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DIFFERENTIAL OPERATORS AND BOUNDARY VALUE PROBLEMS ON HYPERSURFACES

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We explore the extent to which basic differential operators (such as Laplace-Beltrami, Lamé, Navier-Stokes, etc.) and boundary value problems on a hypersurface $\mathcal{S}$ in $\mathbb{R}^n$ can be expressed globally, in terms of the standard spatial coordinates in $\mathbb{R}^n$. The approach we develop also provides, in some important cases, useful simplifications as well as new interpretations of classical operators and equations.

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1

Introduction

Boundary value problems (BVP's) for partial differential equations (PDE's) on surfaces arise in a variety of situations and have many practical applications. See, for example, [Ha, §72] for the heat conduction by surfaces, [Ar, §10] for the equations of surface flow, [Ci], [Ci2], [Go] for shell problems in elasticity, [AC] for the vacuum Einstein equations describing gravitational fields, [TZ] for the Navier-Stokes equations on spherical domains, [AMM1, AMM2] for Stokes equations, [MaMi] for minimal surfaces, as well as the references therein. Furthermore, while studying the asymptotic behavior of solutions to elliptic boundary value problems in the

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neighborhood of a conical point one is led to considering a one-parameter family of boundary value problems in a subdomain $\mathcal{S}$ of S^{n-1}, the unit sphere in $\mathbb{R}^n$, naturally associated (via the Mellin transform) with the original elliptic problem. A classical reference in this regard is [Ko]. Finally, PDE's on surfaces also turn up naturally in the limit case, as the thickness goes to zero, of equations in thin layers or shells. Cf. [Ci, §3] for the case of elasticity, and [TW], [TZ] for the case of Navier-Stokes equations.

A hypersurface $\mathcal{S}$ in $\mathbb{R}^n$ has the natural structure of a $(n-1)$-dimensional Riemannian manifold and the aforementioned PDE's are not the immediate analogues of the ones corresponding to the flat, Euclidean case, since they have to take into consideration geometric characteristics of $\mathcal{S}$ such as curvature. Inherently, these PDE's are originally written in local coordinates, intrinsic to the manifold structure of $\mathcal{S}$.

The main aim of this paper is to explore the extent to which the most basic partial differential operators (PDO's), as well as their associated boundary value problems, on a hypersurface $\mathcal{S}$ in $\mathbb{R}^n$, can be expressed globally, in terms of the standard spatial coordinates in $\mathbb{R}^n$. It turns out that a convenient way to carry out this program is by employing the so-called Günter derivatives (cf. [Gu], [KGBB], [Du]):

$$\mathcal{D} := (\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_n). \tag{0.1}$$

Here, for each $1 \leq j \leq n$, the first-order differential operator $\mathcal{D}_j$ is the directional derivative along $\pi\, e_j$, where $\pi : \mathbb{R}^n \to T\mathcal{S}$ is the orthogonal projection onto the tangent plane to $\mathcal{S}$ and, as usual, $e_j = (\delta_{jk})_{1\leq k\leq n} \in \mathbb{R}^n$, with δ_{jk} denoting the Kronecker symbol. The operator $\mathcal{D}$ is globally defined on $\mathcal{S}$ by means of the unit normal vector field, and has a relatively simple structure. In terms of (0.1), the Laplace-Beltrami operator on $\mathcal{S}$ simply becomes

$$\Delta_{\mathcal{S}} = \mathcal{D} \bullet \mathcal{D} = \sum_{j=1}^{n} \mathcal{D}_j^2 \quad \text{on} \quad \mathcal{S}. \tag{0.2}$$

(Cf. [MaMi, pp. 2 ff and p. 8].) Moreover, $\Delta_{\mathcal{S}}$ is the natural operator associated with the Euler-Lagrange equations for the variational integral

$$\mathcal{E}[u] = -\frac{1}{2} \int_{\mathcal{S}} \|\mathcal{D}u\|^2 \, dS \tag{0.3}$$

(Cf. [MaMi, pp. 9 ff]). A similar approach, based on the principle that, at equilibrium, the displacement minimizes the potential energy, leads to the following expression for the Lamé operator $\mathcal{L}$ on $\mathcal{S}$:

$$\mathcal{L}u = \mu\, \pi\, (\mathcal{D} \bullet \mathcal{D})u + (\lambda + \mu)\, \mathcal{D}(\mathcal{D} \bullet u) - \mu\, (n-1)\mathcal{H}\mathcal{W}u, \tag{0.4}$$

for arbitrary vector fields u on $\mathcal{S}$, which are tangent to $\mathcal{S}$. Above, $\lambda, \mu \in \mathbb{R}$ are the Lamé moduli, whereas $\mathcal{H}$, $\mathcal{W}$ stand, respectively, for the the mean curvature and the Weingarten map of $\mathcal{S}$. In particular, when combined with the recent work from [MMT] (dealing with general elliptic BVP's on Lipschitz subdomains of Riemannian manifolds), this identification ensures the well-posedness of the boundary-value problem

$$\begin{cases} u = (u_1, ..., u_n) \in H^{s+1/2,2}(\mathcal{S}, \mathbb{R}^n), \\ \langle u, N \rangle = 0 \quad \text{in} \quad \mathcal{S}, \\ \mu\, \pi\, (\mathcal{D} \bullet \mathcal{D})u + (\lambda + \mu)\, \mathcal{D}(\mathcal{D} \bullet u) - \mu\, (n-1)\mathcal{H}\mathcal{W}u = 0 \quad \text{in} \quad \mathcal{S}, \\ u\Big|_{\partial\mathcal{S}} = \vec{f} \in H^{s,2}(\partial\mathcal{S}, \mathbb{R}^n), \quad \langle \vec{f}, N \rangle = 0 \text{ on } \partial\mathcal{S}, \end{cases} \tag{0.5}$$

whenever $\mu > 0$, $2\mu + \lambda > 0$, and $0 \leq s \leq 1$. Here $H^{s,2}$ stands for the usual L^2-based Sobolev scale. Other operators discussed in this paper are the Hodge-Laplacian

$$\Delta_{HL} := -d_{\mathcal{S}} d^*_{\mathcal{S}} - d^*_{\mathcal{S}} d_{\mathcal{S}}, \tag{0.6}$$

where $d_{\mathcal{S}}$ is the exterior derivative operator on $\mathcal{S}$, and $d^*_{\mathcal{S}}$ its formal adjoint, and the Navier-Stokes system on $\mathcal{S} \times (0, \infty)$; see §7 for details.

These results are useful in numerical and engineering applications (cf. [AN], [Be], [Ce], [Co], [DL], [BGS], [Sm]) and we plan to treat a number of special hypersurfaces in greater detail in a subsequent publication.

The layout of the paper is as follows. In §2 we review some basic differential-geometric concepts which are relevant for the work at hand. In §3, based on variational methods (minimization of energy functional), we identify the natural Lamé operator on a general (elastic, linear, isotropic) manifold M. Starting with §4, we specialize our discussion to the case of a hypersurface $\mathcal{S}$, viewed as a Riemannian manifold with the metric inherited from the ambient Euclidean space. In particular, here we discuss the possibility of extending the unit normal to $\mathcal{S}$, i.e. $N : \mathcal{S} \to S^{n-1}$ in a neighborhood of $\mathcal{S}$ under additional assumptions. In §5 we derive some very useful 'integration by parts' formulas for first order operators which are tangent to a hypersurface $\mathcal{S}$.

The proof of the identification (0.2) is given in §6, via a method which is interesting in its own right. In fact, this is flexible enough to apply to the case of systems of equations, such as the Lamé operators. This yields (0.4), in §7. Finally, applications to elliptic BVP's on smooth hypersurfaces with Lipschitz boundaries (such as (0.5)), are presented in §8.

1 Brief review of classical differential geometry of manifolds

Let M be a smooth manifold, possibly with boundary, of (real) dimension n. As usual, by TM and T^*M we denote, respectively, the tangent and cotangent bundle on M. Throughout, we shall also denote by TM global (C^∞) sections in TM (i.e., $TM \equiv C^\infty(M, TM)$); similarly, $T^*M \equiv C^\infty(M, T^*M)$. More generally, if $\Lambda^\ell TM$ stands for the corresponding exterior power bundle (differential forms of degree ℓ), then we shall use the abbreviation $\Lambda^\ell TM \equiv C^\infty(M, \Lambda^\ell TM)$.

We shall assume that M is equipped with a smooth Riemannian metric tensor $g = \sum_{j,k} g_{jk} dx_j \otimes dx_k$, denote by $(g^{jk})_{jk}$ the inverse matrix to (g_{jk}) and set $g := \det(g_{jk})_{jk}$. In particular, $d\mathrm{Vol}$, the volume element in M is locally given by $d\mathrm{Vol} = \sqrt{g}\, dx_1 ... dx_n$. Recall next that

$$\operatorname{div} X := \sum_j \sqrt{g}^{-1} \partial_j (\sqrt{g} X_j) \quad \text{if} \quad X = \sum_j X_j \partial/\partial x_j \in TM, \tag{1.1}$$

and

$$\operatorname{grad} f = \sum_{j,k} (g^{jk} \partial_j f)\, \partial/\partial x_k \tag{1.2}$$

are, respectively, the usual divergence and gradient operators. Accordingly, the Laplace-Beltrami operator Δ becomes

$$\Delta := \operatorname{div} \operatorname{grad} = \sqrt{g}^{-1} \sum_{j,k=1}^{n} \partial_j (g^{jk} \sqrt{g} \partial_k \cdot\,)\,. \tag{1.3}$$

The pairing $\langle dx_j, dx_k \rangle := g^{jk}$ defines an inner product in $\Lambda^1 TM$. With respect to this, grad and $-\operatorname{div}$ are adjoint to each other. We shall also abbreviate $dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_\ell}$ by dx_I, where $I = (i_1, i_2, ..., i_\ell)$ and let wedge stand for the ordinary exterior product of forms. Then, $\langle dx_I, dx_J \rangle := \det\Big((g^{ij})_{\substack{i\in I\\ j\in J}} \Big)$, $|I| = |J| = \ell$, defines an inner product in $\Lambda^\ell TM$ for each $1 \le \ell \le n$.

If, as usual, we let $d = \sum_j \partial/\partial x_j\, dx_j \wedge$ stand for the exterior derivative operator on M, and denote by δ its formal adjoint (with respect to the above metric), then the Hodge-Laplacian on M becomes

$$\Delta := -d\delta - \delta d. \tag{1.4}$$

As is customary, we may identify vector fields with one-forms, i.e., $TM \cong T^*M = \Lambda^1 TM$, via $\partial/\partial x_j \mapsto g_{jk}dx_k$ (lowering indices). This mapping is an isometry whose inverse is given by $dx_j \mapsto g^{jk}\partial/\partial x_k$ (raising indices). In the sequel, we shall not make any notational distinction between a vector field and its associated 1-form (i.e., we shall tacitly identify $TM \equiv T^*M$). Under this identification, $\mathrm{grad} : C^\infty(M) \to C^\infty(M, TM)$ becomes $d : C^\infty(M) \to C^\infty(M, \Lambda^1 TM)$ and $\mathrm{div} : C^\infty(M, TM) \to C^\infty(M)$ becomes $-\delta : C^\infty(M, \Lambda^1 TM) \to C^\infty(M)$.

A tensor of type (k, j) is a map

$$F : (TM \times ... \times TM) \times (\Lambda^1 TM \times ... \times \Lambda^1 TM) \to C^\infty(M) \tag{1.5}$$

(with j factors of TM and k factors of $\Lambda^1 TM$) which is linear in each factor over the ring $C^\infty(M)$. There is a natural inner product at the level of (k, j) tensors defined by

$$\begin{aligned}\langle F, G\rangle \quad := \quad & \sum F(X_{\alpha_1}, X_{\alpha_2}, ..., X_{\alpha_j}, \omega_{\beta_1}, \omega_{\beta_2}, ..., \omega_{\beta_k}) \\ & \cdot G(X_{\alpha_1}, X_{\alpha_2}, ..., X_{\alpha_j}, \omega_{\beta_1}, \omega_{\beta_2}, ..., \omega_{\beta_k}),\end{aligned} \tag{1.6}$$

where X_α's are an orthonormal frame for TM and ω_β's are the dual basis in $\Lambda^1 TM$ (summation over all possible choices of indices).

Next, let ∇ be the associated Levi-Civita connection. Among other things, the metric property

$$Z\langle X, Y\rangle = \nabla_Z \langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle, \qquad \forall\, X, Y, Z \in TM, \tag{1.7}$$

holds. This, in concert with

$$X^* = -X - \mathrm{div}\, X, \qquad \forall\, X \in TM. \tag{1.8}$$

further entails that

$$(\nabla_X)^* = -\nabla_X - \mathrm{div}\, X, \qquad \forall\, X \in TM. \tag{1.9}$$

For each $X \in TM$, ∇X is the tensor of type $(0, 2)$ defined by

$$(\nabla X)(Y, Z) := \langle \nabla_Z X, Y\rangle, \qquad \forall\, Y, Z \in TM. \tag{1.10}$$

with trace

$$\mathrm{Tr}(\nabla X) = \sum_{j=1}^{n} \langle \nabla_{T_j} X, T_j\rangle = \mathrm{div}\, X \tag{1.11}$$

for any orthonormal frame $\{T_j\}_j$ in TM. For any $X \in TM$, the antisymmetric part of ∇X is simply dX, i.e.

$$dX(Y, Z) = \langle dX, Y \wedge Z\rangle = \tfrac{1}{2}\Big\{\langle \nabla_Y X, Z\rangle - \langle \nabla_Z X, Y\rangle\Big\}, \qquad \forall\, Y, Z \in TM, \tag{1.12}$$

whereas the symmetric part of ∇X is $\mathrm{Def}\, X$, the deformation of X, i.e.

$$(\mathrm{Def}\, X)(Y,Z) = \tfrac{1}{2}\Big\{\langle\nabla_Y X, Z\rangle + \langle\nabla_Z X, Y\rangle\Big\}, \qquad \forall\, Y,Z \in TM. \tag{1.13}$$

Thus, $\mathrm{Def}\, X$ is a symmetric tensor field of type $(0,2)$. In coordinate notation,

$$(\mathrm{Def}\, X)_{jk} = \tfrac{1}{2}(X_{j;k} + X_{k;j}), \qquad \forall\, j,k. \tag{1.14}$$

Here, as usual, for a vector field $X = \sum_j X^j \partial_j$, we set $X_{k;j} := \partial_j X_k + \sum_l \Gamma^l_{kj} X_l$, where $X_k = \sum_l g_{kl} X^l$ and Γ^l_{kj} are the Christoffel symbols associated with the metric. Deformation-free vector fields X are usually referred to as *Killing fields*. They satisfy

$$\sum_\ell \Big[g_{k\ell} X^\ell_{\ ;j} + g_{j\ell} X^\ell_{\ ;k}\Big] = X_{k;j} + X_{j;k} = 0, \qquad \forall\, j,k. \tag{1.15}$$

The adjoint of Def is Def^* defined in local coordinates by $(\mathrm{Def}^* w)^j = -w^{jk}{}_{;k}$ for each symmetric tensor field w of type $(0,2)$. In particular, if $\nu \in TM$ is the outward unit normal to $\partial M \hookrightarrow M$, then the integration by part formula

$$\int_M \langle \mathrm{Def}\, u, w\rangle \, d\mathrm{Vol} = \int_M \langle u, \mathrm{Def}^* w\rangle \, d\mathrm{Vol} + \int_{\partial M} w(\nu, u)\, d\mathrm{vol}, \tag{1.16}$$

holds for any $u \in TM$, and any symmetric tensor field w of type $(0,2)$. Here and elsewhere, $d\mathrm{vol}$ will denote the volume element on ∂M.

Formula (1.16) is a particular case of a more general integration by parts identity, to the effect that

$$\int_M \langle Pu, w\rangle \, d\mathrm{Vol} = \int_M \langle u, P^* w\rangle \, d\mathrm{Vol} + \int_{\partial M} \langle \sigma(P;\nu)u, w\rangle \, d\mathrm{vol}, \tag{1.17}$$

valid for a general first-order differential operator $P = \sum_j A_j \partial_j +$ zero order terms (acting between two hermitian vector bundles on M), with principal symbol $\sigma(P;\xi) = \sum_j A_j \xi_j$, for $\xi \in T^*M$.

For further reference, let us note here that $\sigma(\nabla_X;\xi) = \langle X, \xi\rangle$, and that $\sigma(d;\xi) = \xi \wedge \cdot$, $\sigma(\delta;\xi) = \xi \vee \cdot$, where we have denoted by $\vee$ the adjoint of the exterior product, in the sense that

$$\langle \xi \vee u, w\rangle = \langle u, \xi \wedge w\rangle, \qquad \xi \in \Lambda^1 TM,\ u \in \Lambda^\ell TM,\ w \in \Lambda^{\ell-1} TM. \tag{1.18}$$

The Riemann curvature tensor $\mathcal{R}$ of M is given by

$$\mathcal{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \qquad X,Y,Z \in TM, \tag{1.19}$$

where $[X,Y] := XY - YX$ is the usual commutator bracket. It is convenient to change this into a $(0,4)$-tensor by setting

$$\mathcal{R}(X,Y,Z,W) := \langle \mathcal{R}(X,Y)Z, W\rangle, \qquad X,Y,Z,W \in TM. \tag{1.20}$$

The Ricci curvature Ric on M is a $(0,2)$-tensor defined as a contraction of $\mathcal{R}$:

$$\mathrm{Ric}\,(X,Y) := \sum_{j=1}^n \langle \mathcal{R}(T_j,Y)X, T_j\rangle = \sum_{j=1}^n \langle \mathcal{R}(Y,T_j)T_j, X\rangle, \qquad \forall\, X,Y \in TM, \tag{1.21}$$

where $T_1, \ldots, T_n$ is an orthonormal frame in TM. Thus, Ric is a symmetric bilinear form.

Under the identification $TM \equiv \Lambda^1 TM$, the Bochner Laplacian and the Hodge Laplacian are related by

$$\nabla^* \nabla \equiv -\Delta - \mathrm{Ric}, \tag{1.22}$$

a special case of the Weitzenbock identity.

Consider now $\mathcal{S} \hookrightarrow M$, a smooth, orientable submanifold of codimension one in M, and fix some $\nu \in TM$ such that $\nu|_{\mathcal{S}}$ becomes the outward unit normal to $\mathcal{S}$. If $\nabla^{\mathcal{S}}$ is the induced Levi-Civita connection on $\mathcal{S}$ (from the metric inherited from M) it is then well-known that

$$\nabla^{\mathcal{S}}_X Y = \pi(\nabla_X Y), \qquad \forall\, X, Y \in T\mathcal{S}, \tag{1.23}$$

where

$$\pi : TM \longrightarrow T\mathcal{S}, \qquad \pi = I - \langle \cdot, \nu \rangle \nu = \nu \vee (\nu \wedge \,\cdot\,), \tag{1.24}$$

is the canonical orthogonal projection onto $T\mathcal{S}$. In particular, the second fundamental form of $\mathcal{S}$ becomes

$$II(X, Y) := \nabla_X Y - \nabla^{\mathcal{S}}_X Y = \pi(\nabla_X Y), \qquad \forall\, X, Y \in T\mathcal{S}. \tag{1.25}$$

In this setting, the Weingarten map

$$\mathcal{W} : T\mathcal{S} \longrightarrow T\mathcal{S}, \tag{1.26}$$

originally defined uniquely by the requirement that

$$\langle \mathcal{W}X, Y \rangle = \langle \nu, II(X, Y) \rangle, \qquad \forall\, X, Y \in T\mathcal{S}, \tag{1.27}$$

reduces to

$$\mathcal{W}X = -\nabla_X \nu \quad \text{on } \ \mathcal{S}, \qquad \forall\, X \in T\mathcal{S}, \tag{1.28}$$

known as Weingarten formula.

An excellent reference for the material in this section is [Ta2]. Here we only want to point out that, whenever necessary in order to avoid confusion, we shall write d_M, grad_M, div_M, Δ_M, etc., in place of d, grad, div, Δ, etc.

2 The derivation of the Lamé operator on manifolds

In the present paragraph ∇u, for $u \in C^\infty(M)$ a scalar function, is naturally identified with the gradient $\mathrm{grad}\, u$ and is identified with the Jakobi matrix $(\partial_k u_j)_{j,k}$ if $u = (u_1, \ldots, u_n)^\top$ is a vector-function.

One way of understanding the genesis of the Laplace-Beltrami operator (1.3) is to consider the energy functional

$$\mathcal{E}[u] := \int_M \|\mathrm{grad}\, u\|^2 \, d\mathrm{Vol}, \qquad u \in C^\infty(M). \tag{2.1}$$

Then any minimizer u of (2.1) should satisfy

$$\frac{d}{dt}\mathcal{E}[u+tv]\Big|_{t=0} = 0, \qquad \forall\, v \in C_o^\infty(M), \tag{2.2}$$

thus, after an integration by parts,

$$\Delta u = 0 \quad \text{on} \quad M. \tag{2.3}$$

In other words, (2.3) is the Euler-Lagrange equation associated with the integral functional (2.1).
Similarly, minimizers of the energy functional

$$\mathcal{E}[u] := -\frac{1}{2}\int_M \Big[\|du\|^2 + \|\delta u\|^2\Big]\, d\mathrm{Vol}, \qquad u \in \Lambda^\ell TM, \tag{2.4}$$

are null-solutions of the Hodge-Laplacian (1.4), while minimizers of the energy functional

$$\mathcal{E}[u] := -\frac{1}{2}\int_M \|\nabla u\|^2\, d\mathrm{Vol}, \qquad u \in TM, \tag{2.5}$$

are null-solutions of the Bochner-Laplacian (1.22).

Our aim is to adopt a similar point of view in the case of the (possibly anisotropic) Lamé system of elasticity on M. The departure point is to consider the total free elastic energy

$$\mathcal{E}[u] := -\frac{1}{2}\int_M E(x, \nabla u(x))\, d\mathrm{Vol}_x, \qquad u \in TM, \tag{2.6}$$

ignoring at the moment the displacement boundary conditions. As before, equilibria states correspond to minimizers of the above variational integral. The first order of business is to identify the correct form of the stored energy density $E(x, \nabla u(x))$. We shall restrict attention to the case of linear elasticity. In this scenario, E depends bilinearly on the stress tensor $\sigma = (\sigma_{jk})_{jk}$ and the deformation (strain) tensor $\varepsilon = (\varepsilon_{jk})_{jk}$ which, according to Hooke's law, satisfy $\sigma = \mathcal{T}\varepsilon$, for some linear, fourth-order tensor $\mathcal{T}$. If the medium is also homogeneous (i.e. the density and elastic parameters are position-independent), it follows that E depends quadratically on ∇u, i.e.

$$E(x, \nabla u(x)) = \langle \mathcal{T}\nabla u(x), \nabla u(x)\rangle \tag{2.7}$$

for some linear operator

$$\mathcal{T} : M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R}), \tag{2.8}$$

where $M_{n,n}(\mathbb{R})$ stands for the vector space of all $n \times n$ matrices with real entries. Hereafter, we organize $M_{n,n}(\mathbb{R})$ as a real Hilbert space with respect to the inner product

$$\langle A, B\rangle := \mathrm{Tr}(AB^\top), \qquad \forall\, A, B \in M_{n,n}(\mathbb{R}), \tag{2.9}$$

where the superscript $\top$ denotes transposition, and Tr is the usual trace operator for square matrices. It is customary to assume that the linear operator (2.8) is self-adjoint, that is

$$\langle \mathcal{T}A, B\rangle = \langle A, \mathcal{T}B\rangle\,, \qquad A, B \in M_{n,n}(\mathbb{R})\,. \tag{2.10}$$

The operator $\mathcal{T} = (c_{ijk\ell})_{ijk\ell}$, i.e.,

$$\mathcal{T}A = \Big(\sum_{k,\ell} c_{ijk\ell}a_{k\ell}\Big)_{ij}, \qquad \text{for } \ A = (a_{k\ell})_{k\ell} \in M_{n,n}(\mathbb{R}), \tag{2.11}$$

will be referred to in the sequel as the elasticity tensor. The condition (2.10), written in coordinate notation, is equivalent to the following equality

$$c_{ijk\ell} = c_{k\ell ij}, \qquad \forall\, i,j,k,\ell\,. \tag{2.12}$$

It is also customary to impose a symmetry condition and one is presented with two natural options, namely

$$\mathcal{T}(A^\top) = \mathcal{T}A, \qquad \forall\, A \in M_{n,n}(\mathbb{R}), \tag{2.13}$$

and

$$(\mathcal{T}A)^\top = \mathcal{T}A, \qquad \forall\, A \in M_{n,n}(\mathbb{R}), \tag{2.14}$$

where the second one (2.14) follows from (2.11) and from (2.13).
Then (2.13) amounts to symmetry in the second pair of indices, i.e.

$$c_{ijk\ell} = c_{ij\ell k}, \qquad \forall\, i,j,k,\ell, \tag{2.15}$$

whereas (2.14) is equivalent to symmetry in the first pair of indices, i.e.

$$c_{ijk\ell} = c_{jik\ell}, \qquad \forall\, i,j,k,\ell. \tag{2.16}$$

To sum up our discussion so far, we note that a linear operator $\mathcal{T}$ (as in (2.8)), which corresponds to the energy functional of anisotropic elasticity (cf. (2.7)), satisfies the symmetry conditions (2.10), (2.13), (2.14). Thus, for the corresponding matrix $\mathcal{T} = (c_{ijk\ell})_{ijk\ell=1}^n$, the symmetry conditions (2.12), (2.15) and (2.16) hold, so this matrix may have at most $n + n^2(n-1)^2/2$ different entries.

The isotropic media is further assumed to satisfy

$$\mathcal{T}(UAU^{-1}) = U(\mathcal{T}A)U^{-1}, \qquad \forall\, A, U \in M_{n,n}(\mathbb{R}),\ U \text{ unitary}. \tag{2.17}$$

As we shall see *a posteriori*, the conditions (2.13), (2.14) and (2.17) imply the linear operator (2.8) is self adjoint, i.e., imply the condition (2.10). Indeed, we have:

Proposition 2.1 *Consider a linear operator $\mathcal{T}$, as in (2.8), such that the isotropy condition (2.17) holds. Then $\mathcal{T}$ satisfies (2.13) if and only if it satisfies (2.14). Furthermore, any linear operator $\mathcal{T}$ which satisfies (2.17) along with either (2.13) or (2.14) has the form*

$$\mathcal{T}A = \lambda\,(\mathrm{Tr}\,A)I + \mu\,(A + A^\top), \qquad A \in M_{n,n}(\mathbb{R}), \tag{2.18}$$

for some constants $\lambda, \mu \in \mathbb{R}$.

P r o o f. Let us first show that any linear operator (2.8) satisfying (2.17), (2.13) can be represented in the form (2.18). By the previous discussion, it suffices to prove that the space of linear operators (2.8) satisfying (2.17), (2.13) has dimension two.

Since for any $A \in M_{n,n}(\mathbb{R})$ we have $\mathcal{T}A = \frac{1}{2}\mathcal{T}(A + A^\top)$, thanks to (2.13), and since $A + A^\top$ can be diagonalized (via a suitable conjugation with an unitary matrix U), it suffices to show that

$$\mathcal{T}D = \begin{pmatrix} a & 0 & 0 & ... & 0 \\ 0 & b & 0 & ... & 0 \\ 0 & 0 & b & ... & 0 \\ & & ... & & \\ 0 & 0 & 0 & ... & b \end{pmatrix}, \text{ where } D := \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ & & ... & \\ 0 & 0 & ... & 0 \end{pmatrix}$$

for two numbers $a, b \in \mathbb{R}$. To this end, consider the following types of unitary matrices:

$$U_{i_o,j_o} := \begin{pmatrix} 1 & 0 & ... & ... & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & ... & 0 \\ 0 & ... & 0 & ... & 1 & ... & 0 \\ & & ... & ... & ... & & \\ 0 & ... & 1 & ... & 0 & ... & 0 \\ & & ... & ... & ... & & \\ 0 & 0 & 0 & ... & ... & 1 & 0 \\ 0 & 0 & 0 & ... & ... & ... & 1 \end{pmatrix}, \quad W_{i_o,j_o} := \begin{pmatrix} 1 & 0 & ... & ... & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & ... & 0 \\ 0 & ... & 0 & ... & 1 & ... & 0 \\ & & ... & ... & ... & & \\ 0 & ... & -1 & ... & 0 & ... & 0 \\ & & ... & ... & ... & & \\ 0 & 0 & 0 & ... & ... & 1 & 0 \\ 0 & 0 & 0 & ... & ... & ... & 1 \end{pmatrix}$$

where the only non-zero, off the diagonal entries are at (i_o, j_o) and (j_o, i_o). Next, set

$$A := \mathcal{T}D, \qquad A = \Big(a_{ij}\Big)_{1 \leq i,j \leq n}$$

and observe that D is invariant under conjugation by W_{i_o,j_o}, i.e. $W_{i_o,j_o} D W_{i_o,j_o}^\top = D$, as long as $i_o \neq 1$ and $j_o \neq 1$. Thus, by (2.17), the same is true for $A = \mathcal{T}D$ which, in turn, eventually implies that

$$a_{i_o i_o} = a_{j_o j_o}, \qquad \forall\, i_o, j_o \neq 1. \tag{2.19}$$

The next observation is that D is invariant under conjugation by the product $U_{i_o,j_o} W_{i_o,j_o}$, i.e.

$$U_{i_o,j_o} W_{i_o,j_o} D U_{i_o,j_o}^\top W_{i_o,j_o}^\top = D\,,$$

this time for every $1 \leq i_o \neq j_o \leq n$. Hence, by (2.17), the same holds for $A = \mathcal{T}D$, which ultimately implies that $a_{i_o j_o} = -a_{j_o i_o}$ for every pair of indices $1 \leq i_o \neq j_o \leq n$. Consequently,

$$a_{i_o j_o} = 0, \quad \text{ for every } \; 1 \leq i_o \neq j_o \leq n. \tag{2.20}$$

Under the current assumptions, i.e. (2.17), (2.13), the desired conclusion, i.e. that $\mathcal{T}D$ has the two-parameter diagonal form indicated above, now follows readily from (2.19) and (2.20).

There remains to analyze the case when the linear operator $\mathcal{T}$ satisfies (2.17) along with (2.14). In this situation, it can be readily checked that $\mathcal{T}^*$, the adjoint of $\mathcal{T}$ with respect to the inner product (2.9), satisfies (2.17), (2.13), so the previous reasoning applies. Consequently, $\mathcal{T}^*$ can be represented in the form (2.18), which is invariant under adjunction. Hence $\mathcal{T}$ can be written in the form (2.18) also. In particular, (2.18) holds in this case as well, and this finishes the proof of the proposition. □

Remark. The above proof can be modified to hold in the case when (2.17) is (seemingly) weakened to allow only *orientation preserving* unitary matrices U. All one has to do in this later case is to employ the invariance of D under conjugation by $U_{k_o,\ell_o} U_{i_o,j_o} W_{i_o,j_o}$ (with $k_o, \ell_o \neq 1$), in place of conjugation by (the inversion) $U_{i_o,j_o} W_{i_o,j_o}$ as in the original proof.

We are now ready to derive the Lamé equations of elasticity on M.

Theorem 2.2 *On a Riemannian manifold M, modeling a homogeneous, linear, isotropic, elastic medium, the Lamé operator $\mathcal{L}$ is given by*

$$\mathcal{L} = -2\mu\,\mathrm{Def}^*\mathrm{Def} + \lambda\,\mathrm{grad}\,\mathrm{div}. \tag{2.21}$$

In particular, $\mathcal{L}$ is strongly elliptic, formally self-adjoint.

If $u \in TM$ denotes the displacement, natural boundary conditions for $\mathcal{L}$ include prescribing $u|_{\partial M}$, Dirichlet type, and

$$\mathrm{Traction}\,u := 2\mu\,(\mathrm{Def}\,u)\nu + \lambda\,(\mathrm{div}\,u)\nu \quad \textit{on} \;\; \partial M, \tag{2.22}$$

Neumann type.

Here $\nu \in TM$ is the outward unit normal to ∂M and we identify $(\mathrm{Def}\,u)\nu$ with the vector field uniquely determined by the requirement that $\langle(\mathrm{Def}\,u)\nu, X\rangle = (\mathrm{Def}\,u)(\nu, X)$ for each $X \in TM$.

Proof. For elastic materials, it is known that the elasticity tensor (2.8) is symmetric matrix-valued, i.e. it satisfies (2.13). Thus, according to the discussion in the first part of this section, the elasticity tensor in the case of linear, isotropic, elastic media is given by (2.18), where λ, μ are the Lamé moduli. Consequently, the stored energy density we need to consider is

$$E(A) = \langle \mathcal{T}A, A\rangle = \lambda\,(\mathrm{Tr}\,A)^2 + \frac{\mu}{2}\,\mathrm{Tr}\,((A + A^\top)^2). \tag{2.23}$$

Further substituting $A := \nabla u$ in (2.23) yields

$$E(x, \nabla u(x)) = \lambda\,(\mathrm{div}\,u)^2(x) + 2\mu\,\langle(\mathrm{Def}\,u)(x), (\mathrm{Def}\,u)(x)\rangle, \tag{2.24}$$

by (1.11) and (1.13). Thus, we are led to considering the variational integral

$$\mathcal{E}[u] = -\frac{1}{2}\int_M \Big[\lambda\,(\mathrm{div}\,u)^2 + 2\mu\,\langle \mathrm{Def}\,u, \mathrm{Def}\,u\rangle\Big]\,d\mathrm{Vol}, \qquad u \in TM. \tag{2.25}$$

To determine the associated Euler-Lagrange equation, for an arbitrary $v \in TM$, smooth and compactly supported, we compute

$$\begin{aligned} \frac{d}{dt}\mathcal{E}[u+tv]\Big|_{t=0} &= -\int_M \Big[\lambda\,(\mathrm{div}\,u)(\mathrm{div}\,v) + 2\mu\,\langle \mathrm{Def}\,u, \mathrm{Def}\,v\rangle\Big]\,d\mathrm{Vol} \\ &= \int_M \langle(\lambda\,\mathrm{grad}\,\mathrm{div} - 2\mu\,\mathrm{Def}^*\mathrm{Def}\,)u, v\rangle\,d\mathrm{Vol} \\ &\quad -\int_{\partial M} \langle 2\mu\,(\mathrm{Def}\,u)\nu + \lambda\,(\mathrm{div}\,u)\nu\,,\, v\rangle\,d\mathrm{vol} \end{aligned} \tag{2.26}$$

after integrating by parts, based on (1.16) and the Divergence Theorem (i.e. (1.17) with $P = \mathrm{div}$). ¿From (2.26), the desired conclusions follow without difficulty. □

Remark. In the (linear) anisotropic case, the Lamé operator takes the less explicit form

$$\mathcal{L}u = -\nabla^*\mathcal{T}\nabla u, \qquad u \in TM, \tag{2.27}$$

where, as before, $\mathcal{T}$ is the elasticity tensor; cf. (2.11).

3 Distinguished extensions of the unit normal to a hypersurface

We retain the notation adopted in §2, albeit specialized to the case $M = \mathbb{R}^n$. In particular, ∇ and div are the usual gradient and divergence operators, $\nabla := (\partial_1, \ldots, \partial_n)$ and $\mathrm{div} = \nabla \bullet$, respectively, in $\mathbb{R}^n$. Also, if X is a vector field in $\mathbb{R}^n$, we denote by ∂_X the directional derivative $X \bullet \nabla$. Hereafter, $X \bullet Y = \langle X, Y \rangle$ stands for the standard inner product in $\mathbb{R}^n$. Thus,

$$\nabla_X Y = (X \bullet \nabla) Y = (X_k(\partial_k Y_j))_j, \qquad \text{if} \quad Y = (Y_j)_j,\ X = (X_k)_k.$$

Also, ∇X is naturally identified with the matrix $(\partial_k X_j)_{j,k}$.

Next, let $\mathcal{S}$ be a C^k-hypersurface, i.e. an orientable, submanifold of class C^k, $k \geq 1$, of codimension one in $\mathbb{R}^n$, with unit normal $N(x) = (N_1(x), \ldots, N(x))$, $x \in \mathcal{S}$. Also let $\partial\mathcal{S}$ denote the boundary of $\mathcal{S}$ ($\partial\mathcal{S} = \emptyset$ if $\mathcal{S}$ is closed) and $\gamma = (\gamma_1, ..., \gamma_n)$ stand for the outward unit vector to $\partial\mathcal{S}$, relative to $\mathcal{S}$.

Hereafter, we let dS and ds denote the 'volume' element on $\mathcal{S}$ and $\partial\mathcal{S}$, respectively.
The typical situation is when $\mathcal{S}$ is an open region of the boundary $\partial\Omega$ of a smoothly bounded domain Ω in $\mathbb{R}^n$.

The following propositions describe an extension of the unit normal to a hypersurface enjoying a number of properties that will prove very useful in the sequel. Before stating our first result, we remind the reader that

$$\pi : \mathbb{R}^n \longrightarrow T\mathcal{S}, \quad \pi = (\delta_{jk} - N_j N_k)_{j,k}, \tag{3.1}$$

denotes the canonical orthogonal projection onto the tangent space to $\mathcal{S}$.

Proposition 3.1 *For any unitary extension $\nu \in C^1(\mathcal{U})$ of N, in a neighborhood $\mathcal{U}$ of $\mathcal{S}$, the following conditions are equivalent:*

(i) $\nabla_\nu \nu = 0$ *on $\mathcal{S}$, i.e., $\partial_\nu \nu_j = 0$ on $\mathcal{S}$ for $j = 1, 2, ..., n$;*

(ii) $d\nu = 0$ *on $\mathcal{S}$, i.e., $\partial_k \nu_j - \partial_j \nu_k = 0$ on $\mathcal{S}$, for $k,\ j = 1,\ 2, \ldots, n$.*

P r o o f. That $(ii) \Rightarrow (i)$ follows readily by writing

$$\langle \nabla_\nu \nu, X \rangle = 2\, d\nu(\nu, X) + \langle \nabla_X \nu, \nu \rangle = 2\, d\nu(\nu, X) + \tfrac{1}{2} \nabla_X \|\nu\|^2$$

for any vector field X in $\mathbb{R}^n$.

As for the opposite implication, we first observe that, in general,

$$\nabla_\nu \nu \Big|_{\mathcal{S}} = 0 \quad \& \quad \nu \Big|_{\mathcal{S}} = N \implies \nabla \nu \Big|_{\mathcal{S}} \text{ depends only on } \mathcal{S} \text{ and not on } \nu. \tag{3.2}$$

Indeed, for any field X,

$$\nabla_X \nu \Big|_{\mathcal{S}} = \nabla_{\pi X}\, \nu \Big|_{\mathcal{S}} + \langle X, \nu \rangle \nabla_\nu \nu \Big|_{\mathcal{S}} = \nabla_{\pi X} N \quad \text{on } \mathcal{S},$$

since $\nabla_{\pi X}$ is a tangential derivative operator.

In particular, given a field ν which satisfies

$$\|\nu\| = 1 \quad \text{near} \quad \mathcal{S}, \quad \nu = N \quad \text{on} \quad \mathcal{S}, \quad \text{and} \quad \nabla_\nu \nu = 0 \quad \text{on} \quad \mathcal{S}, \tag{3.3}$$

it follows that $d\nu\Big|_{\mathcal{S}}$ depends intrinsically on the hypersurface $\mathcal{S}$.

Let us now consider a particular extension. To set the stage, let $r : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^k with the property that $r = 0$ on $\mathcal{S}$ and $dr \neq 0$ in some neighborhood $\mathcal{U}$ of $\mathcal{S}$. For example, r can be taken to be the "signed" distance to $\mathcal{S}$, defined as $\mathrm{dist}\,(x, \mathcal{S})$ for x above $\mathcal{S}$ and $-\mathrm{dist}\,(x, \mathcal{S})$ for x below $\mathcal{S}$, first in some neighborhood of $\mathcal{S}$ then extended to the whole $\mathbb{R}^n$.

In particular, $dr/\|dr\|$ is a unitary extension of N, the normal to $\mathcal{S}$. Upon noting that $dr/\|dr\| = d(r/\|dr\|)$ on $\mathcal{S}$, we finally, take

$$\nu := d\Big(\tfrac{r}{\|dr\|}\Big)\Big/\Big\|d\Big(\tfrac{r}{\|dr\|}\Big)\Big\|$$

in some neighborhood $\mathcal{U}$ of $\mathcal{S}$. Clearly, $\|\nu\| = 1$ in $\mathcal{U}$ and

$$d\nu = d^2\Big(\tfrac{r}{\|dr\|}\Big)\Big/\Big\|d\Big(\tfrac{r}{\|dr\|}\Big)\Big\| - d\Big(\tfrac{r}{\|dr\|}\Big)\wedge d\,\Big\|d\Big(\tfrac{r}{\|dr\|}\Big)\Big\|^{-1}.$$

Recall now that $d^2 = 0$ in $\mathbb{R}^n$ so that the first term in the right-side above is zero. Next, $d(r/\|dr\|) = N$ on $\mathcal{S}$, so that $\Big\|d\Big(\tfrac{r}{\|dr\|}\Big)\Big\|^{-1} = 1$ on $\mathcal{S}$. Since the differential operator $N\wedge d = \sum_{j<k}(N_j\partial_k - N_k\partial_j)dx_j\wedge dx_k$ contains only tangential derivatives (cf. (4.10) and the discussion in §5), it follows that the second term in the right-side above is also zero. All in all, $d\nu = 0$ on $\mathcal{S}$.

In particular, by the implication already proved (i.e. $(ii)\Rightarrow(i)$) it follows that ν satisfies (i). Hence, since this particular field satisfies $d\nu = 0$ on $\mathcal{S}$, the above discussion implies that any other extension of N as in (3.3) has this property.

The proof of the proposition is therefore finished. □

Remark. A unit vector field ν, $\|\nu(x)\| = 1$ for all $x\in\Omega\subset\mathbb{R}^n$, is called *integrable* if there exists a family of surfaces filling up Ω which are orthogonal to the given vector field ν.

In [MaMi, § 1.1.4] it is proved that the necessary and sufficient condition for $\nu = \big(\nu_1,\dots,nu_n\big)^\top$ to be integrable is the symmetry of the matrix

$$\mathcal{R}(x) := \nabla\nu(x) = (\mathcal{D}_k\nu_j(x))_{j,k}\,, \tag{3.4}$$

where $\mathcal{D}_k := \partial_k - \nu_k\nu\cdot\nabla$ are the Gunter's derivatives (cf. the forthcoming § 4). Note, that the matrix $\mathcal{R}(x)$ coincides with

$$R(x) := \nabla\nu(x) = (\partial_k\nu_j(x))_{j,k}, \qquad x\in\mathcal{U} \tag{3.5}$$

(see § 4) and Proposition 3.1 is not a direct consequence of the formulated result, although they are related.

Concerning the particular extension of the normal vector field with the distance function, exploited above. This extension is well known and used often to define non-smooth (simply integrable) mean curvature to a rough surface (cf. (3.6)). We have exposed the proof of this part just for the convenience of a reader.

In the sequel, given a hypersurface $\mathcal{S}$ and an extended unit vector ν in a neighborhood $\mathcal{U}$ of $\mathcal{S}$, we shall tacitly assume that the projection π has been extended to $\mathcal{U}$ by setting $\pi = I - \langle\cdot,\nu\rangle\nu = \nu\vee(\nu\wedge\cdot)$.

The forthcoming Proposition 3.2 and Proposition 3.4 are folklore. Being unable to find a comprehensive reference to their proofs in literature, we expose short proofs for the convenience of a reader.

Proposition 3.2 *For any unitary extension $\nu\in C^1(\mathcal{U})$ of N, in a neighborhood $\mathcal{U}$ of the hypersurface $\mathcal{S}$, the quantity* $\operatorname{div}\nu|_{\mathcal{S}}$ *depends only on $\mathcal{S}$ and not on the particular ν itself. In fact,*

$$\mathcal{G} := \operatorname{div}\nu \quad \textit{satisfies} \quad \mathcal{G}\Big|_{\mathcal{S}} = (n-1)\mathcal{H}, \tag{3.6}$$

where $\mathcal{H}$ stands for the mean curvature of $\mathcal{S}$.

P r o o f. Fix a local ortho-normal frame $T_1,\dots,T_{n-1}$ in some open subset $\mathcal{O}$ of $\mathcal{S}$. In particular,

$$T_1,\dots,T_{n-1},\nu \quad \text{is a local orthonormal basis for} \quad \mathbb{R}^n \tag{3.7}$$

at each point on $\mathcal{O}$. Next, recall that

$$\operatorname{div}\nu\Big|_{\mathcal{O}} = \operatorname{Tr}(\nabla\nu)\Big|_{\mathcal{O}} = \sum_{j=1}^{n}\langle\nabla_{T_j}\nu, T_j\rangle|_{\mathcal{O}} = \sum_{j=1}^{n-1}\langle\nabla_{T_j}N, T_j\rangle \text{ on } \mathcal{O},$$

since $\langle \nabla_\nu \nu, \nu\rangle = \frac{1}{2}\nabla_\nu \|\nu\|^2 = 0$ in $\mathcal{U}$. Now, the last expression above is clearly independent of ν. Since $\mathcal{O}$ is arbitrary, this justifies the first claim of the proposition.

To prove the second part, i.e. the identity (3.6), it suffices to perform calculations for a particular choice of the unitary extension ν of N. If, locally, $\mathcal{S}$ is given as the graph of a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, pick

$$\nu(x) := \frac{(\nabla\varphi(x'), -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}, \qquad \forall\, x = (x', x_n) \in \mathbb{R}^n \equiv \mathbb{R}^{n-1} \times \mathbb{R}.$$

Then

$$\Big(\mathrm{div}\, \nu\Big)(x', \varphi(x')) = \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j}\Big[\frac{\partial_j \varphi(x')}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}\Big], \quad \forall\, x' \in \mathbb{R}^{n-1}.$$

As is well-known, the right hand-side above is $n-1$ times the mean-curvature at the point $(x', \varphi(x'))$ on the graph of φ. $\square$

Definition 3.3 Let $\mathcal{S}$ be a hypersurface in $\mathbb{R}^n$ with unit normal N. A vector filed $\nu \in C^1(\mathcal{U})$, where $\mathcal{U}$ is a neighborhood of $\mathcal{S}$, will be called an **extended unit field** for $\mathcal{S}$ if $\nu\Big|_{\mathcal{S}} = N$, $\|\nu\| = 1$ in $\mathcal{U}$, and if ν satisfies either one of the (equivalent) conditions listed in Proposition 3.1.

It is implicit in the course of the proof of Proposition 3.1 that each C^k, $k \geq 2$, hypersurface $\mathcal{S}$ has a C^{k-1} extended unit field (which, nonetheless, is not unique).

Proposition 3.4 *Let $\mathcal{S}$ be a hypersurface in $\mathbb{R}^n$ and fix an extended unit field ν in a neighborhood $\mathcal{U}$ of $\mathcal{S}$. Then, for the $n \times n$ matrix valued function $R(x)$ in (3.5) the following are true:*

(i) $R\nu = 0$ in $\mathcal{U}$;

(ii) Tr $(R) = \mathcal{G}$ in $\mathcal{U}$.

Moreover, when restricted to the hypersurface $\mathcal{S}$, R has the following additional properties:

(iii) R depends only on $\mathcal{S}$ and not on the choice of the extended unit ν.

(iv) $R^\top = R$ on $\mathcal{S}$;

(v) $(Ru)|_{\mathcal{S}}$ is tangential to $\mathcal{S}$ for any vector field $u : \mathcal{S} \to \mathbb{R}^n$. In fact,

$$R\Big|_{T\mathcal{S}} = -\mathcal{W} \tag{3.8}$$

the opposite of the Weingarten map of $\mathcal{S}$. In particular, the eigenvalues $\{\kappa_j\}_{1\leq j\leq n-1}$ of $-R$ (at points on $\mathcal{S}$) as an operator on $T\mathcal{S}$ are the principal curvatures of $\mathcal{S}$, whereas its determinant is Gauss's total curvature of $\mathcal{S}$;

Proof. First, $R\nu = \nabla\|\nu\|^2 = 0$ in $\mathcal{U}$, justifying (i). Next, (ii) follows from (3.6) and (3.5), whereas $(iii) - (iv)$ are direct consequences of (3.2) and (i) in Proposition 3.1. Next, the first part of(v) is a consequence (i) and (iii). As for (3.8), for each $X \in T\mathcal{S}$ we write

$$\mathcal{W}X = -\nabla^{\mathcal{S}}_X \nu = -\pi(\nabla_X \nu) = -\nabla_X \nu = -RX$$

since, as we have just seen, $\nabla_X \nu = RX$ is tangential to $\mathcal{S}$. $\square$

Remark. If $v_j \in T\mathcal{S}$, $1 \leq j \leq n-1$, form an orthonormal system of $T\mathcal{S}$ and are eigenvectors for the matrix $-R$, i.e. $-Rv_j = \kappa_j v_j$, $1 \leq j \leq n-1$, it follows that

$$\partial_{v_j} \nu = -\kappa_j\, v_j, \qquad 1 \leq j \leq n-1.$$

4 Calculus of tangential differential operators

Let

$$Pu = \Big(\sum_{j,\beta} a_j^{\alpha\beta}\partial_j u_\beta + \sum_\beta b^{\alpha\beta}u_\beta\Big)_\alpha \tag{4.1}$$

be a first-order differential operator acting on vector-valued functions $u = (u_\beta)_\beta$ in $\mathbb{R}^n$. Its adjoint (in $\mathbb{R}^n$) is then defined by

$$P^*v = \Big(-\sum_{j,\alpha}\partial_j(a_j^{\alpha\beta}v_\alpha) + \sum_\alpha b^{\alpha\beta}v_\alpha\Big)_\beta \tag{4.2}$$

and its symbol is given by the matrix-valued function

$$\sigma(P;\xi) := \Big(\sum_j a_j^{\alpha\beta}\xi_j\Big)_{\alpha\beta}, \qquad \xi = (\xi_j)_{1\leq j\leq n}. \tag{4.3}$$

Let henceforth $\mathcal{S}$ be a fixed, C^2-hypersurface in $\mathbb{R}^n$, with unit normal N.
We say that P is a **weakly tangential operator** to the hypersurface $\mathcal{S}$, with unit normal N, provided

$$\sigma(P;N) = 0 \quad \text{on the hypersurface } \mathcal{S}. \tag{4.4}$$

If $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain, and if P is a first-order operator, weakly tangential to $\partial\Omega$, then P can be integrated by parts over Ω without boundary terms, i.e.

$$\int_\Omega \langle Pu, v\rangle\, dx = \int_\Omega \langle u, P^*v\rangle\, dx, \tag{4.5}$$

for any C^1, vector-valued functions u, v in $\bar{\Omega}$.

Next, call P a **strongly tangential operator** to $\mathcal{S}$ provided there exists an extended unit field ν such that

$$\sigma(P;\nu) = 0 \quad \text{in an open neighborhood of } \mathcal{S} \text{ in } \mathbb{R}^n. \tag{4.6}$$

Since $\sigma(P^*;\xi) = -\sigma(P;\xi)^\top$, for each $\xi \in \mathbb{R}^n$, it follows that P is weakly/strongly tangential if and only if P^* is so.

Recall that dS, ds stand for the volume elements on $\mathcal{S}$, $\partial\mathcal{S}$, and that N, γ denote the outward unit vectors to $\mathcal{S}$ and its boundary $\partial\mathcal{S}$, respectively (Cf. § 3).

Theorem 4.1 *Let P be a first-order differential operator as in (4.1) with coefficients of class C^1 in $\mathbb{R}^n$. If P is weakly tangential to $\mathcal{S}$ then P extends uniquely to an operator (still denoted by P) which acts on C^1 vector-valued functions defined on $\mathcal{S}$. In fact, this extension satisfies*

$$(Pu)\Big|_{\mathcal{S}} = P\Big(u|_{\mathcal{S}}\Big) \tag{4.7}$$

for every C^1 function u defined in a neighborhood of $\mathcal{S}$. Furthermore, similar considerations apply to P^*, and

$$\begin{aligned}\int_{\mathcal{S}} \langle Pu, v\rangle\, dS &= \int_{\mathcal{S}} \langle u, P^*v\rangle\, dS + \sum_{j,\alpha,\beta}\int_{\mathcal{S}} (\partial_N a_j^{\alpha\beta})N_j u_\beta v_\alpha\, dS \\ &\quad + \oint_{\partial\mathcal{S}} \langle \sigma(P;\gamma)u, v\rangle\, ds,\end{aligned} \tag{4.8}$$

for any C^1, vector-valued functions u, v defined on $\mathcal{S}$.

If in fact P is strongly tangential to $\mathcal{S}$ then the following integration by parts formula holds:

$$\int_{\mathcal{S}} \langle Pu, v\rangle \, dS = \int_{\mathcal{S}} \langle u, P^* v\rangle \, dS + \oint_{\partial\mathcal{S}} \langle \sigma(P;\gamma)u, v\rangle \, ds, \tag{4.9}$$

for any C^1, vector-valued functions u, v defined on $\mathcal{S}$.

As a corollary, for a first-order differential operator P which is weakly tangential to a hypersurface $\mathcal{S}$, its transposed in $\mathbb{R}^n$ differs from its transposed in the sense of integration by parts along $\mathcal{S}$ by a zero-order term. In fact, this term disappears if P is strongly tangential to $\mathcal{S}$.

Let us consider the first-order, tangential differential operators (Stokes's derivatives)

$$\mathcal{M}_{jk} := N_j \partial_k - N_k \partial_j, \qquad 1 \le j, k \le n\,, \tag{4.10}$$

where, we remind, N is the unit normal to $\mathcal{S}$.

If ν is an extended unit field for $\mathcal{S}$, then each surface operator $\mathcal{M}_{jk}$ extends accordingly by setting $\mathcal{M}_{jk} = \nu_j \partial_k - \nu_k \partial_j$. In the sequel, we shall make no distinction between this operator in $\mathbb{R}^n$ and (4.10).

Lemma 4.2 *The following formulas hold:*
(i) $\mathcal{M}_{jk} = -\mathcal{M}_{kj}$, for each $1 \le j, k \le n$;
(ii) $\partial_k = \sum_{j=1}^n \nu_j \mathcal{M}_{jk} + \nu_k \partial_\nu$, for each $1 \le k \le n$;
(iii) $\sum_{k=1}^n \mathcal{M}_{jk}\nu_k = \nu_j \mathcal{G}$, for each $1 \le j \le n$.

P r o o f. The first two identities are immediate from definitions. To see the last one, we write

$$\sum_{k=1}^n \mathcal{M}_{jk}\nu_k = \sum_{k=1}^n (\nu_j \partial_k - \nu_k \partial_j)\nu_k = \nu_j \operatorname{div}\nu - \tfrac{1}{2}\partial_j(\|\nu\|^2) = \nu_j \mathcal{G},$$

as desired. □

Lemma 4.3 *For any C^1, real-valued functions f, g on $\mathcal{S}$ and any $1 \le j < k \le n$, there holds*

$$\int_{\mathcal{S}} \Big[(\mathcal{M}_{jk} f) g + f(\mathcal{M}_{jk} g)\Big] \, dS = \oint_{\partial\mathcal{S}} (N_j \gamma_k - N_k \gamma_j) f g \, ds. \tag{4.11}$$

P r o o f. Let $\iota : \mathcal{S} \hookrightarrow \mathbb{R}^n$, $\jmath : \partial\mathcal{S} \hookrightarrow \mathcal{S}$, and $i : \partial\mathcal{S} \hookrightarrow \mathbb{R}^n$ be the natural inclusion operators. In particular, $i = \iota \circ \jmath$. It is then essentially well-known that, with the superscript 'star' denoting pull-back,

$$\iota^*(dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_n) = (-1)^{j+1} N_j \, dS, \qquad j = 1, 2, ..., n, \tag{4.12}$$

and, for $1 \le j < k \le n$,

$$i^*(dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge \widehat{dx_k} \wedge ... \wedge dx_n) = (-1)^{j+k+1}(N_j \gamma_k - N_k \gamma_j) \, ds. \tag{4.13}$$

Here, as usual, the 'hat' symbol indicates omission. Now, if we set

$$\omega := (-1)^{j+k+1} f g \, dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge \widehat{dx_k} \wedge ... \wedge dx_n,$$

then

$$\begin{aligned} d\omega &= (-1)^k \partial_j(fg)\, dx_1 \wedge ... \wedge \widehat{dx_k} \wedge ... \wedge dx_n \\ &\quad -(-1)^j \partial_k(fg)\, dx_1 \wedge ... \wedge \widehat{dx_k} \wedge ... \wedge dx_n. \end{aligned} \tag{4.14}$$

Consequently,

$$d_{\mathcal{S}}\iota^*(\omega) = \iota^*(d_{\mathbb{R}^n}\omega) = [N_j\partial_k(fg) - N_k\partial_j(fg)]\, dS \tag{4.15}$$

whereas

$$\jmath^*(\iota^*\omega) = (\iota \circ \jmath)^*\omega = i^*\omega = (N_j\gamma_k - N_k\gamma_j) fg\, ds. \tag{4.16}$$

The desired conclusion then follows from Stokes's classical formula

$$\int_{\mathcal{S}} d_{\mathcal{S}}(\iota^*\omega) = \oint_{\partial\mathcal{S}} \jmath^*(\iota^*\omega) \tag{4.17}$$

with the help of (4.15), (4.16). □

After this preamble, we are ready to present the

P r o o f of Theorem 4.1. Fix two arbitrary C^1, vector-valued functions defined in a neighborhood of $\mathcal{S}$. For starters we note that, on $\mathcal{S}$,

$$\begin{aligned} (Pu)_\alpha &= \sum_{j,\beta} a_j^{\alpha\beta}\partial_j u_\beta + \sum_\beta b^{\alpha\beta}u_\beta \\ &= -\sum_{j,k,\beta} a_j^{\alpha\beta} N_k \mathcal{M}_{jk} u_\beta + \sum_{j,\beta} a_j^{\alpha\beta} N_j \partial_N u_\beta + \sum_\beta b^{\alpha\beta}u_\beta \\ &= -\sum_{j,k,\beta} a_j^{\alpha\beta} N_k \mathcal{M}_{jk} u_\beta + \sum_\beta b^{\alpha\beta}u_\beta + (\sigma(P;N)\nabla_N u)_\alpha \\ &= -\sum_{j,k,\beta} a_j^{\alpha\beta} N_k \mathcal{M}_{jk} u_\beta + \sum_\beta b^{\alpha\beta}u_\beta \end{aligned} \tag{4.18}$$

on account of the weak tangentiality of P. If we now take the last expression above as a definition of P on functions which are defined only on $\mathcal{S}$, then (4.7) holds.

Next, fix an extended unit field ν for $\mathcal{S}$. Using (4.11) and integrating by parts we get

$$\begin{aligned} \int_{\mathcal{S}} \langle Pu,\, v\rangle\, dS &= -\sum_{j,k}\sum_{\alpha,\beta}\int_{\mathcal{S}} a_j^{\alpha\beta}\nu_k(\mathcal{M}_{jk}u_\beta)v_\alpha\, dS + \sum_{\alpha,\beta}\int_{\mathcal{S}} b^{\alpha\beta}u_\beta v_\alpha\, dS \\ &= \sum_{j,k}\sum_{\alpha,\beta}\int_{\mathcal{S}} u_\beta[\mathcal{M}_{jk}(a_j^{\alpha\beta}\nu_k v_\alpha)]\, dS + \sum_{\alpha,\beta}\int_{\mathcal{S}} b^{\alpha\beta}u_\beta v_\alpha\, dS \\ &\quad -\oint_{\partial\mathcal{S}}\sum_{j,k}\sum_{\alpha,\beta}(\nu_j\gamma_k - \nu_k\gamma_j)a_j^{\alpha\beta}\nu_k u_\beta v_\alpha\, ds. \end{aligned} \tag{4.19}$$

In the boundary integral, we write

$$
\begin{aligned}
-\sum_{j,k}\sum_{\alpha,\beta}&[\nu_j\gamma_k a_j^{\alpha\beta}\nu_k u_\beta v_\alpha - \nu_k\gamma_j a_j^{\alpha\beta}\nu_k u_\beta v_\alpha] \\
&= -\langle\gamma,\nu\rangle\langle\sigma(P;\nu)u,v\rangle + \|\nu\|^2\langle\sigma(P;\gamma)u,v\rangle \\
&= \langle\sigma(P;\gamma)u,v\rangle, \qquad (4.20)
\end{aligned}
$$

since $\langle\gamma,\nu\rangle = 0$ on $\mathcal{S}$; this term is in agreement with (4.8).

As for the first surface integral in the rightmost expression in (4.19), use Leibnitz's rule to expand

$$
\begin{aligned}
\sum_{j,k}\sum_{\alpha,\beta} u_\beta[\mathcal{M}_{jk}(a_j^{\alpha\beta}\nu_k v_\alpha)] &= \sum_{j,k}\sum_{\alpha,\beta} u_\beta a_j^{\alpha\beta} v_\alpha \mathcal{M}_{jk}\nu_k + \sum_{j,k}\sum_{\alpha,\beta} u_\beta\nu_k\mathcal{M}_{jk}(a_j^{\alpha\beta}v_\alpha) \\
&= : I + II. \qquad (4.21)
\end{aligned}
$$

With regard to I above, we invoke (iii) in Lemma 4.2 to write

$$
I = \sum_{j,\alpha,\beta} u_\beta a_j^{\alpha\beta} v_\alpha \nu_j \mathcal{G} = \langle\sigma(P;\nu)u,v\rangle\mathcal{G} = 0, \qquad (4.22)
$$

once again due to (4.6). Turning our attention to II, recall from (ii) in Lemma 4.2 that $\sum_{k=1}^n \nu_k\mathcal{M}_{jk} = \nu_j\partial_\nu - \partial_j$. Thus,

$$
\begin{aligned}
II + \sum_{\alpha,\beta} b^{\alpha\beta}u_\beta v_\alpha &= -\sum_{j,\alpha,\beta} u_\beta\partial_j(a_j^{\alpha\beta}v_\alpha) + \sum_{j,\alpha,\beta} u_\beta\nu_j\partial_\nu(a_j^{\alpha\beta}v_\alpha) + \sum_{\alpha,\beta} b^{\alpha\beta}u_\beta v_\alpha \\
&= \langle u, P^*v\rangle + \sum_{j,\alpha,\beta}\Big[u_\beta\nu_j(\partial_\nu a_j^{\alpha\beta})v_\alpha + u_\beta\nu_j a_j^{\alpha\beta}\partial_\nu v_\alpha\Big] \\
&= \langle u, P^*v\rangle + \sum_{j,\alpha,\beta}\Big[u_\beta\partial_\nu(\nu_j a_j^{\alpha\beta})v_\alpha - u_\beta(\partial_\nu\nu_j)a_j^{\alpha\beta}v_\alpha\Big] \\
&\quad + \langle\sigma(P;\nu)u, \nabla_\nu v\rangle \\
&= \langle u, P^*v\rangle + \langle[\partial_\nu\sigma(P;\nu)]u,v\rangle - \langle\sigma(P;\nabla_\nu\nu)u,v\rangle \\
&\quad + \langle\sigma(P;\nu)u,\nabla_\nu v\rangle. \qquad (4.23)
\end{aligned}
$$

Thanks to the weak tangentiality of P, the last term in the last line above vanishes on $\mathcal{S}$; in fact, since $\nabla_\nu\nu = 0$ on $\mathcal{S}$ so does the next-to-the-last term. Finally, $\langle[\partial_\nu\sigma(P;\nu)]u,v\rangle = \sum_{j,\alpha,\beta}(\partial_N a_j^{\alpha\beta})N_j u_\beta v_\alpha$ on $\mathcal{S}$, after some simple algebra. Thus, ultimately,

$$
II + \sum_{\alpha,\beta} b^{\alpha\beta}u_\beta v_\alpha = \langle u, P^*v\rangle + \sum_{j,\alpha,\beta}(\partial_N a_j^{\alpha\beta})N_j u_\beta v_\alpha \quad \text{on} \quad \mathcal{S}.
$$

This, in concert with (4.19)-(4.20) and (4.22), finishes the proof of (4.8).

In the case when P is strongly tangential the identity also follows from what we have proved so far since, in this scenario, $\partial_\nu\sigma(P;\nu) = 0$ by definition. $\square$

Remark. By iteration, an identity similar in spirit to (4.9) holds for higher order differential operators which are successive compositions of first-order differential operators, strongly tangential to $\mathcal{S}$.

The Stokes derivative operators $\mathcal{M}_{jk} = \nu_j\partial_k - \nu_k\partial_j$, introduced in connection with a hypersurface $\mathcal{S}$ (with ν denoting an extended unit field for $\mathcal{S}$), are clearly strongly tangential to $\mathcal{S}$, since $\sigma(\mathcal{M}_{jk};\xi) = \nu_j\xi_k - \nu_k\xi_j$. As

$\mathcal{M}^*_{jk} = \partial_k(\nu_j\,\cdot\,) - \partial_j(\nu_k\,\cdot\,)$ which becomes $-\mathcal{M}_{jk}$ on $\mathcal{S}$, it follows that, *a posteriori*, Lemma 4.3 is a special case of Theorem 4.1.

In this connection, let us also point out that $\nu\wedge d$, acting on scalar functions on $\mathcal{S}$, is naturally identified with the skew-symmetric matrix whose entries are the Stokesian derivatives, in the sense that

$$\nu\wedge d = \tfrac{1}{2}\sum_{j,k=1}^{n}\mathcal{M}_{jk}\,dx_j\wedge dx_k = \sum_{1\leq j<k\leq n}\mathcal{M}_{jk}\,dx_j\wedge dx_k. \tag{4.24}$$

Of special interest for us in this paper are the so-called Günter derivatives

$$\mathcal{D}_j := \partial_j - \nu_j\partial_\nu = \partial_j - \sum_{k=1}^{n}\nu_j\nu_k\partial_k\,, \qquad j=1,2,...,n. \tag{4.25}$$

We set

$$\mathcal{D}f := \Big(\mathcal{D}_1 f, \mathcal{D}_2 f, ..., \mathcal{D}_n f\Big), \quad \text{for scalar-valued functions,} \tag{4.26}$$

$$\mathcal{D}\bullet u := \sum_{j=1}^{n}\mathcal{D}_j u_j, \quad \text{if} \quad u=(u_1,...,u_n). \tag{4.27}$$

For further reference, below we collect some of the most basic properties of this system of differential operators.

Proposition 4.4 *The following relations are valid:*
(i) $\mathcal{D}_j = \sum\limits_{k=1}^{n}\nu_k\mathcal{M}_{kj}$*, for each* $1\leq j\leq n$*;*
(ii) $\mathcal{M}_{jk} = \nu_j\mathcal{D}_k - \nu_k\mathcal{D}_j$ *for each* $1\leq j,k\leq n$*;*
(iii) $\sum\limits_{j=1}^{n}\nu_j\mathcal{D}_j = 0$ *and* $\sum\limits_{j=1}^{n}\mathcal{D}_j\nu_j = \mathcal{G}$*.*
(iv) $[\mathcal{D}_j,\mathcal{D}_k] = \nu_j(\nabla\nu_k\bullet\nabla) - \nu_k(\nabla\nu_j\bullet\nabla)$ *on* $\mathcal{S}$*, for each* $1\leq j,k\leq n$*;*
(v) for every C^1 *functions* f,g *on* $\mathcal{S}$*, and every* $1\leq j\leq n$*,*

$$\int_{\mathcal{S}}(\mathcal{D}_j f)g\,dS = \int_{\mathcal{S}}\Big[-f(\mathcal{D}_j g) + N_j\mathcal{G}fg\Big]\,dS + \oint_{\partial\mathcal{S}}\gamma_j fg\,ds. \tag{4.28}$$

P r o o f. The first three identities are simple consequences of definitions. To prove (iv) we first note that

$$\begin{aligned}\mathcal{D}_j\mathcal{D}_k &= (\partial_j-\nu_j\partial_\nu)(\partial_k-\nu_k\partial_\nu)\\ &= \partial_j\partial_k - (\partial_j\nu_k)\partial_\nu - \sum_{l=1}^{n}\Big[\nu_k(\partial_j\nu_l)\partial_l + \nu_k\nu_l\partial_j\partial_l + \nu_j\nu_l\partial_l\partial_k\Big] + \nu_j\nu_k\partial_\nu^2\end{aligned} \tag{4.29}$$

where the second equality utilizes the fact that $\partial_\nu\nu_k = 0$ on $\mathcal{S}$ (cf. (i) in Proposition 3.1). If we now observe, with the aid of (ii) in Proposition 3.1, that the expression

$$\partial_j\partial_k - (\partial_j\nu_k)\partial_\nu - \sum_{l=1}^{n}\Big[\nu_k\nu_l\partial_j\partial_l + \nu_j\nu_l\partial_l\partial_k\Big] + \nu_j\nu_k\partial_\nu^2$$

is symmetric in j and k, then the desired commutator identity follows from (4.29).

Turning to (v), note that $\mathcal{D}_j$ is a first-order differential operator defined in a neighborhood of $\mathcal{S}$ in $\mathbb{R}^n$, and whose principal symbol is $\sigma(\mathcal{D}_j;\xi) = \xi_j - \nu_j\langle\xi,\nu\rangle$, for $\xi \in \mathbb{R}^n$. In particular, $\mathcal{D}_j$ is strongly tangential to $\mathcal{S}$, and $\sigma(\mathcal{D}_j;\gamma) = \gamma_j$. Thus, (4.28) will follow from Theorem 4.1 as soon as the transposed of $\mathcal{D}_j$ in $\mathbb{R}^n$ is properly identified. To this end, we compute

$$(\mathcal{D}_j)^* = \Big(\partial_j - \nu_j\sum_{k=1}^n \nu_k\partial_k\Big)^* = -\partial_j + \sum_{k=1}^n \partial_k\Big[(\nu_k\nu_j)\cdot\Big] = -\mathcal{D}_j + \nu_j\mathcal{G} + \partial_\nu\nu_j \tag{4.30}$$

and observe that, when further restricted to $\mathcal{S}$, the last term vanishes, as desired. □

Other important examples of strongly tangential, first-order differential operators are offered by

$$\begin{aligned}
&P_1 := \operatorname{div} - \partial_\nu\langle\cdot,\nu\rangle, \quad \text{with} \quad P_1^* = -\nabla + (\partial_\nu\,\cdot\,)\nu + \mathcal{G}\nu,\\
&P_2 := \nabla_\nu\pi - \nu\vee d, \quad \text{with} \quad P_2^* = -\pi\nabla_\nu + \mathcal{G}\pi - \delta(\nu\wedge\,\cdot\,),\\
&P_3 := \operatorname{div}\pi(\cdot), \quad \text{with} \quad P_3^* = -\pi\nabla.
\end{aligned} \tag{4.31}$$

Indeed,

$$\sigma(P_1;\xi) = \langle\xi,\cdot\rangle - \langle\nu,\xi\rangle\langle\nu,\cdot\rangle, \quad \sigma(P_2;\xi) = \langle\xi,\nu\rangle\,\pi - \nu\vee(\xi\wedge\cdot) \quad \text{and} \quad \sigma(P_3;\xi) = \langle\xi,\pi(\cdot)\rangle,$$

so that (4.6) is easily verified in each case.

We are interested to express these operators in terms of the Günter derivatives (4.25) rather than the ordinary ∂_j's. A general result to this effect is as follows.

Proposition 4.5 *Let*

$$Pu = \Big(\sum_{j,\beta} a_j^{\alpha\beta}\partial_j u_\beta + \sum_\beta b^{\alpha\beta}u_\beta\Big)_\alpha \tag{4.32}$$

be a first-order differential operator which is strongly tangential to a hypersurface $\mathcal{S}$. Then P remains unchanged (in a neighborhood of $\mathcal{S}$) if one formally replaces ∂_j by $\mathcal{D}_j$, i.e.

$$Pu = \Big(\sum_{j,\beta} a_j^{\alpha\beta}\mathcal{D}_j u_\beta + \sum_\beta b^{\alpha\beta}u_\beta\Big)_\alpha. \tag{4.33}$$

P r o o f. The two right sides in (4.32) and (4.33) differ by $\sigma(P;\nu)(\nabla_\nu u)$. □

When used in conjunction with the operators (4.31), the above proposition gives

Proposition 4.6 *For an arbitrary vector field u we have*

$$\begin{aligned}
&P_1 u = \mathcal{D}\bullet(\pi\,u) + \mathcal{G}\langle\nu,u\rangle - \langle Ru,\nu\rangle,\\
&P_2 u = \mathcal{D}(\langle\nu,u\rangle) + \langle\nu,u\rangle\nabla_\nu\nu - R^\top u,\\
&P_3 u = \mathcal{D}\bullet(\pi\,u) - \langle Ru,\nu\rangle,
\end{aligned} \tag{4.34}$$

where $R := \Big(\partial_k\nu_j\Big)_{j,k}$ has been introduced in Proposition 3.4.

P r o o f. These follow by invoking Proposition 4.5 plus a straightforward calculation. In the case of P_2, it helps to first notice that, if $u = (u_1, ..., u_n) \equiv u_1 dx_1 + ... + u_n dx_n$, then

$$\begin{aligned} \nu \vee du &= \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} \nu_k \partial_k u_j \Big) dx_j - \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} \nu_k \partial_j u_k \Big) dx_j \\ &= \nabla_\nu u - \nabla \langle \nu, u \rangle + R^\top u. \end{aligned} \tag{4.35}$$

We leave the details to the interested reader. □

Remark. Although the tangential derivatives $\mathcal{M}_{jk}$ and $\mathcal{D}_j$ are known for a long time ($\mathcal{M}_{jk}$ was introduced by Stokes and $\mathcal{D}_j$ by the soviet mechanist Günter in [Gu]) and were applied by several authors (see [Ce, Co, DL, Gu, KGBB, MaMi, NDS, Ne] etc.), we have not found most of the properties listed above in the literature. In [Gu, KGBB] these differential operators, in 3D case, are applied to problems of mechanics, while in [MaMi] to differential geometry and minimal surfaces.

5 Expressing surface differential operators in spatial coordinates

Throughout this section we shall regard the hypersurface $\mathcal{S}$ as a Riemannian manifold with the natural metric inherited from $\mathbb{R}^n$. The goal is to describe the action of $\mathrm{div}_{\mathcal{S}}$ on $T\mathcal{S}$, as well as that of $\mathrm{grad}_{\mathcal{S}}$ and $\Delta_{\mathcal{S}}$ on scalar functions on $\mathcal{S}$, it terms of the Günter derivatives (4.25). Our main result in this regard is as follows.

Theorem 5.1 *For any smooth, tangential field $u = (u_1, ..., u_n)$ on $\mathcal{S}$ we have*

$$\mathrm{div}_{\mathcal{S}}\, u = \mathcal{D} \bullet u = \sum_{j=1}^{n} \mathcal{D}_j u_j. \tag{5.1}$$

Also, for any smooth, real-valued function f on $\mathcal{S}$,

$$\mathrm{grad}_{\mathcal{S}}\, f = \mathcal{D} f = \Big(\mathcal{D}_1 f, \mathcal{D}_2 f, ..., \mathcal{D}_n f \Big). \tag{5.2}$$

In particular, the Laplace-Beltrami operator $\Delta_{\mathcal{S}}$ on $\mathcal{S}$ takes the form

$$\Delta_{\mathcal{S}}\, f = \mathrm{div}_{\mathcal{S}} \mathrm{grad}_{\mathcal{S}}\, f = \mathcal{D} \bullet \Big(\mathcal{D} f \Big) = \sum_{j=1}^{n} \mathcal{D}_j^2 f. \tag{5.3}$$

P r o o f. The operators in question have local character, so it suffices to carry out calculations in some fixed, small open subset $\mathcal{O}$ of $\mathcal{S}$. With π denoting the orthogonal projection onto $T\mathcal{S}$, we consider the $n \times n$ matrix $A(u)$ uniquely defined by the requirement that

$$\langle A(u) X, Y \rangle = \langle \nabla_{\pi X}\, u, \pi Y \rangle, \qquad \forall\, X, Y \in \mathbb{R}^n. \tag{5.4}$$

Also, fix an ortho-normal frame $T_1, \dots, T_{n-1}$ to $T\mathcal{S}$ in $\mathcal{O}$, so that

$$T_1, \dots, T_{n-1}, \nu \quad \text{is an orthonormal basis for} \quad \mathbb{R}^n \tag{5.5}$$

at each point in $\mathcal{O}$. Thus, if we set $T_n := \nu$ then the vectors $T_1, \dots, T_n$ form an ortho-normal basis in $\mathbb{R}^n$, when evaluated at points in $\mathcal{O}$.

Going further, if $\nabla^{\mathcal{S}}$ stands for the Levi-Civita connection on $\mathcal{S}$ then, according to (1.23), $\nabla^{\mathcal{S}}_X Y = \pi(\nabla_X Y)$ for any $X, Y \in T\mathcal{S}$. For any $u \in T\mathcal{S}$ supported in $\mathcal{O}$, we may therefore write

$$\mathrm{div}_{\mathcal{S}} u = \sum_{j=1}^{n-1} \langle \nabla^{\mathcal{S}}_{T_j} u, T_j \rangle = \sum_{j=1}^{n} \langle \nabla_{T_j} u, T_j \rangle = \sum_{j=1}^{n} \langle A(u) T_j, T_j \rangle. \tag{5.6}$$

At this point we claim that the last term in (5.6) does not depend on the particular basis of $\mathbb{R}^n$. Indeed, this is folklore and, for the reader's convenience, a couple of such statements are collected below.

Lemma 5.2 *If A is a $n \times n$ matrix with real entries then*

$$\mathit{Tr}\, A = \sum_{j=1}^{n} \langle A T_j, T_j \rangle \tag{5.7}$$

for any orthonormal basis $\{T_j\}_j$ in $\mathbb{R}^n$. In particular, the sum in the right side above is independent of the particular orthonormal basis $\{T_j\}_j$.

Also, if B is another $n \times n$ matrix with real entries, then

$$\mathit{Tr}\,(AB^\top) = \sum_{j,k=1}^{n} \langle A T_j, T_k \rangle \langle B T_j, T_k \rangle \tag{5.8}$$

is independent of the particular orthonormal basis $\{T_j\}_j$.

Returning to the task of carrying on the calculation initiated in (5.6) we denote by $e_j := (0, \ldots, 1, \ldots, 0)$, for $j = 1, \ldots, n$, the usual canonical basis in $\mathbb{R}^n$, and by

$$\bar{e}_j := \pi e_j = e_j - \nu_j \nu = \sum_{k=1}^{n} (\delta_{jk} - \nu_j \nu_k) e_k, \qquad j = 1, \ldots, n. \tag{5.9}$$

the projection of e_j onto $T\mathcal{S}$. One simple yet important observation is the fact that the Günter operators are directional derivatives corresponding to the $\bar{e}_j$'s, i.e.

$$\mathcal{D}_j = \nabla_{\bar{e}_j}, \qquad j = 1, 2, ..., n.$$

Relying on these observations, we may now continue –invoking Lemma 5.2– with

$$\begin{aligned} \mathrm{div}_{\mathcal{S}} u &= \sum_{j=1}^{n} \langle A(u) e_j, e_j \rangle = \sum_{j=1}^{n} \langle \nabla_{\bar{e}_j} u, \bar{e}_j \rangle = \sum_{j,k=1}^{n} (\mathcal{D}_j u_k) \langle e_k, \bar{e}_j \rangle \\ &= \sum_{j=1}^{n} \mathcal{D}_j u_j - \sum_{j,k=1}^{n} \nu_j (\mathcal{D}_j u_k) \nu_k = \sum_{j=1}^{n} \mathcal{D}_j u_j, \end{aligned} \tag{5.10}$$

justifying (5.1).

Turning attention to the operator $\mathrm{grad}_{\mathcal{S}}$, we note that if f is scalar and $u \in T\mathcal{S}$, both smooth and supported away from $\partial\mathcal{S}$, then

$$\begin{aligned}\int_{\mathcal{S}}\langle \mathrm{grad}_{\mathcal{S}} f, u\rangle\, dS &= -\int_{\mathcal{S}} f \,\mathrm{div}_{\mathcal{S}} u\, dS = -\int_{\mathcal{S}} f\sum_{j=1}^{n}\mathcal{D}_j u_j \\ &= -\int_{\mathcal{S}}\sum_{j=1}^{n}(\mathcal{D}_j^* f)u_j\, dS = \int_{\mathcal{S}}\sum_{j=1}^{n} u_j\mathcal{D}_j f\, dS. \\ &= \int_{\mathcal{S}}\langle \mathcal{D}f, u\rangle\, dS. \end{aligned} \tag{5.11}$$

Here we have used (4.28) and the tangentiality of u. Since both $\mathrm{grad}_{\mathcal{S}} f$ and $\mathcal{D}f$ are tangential, it follows that (5.11) holds for arbitrary smooth vector fields $u : \mathcal{S} \to \mathbb{R}^n$ (not just tangential). We may therefore conclude that $\mathrm{grad}_{\mathcal{S}} f = \mathcal{D}f$, as desired.

Finally, (5.3) follows from (1.3) and what we have proved so far. This finishes the proof of the theorem. □

Define

$$(\partial_N^2 f)|_{\mathcal{S}} := \sum_{j,k} N_j N_k(\partial_j\partial_k f)|_{\mathcal{S}} = (\partial_\nu^2 f)|_{\mathcal{S}}. \tag{5.12}$$

Corollary 5.3 *For any smooth scalar function f, defined in a neighborhood of $\mathcal{S}$, there holds*

$$\left(\Delta_{\mathbb{R}^n} f\right)\Big|_{\mathcal{S}} = \Delta_{\mathcal{S}}(f|_{\mathcal{S}}) + \mathcal{G}(\partial_N f)|_{\mathcal{S}} + (\partial_N^2 f)|_{\mathcal{S}}. \tag{5.13}$$

To keep matters in perspective, it is illuminating to work out in detail the case when $\mathcal{S} = S^{n-1}$, the unit sphere in $\mathbb{R}^n$. In this scenario, one can choose $\nu(x) := x/\|x\|$, $x \in \mathbb{R}^n \setminus 0$, so that $\mathcal{G} := \mathrm{div}\,\nu = (n-1)/\|x\|$, and $\partial_\nu = \sum(x_j/\|x\|)\partial_j = \partial_r$, the radial derivative in $\mathbb{R}^n$. Then (5.13) becomes, after a rescaling, the classical formula

$$\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}.$$

P r o o f. The identity (5.13) follows by expanding

$$\Delta_{\mathcal{S}} = \sum_{j=1}^{n}\mathcal{D}_j^2 = \sum_{j=1}^{n}(\partial_j - \nu_j\partial_\nu)(\partial_j - \nu_j\partial_\nu)$$

and performing straightforward algebraic manipulations based on Proposition 3.1. We omit the straightforward details. □

Corollary 5.4 *For any smooth scalar function f, and any smooth vector field u defined in a neighborhood of $\mathcal{S}$, the following identities hold:*

$$\mathrm{grad}_{\mathcal{S}}\left(\langle N, u|_{\mathcal{S}}\rangle\right) = [\nabla_\nu(\pi\, u)]|_{\mathcal{S}} - N \vee (du)|_{\mathcal{S}} + R(u|_{\mathcal{S}}), \tag{5.14}$$

$$(\mathrm{div}\, u)\big|_{\mathcal{S}} = \mathrm{div}_{\mathcal{S}}(\pi\, u|_{\mathcal{S}}) + \mathcal{G}\langle u|_{\mathcal{S}}, N\rangle + \langle(\nabla_N u)|_{\mathcal{S}},\, N\rangle, \tag{5.15}$$

$$\mathrm{div}_{\mathcal{S}}(\pi\, u|_{\mathcal{S}}) = \mathrm{div}(\pi\, u)|_{\mathcal{S}}, \tag{5.16}$$

$$\mathrm{grad}_{\mathcal{S}}(f|_{\mathcal{S}}) = [\pi\nabla f]|_{\mathcal{S}}, \tag{5.17}$$

where R is as in (3.5).

P r o o f. To justify (5.14)-(5.16), in the light of (4.7), we simply point to the alternative representations of the operators P_1, P_2, P_3 in (4.31) and (4.34), keeping in mind that $\nabla_\nu \nu = 0$ on $\mathcal{S}$. Finally, (5.17) is the adjoint of (5.16). □

Corollary 5.5 *For the Stokes derivatives from (4.10), here holds*

$$\tfrac{1}{2}\sum_{j,k=1}^{n} \mathcal{M}_{jk}^2 = \sum_{1\leq j<k\leq n} \mathcal{M}_{jk}^2 = \Delta_{\mathcal{S}} \quad \textit{on} \quad \mathcal{S}. \tag{5.18}$$

P r o o f. Denote by Q the second-order differential operator in the left side of (5.18), and fix two scalar functions f, g on $\mathcal{S}$, supported away from the boundary. We may then write

$$\begin{aligned}\int_{\mathcal{S}} (Qf)g\,dS &= -\sum_{1\leq j<k\leq n} \int_{\mathcal{S}} (\mathcal{M}_{jk}f)(\mathcal{M}_{jk}g)\,dS = -\int_{\mathcal{S}} \langle \nu\wedge df, \nu\wedge dg\rangle\,dS \\ &= \int_{\mathcal{S}} [\mathrm{div}\,(\pi\nabla f)]g\,dS \end{aligned}\tag{5.19}$$

thanks to (4.11), (4.24), and the fact that $P := \nu \wedge d$ is strongly tangential to $\mathcal{S}$, with adjoint $P^* = \delta(\nu \vee \cdot)$. Since f and g are arbitrary, it follows that

$$Q|_{\mathcal{S}} = \mathrm{div}\,(\pi\nabla\cdot)|_{\mathcal{S}} = \mathrm{div}_{\mathcal{S}}\,(\pi\nabla\cdot|_{\mathcal{S}}) = \mathrm{div}_{\mathcal{S}}\,(\mathrm{grad}_{\mathcal{S}}\cdot|_{\mathcal{S}}) = \Delta_{\mathcal{S}} \quad \text{on} \quad \mathcal{S},$$

by (5.16)-(5.17) and (5.3). □

A number of related identities, at least for $n = 3$ and special extensions of the unit normal, can be found in [DL], [Ce], [Co], [KGBB], [NDS], [Ne] and the references therein.

Moreover, in [MaMi] it is proved that the solution to the Laplace-Beltrami equation

$$\nabla_{\mathcal{S}}\nu + c^2\nu = 0\,, \qquad c^2 := \sum_{j=1}^{n} \big(\mathcal{D}_j\nu_j\big)^2 = \sum_{j=1}^{n} \big(\partial_j\nu_j\big)^2 \tag{5.20}$$

describes a surface with constant mean curvature.

6 The identification of the surface Lamé operator and related PDO's

Recall from §3 that the geometric-differential definition of the deformation tensor $\mathrm{Def}_{\mathcal{S}}$ and the Lamé operator $\mathcal{L}$ on $\mathcal{S}$ are, respectively,

$$\mathrm{Def}_{\mathcal{S}}(X)(Y,Z) := \tfrac{1}{2}\{\langle \nabla^{\mathcal{S}}_Y X, Z\rangle + \langle \nabla^{\mathcal{S}}_Z X, Y\rangle\}, \qquad \forall\, X, Y, Z, \in T\mathcal{S}, \tag{6.1}$$

and

$$\mathcal{L} := -2\mu\,\mathrm{Def}^*_{\mathcal{S}}\mathrm{Def}_{\mathcal{S}} + \lambda\,\mathrm{grad}_{\mathcal{S}}\mathrm{div}_{\mathcal{S}}. \tag{6.2}$$

The main result in this section, dealing with the identification of Lamé operator (6.2), is as follows.

Theorem 6.1 *The following identities hold on $\mathcal{S}$:*

$$\mathcal{L} = \mu\,\pi\Delta_{\mathcal{S}} + (\lambda+\mu)\,\mathrm{grad}_{\mathcal{S}}\mathrm{div}_{\mathcal{S}} + \mu\,\mathcal{G}R = \mu\,\pi\,(\mathcal{D}\bullet\mathcal{D}) + (\lambda+\mu)\,\mathcal{D}(\mathcal{D}\bullet) + \mu\,\mathcal{G}R. \tag{6.3}$$

In particular, $\mathcal{L} : T\mathcal{S} \longrightarrow T\mathcal{S}$ is a second-order, strongly-elliptic, formally self-adjoint, differential operator on $\mathcal{S}$.

Proof. Given the local nature of the identities we seek to prove, it suffices to work locally, in a small open subset $\mathcal{O}$ of $\mathcal{S}$, where an ortho-normal frame $T_1, \dots, T_{n-1}$ to $T\mathcal{S}$ has been fixed. As before, we set $T_n := \nu$ so that $\{T_j\}_{1\leq j\leq n}$ is an ortho-normal basis for $\mathbb{R}^n$, at points in $\mathcal{O}$.

Next, fix u a tangent field to $\mathcal{S}$ supported in $\mathcal{O}$, and let $A(u) = (a_{jk}(u))_{j,k}$ be the $n \times n$ matrix uniquely defined by the requirement that

$$\langle A(u)X, Y\rangle := \mathrm{Def}_{\mathcal{S}}(u)(\pi X, \pi Y), \qquad \forall\, X, Y \in \mathbb{R}^n. \tag{6.4}$$

It is then clear that

$$[A(u)]^\top = A(u) \quad \text{and} \quad A(u)\nu = 0. \tag{6.5}$$

For each j, k we can write

$$\begin{aligned} a_{jk}(u) &= \langle A(u)e_k, e_j\rangle = \mathrm{Def}_{\mathcal{S}}(u)(\bar{e}_k, \bar{e}_j) \\ &= \tfrac{1}{2}\Big(\langle \nabla^{\mathcal{S}}_{\bar{e}_k} u, \bar{e}_j\rangle + \langle \nabla^{\mathcal{S}}_{\bar{e}_j} u, \bar{e}_k\rangle\Big) = \tfrac{1}{2}\big(\langle \nabla_{\bar{e}_k} u, \bar{e}_j\rangle + \langle \nabla_{\bar{e}_j} u, \bar{e}_k\rangle\big) \\ &= \tfrac{1}{2}\sum_{r=1}^{n}\Big[(\mathcal{D}_k u_r)(\delta_{jr} - \nu_j\nu_r) + (\mathcal{D}_j u_r)(\delta_{kr} - \nu_k\nu_r)\Big]. \end{aligned} \tag{6.6}$$

To further simplify (6.6) we note that, on $\mathcal{S}$,

$$\sum_{r=1}^{n}(\mathcal{D}_k u_r)\nu_j\nu_r = \mathcal{D}_k\Big(\sum_{r=1}^{n} u_r\nu_r\Big)\nu_j - \sum_{r=1}^{n} u_r(\mathcal{D}_k\nu_r)\nu_j = -\sum_{r=1}^{n} u_r(\partial_r\nu_k)\nu_j, \tag{6.7}$$

for every j, k. In the last step above we have used the fact that $\langle u, \nu\rangle = 0$ on $\mathcal{S}$, and (i) in Proposition 3.1. Combining (6.6) and (6.7) we eventually arrive at

$$a_{jk}(u) = \tfrac{1}{2}\Big[\mathcal{D}_k u_j + \mathcal{D}_j u_k + \nabla_u(\nu_j\nu_k)\Big]. \tag{6.8}$$

Now if v is also a smooth vector field, tangent to $\mathcal{S}$, we get –keeping in mind that $\mathrm{Def}_{\mathcal{S}}(u)$ is symmetric– that

$$\begin{aligned} &\langle \mathrm{Def}_{\mathcal{S}}(u), \mathrm{Def}_{\mathcal{S}}(v)\rangle = \mathrm{Tr}\,\Big(\mathrm{Def}_{\mathcal{S}}(u)\mathrm{Def}_{\mathcal{S}}(v)\Big) \\ &= \sum_{j,k=1}^{n-1} \mathrm{Def}_{\mathcal{S}}(u)(T_j, T_k)\mathrm{Def}_{\mathcal{S}}(v)(T_j, T_k) \\ &= \sum_{j,k=1}^{n-1} \langle A(u)T_j, T_k\rangle\langle A(v)T_j, T_k\rangle = \sum_{j,k=1}^{n} \langle A(u)T_j, T_k\rangle\langle A(v)T_j, T_k\rangle \\ &= \sum_{j,k=1}^{n} \langle A(u)e_j, e_k\rangle\langle A(v)e_j, e_k\rangle = \sum_{j,k=1}^{n} a_{jk}(u)a_{jk}(v). \end{aligned} \tag{6.9}$$

The second to the last equality in (6.9) relies on the second part of Lemma 5.2. Integrating, this leads to

$$
\begin{aligned}
&4\sum_{j,k=1}^{n}\int_{\mathcal{S}} a_{jk}(u)a_{jk}(v)\,dS \\
&\qquad = \int_{\mathcal{S}}\Big[\mathcal{D}_k u_j + \mathcal{D}_j u_k + \nabla_u(\nu_j\nu_k)\Big]\Big[\mathcal{D}_k v_j + \mathcal{D}_j v_k + \nabla_v(\nu_j\nu_k)\Big]\,dS. \qquad (6.10)
\end{aligned}
$$

To proceed, we first consider

$$
\begin{aligned}
&\int_{\mathcal{S}}\sum_{j,k=1}^{n}(\mathcal{D}_j u_k + \mathcal{D}_k u_j)(\mathcal{D}_j v_k + \mathcal{D}_k v_j)\,dS = 2\int_{\mathcal{S}}\sum_{j,k=1}^{n}\mathcal{D}_j^*(\mathcal{D}_j u_k + \mathcal{D}_k u_j)v_k\,dS \\
&= 2\int_{\mathcal{S}}\sum_{j,k=1}^{n}\Big[-v_k\mathcal{D}_j^2 u_k - v_k\mathcal{D}_j\mathcal{D}_k u_j + \mathcal{G}\nu_j(\mathcal{D}_j u_k)v_k + \mathcal{G}\nu_j(\mathcal{D}_k u_j)v_k\Big]\,dS \\
&=: 2(I + II + III + IV). \qquad (6.11)
\end{aligned}
$$

It is immediate that $I = -\int_{\mathcal{S}}\langle\Delta_{\mathcal{S}}u, v\rangle\,dS$, while $\sum\limits_{j=1}^{n}\nu_j\mathcal{D}_j = 0$ on $\mathcal{S}$ forces $III = 0$. Next we concentrate on IV. By using Proposition 3.1 and the tangentiality of u we get

$$
\begin{aligned}
IV &= \int_{\mathcal{S}}\mathcal{G}\sum_{k=1}^{n}v_k\left(\mathcal{D}_k\Big(\sum_{j=1}^{n}\nu_j u_j\Big) - \sum_{j=1}^{n}u_j(\partial_k\nu_j - \nu_k\partial_\nu\nu_j)\right)\,dS \\
&= -\int_{\mathcal{S}}\mathcal{G}\langle Ru, v\rangle\,dS. \qquad (6.12)
\end{aligned}
$$

As for II, we employ the commutator identity from (iv) in Proposition 4.4 plus the fact that u and v are tangential to write

$$
\begin{aligned}
\sum_{j,k=1}^{n}v_k\mathcal{D}_j\mathcal{D}_k u_j &= \sum_{j,k=1}^{n}v_k\mathcal{D}_k\mathcal{D}_j u_j + \sum_{j,k=1}^{n}v_k[\mathcal{D}_j, \mathcal{D}_k]u_j \\
&= \sum_{j,k=1}^{n}v_k\mathcal{D}_k\mathcal{D}_j u_j + \sum_{j,k,l=1}^{n}(\partial_k\nu_l)(\partial_l\nu_j)u_j v_k \qquad (6.13)
\end{aligned}
$$

on $\mathcal{S}$. Thus,

$$
\begin{aligned}
-II &= \int_{\mathcal{S}}\Big(\sum_{j,k=1}^{n}v_k\mathcal{D}_k\mathcal{D}_j u_j - \sum_{l,j,k=1}^{n}(\partial_k\nu_l)(\partial_l\nu_j)u_j v_k\Big)\,dS \\
&= \int_{\mathcal{S}}\langle\mathrm{grad}_{\mathcal{S}}\mathrm{div}_{\mathcal{S}}u, v\rangle\,dS - \int_{\mathcal{S}}\langle R^2 u, v\rangle\,dS. \qquad (6.14)
\end{aligned}
$$

At this point, we may therefore conclude that

$$\int_{\mathcal{S}}\sum_{j,k=1}^{n}(\mathcal{D}_j u_k+\mathcal{D}_k u_j)(\mathcal{D}_j v_k+\mathcal{D}_k v_j)\,dS$$
$$=2\int_{\mathcal{S}}\langle -\Delta_{\mathcal{S}}u-\mathrm{grad}_{\mathcal{S}}\mathrm{div}_{\mathcal{S}}u+R^2u-\mathcal{G}Ru,v\rangle\,dS. \tag{6.15}$$

We now proceed to analyze the remaining terms in (6.10). More precisely, we still have to take into account the terms containing either $\nabla_u(\nu_j\nu_k)$ or $\nabla_v(\nu_j\nu_k)$. We start with the identity

$$\sum_{j,k=1}^{n}(\mathcal{D}_k u_j)\nabla_v(\nu_j\nu_k)$$
$$=\sum_{j,k=1}^{n}\nu_k(\mathcal{D}_k u_j)\nabla_v\nu_j+\sum_{k=1}^{n}(\nabla_v\nu_k)\Big(\mathcal{D}_k\big(\sum_{j=1}^{n}u_j\nu_j\big)-\sum_{j=1}^{n}u_j\partial_k\nu_j\Big)$$
$$=-\sum_{k,j=1}^{n}(\nabla_v\nu_k)(\nabla_u\nu_k)=-\langle R^2u,v\rangle, \tag{6.16}$$

valid at points on $\mathcal{S}$. There are four such terms in (6.10), i.e. containing either $\nabla_u(\nu_j\nu_k)$ or $\nabla_v(\nu_j\nu_k)$, but not both. An inspection of the above calculation shows that, on $\mathcal{S}$, they are all equal to $-\langle R^2u,v\rangle$.

We are still left with computing

$$\sum_{j,k=1}^{n}\nabla_u(\nu_j\nu_k)\nabla_v(\nu_j\nu_k)=\sum_{j,k,r,l=1}^{n}\Big[u_r(\partial_r\nu_j)\nu_k+u_r(\partial_r\nu_k)\nu_j\Big]\Big[v_l(\partial_l\nu_j)\nu_k+v_l(\partial_l\nu_k)\nu_j\Big]$$
$$=2\langle R^2u,v\rangle+2\sum_{k,r,l=1}^{n}u_r(\partial_r\nu_k)v_l\nu_k\tfrac{1}{2}\partial_l\Big(\sum_{j=1}^{n}\nu_j^2\Big)=2\langle R^2u,v\rangle, \tag{6.17}$$

on $\mathcal{S}$. At this point we combine all the above to get

$$4\sum_{j,k=1}^{n}\int_{\mathcal{S}}a_{jk}(u)a_{jk}(v)\,dS=2\int_{\mathcal{S}}\langle -\Delta_{\mathcal{S}}u-\mathrm{grad}_{\mathcal{S}}\mathrm{div}_{\mathcal{S}}u-\mathcal{G}Ru,v\rangle\,dS. \tag{6.18}$$

Having deduced (6.18), we may now compute

$$\begin{aligned}4\int_{\mathcal{S}}\langle \mathrm{Def}_{\mathcal{S}}^*\mathrm{Def}_{\mathcal{S}}(u),v\rangle\,dS &= \int_{\mathcal{S}}\langle \mathrm{Def}_{\mathcal{S}}(u),\mathrm{Def}_{\mathcal{S}}(v)\rangle\,dS\\ &= 4\sum_{j,k=1}^{n}\int_{\mathcal{S}}a_{jk}(u)a_{jk}(v)\,dS \qquad (6.19)\\ &= 2\int_{\mathcal{S}}\langle -\Delta_{\mathcal{S}}u-\mathrm{grad}_{\mathcal{S}}\mathrm{div}_{\mathcal{S}}u-\mathcal{G}Ru\,,\,v\rangle\,dS.\end{aligned}$$

Thus,

$$4\,\mathrm{Def}_{\mathcal{S}}^*\mathrm{Def}_{\mathcal{S}}=-2\,\pi\Delta_{\mathcal{S}}-2\,\mathrm{grad}_{\mathcal{S}}\mathrm{div}_{\mathcal{S}}-2\,\mathcal{G}R, \tag{6.20}$$

since the tangential vectors fields u, v are arbitrary (note that here we use the fact that the image of R is a subspace of $T\mathcal{S}$).

The first identity in (6.3) now follows easily from (6.20) and (6.2). The remaining identity in (6.3) then follows from what we have just proved and Theorem 5.1. □

Next recall the definition of the Hodge-Laplacian acting on 1-forms, i.e.

$$\Delta_{HL} := -d_{\mathcal{S}} d_{\mathcal{S}}^* - d_{\mathcal{S}}^* d_{\mathcal{S}} : \Lambda^1 T\mathcal{S} \longrightarrow \Lambda^1 T\mathcal{S} \tag{6.21}$$

where $d_{\mathcal{S}}$ is the exterior derivative operator on $\mathcal{S}$, and $d_{\mathcal{S}}^*$ its formal adjoint. As explained in §2, 1-forms on $\mathcal{S}$ are naturally identified with tangent fields to $\mathcal{S}$ so, from now on, we shall think of Δ_{HL} as mapping $T\mathcal{S}$ into itself.

As pointed out in §2, the Hodge-Laplacian (6.21) is related to

$$\Delta_{BL} := -(\nabla^{\mathcal{S}})^* \nabla^{\mathcal{S}}, \tag{6.22}$$

the Bochner-Laplacian on $\mathcal{S}$, via the Weitzenbock identity

$$\Delta_{BL} = \Delta_{HL} + \mathrm{Ric}_{\mathcal{S}}. \tag{6.23}$$

Our aim is to find alternative expressions for all these objects, starting with the Ricci tensor.

Theorem 6.2 *On $\mathcal{S}$, there holds*

$$\mathrm{Ric}_{\mathcal{S}} = -R^2 + \mathcal{G}R. \tag{6.24}$$

In particular, when $n = 3$ –i.e. for a two-dimensional surface $\mathcal{S}$ in $\mathbb{R}^3$– the above identity reduces to

$$\mathrm{Ric}_{\mathcal{S}} = -\det \mathcal{W} = -\mathcal{K}, \tag{6.25}$$

where $\mathcal{K}$ is the Gaussian curvature of the surface $\mathcal{S}$.

Proof. Let us denote by $\mathcal{R}_{\mathcal{S}}$ the Riemann curvature tensor of $\mathcal{S}$. Since $\mathbb{R}^n$ has zero curvature, it follows from Gauss's Theorema Egregium that, if X, Y, Z, W are tangent vector fields to $\mathcal{S}$, then

$$\langle \mathcal{R}_{\mathcal{S}}(X,Y)Z, W\rangle = \langle II_{\mathcal{S}}(X,W), II_{\mathcal{S}}(Y,Z)\rangle - \langle II_{\mathcal{S}}(Y,W), II_{\mathcal{S}}(X,Z)\rangle. \tag{6.26}$$

See, e.g., [Ta2], Vol. II, p. 481. In this context, the second fundamental form of $\mathcal{S}$ becomes $II_{\mathcal{S}}(X,Y) = \nabla_X Y - \nabla_X^{\mathcal{S}} Y = \langle \nabla_X Y, \nu\rangle\nu$, by (1.23). Thus, on $\mathcal{S}$,

$$\begin{aligned} \langle \mathcal{R}_{\mathcal{S}}(X,Y)Z, W\rangle &= \langle \nabla_X W, \nu\rangle\langle \nabla_Y Z, \nu\rangle - \langle \nabla_Y W, \nu\rangle\langle \nabla_X Z, \nu\rangle \\ &= \langle W, \nabla_X \nu\rangle\langle Z, \nabla_Y \nu\rangle - \langle W, \nabla_Y \nu\rangle\langle Z, \nabla_X \nu\rangle \\ &= \langle RW, X\rangle\langle RZ, Y\rangle - \langle RW, Y\rangle\langle RZ, X\rangle. \end{aligned} \tag{6.27}$$

For the second equality in (6.27) we have used the fact that X, Y, Z, and W are tangential, so in particular, $\nabla_X\langle W, \nu\rangle = 0$, $\nabla_Y\langle Z, \nu\rangle = 0$, $\nabla_Y\langle W, \nu\rangle = 0$, and $\nabla_X\langle Z, \nu\rangle = 0$ on $\mathcal{S}$.

Next, recall from (1.21) the definition of the Ricci tensor, i.e.

$$\mathrm{Ric}\,(X,Y)_{\mathcal{S}} := \sum_{j=1}^{n-1} \langle \mathcal{R}_{\mathcal{S}}(T_j, Y)X, T_j\rangle, \tag{6.28}$$

where $T_1, \dots, T_{n-1}$ is, locally, an orthonormal basis in $T\mathcal{S}$, and X, Y are arbitrary tangential vector fields to $\mathcal{S}$. If we set $T_n := \nu$, and employ (6.27) together with $R\nu = 0$, we obtain

$$\sum_{j=1}^{n-1} \langle \mathcal{R}_{\mathcal{S}}(T_j, Y)X, T_j \rangle = \sum_{j=1}^{n} [\langle RT_j, T_j \rangle \langle RX, Y \rangle - \langle RT_j, Y \rangle \langle RX, T_j \rangle]$$
$$= \mathcal{G}\langle RX, Y \rangle - \langle RY, \sum_{j=1}^{n} \langle T_j, RX \rangle T_j \rangle = -\langle (R^2 - \mathcal{G}R)X, Y \rangle, \tag{6.29}$$

which takes care of (6.24).

Finally, (6.25) is a consequence of what we have proved so far, (3.8), and the elementary identity $A^2 - (\mathrm{Tr}\, A)A = -(\det A)I$, valid for any 2×2 matrix A. □

Theorem 6.3 *The following identities are valid:*

$$\Delta_{BL} = \pi \Delta_{\mathcal{S}} + R^2, \tag{6.30}$$
$$\Delta_{HL} = \pi \Delta_{\mathcal{S}} + 2R^2 - \mathcal{G}R. \tag{6.31}$$

P r o o f. In order to identify the Bochner-Laplacian operator Δ_{BL} on $\mathcal{S}$ we observe that, with u tangential field fixed, if the matrix $A(u)$ satisfies $\langle A(u)X, Y \rangle = \langle \nabla^{\mathcal{S}}_{\pi X} u, \pi Y \rangle$, for each $X, Y \in \mathbb{R}^n$ then, much as in the proof of Theorem 5.1,

$$a_{jk}(u) := \langle A(u)e_k, e_j \rangle = \langle \nabla_{\bar{e}_k} u, \bar{e}_j \rangle = \mathcal{D}_k u_j - \sum_{r=1}^{n} \nu_j \nu_r \mathcal{D}_k(u_r). \tag{6.32}$$

On account of this and Lemma 5.2 we can now write

$$\int_{\mathcal{S}} \langle (\nabla^{\mathcal{S}})^* \nabla^{\mathcal{S}} u, v \rangle \, dS = \int_{\mathcal{S}} \langle \nabla^{\mathcal{S}} u, \nabla^{\mathcal{S}} v \rangle \, dS = \sum_{j,k=1}^{n-1} \int_{\mathcal{S}} \langle \nabla^{\mathcal{S}}_{T_j} u, T_k \rangle \langle \nabla^{\mathcal{S}}_{T_j} v, T_k \rangle \, dS$$
$$= \sum_{j,k=1}^{n} \int_{\mathcal{S}} \langle A(u)T_j, T_k \rangle \langle A(v)T_j, T_k \rangle \, dS = \sum_{j,k=1}^{n} \int_{\mathcal{S}} a_{jk}(u) a_{jk}(v) \, dS$$
$$= \sum_{j,k=1}^{n} \int_{\mathcal{S}} \Big[\mathcal{D}_k u_j \mathcal{D}_k v_j$$
$$- \sum_{r=1}^{n} \nu_j \nu_r \mathcal{D}_j u_r \mathcal{D}_k v_j - \sum_{l=1}^{n} \nu_j \nu_l \mathcal{D}_k u_j \mathcal{D}_k v_l + \sum_{r,l=1}^{n} \nu_r \nu_l \mathcal{D}_k u_r \mathcal{D}_k v_l \Big] \, dS$$
$$= \sum_{j,k=1}^{n} \int_{\mathcal{S}} \Big[(\mathcal{D}_k^* \mathcal{D}_k u_j) v_j - \sum_{r=1}^{n} u_r v_j (\partial_k \nu_r)(\partial_k \nu_j) \Big] \, dS$$
$$= \int_{\mathcal{S}} \langle -\Delta_{\mathcal{S}} u - R^2 u, v \rangle \, dS. \tag{6.33}$$

In the next-to-the-last equality, we have applied the following identity to the terms under the integral sign:

$$\sum_{r=1}^{n} \nu_r \mathcal{D}_s w_r = \mathcal{D}_s\Big(\sum_{r=1}^{n} \nu_r w_r\Big) - \sum_{r=1}^{n} w_r \mathcal{D}_s \nu_r = -\sum_{r=1}^{n} w_r \partial_s \nu_r, \quad \text{on} \quad \mathcal{S}, \tag{6.34}$$

valid for any tangential vector field w, and any index $s \in \{1, \dots, n\}$. In turn, the identity (6.34) can be seen from a direct computation (recall that $\partial_\nu \nu_r = 0$ on $\mathcal{S}$). Finally, to justify the last equality in (6.33), it suffices to recall (4.30), (5.3) and the fact that $\sum\limits_{k=1}^{n} \nu_k \mathcal{D}_k = 0$.

The conclusion is that (6.30) holds. Finally, the identity (6.30) in concert with (6.23) and (6.24) implies (6.31). □

Recall now from [EM, *Note Added in Proof*, pp.161-162], [Ta] (cf. also the remark at the end of this paper), and [Ta2, Vol. III], that the Navier-Stokes system for a velocity field u, tangent to $\mathcal{S}$, and a (scalar-valued) pressure function p on $\mathcal{S}$ reads

$$\begin{aligned} &\frac{\partial u}{\partial t} - 2\,\mathrm{Def}^*_{\mathcal{S}}\mathrm{Def}_{\mathcal{S}}(u) + \nabla^{\mathcal{S}}_u u - \mathrm{grad}_{\mathcal{S}} p = \vec{f} \quad \text{in} \quad \mathcal{S}\times(0,\infty), \\ &\qquad \mathrm{div}_{\mathcal{S}} u = 0, \quad \text{in} \quad \mathcal{S}. \end{aligned} \tag{6.35}$$

Theorem 6.4 *The Navier-Stokes system (6.35) is equivalent to*

$$\begin{aligned} &\frac{\partial u}{\partial t} - \pi\nabla_u u + \pi\Delta_{\mathcal{S}} u + \mathcal{G}Ru - \mathrm{grad}_{\mathcal{S}} p = \vec{f} \quad \textit{in} \quad \mathcal{S}\times(0,\infty), \\ &\qquad \mathrm{div}_{\mathcal{S}} u = 0 \quad \textit{in} \quad \mathcal{S}. \end{aligned} \tag{6.36}$$

Proof. This is a direct consequence of (6.20) and (1.23). □

7 Further applications

We debut by briefly discussing a number of boundary value problems for tangential operators to a smooth hypersurface $\mathcal{S}$ in $\mathbb{R}^n$, for which $\partial\mathcal{S}$ is a Lipschitz submanifold of codimension one in $\mathcal{S}$.

For starters, the treatment of the classical Dirichlet and Neumann boundary problems for the Laplace-Beltrami operator in Lipschitz subdomains of Riemannian manifolds from [MT] translate into the well-posedness results about

$$\begin{cases} (\mathcal{D}\bullet\mathcal{D})u = f \quad \text{in} \quad \mathcal{S}, \\ u\Big|_{\partial\mathcal{S}} \quad \text{or} \quad \nabla_\gamma u \quad \text{prescribed on} \quad \partial\mathcal{S}. \end{cases}$$

In order to be more specific, consider the case of the Lamé system on $\mathcal{S}$ and, to set the stage, let $H^{s,p}$ stand for the scale of L^p-based Sobolev spaces, $1 < p < \infty$, $s \in \mathbb{R}$.

Theorem 7.1 *Assume that $\mathcal{S}$ is a C^∞ hypersurface in $\mathbb{R}^n$ with unit normal N and a Lipschitz boundary $\partial\mathcal{S}$. If the Lamé moduli $\lambda, \mu \in \mathbb{R}$ satisfy*

$$\mu > 0, \quad 2\mu + \lambda > 0, \tag{7.1}$$

then the boundary value problem

$$\begin{cases} u=(u_1,...,u_n)\in H^{s+1/2,2}(\mathcal{S},\mathbb{R}^n), \\ \langle u,N\rangle=0 \quad in \quad \mathcal{S}, \\ \mu\,\pi\,(\mathcal{D}\bullet\mathcal{D})u+(\lambda+\mu)\,\mathcal{D}(\mathcal{D}\bullet u)+\mu\,\mathcal{G}Ru=0 \quad in \quad \mathcal{S}, \\ u\Big|_{\partial\mathcal{S}}=\vec{f}\in H^{s,2}(\partial\mathcal{S},\mathbb{R}^n), \quad \langle\vec{f},N\rangle=0 \; on \; \partial\mathcal{S}, \end{cases} \tag{7.2}$$

is well-posed for each $0\le s\le 1$.

Furthermore, if the traction operator is defined by

$$\text{Traction}\, u=2\mu\,\mathcal{D}ef\,(u)\gamma+\mu\,\langle Ru,\gamma\rangle\nu+\lambda\,(\mathcal{D}\bullet u)\gamma \tag{7.3}$$

where

$$\mathcal{D}ef\,(u):=\Big(\mathcal{D}_k u_j+\mathcal{D}_j u_k\Big)_{1\le j,k\le n}$$

(recall that $\gamma\in T\mathcal{S}$ *is the outward unit normal to* $\partial\mathcal{S}$*), then*

$$\begin{cases} u=(u_1,...,u_n)\in H^{3/2-s,2}(\mathcal{S},\mathbb{R}^n), \\ \langle u,N\rangle=0 \quad in \quad \mathcal{S}, \\ \mu\,\pi\,(\mathcal{D}\bullet\mathcal{D})u+(\lambda+\mu)\,\mathcal{D}(\mathcal{D}\bullet u)+\mu\,\mathcal{G}Ru=0 \quad in \quad \mathcal{S}, \\ \text{Traction}\, u\Big|_{\partial\mathcal{S}}=\vec{f}\in H^{-s,2}(\partial\mathcal{S},\mathbb{R}^n), \end{cases} \tag{7.4}$$

is Fredholm solvable, of index zero, for each $0\le s\le 1$.

For a discussion pertaining to the physical significance of (7.1) see [LL], p. 11. The proof relies on the corresponding statement for the boundary problem for the intrinsic Lamé operator for $\mathcal{S}$ (viewed as an abstract Riemannian manifold) from [Mi], and the identifications (6.3).

Theorem 7.2 *Let* $\mathcal{S}$ *be a* C^∞ *hypersurface in* $\mathbb{R}^n$ *with unit normal* N *and Lipschitz boundary* $\partial\mathcal{S}$*. Then the boundary value problem*

$$\begin{cases} u=(u_1,...,u_n)\in H^{s+1/2,2}(\mathcal{S},\mathbb{R}^n), \quad p\in H^{s-1/2,2}(\mathcal{S}), \\ \langle u,N\rangle=0 \quad in \quad \mathcal{S}, \quad \int_{\mathcal{S}} p\,dS=0, \\ \pi\,(\mathcal{D}\bullet\mathcal{D})u+\mathcal{G}Ru-\mathcal{D}p=0 \quad in \quad \mathcal{S}, \\ \mathcal{D}\bullet u=0 \quad in \quad \mathcal{S}, \\ u\Big|_{\partial\mathcal{S}}=\vec{f}\in H^{s,2}(\partial\mathcal{S},\mathbb{R}^n), \quad \langle\vec{f},N\rangle=0 \;\; on \;\; \partial\mathcal{S}, \quad \oint_{\partial\mathcal{S}}\langle\vec{f},\gamma\rangle\,ds=0, \end{cases} \tag{7.5}$$

is well-posed for each $0\le s\le 1$.

Once again, this follows by translating the main result in [MT2] by means of the identifications (6.3).

Finally, natural boundary value problems for the Hodge-Laplacian on Lipschitz subdomains of (general) Riemannian manifolds have been recently treated in [MMT]. When phrased in terms of the operators studied in this paper, these yield the following sample result:

Theorem 7.3 *Let* $\mathcal{S}$ *be a* C^∞ *hypersurface in* $\mathbb{R}^n$ *with unit normal* N *and Lipschitz boundary* $\partial\mathcal{S}$*. Then the boundary value problem*

$$\begin{cases} u=(u_1,...,u_n)\in H^{1/2,2}(\mathcal{S},\mathbb{R}^n), \quad \langle u,N\rangle=0 \quad in \quad \mathcal{S}, \\ \pi\,(\mathcal{D}\bullet\mathcal{D})u+(2R^2-\mathcal{G}R)u=0 \quad in \quad \mathcal{S}, \\ (N\wedge d)u=0 \quad in \quad \mathcal{S}, \\ \langle\gamma,u|_{\partial\mathcal{S}}\rangle=\vec{f}\in L^2(\partial\mathcal{S},\mathbb{R}^n), \quad \langle\vec{f},N\rangle=0, \quad \langle\vec{f},\gamma\rangle=0 \quad on \quad \partial\mathcal{S}, \end{cases} \tag{7.6}$$

is Fredholm solvable, of index zero.

We conclude with a regularity result, very useful in electromagnetic theory (see, e.g., [Ne] for a discussion in a lower dimensional case).

Theorem 7.4 *Let Ω be a smooth, bounded subdomain of $\mathbb{R}^n$ and set $\mathcal{S} := \partial\Omega$. As usual, we let N stand for the outward unit normal to $\partial\Omega$ and denote by π the orthogonal projection onto $T\mathcal{S}$. Then for each field $u = (u_1, ..., u_n) \in L^2(\Omega, \mathbb{R}^n)$ such that*

$$\sum_{j=1}^{n} \partial_j u_j \in L^2(\Omega) \quad \textit{and} \quad \partial_j u_k - \partial_k u_j \in L^2(\Omega) \quad \textit{for each} \quad 1 \leq j, k \leq n, \tag{7.7}$$

the following statements are equivalent:
(i) $u \in H^{1,2}(\Omega, \mathbb{R}^n)$*;*
(ii) $\pi u|_{\mathcal{S}} \in H^{-1/2,2}(\mathcal{S}, \mathbb{R}^n)$ *and* $\mathrm{div}_{\mathcal{S}}(\pi u|_{\mathcal{S}}) \in H^{-1/2,2}(\mathcal{S})$*;*
(iii) $\langle N, (u|_{\mathcal{S}})\rangle \in H^{1/2,2}(\mathcal{S})$.

P r o o f. The departure point is to write, via (1.22) and repeated integrations by parts (cf. (1.17)), that

$$\begin{aligned}
\int_\Omega \|\nabla u\|^2 \, dx &= \int_\Omega \langle \nabla^* \nabla u, u\rangle \, dx + \int_{\mathcal{S}} \langle \nabla_N u, u\rangle \, dS \\
&= \int_\Omega \langle (d\delta + \delta d)u, u\rangle \, dx + \int_{\mathcal{S}} \langle \nabla_N u, u\rangle \, dS \\
&= \int_\Omega \Big[\|du\|^2 + \|\delta u\|^2\Big] \, dx + \int_{\mathcal{S}} \langle Pu, u\rangle \, dS
\end{aligned} \tag{7.8}$$

where

$$Pu := \nabla_\nu u - \nu \vee du - (\mathrm{div}\, u)\nu. \tag{7.9}$$

Next, we decompose $\nabla_\nu u = \nabla_\nu(\pi u) + \nabla_\nu(\langle u, \nu\rangle\nu)$, and use (5.14)-(5.15) to replace the last two terms in (7.9). This procedure yields

$$\langle Pu, u\rangle = \langle \mathrm{grad}_{\mathcal{S}}(\langle \nu, u\rangle), \pi u\rangle - \mathrm{div}_{\mathcal{S}}(\pi u)\langle \nu, u\rangle - \mathcal{G}\langle \nu, u\rangle^2 - \langle R\pi u, \pi u\rangle, \tag{7.10}$$

so that, all in all,

$$\begin{aligned}
\int_\Omega \|\nabla u\|^2 \, dx &= \int_\Omega \Big[\|du\|^2 + \|\delta u\|^2\Big] \, dx \\
&\quad - \int_{\mathcal{S}} \Big[2\langle \nu, u\rangle \mathrm{div}_{\mathcal{S}}(\pi u) + \mathcal{G}\langle \nu, u\rangle^2 + \langle R\pi u, \pi u\rangle\Big] \, dS.
\end{aligned} \tag{7.11}$$

The boundary integral in (7.11) can be estimated by a (fixed) multiple of

$$\begin{aligned}
&\|\langle \nu, u\rangle\|_{H^{1/2,2}(\mathcal{S})} \cdot \|\mathrm{div}_{\mathcal{S}}(\pi u)\|_{H^{-1/2,2}(\mathcal{S})} \\
&\quad + \|\langle \nu, u\rangle\|_{H^{1/2,2}(\mathcal{S})} \cdot \|\langle \nu, u\rangle\|_{H^{-1/2,2}(\mathcal{S})} \\
&\quad + \|\pi u\|_{H^{1/2,2}(\mathcal{S})} \cdot \|\pi u\|_{H^{-1/2,2}(\mathcal{S})}
\end{aligned} \tag{7.12}$$

In concert with standard trace theorems, to the effect that

$$
\begin{aligned}
\|u\|_{H^{1/2,2}(\mathcal{S},\mathbb{R}^n)} &\leq C \quad \Big[\|\nabla u\|_{L^2(\Omega,\mathbb{R}^{n^2})} + \|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big],\\
\|\langle N,u\rangle\|_{H^{-1/2,2}(\mathcal{S},\mathbb{R}^n)} &\leq C \quad \Big[\|\operatorname{div} u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big],\\
\|\pi u\|_{H^{-1/2,2}(\mathcal{S},\mathbb{R}^n)} &\leq C \quad \Big[\sum_{j,k}\|\partial_j u_k - \partial_k u_j\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega,\mathbb{R}^n)}\Big],
\end{aligned}
\tag{7.13}
$$

this implies equivalence of norms

$$
\begin{aligned}
\|u\|_{H^{1,2}(\Omega,\mathbb{R}^n)} &\approx \sum_{j,k}\|\partial_j u_k - \partial_k u_j\|_{L^2(\Omega)} + \sum_j \|\partial_j u_j\|_{L^2(\Omega)}\\
&\quad +\|\pi u\|_{H^{-1/2,2}(\mathcal{S})} + \|\operatorname{div}_{\mathcal{S}}(\pi u)\|_{H^{-1/2,2}(\mathcal{S})}\\
&\approx \sum_{j,k}\|\partial_j u_k - \partial_k u_j\|_{L^2(\Omega)} + \sum_j \|\partial_j u_j\|_{L^2(\Omega)} + \|\langle N,u\rangle\|_{H^{1/2,2}(\mathcal{S})}.
\end{aligned}
\tag{7.14}
$$

With this *a priori* estimate at hand, the desired conclusion follows from a standard density argument. □

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