a general point and assume  $II_{X,x} \simeq II_{Z,z}$ . We determine sufficient conditions that imply  $X$  is projectively isomorphic to  $Z$ .

We have a filtration of  $V$ ,  $V_0 = \hat{x} \subset V_1 = \hat{T}_x X \subset V = V_2$ . Write  $L = V_0, T = V_1/V_0, N = V/V_1$ . We have an induced grading of  $\mathfrak{gl}(V)$  where

$$\begin{aligned} \mathfrak{gl}(V)_0 &= \mathfrak{gl}(L) \oplus \mathfrak{gl}(T) \oplus \mathfrak{gl}(N), \\ \mathfrak{gl}(V)_{-1} &= L^* \otimes T \oplus T^* \otimes N, \\ \mathfrak{gl}(V)_1 &= L \otimes T^* \oplus T \otimes N^*, \\ \mathfrak{gl}(V)_{-2} &= L^* \otimes N, \\ \mathfrak{gl}(V)_2 &= L \otimes N^*. \end{aligned}$$

In what follows we can no longer ignore the twist in defining  $II$ , that is, we have  $II \in S^2 T^* \otimes N \otimes L$ .



Let  $\mathfrak{g}_{-1} \subset \mathfrak{gl}(V)_{-1}$  denote the image of

$$\begin{aligned} T &\rightarrow L^* \otimes T + T^* \otimes N \\ e_\alpha &\mapsto e^0 \otimes e_\alpha + q_{\alpha\beta}^\mu e^\beta \otimes e_\mu. \end{aligned}$$

Let  $\mathfrak{g}_0 \subset \mathfrak{gl}(V)_0$  denote the subalgebra annihilating  $II$ . More precisely

$$u = \begin{pmatrix} x_0^0 & & \\ & x_\beta^\alpha & \\ & & x_\nu^\mu \end{pmatrix} \in \mathfrak{gl}(V)_0$$

is in  $\mathfrak{g}_0$  iff

$$u.II := (-x_\nu^\mu q_{\alpha\beta}^\nu + x_\alpha^\gamma q_{\gamma\beta}^\mu + x_\beta^\gamma q_{\alpha\gamma}^\mu - x_0^0 q_{\alpha\beta}^\mu) e^\alpha \circ e^\beta \otimes e_\mu \otimes e_0 = 0. \quad (6.1)$$

Let  $\mathfrak{g}_1 \subset \mathfrak{gl}(V)_1$  denote the maximal subspace such that  $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathfrak{g}_0$ . Note that  $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  coincides with the  $\mathbb{Z}$ -graded semi-simple Lie algebra giving rise to  $Z = G/P$  and that the inclusion  $\mathfrak{g} \subset \mathfrak{gl}(V)$  coincides with the embedding  $\mathfrak{g} \rightarrow \mathfrak{gl}(W)$ . The grading on  $\mathfrak{g}$  induced from the grading of  $\mathfrak{gl}(V)$  agrees with the grading induced by  $P$ . In particular,  $\mathfrak{g}_{\pm 2} = 0$ .

We let  $\mathfrak{g}^\perp = \mathfrak{gl}(V)/\mathfrak{g}$  and note that  $\mathfrak{g}^\perp$  is naturally a  $\mathfrak{g}$ -module. Alternatively, one can work with  $\mathfrak{sl}(V)$  instead of  $\mathfrak{gl}(V)$  and define  $\mathfrak{g}^\perp$  as the Killing-orthogonal complement to  $\mathfrak{g}_0$  in  $\mathfrak{sl}(V)$  (Working with  $\mathfrak{sl}(V)$ -frames does not effect projective geometry since the actual group of projective transformations is  $PGL(V)$ ).

Recall that  $F_3$  arises by applying the Cartan lemma to the equations

$$0 = -\omega_\alpha^\mu \wedge \omega_\beta^\alpha - \omega_\nu^\mu \wedge \omega_\beta^\nu + q_{\alpha\beta}^\mu (\omega_0^\alpha \wedge \omega_0^0 + \omega_\gamma^\alpha \wedge \omega_0^\gamma) \quad \forall \mu, \beta \quad (6.2)$$

i.e., the tensor

$$(-q_{\beta\gamma}^\nu \omega_\nu^\mu + q_{\alpha\gamma}^\mu \omega_\beta^\alpha + q_{\alpha\beta}^\mu \omega_\gamma^\alpha - q_{\gamma\beta}^\mu \omega_0^0) \wedge \omega_0^\gamma \otimes e^\beta \otimes e_\mu \quad (6.3)$$

must vanish. All forms appearing in the term in parenthesis in (6.3) are  $\mathfrak{gl}(V)_0$ -valued. Comparing with (6.1), we see that the  $\mathfrak{g}_0$ -valued part will be zero and  $(\mathfrak{g}^\perp)_0$  bijects to the image in parenthesis.

Thus we may think of obtaining the coefficients of  $F_3$  at  $x$  in two stages, first we write the  $(\mathfrak{g}^\perp)_0$  component of the Maurer-Cartan form of  $GL(V)$  as an arbitrary linear combination of semi-basic forms, i.e., we choose a map  $T \rightarrow (\mathfrak{g}^\perp)_0$ . Once we have chosen such a map, substituting the image into (6.3) yields a  $(\Lambda^2 T^* \otimes \mathfrak{gl}(V)_{-1})$ -valued tensor. But by the definition of  $\mathfrak{g}$ , it is actually  $(\Lambda^2 T^* \otimes (\mathfrak{g}^\perp)_{-1})$ -valued. Then we require moreover that that this tensor is zero. In other words, pointwise we have a map

$$\partial^{1,1} : T^* \otimes (\mathfrak{g}^\perp)_0 \rightarrow \Lambda^2 T^* \otimes (\mathfrak{g}^\perp)_{-1}$$

and the (at this stage) admissible coefficients of  $F_3$  are determined by a choice of map  $T \rightarrow (\mathfrak{g}^\perp)_0$  which is in the kernel of  $\partial^{1,1}$ .

Now the variation of  $F_3$  as one moves in the fiber is given by a map  $T \rightarrow (\mathfrak{g}^\perp)_0$  induced from the action of  $(\mathfrak{g}^\perp)_1$  on  $(\mathfrak{g}^\perp)_0$ . We may express it as:

$$\begin{aligned} x_\beta^0 e_0 \otimes e^\beta + x_\nu^\alpha e_\alpha \otimes e^\nu &\mapsto x_\beta^0 \omega_\beta^\beta e_0 \otimes e^0 + (x_\beta^0 \omega_0^\alpha \\ &\quad + x_\nu^\alpha \omega_\beta^\nu) \otimes e^\beta \otimes e_\alpha \\ &\quad + x_\nu^\alpha \omega_\alpha^\mu \otimes e^\nu \otimes e_\mu \end{aligned} \quad (6.4)$$

where

$$\begin{pmatrix} 0 & x_\beta^0 & 0 \\ 0 & 0 & x_\nu^\gamma \\ 0 & 0 & 0 \end{pmatrix}$$

is a general element of  $\mathfrak{gl}(V)_1$ . The kernel of this map is  $\mathfrak{g}_1$  so it induces a linear map with source  $(\mathfrak{g}^\perp)_1$ . Similarly, the image automatically takes values in  $T^* \otimes (\mathfrak{g}^\perp)_0$ . In summary, (6.4) may be expressed as a map

$$\partial^{2,0} : (\mathfrak{g}^\perp)_1 \rightarrow T^* \otimes (\mathfrak{g}^\perp)_0.$$

Let  $C^{p,q} := \Lambda^q T^* \otimes (\mathfrak{g}^\perp)_{p-1}$ . Considering  $\mathfrak{g}^\perp$  as a  $T$ -module (via the embedding  $T \rightarrow \mathfrak{g}$ ) we have the Lie algebra cohomology groups

$$H^{p,1}(T, \mathfrak{g}^\perp) := \frac{\ker \partial^{p,1} : C^{p,1} \rightarrow C^{p-1,2}}{\text{Image } \partial^{p+1,0} : C^{p+1,0} \rightarrow C^{p,1}}.$$

We summarize the above discussion:

**PROPOSITION 6.1.** *Let  $X \subset \mathbb{P}V$  be an analytic submanifold, let  $x \in X_{\text{general}}$  and suppose  $II_{X,x} \simeq II_{Z,z}$  where  $Z \subset \mathbb{P}W$  is a rank two CHSS in its minimal homogeneous embedding. The choices of  $F_{3,X,x}$  imposed by (6.2), modulo motions in the fiber, is isomorphic to the Lie algebra cohomology group  $H^{1,1}(T, \mathfrak{g}^\perp)$ .*

Now if  $F_3$  is normalizable and normalized to zero, differentiating again, we obtain the equation

$$(q_{\alpha\beta}^\mu \omega_\gamma^0 + q_{\alpha\epsilon}^\mu q_{\beta\gamma}^\nu \omega_\nu^\epsilon) \wedge \omega_0^\gamma \otimes e_\mu \otimes e^\alpha \circ e^\beta = 0$$

which determines the possible coefficients of  $F_4$ .

We conclude any choice of  $F_4$  must be in the kernel of the map

$$\partial^{2,1} : T^* \otimes (\mathfrak{g}^\perp)_1 \rightarrow \Lambda^2 T^* \otimes (\mathfrak{g}^\perp)_0.$$

The variation of  $F_4$  in the fiber of  $\mathcal{F}^1$  is given by the image of the map

$$x_\nu^0 e_0 \otimes e^\nu \mapsto x_\nu^0 \omega_0^\epsilon e_\epsilon \otimes e^\nu$$

as  $v \wedge w$  ranges over the decomposable elements of  $\Lambda^2 T$ . Without indicies, the variation of  $F_4$  is the image of the map

$$\partial^{3,0} : (\mathfrak{g}^\perp)_2 \rightarrow T^* \otimes (\mathfrak{g}^\perp)_1.$$

We conclude that if  $F_3$  has been normalized to zero, then  $F_4$  is normalizable to zero if  $H^{2,1}(T, \mathfrak{g}^\perp) = 0$ .

Finally, if  $F_3, F_4$  are normalized to zero, then the coefficients of  $F_5$  are given by

$$\ker \partial^{3,1} : T^* \otimes (\mathfrak{g}^\perp)_2 \rightarrow \Lambda^2 T^* \otimes (\mathfrak{g}^\perp)_1,$$

and there is nothing to quotient by in this case because  $\mathfrak{gl}(V)_3 = 0$ . In summary

**PROPOSITION 6.2.** *A sufficient condition for second order rigidity to hold for a rank two CHSS in its minimal homogeneous embedding  $Z = G/P \subset \mathbb{P}W$  is that  $H^{1,1}(T, \mathfrak{g}^\perp) = 0, H^{2,1}(T, \mathfrak{g}^\perp) = 0, H^{3,1}(T, \mathfrak{g}^\perp) = 0$ .*

Assume for simplicity that  $G$  is simple and  $P = P_{\alpha_{i_0}}$  is the maximal parabolic subgroup obtained by deleting all root spaces whose roots have a negative coefficient on the simple root  $\alpha_{i_0}$ . By Kostant's results [13], the  $\mathfrak{g}_0$ -module  $H^{*,1}(T, \Gamma)$  for any irreducible  $\mathfrak{g}$ -module  $\Gamma$  of highest weight  $\lambda$  is the irreducible  $\mathfrak{g}_0$ -module with highest weight  $\sigma_{\alpha_{i_0}}(\lambda + \rho) - \rho$ , where  $\sigma_{\alpha_{i_0}}$  is the simple reflection in the Weyl group corresponding to  $\alpha_{i_0}$  and  $\rho$  is half the sum of the simple roots.

But now as long as  $G/P_{\alpha_{i_0}}$  is not projective space or a quadric hypersurface, Hwang and Yamaguchi, following [33], observe that any non-trivial  $\mathfrak{g}$ -module  $\Gamma$  yields a  $\mathfrak{g}_0$ -module in  $H^{*,1}$  that has a non-positive grading, so one concludes that the above groups are *a priori* zero. Thus one only need show that the trivial representation is not a submodule of  $\mathfrak{g}^\perp$ .

**REMARK 6.1.** *We note that the condition in Proposition 6.2 is not necessary for second order rigidity. It holds in all rigid rank two cases but one,  $Seg(\mathbb{P}^1 \times \mathbb{P}^n)$ , with  $n > 1$ , which we saw, in §1, is indeed rigid to order two. Note that in that case the naïve moving frames approach is significantly more difficult as one must prolong several times before obtaining the vanishing of the normalized  $F_3$ .*

To prove the general case of the Hwang-Yamaguchi theorem, say the last nonzero fundamental form is the  $k$ -th. Then one must show  $H^{1,1}, \dots, H^{k+1,1}$  are all zero. But again, Kostant's theory applies to show all groups  $H^{p,1}$  are zero for  $p > 0$ . Note that in this case,  $H^{1,1}$  governs the vanishing  $F_{3,2}, \dots, F_{k,k-1}$ , where  $F_{k,l}$  denotes the component of  $F_k$  in  $S^k T^* \otimes N_l$ . In general,  $H^{l,1}$  governs  $F_{2+l,2}, \dots, F_{k+l,k-1}$ .

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