Differential and Integral Equations, Volume 1, Number 1, January 1988, pp. 97–116.

DIFFERENTIAL OPERATORS AND THE LEGENDRE TYPE POLYNOMIALS

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Abstract. In 1938, H.L. Krall discovered a fourth-order differential equation that has orthogonal polynomial solutions, called the Legendre type polynomials. Properties of these polynomials and the right-definite problem generated by this fourth-order equation were studied by A.M. Krall in 1981. In this paper, we shall consider this right-definite problem from a different point of view which will enable us to study the fourth-order equation and the polynomials in the left-definite case. As a particular consequence of this study, we shall produce the orthogonality of the derivatives of the Legendre type polynomials. The work in this paper extends earlier work of Titchmarsh, Pleijel and Everitt who studied the right- and left-definite problems associated with the classical Legendre polynomials.

1. Introduction. The second-order differential equations which have the classical orthogonal polynomials as solutions are some of the best examples available to illustrate the well-developed theory of self-adjoint extensions of formally symmetric differential expressions. For a comprehensive study of these second order self-adjoint operators, the reader is referred to the survey paper of Littlejohn and Krall [10] and also to the thesis of Otieno [12]. Besides doing a thorough study of the right-definite boundary value problems associated with the differential equations for these classical orthogonal polynomials, Otieno also considers the left-definite boundary value problems for these equations, extending work of Titchmarsh [15], Pleijel [13] and Everitt [4]. For a detailed analysis of self-adjoint extensions of symmetric operators, the texts of Naimark [11] and Akhiezer and Glazman [1] are recommended. The books of Szegö [14] and Chihara [2] are two excellent sources for properties of orthogonal polynomials in general.

The theory of self-adjoint extensions of formally symmetric differential expressions of order greater than two is considerably more complicated than that of second-order expressions. To illustrate this extension theory with some concrete examples, are there higher order

Received September 24, 1987.
AMS(MOS) Subject Classifications: 33A65, 34B20.

differential equations which have orthogonal polynomial solutions? The answer is yes, in fact, the study of orthogonal polynomial solutions to higher order equations was launched by H.L. Krall [8, 9] from 1938-1940. During the 1930's, there was a wide search for orthogonal polynomial solutions to higher order equations and it was Krall who first succeeded. In fact, he found all fourth order differential equations of the form:

$$\sum_{k=1}^{4} a_k(x)y^{(k)}(x) = \lambda_n y(x) \tag{1.1}$$

that have orthogonal polynomial solutions. He produced three fourth-order equations of the form (1.1) which have nonclassical orthogonal polynomial solutions. Furthermore, Krall showed that the weight functions for these polynomials all have jump discontinuties at one or more endpoints of the interval of orthogonality, contrary to the situation for the classical weight functions.

In 1981, A.M. Krall studied these polynomials in detail [7], naming them the Legendre type, Jacobi type and the Laguerre type polynomials. Like the classical orthogonal polynomials, these polynomials can be found through a three-term recurrence relation, various generating functions or a Rodrigues-type formula. Besides a thorough study of these polynomials, Krall also studied the self-adjoint differential operators, associated with each equation, that produce the orthogonal polynomials as the eigenfunctions. As Krall showed, the domains of these self-adjoint operators are obtained by applying appropriate boundary conditions to functions in the maximal domains of the operators.

In this paper, we shall focus our attention on the fourth-order equation for the Legendre type polynomials. The Legendre type polynomials

$$P_n^{\alpha}(x) := \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!(\alpha + \frac{n(n-1)}{2} + 2k)x^{n-2k}}{2^n k!(n-k)!(n-2k)!} \tag{1.2}$$

satisfy the fourth-order differential equation:

$$M_k[y] = (\lambda_n + k)y,$$

where

$$M_k[y] := (x^2-1)^2 y^{(4)} + 8x(x^2-1)y^{(3)} + (4\alpha+12)(x^2-1)y'' + 8\alpha x y' + ky \tag{1.3}$$

and $\lambda_n = n(n+1)(n^2+n+4\alpha-2)$. Here, the numbers α and k are, respectively, fixed positive and nonnegative parameters. Observe that $M_k[y]$ is formally symmetric; i.e.

$$M_k[y] = \left((1-x^2)^2 y''(x)\right)'' - \left([8+4\alpha(1-x^2)]y'(x)\right)' + ky(x).$$

We remark that the Kralls studied the operator $M_0[y]$, i.e. equation (1.3) when $k=0$. For reasons that will be clarified later, we shall study (1.3) for $k \geq 0$.

Let $\hat{\mu}(x)$ denote the monotonic increasing function defined by:

$$\hat{\mu}(x) := \begin{cases} -(\alpha+1)/2 & \text{if } x \in (-\infty, -1] \\ \alpha x/2 & \text{if } x \in (-1, 1) \\ (\alpha+1)/2 & \text{if } x \in [1, \infty) \end{cases}$$

Then, as is well-documented (see [5], for example), $\hat{\mu}$ generates a regular positive measure μ on the Borel sets of the real line. The Legendre polynomials are orthogonal in $L^2_\mu[-1,1]$, where

$$L^2_\mu[-1,1] := \{f : [-1,1] \to \mathbf{C} | f \text{ is Lebesgue measurable and } \int_{[-1,1]} |f|^2\, d\mu < \infty\} \tag{1.4}$$

is the Hilbert space with inner product:

$$\begin{aligned}(f,g)_\mu : &= \int_{[-1,1]} f(t)\bar{g}(t)\, d\mu(t) \\ &= \frac{f(1)\bar{g}(1)}{2} + \frac{a}{2}\int_{-1}^{1} f(t)\bar{g}(t)\, dt + \frac{f(-1)\bar{g}(-1)}{2}\end{aligned} \tag{1.5}$$

and norm $\|f\|_\mu := (f,f)_\mu^{1/2}$. More precisely, the Legendre type polynomials satisfy the orthogonality relationship

$$\int_{[-1,1]} P_n^\alpha(t) P_m^\alpha(t)\, d\mu(t) = \left[\alpha\left(\alpha + \frac{n(n-1)}{2}\right)\left(\alpha + \frac{(n+1)(n+2)}{2}\right)/(2n+1)\right]\delta_{nm} \tag{1.6}$$

where δ_{nm} is the Kronecker delta function.

From a different point of view than that taken by A.M. Krall in [7], we shall also consider the self-adjoint operator associated with (1.3) in the space $L^2_\mu[-1,1]$. This is the so-called right-definite boundary value problem associated with the Legendre type polynomials. From this, we can study the left-definite boundary value problem. That is to say, we shall study the operator M_k in a Hilbert space H, to be properly defined later, endowed with the inner product:

$$(f,g)_H := \frac{\alpha}{2}\int_{-1}^{1} \{(1-t^2)^2 f''(t)\bar{g}''(t) + \left(8 + 4\alpha(1-t^2)\right) f'(t)\bar{g}'(t)\}\, dt + k(f,g)_\mu \tag{1.7}$$

Here, it is essential that $k > 0$.

This work is a continuation of earlier work of Everitt [4] who considered the right- and left-definite boundary value problems for the Legendre polynomials. The methods used in this paper to obtain our results are significantly different from those in [4]. Indeed, the Legendre equation can be studied by classical means following the treatment of such problems in Titchmarsh [15, Sections 4.3-4.7] and Akhiezer and Glazman [1, Appendix 2, §3] for example. However, because the spaces where the right- and left-definite problems associated with the fourth-order equation for the Legendre type polynomials are different from those considered by these classical texts, the analysis involved in this paper to obtain our desired results will be quite different. As a consequence of studying the left-definite problem, we will establish the orthogonality of the derivatives of the Legendre type polynomials in the space H. In fact, we shall show the orthogonality relationship:

$$\begin{aligned}&\left(P_n^\alpha, P_m^\alpha\right)_H = \\ &\left\{\left[n(n+1)(n^2+n+4\alpha-2)+k\right]\alpha\left(\alpha + \frac{n(n-1)}{2}\right)\left(\alpha + \frac{(n+1)(n+2)}{2}\right)/(2n+1)\right\}\delta_{nm}.\end{aligned} \tag{1.8}$$

The organization of this paper is as follows. In §2, we develop some essential properties of functions in the maximal domain Δ_k of M_k in $L^2(-1,1)$ and we also list Green's formula and Dirichlet's formula. In §3, we define the operator T_k, with domain Δ_k, associated with M_k and show that T_k is symmetric in $L^2_\mu[-1,1]$. In fact, T_k, with domain Δ_k, is actually self-adjoint in $L^2_\mu[-1,1]$ but, in order to prove self-adjointness of T_k, we need to show self-adjointness of three related operators. We define these operators in §4 and study their self-adjointness there. In §5, we prove the self-adjointness of T_k in $L^2_\mu[-1,1]$. The left-definite problem associated with the Legendre type polynomials is discussed in §6. Lastly, in §7, the self-adjoint operators associated with the half-range Legendre type series are discussed.

2. Properties of the maximal domain of $\mathbf{M}_k$. The maximal domain Δ_k of M_k in $L^2(-1,1)$ is defined to be

$$\Delta_k := \{f:(-1,1)\to \mathbf{C} \mid f, f', f'', f^{(3)} \in AC_{\text{loc}}(-1,1) \text{ and } f, M_k[f]\in L^2(-1,1)\} \tag{2.1}$$

Here, $AC_{\text{loc}}(-1,1)$ refers to the set of functions $f:(-1,1)\to\mathbf{C}$ that are locally absolutely continuous on $(-1,1)$, i.e., f is absolutely continuous on all compact subintervals of $(-1,1)$. Since $C_0^\infty(-1,1)\subset\Delta_k$, it follows that Δ_k is dense in $L^2(-1,1)$.

For f, $g\in\Delta_k$ and $[\alpha,\beta]\subset(-1,1)$, we have Green's formula

$$\int_\alpha^\beta \{M_k[f](t)\bar{g}(t) - \overline{M_k[g]}(t)f(t)\}\,dt = [f,g](t)\Big|_\alpha^\beta, \tag{2.2}$$

where $[f,g](\cdot)$ is the skew-symmetric sesquilinear form defined by

$$\begin{aligned}[f,g](x) :&= \{((1-x^2)^2 f''(x))' - (8+4\alpha(1-x^2))f'(x)\}\bar{g}(x)\\ &\quad - \{((1-x^2)^2\bar{g}''(x))' - (8+4\alpha(1-x^2))\bar{g}'(x)\}f(x)\\ &\quad - (1-x^2)^2 f''(x)\bar{g}'(x) + (1-x^2)^2\bar{g}''(x)f'(x),\quad -1<x<1,\end{aligned} \tag{2.3}$$

and Dirichlet's formula

$$\begin{aligned}&\int_\alpha^\beta \{(1-t^2)^2 f''(t)\bar{g}''(t) + (8+4\alpha(1-t^2))f'(t)\bar{g}'(t) + kf(t)\bar{g}(t)\}\,dt\\ &= (1-t^2)^2 f''(t)\bar{g}'(t)\Big|_\alpha^\beta - \{((1-t^2)^2 f''(t))' - (8+4\alpha(1-t^2))f'(t)\}\bar{g}(t)\Big|_\alpha^\beta\\ &\quad + \int_\alpha^\beta M_k[f](t)\bar{g}(t)\,dt.\end{aligned} \tag{2.4}$$

Of particular importance later will be Dirchlet's formula when $f=g$:

$$\begin{aligned}&\int_\alpha^\beta \{(1-t^2)^2|f''(t)|^2 + (8+4\alpha(1-t^2))|f'(t)|^2 + k|f(t)|^2\}\,dt\\ &= (1-t^2)^2 f''(t)\bar{f}'(t)\Big|_\alpha^\beta - \{(1-t^2)^2 f''(t))' - (8+4\alpha(1-t^2))f'(t)\}\bar{f}(t)\Big|_\alpha^\beta\\ &\quad + \int_\alpha^\beta M_k[f](t)\bar{f}(t)\,dt.\end{aligned} \tag{2.5}$$

From the definition of Δ_k, we see that the limits

$$\lim_{x\to\pm 1}[f,g](x) \tag{2.6}$$

exist and are finite, for all $f,\ g \in \Delta_k$. Note also, and we shall use this later in our definition of T_k, that the function $1 \in \Delta_k$ and

$$[f,1](x) = \big((1-x^2)^2 f''(x)\big)' - \big(8 + 4\alpha(1-x^2)\big)f'(x), \quad -1 < x < 1 \tag{2.7}$$

for all $f \in \Delta_k$. The main result of this section is the following theorem:

Theorem 2.1. *Let $f,\ g \in \Delta_k$. Then*

(i) $\lim\limits_{x\to\pm 1}\left\{\big((1-x^2)^2 f''(x)\big)' - \big(8+4\alpha(1-x^2)\big)f'(x)\right\}$ *exist and are finite.*

(ii) $f' \in L^2(-1,1)$

(iii) $f \in AC[-1,1]$

(iv) $\big((1-x^2)^2 f''\big)' \in L^2(-1,1)$

(v) $(1-x^2)f'' \in L^2(-1,1)$

(vi) $\lim_{x\to\pm1}(1-x^2)^2 f''(x)\bar{g}'(x) = 0.$

In order to prove Theorem 2.1, which is fundamental to all that follows, we first have to digress to consider some properties of the second-order symmetric differential expression:

$$N[f](x) := -\big((1-x^2)^2 f'(x)\big)' + 4\alpha(1-x^2)f(x), \;\; x \in (-1,1).$$

We shall consider these properties of $N[\cdot]$ on $[0,1)$; there are similar results for $(-1,0]$. Recall that the maximal domain Δ_N of $N[\cdot]$ in $L^2[0,1)$ is defined to be:

$$\Delta_N := \left\{f : [0,1) \to \mathbf{C} | f, f' \in AC_{\mathrm{loc}}[0,1) \text{ and } f, N[f] \in L^2[0,1)\right\}.$$

Lemma 2.2.

(i) $(1-x^2)f' \in L^2[0,1)$ *for all* $f \in \Delta_N$

(ii) $\lim\limits_{x\to 1}(1-x^2)^2 f'(x)\bar{g}(x) = 0$ *for all* $f, g \in \Delta_N$

Proof: For $f,\ g \in \Delta_N,\ \ 0 \le x < 1$, we see that

$$\begin{aligned}&\int_0^x \left\{(1-t^2)^2 f'(t)\bar{g}'(t) + 4\alpha(1-t^2)f(t)\bar{g}(t)\right\} dt = \\ &\int_0^x N[f](t)\bar{g}(t)\,dt + (1-x^2)^2 f'(x)\bar{g}(x) - f'(0)\bar{g}(0).\end{aligned} \tag{2.8}$$

It suffices to prove (i) for real-valued $f \in \Delta_N$. For such f and $g = f$, (2.8) reads:

$$\begin{aligned}&\int_0^x \left\{(1-t^2)^2 (f'(t))^2 + 4\alpha(1-t^2)(f(t))^2\right\} dt = \\ &\int_0^x N[f](t)f(t)\,dt + (1-x^2)^2 f'(x)f(x) - f'(0)f(0).\end{aligned}$$

Since the integrand on the left-hand side of this equation is non-negative, either we must have

(a) $\lim_{x\to 1}\int_0^x \left\{(1-t^2)^2(f'(t))^2+4\alpha(1-t^2)(f(t))^2\right\}dt$ exists and is finite, or

(b) $\lim_{x\to 1}\int_0^x \left\{(1-t^2)^2(f'(t))^2+4\alpha(1-t^2)(f(t))^2\right\}dt=\infty$.

If case (a) occurs, clearly $N[\cdot]$ is Dirichlet, i.e., $(1-x^2)f'\in L^2[0,1)$. Suppose then that situation (b) occurs. Since $\lim_{x\to 1}\int_0^x N[f](t)f(t)\,dt$ exists and is finite, we see that for any $c>0$ and x near 1, we have $(1-x^2)^2f'(x)f(x)\geq c>0$. Integrating the inequality $f'(x)f(x)\geq c/(1-x^2)^2$ yields $(f(x))^2\geq c/(2(1-x))+g(x)$, for x close to 1 and for some $g\in L[0,1)$. However, this contradicts our assumption that $f\in L^2[0,1)$. This proves (i). Observe now, from (2.8), that $\lim_{x\to 1}(1-x^2)^2f'(x)\bar{g}(x)$ exists and is finite. If this limit is not 0, there exists $c>0$ such that

$$(1-x^2)|f'(x)\bar{g}(x)|\geq c/(1-x^2)\quad \text{for } x \text{ near } 1.$$

However this contradicts the fact, from (i), that

$$(1-x^2)|f'(x)\bar{g}(x)|\in L[0,1).$$

Hence $\lim_{x\to 1}(1-x^2)^2f'(x)\bar{g}(x)=0$, for all $f,\ g\in\Delta_N$. ∎

Applying Frobenius' method [6, Sections 16.1-16.33], the indicial equation of $N[\cdot]$ at $x=1$ is $I(r)=4r(r+1)$. Consequently, it follows that $N[\cdot]$ is limit point at $x=1$. We now define a self-adjoint operator A in $L^2[0,1)$ by

$$\begin{gathered}A[f]:=N[f]\\ \text{for}\quad f\in D(A):=\{f\in\Delta_N \mid f(0)=0\}\end{gathered}$$

The fact that A is self-adjoint in $L^2[0,1)$ follows from the analysis in Naimark [11]. Since

$$(Af,f)=\int_0^1 A[f](t)\bar{f}(t)\,dt=\int_0^1\left\{(1-t^2)^2\,|f'(t)|^2+4\alpha(1-t^2)|f(t)|^2\right\}dt\geq 0,$$

we see that A is bounded below in $L^2[0,1)$ by zero. In particular, we have that $-8\in\rho(A)$, the resolvent of A, and hence the resolvent operator $R_\lambda(A):=(A-\lambda I)^{-1}$, for $\lambda=-8$, is a bounded operator from $L^2[0,1)$ onto $D(A)$, where I is the identity operator on $L^2[0,1)$. It is easy to check that, for $f\in L^2[0,1)$,

$$N[(R_\lambda(A)f)(x)]=-8(R_\lambda(A)f)(x)+f(x),\ \ x\in[0,1) \tag{2.9}$$

i.e., $R_\lambda(A)f$, for $\lambda=-8$, is a solution of the nonhomogeneous equation

$$N[y](x)=-\left((1-x^2)^2y'(x)\right)'+4\alpha(1-x^2)y(x)=-8y(x)+f(x), \tag{2.10}$$

with $y\in D(A)$, i.e., $y\in L^2[0,1)$ and $y(0)=0$.

We can give, and require, an explicit representation of $R_\lambda(A)$ when $\lambda = -8$. Consider the differential equation

$$-\big((1-x^2)^2 y'(x)\big)' + 4\alpha(1-x^2)y(x) = -8y(x), \quad x \in [0,1), \tag{2.11}$$

$$\text{i.e., } N[y] = -8y.$$

The indicial equation of (2.11), at $x = 1$, is $I(r) = 4(r-1)(r+2)$. Thus, from Frobenius' method, one solution of (2.11) has the form

$$\varphi(x) := (x-1)\sum_{n=0}^{\infty} a_n (x-1)^n, \quad a_0 \neq 0 \tag{2.12}$$

where this series converges for $|x-1| < 2$. Consequently, $\varphi \in L^2[0,1)$. By reduction of order [6, Section 5.22], another linearly independent solution is given by

$$\psi(x) := \varphi(x)\int_0^x \frac{dt}{\varphi^2(t)(1-t^2)^2} \qquad 0 \le x < 1 \tag{2.13}$$

Since equation (2.11) is limit point at $x = 1$ in $L^2[0,1)$ and since $\varphi \in L^2[0,1)$, we see that $\psi \notin L^2[0,1)$. We also note, again using Frobenius' method, that there exist constants α, $\beta \in \mathbf{C}$ such that

$$\psi(x) = \alpha\varphi(x) + \beta\varphi(x)\ln(1-x) + \frac{b_0}{(1-x)^2} + \frac{b_1}{(1-x)} + \sum_{n=2}^{\infty} b_n (x-1)^{n-2}, \tag{2.14}$$

where $b_0 \neq 0$. We now claim that $\varphi(0) \neq 0$. For, if $\varphi(0) = 0$, then $\varphi \in D(A)$. This implies, however, that φ is an eigenfunction of A with eigenvalue $\lambda = -8$, contradicting the fact that A is bounded below by zero. Hence $\varphi(0) \neq 0$. We are now in a position to prove:

Lemma 2.3. *For any $f \in L^2[0,1)$, we have*

$$(R_{-8}(A)f)(x) \equiv \varphi(x)\int_0^x \psi(t)f(t)\,dt + \psi(x)\int_x^1 \varphi(t)f(t)\,dt, \qquad x \in [0,1).$$

Proof: Define, for $f \in L^2[0,1)$,

$$\Phi(x;f) := \varphi(x)\int_0^x \psi(t)f(t)\,dt + \psi(x)\int_x^1 \varphi(t)f(t)\,dt, \qquad x \in [0,1). \tag{2.15}$$

From (2.13), it follows that $\Phi(0;f) = 0$ for all $f \in L^2[0,1)$. A direct calculation yields $N[\Phi] = -8\Phi + f$ on $[0,1)$; combining this with (2.9) gives us

$$N\big[R_{-8}(A)f - \Phi(\cdot\,;f)\big] = -8\big(R_{-8}(A)f - \Phi(\cdot\,;f)\big).$$

Hence there exists constants c_1, $c_2 \in \mathbf{C}$ such that

$$(R_{-8}(A)f)(x) - \Phi(x;f) = c_1\varphi(x) + c_2\psi(x), \qquad x \in [0,1). \tag{2.16}$$

Evaluating this identity at $x = 0$ immediately yields $c_1 = 0$. To show that $c_2 = 0$, we apply a result of Chisholm and Everitt [3] which shows $\Phi(\cdot\,;\, f) \in L^2[0,1)$ for all $f \in L^2[0,1)$ if and only if there exists a constant $K > 0$ such that

$$\int_0^x |\psi(t)|^2\, dt \int_x^1 |\varphi(t)|^2\, dt \leq K \quad \text{for all } x \in [0,1). \tag{2.17}$$

By (2.12), (2.13) and (2.14), it follows that (2.17) does hold so that $\Phi(\cdot\,;\, f) \in L^2[0,1)$ for all $f \in L^2[0,1)$. Consequently, the left-hand side of (2.16) is in $L^2[0,1)$ while the right-hand side of (2.16) will be in $L^2[0,1)$ only when $c_2 = 0$. This establishes the lemma. ∎

Proof of Theorem 2.1. By (2.6) and (2.7), property (i) follows immediately. Alternatively, it is easy to check that if, for $-1 < x < 1$, $\Lambda(\cdot)$ is defined by:

$$\Lambda(x) = \Lambda(x; f) := -\Big\{ \int_0^x M_k[f](t)\, dt + f^{(3)}(0) - (8 + 4\alpha) f'(0) - k \int_0^x f(t)\, dt \Big\} \tag{2.18}$$

then

$$-\big((1 - x^2)^2 f''(x)\big)' + \big(8 + 4\alpha(1 - x^2)\big) f'(x) = \Lambda(x), \qquad -1 < x < 1. \tag{2.19}$$

Since f, $M_k[f] \in L^2(-1,1) \subset L(-1,1)$, it follows from (2.18) that

$$\lim_{x \to \pm 1} \Lambda(x)$$

exist and are finite. Notice, in fact, that $\Lambda \in AC[-1,1] \subset L^2(-1,1)$. If we write $y := f'$, $f \in \Delta_k$, observe that (2.19) may be written as

$$N[y](x) = -8y(x) + \Lambda(x), \qquad x \in [0,1). \tag{2.20}$$

Since $\Lambda \in AC[0,1)$, we can say that every solution of (2.20) can be represented in the form, for some constants k_1, $k_2 \in \mathbf{C}$,

$$y(x) = k_1 \varphi(x) + k_2 \psi(x) + \Phi(x; \Lambda), \qquad x \in [0,1),$$

where Φ is defined by (2.15). Accordingly then we have, for any $f \in \Delta_k$, the identity:

$$f'(x) = k_1 \varphi(x) + k_2 \psi(x) + \Phi(x; \Lambda), \qquad x \in [0,1). \tag{2.21}$$

Integrating (2.21) yields:

$$f(x) = k_1 \int_0^x \varphi(t)\, dt + k_2 \int_0^x \psi(t)\, dt + \int_0^x \Phi(t; \Lambda)\, dt + d, \qquad x \in [0,1),$$

where d is some constant. Of the terms on the right-hand side, the first, third and fourth are all in $L^2[0,1)$. The second term gives, from (2.14) and for some constant $c \neq 0$ and $g \in L^2[0,1)$,

$$\int_0^x \psi(t)\, dt = \frac{c}{(x-1)} + g(x),$$

which however implies $k_2 \int_0^x \psi(t)\,dt \notin L^2[0,1)$ unless $k_2 = 0$; this is now required since $f \in L^2[0,1)$. Hence, from (2.21), we now have the following representation:

$$f'(x) = k_1\varphi(x) + \Phi(x;\Lambda), \qquad x \in [0,1),$$

and it follows that $f' \in L^2[0,1)$. A similar argument shows that f' is also in $L^2(-1,0]$ and this completes the proof of (ii).

From the fact that $f' \in AC_{\mathrm{loc}}(-1,1)$, i.e. that $f(x) = f(0) + \int_0^x f'(t)\,dt$, $-1 < x < 1$, and knowing that $f' \in L^2(-1,1) \subset L(-1,1)$, we see that $\lim_{x\to\pm 1} f(x)$ exist and are finite. If we define $f(\pm 1) = \lim_{x\to\pm 1} f(x)$, and we shall henceforth assume this, then we see that $f \in AC[-1,1]$. This proves (iii). Because $f' \in L^2(-1,1)$ and $\big(8+4\alpha(1-x^2)\big)$ is bounded on $(-1,1)$, clearly, we have $\big(8+4\alpha(1-x^2)\big)f' \in L^2(-1,1)$. Since $\Lambda \in L^2(-1,1)$, it follows from (2.19) that $\big((1-x^2)^2 f''\big)' \in L^2(-1,1)$, proving (iv). To prove (v), we return to (2.5) in the form:

$$\begin{aligned}
&\int_0^x \left\{(1-t^2)^2|f''(t)|^2 + \big(8+4\alpha(1-t^2)\big)|f'(t)|^2 + k|f(t)|^2\right\} dt \\
&= (1-x^2)^2 f''(x)\bar{f}'(x) - \left\{\big((1-x^2)^2 f''(x)\big)' - (8+4\alpha(1-x^2))f'(x)\right\}\bar{f}(x) \\
&\quad + \int_0^x M_k[f](t)\bar{f}(t)\,dt + K, \quad \text{where } K \text{ is a constant and } 0 \le x < 1.
\end{aligned}$$

By (i) and (iii), we know that the limit

$$\lim_{x\to 1}\left\{\big((1-x^2)^2 f''(x)\big)' - \big(8+4\alpha(1-x^2)\big)f'(x)\right\}\bar{f}(x)$$

exists and is finite. Also since f, $M_k[f] \in L^2[0,1)$, we have that

$$\lim_{x\to 1}\int_0^x M_k[f](t)\bar{f}(t)\,dt \quad \text{and} \quad \lim_{x\to 1}\int_0^x |f(t)|^2\,dt$$

exist and are finite. Since $f' \in L^2[0,1)$ and $\big(8+4\alpha(1-x^2)\big)$ is bounded, clearly,

$$\lim_{x\to 1}\int_0^x \big(8+4\alpha(1-t^2)\big)|f'(t)|^2\,dt$$

exists and is finite. Consequently, if $(1-x^2)f'' \notin L^2[0,1)$, it follows that

$$\lim_{x\to 1}(1-x^2)^2 f''(x)\bar{f}'(x) = \infty\,.$$

Without loss of generality, we may assume that f is real-valued on $[0,1)$. Then, for any constant $c > 0$ and x near 1, we have:

$$(1-x^2)^2 f''(x)f'(x) \ge c > 0.$$

Integrating the inequality $f''(x)f'(x) \ge c/\big((1+x)^2(1-x)^2\big)$, yields:

$$(f'(x))^2 \ge \frac{d}{1-x} + g(x),$$

for some constant $d \neq 0$ and $g \in L[0,1)$. However, this implies

$$f' \notin L^2[0,1),$$

a contradiction. Hence, $(1-x^2)f'' \in L^2[0,1)$. Similarly, we have that

$$(1-x^2)f'' \in L^2(-1,0].$$

This establishes (v). To prove (vi), we consider (2.4) in the form:

$$\begin{aligned}
&\int_0^x \{(1-t^2)^2 f''(t)\bar{g}''(t) + \big(8+4\alpha(1-t^2)\big)f'(t)\bar{g}'(t) + kf(t)\bar{g}(t)\}\,dt \\
&= (1-x^2)^2 f''(x)\bar{g}'(x) - \{\big((1-x^2)^2 f''(x)\big)' - \big(8+4\alpha(1-x^2)\big)f'(x)\}\bar{g}(x) \\
&\quad + \int_0^x M_k[f](t)\bar{g}(t)\,dt + K, \quad K \text{ a constant.}
\end{aligned}$$

For f, $g \in \Delta_k$, the left-hand side of this equation has a finite limit as $x \to 1$; this follows from the definition of Δ_k and properties (ii) and (v). Furthermore, the second and third terms on the right-hand side of the above equation are finite; this follows from the definition of Δ_k, and (i) and (iii) of Theorem 2.1. Hence, we see that

$$\lim_{x\to 1}(1-x^2)^2 f''(x)\bar{g}'(x)$$

exists and is finite. Suppose this limit is not zero. Then, there exists a number $b > 0$ such that

$$(1-x^2)^2|f''(x)\bar{g}'(x)| \geq b > 0 \quad \text{for all } x \text{ near } 1,$$

and hence

$$(1-x^2)|f''(x)\bar{g}'(x)| \geq \frac{b}{(1-x^2)}, \quad \text{for all } x \text{ near } 1. \tag{2.22}$$

Since $(1-x^2)f''$, $\bar{g}' \in L^2[0,1)$, the left-side of (2.22) is in $L[0,1)$. However, the term on the right-hand side of (2.22) is not, giving us a contradiction. Hence, $\lim_{x\to 1}(1-x^2)^2 f''(x)\bar{g}'(x) = 0$. Similarly, the corresponding limit at -1 is zero; this completes the proof of the theorem. ∎

3. Definition of $\mathbf{T}_k$. Recall from (2.6) and (2.7) that the limits

$$\lim_{x\to\pm 1}[f,1](x)$$

both exist and are finite. We define

$$[f,1](\pm 1) := \lim_{x\to\pm 1}[f,1](x).$$

Also, recall in §2 that we defined $f(\pm 1) = \lim_{x\to\pm 1} f(x)$, $f \in \Delta_k$. Hence, from Theorem 2.1 (iii), we see that $\Delta_k \subset L^2_\mu[-1,1]$, where $L^2_\mu[-1,1]$ is the Hilbert space defined in (1.4) with inner product given in (1.5).

Define the operator T_k in $L^2_\mu[-1,1]$ by:

$$T_k[f](x) := \begin{cases} \alpha[f,1](-1) + kf(-1) & \text{if } x = -1 \\ M_k[f](x) & \text{if } -1 < x < 1 \\ -\alpha[f,1](1) + kf(1) & \text{if } x = 1 \end{cases}$$
$$D(T_k) := \Delta_k$$

Theorem 3.1. *T_k is symmetric in $L^2_\mu[-1,1]$. Furthermore, T_k is bounded below in $L^2_\mu[-1,1]$ by kI, where I is the identity operator on $L^2_\mu[-1,1]$.*

Proof: Let $f,\, g \in D(T_k)$. First, notice in light of (2.7), that Green's formula (2.3) may be written as:

$$\begin{aligned}
&\int_{-1}^{1} M_k[f](t)\bar{g}(t)\,dt \\
&= \lim_{x\to 1} \left\{[f,1](x)\bar{g}(x) - \overline{[g,1]}(x)f(x) - (1-x^2)^2 f''(x)\bar{g}'(x) + (1-x^2)^2\bar{g}''(x)f'(x)\right\} \\
&\quad - \lim_{x\to -1} \left\{[f,1](x)\bar{g}(x) - \overline{[g,1]}(x)f(x) - (1-x^2)^2 f''(x)\bar{g}'(x) + (1-x^2)^2\bar{g}''(x)f'(x)\right\} \\
&\quad + \int_{-1}^{1} \overline{M_k[g]}(t)f(t)\,dt,
\end{aligned}$$

where we have written $[\bar{g},1](x) = \overline{[g,1]}(x)$, since the coefficients of $M_k[\cdot]$ are real-valued on $(-1,1)$. By Theorem 2.1, all eight terms in the above limits have individual limits; in fact , we can infer from Theorem 2.1 that the above equation may be simplified to:

$$\begin{aligned}
&\int_{-1}^{1} M_k[f](t)\bar{g}(t)\,dt \\
&= [f,1](1)\bar{g}(1) - \overline{[g,1]}(1)f(1) - [f,1](-1)\bar{g}(-1) + \overline{[g,1]}(-1)f(-1) + \int_{-1}^{1} \overline{M_k[g]}(t)f(t)\,dt.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left(T_k[f],g\right)_\mu &= \frac{T_k[f](1)\bar{g}(1)}{2} + \frac{\alpha}{2}\int_{-1}^{1} M_k[f](t)\bar{g}(t)\,dt + \frac{T_k[f](-1)\bar{g}(-1)}{2} \\
&= -\frac{\alpha[f,1](1)\bar{g}(1)}{2} + \frac{kf(1)\bar{g}(1)}{2} + \frac{\alpha}{2}\Big\{[f,1](1)\bar{g}(1) - \overline{[g,1]}(1)f(1) \\
&\quad - [f,1](-1)\bar{g}(-1) + \overline{[g,1]}(-1)f(-1) + \int_{-1}^{1} \overline{M_k[g]}(t)f(t)\,dt\Big\} \\
&\quad + \frac{\alpha[f,1](-1)\bar{g}(-1)}{2} + \frac{kf(-1)\bar{g}(-1)}{2} \qquad (3.1) \\
&= \frac{kf(1)\bar{g}(1)}{2} - \frac{\alpha}{2}\overline{[g,1]}(1)f(1) + \frac{\alpha}{2}\overline{[g,1]}(-1)f(-1) \\
&\quad + \frac{kf(-1)\bar{g}(-1)}{2} + \frac{\alpha}{2}\int_{-1}^{1} \overline{M_k[g]}(t)f(t)\,dt \\
&= \left(f, T_k[g]\right)_\mu .
\end{aligned}$$

Hence $T_k[\cdot]$ is Hermitian. Since $C^\infty[-1,1] \subset D(T_k)$ and $C^\infty[-1,1]$ is dense in $L^2_\mu[-1,1]$, we have that $D(T_k)$ is dense in $L^2_\mu[-1,1]$. Consequently, T_k is symmetric in $L^2_\mu[-1,1]$. From Dirichlet's formula (2.4) and Theorem 2.1 (vi), we see that

$$\begin{aligned}
&\int_{-1}^{1} \overline{M_k[g]}(t) f(t)\, dt \\
&= \int_{-1}^{1} \Big\{(1-t^2)^2 f''(t)\bar{g}''(t) + \big(8+4\alpha(1-t^2)\big) f'(t)\bar{g}'(t) + k f(t)\bar{g}(t)\Big\}\, dt \\
&\quad + \overline{[g,1]}(1) - \overline{[g,1]}(-1) f(-1).
\end{aligned}$$

Combining this with equation (3.1) yields the identity:

$$\big(T_k[f], g\big)_\mu = \frac{\alpha}{2}\int_{-1}^{1} \Big\{(1-t^2)^2 f''(t)\bar{g}''(t) + \big(8+4\alpha(1-t^2)\big) f'(t)\bar{g}'(t)\Big\}\, dt + k(f,g)_\mu, \tag{3.2}$$

valid for all $f,\ g \in D(T_k)$. In particular, since

$$(1-x^2)^2|f''(x)|^2 + \big(8+4\alpha(1-x^2)\big)|f'(x)|^2 \geq 0 \quad \text{on} \quad (-1,1),$$

we have:

$$\begin{aligned}
\big(T_k[f], f\big)_\mu &= \frac{\alpha}{2}\int_{-1}^{1} \Big\{(1-t^2)^2 |f''(t)|^2 + \big(8+4\alpha(1-t^2)\big)|f'(t)|^2\Big\}\, dt + k(f,f)_\mu \\
&\geq k(f,f)_\mu .
\end{aligned} \tag{3.3}$$

Hence, $T_k[\cdot]$ is bounded below by kI in $L^2_\mu[-1,1]$. This completes the proof. ∎

Remarks. 1. Notice that the right-hand sides of equations (3.2) and (1.7) are the same.
2. Observe that $P_n^\alpha \in \Delta_k$, $n = 0,1,2,\ldots$. Furthermore, by construction, it is easy to see that $T_k[P_n^\alpha] = (\lambda_n + k)P_n^\alpha$; i.e., P_n^α is an eigenfunction of T_k with corresponding eigenvalue $\lambda_n + k$. If we substitute $f = P_n^\alpha$ and $g = P_m^\alpha$ into (3.2) and use (1.6), we arrive at the following orthogonality relationship of the derivatives of the Legendre type polynomials:

$$\begin{aligned}
&\int_{-1}^{1} \Big\{(1-t^2)^2\Big(\frac{d^2P_n^\alpha(t)}{dt^2}\Big)\Big(\frac{d^2P_m^\alpha(t)}{dt^2}\Big) + \big(8+4\alpha(1-t^2)\big)\Big(\frac{dP_n^\alpha(t)}{dt}\Big)\Big(\frac{dP_m^\alpha(t)}{dt}\Big)\Big\}\, dt \\
&= \Big[2n(n+1)(n^2+n+4\alpha-2)\big(\alpha+\frac{n(n-1)}{2}\big)\big(\alpha+\frac{(n+1)(n+2)}{2}\big)/(2n+1)\Big]\delta_{nm},
\end{aligned} \tag{3.4}$$

which is equivalent to the formula given in (1.8). Notice that this orthogonality is with respect to Lebesgue measure and not with respect to the measure μ.
3. The inequality in (3.3) is best possible. Indeed, take $f(x) = P_0^\alpha(x) \equiv \alpha$, $x \in [-1,1]$, to see this result.

4. Three associated operators. In order to prove that T_k is self-adjoint in $L^2_\mu[-1,1]$, we need to show the self-adjointness of three related operators, two of which will sum to T_k. To discuss these operators, it is essential to first discuss, in some detail, the solutions of

$$M_k[y] = 0. \tag{4.1}$$

Since the coefficients of $M_k[\cdot]$ are analytic on $(-1,1)$, we can use the Frobenius method to construct series solutions of (4.1). The endpoints $x=\pm 1$ are regular singular endpoints of $M_k[\cdot]$ and the indicial equation at both $x=\pm 1$ is $I(r)=r(r-1)(r-2)(r+1)$. Following the analysis in [6, pp. 396-403], it can be shown that four linearly independent solutions of (4.1) about $x=1$ are given by:

$$\phi_2(x)=(1-x)^2\sum_{n=0}^{\infty}a_n(x-1)^n, \qquad a_0\neq 0 \tag{4.2}$$

$$\phi_1(x)=(1-x)\sum_{n=0}^{\infty}b_n(x-1)^n+(1-x)\ln|1-x|\sum_{n=0}^{\infty}c_n(x-1)^n, \quad b_0\neq 0 \tag{4.3}$$

$$\phi_0(x)=\sum_{n=0}^{\infty}d_n(x-1)^n+\ln|1-x|\sum_{n=0}^{\infty}e_n(x-1)^n, \quad d_0\neq 0 \tag{4.4}$$

$$\phi_{-1}(x)=(1-x)^{-1}\Big\{\sum_{n=0}^{\infty}f_n(x-1)^n+\ln|1-x|\sum_{n=0}^{\infty}g_n(x-1)^n\Big\}, \quad f_0\neq 0, \tag{4.5}$$

where all series have radius of convergence 2. Here, the subscripts refer to the indicial roots. Note that $\phi_r\in L^2[0,1)$, $r=0,1,2$, but that $\phi_{-1}\notin L^2[0,1)$. Consequently, $M_k[\cdot]$ is in the limit-3 case at $x=1$. Similarly, $M_k[\cdot]$ is in the limit-3 case at $x=-1$, where there are four solutions ψ_r, $r=0,1,2,-1$, similar to the four above satisfying $\psi_r\in L^2(-1,0]$, $r=0,1,2$, but $\psi_{-1}\notin L^2(-1,0]$.

We can, in fact, describe the solutions ϕ_1, ϕ_2 in a little more detail. Indeed, since we have, for $r=0,1,2$,

$$\phi_r\in\Delta_k[0,1):=\Big\{f:[0,1)\to\mathbf{C}\,|\,f,f',f'',f^{(3)}\in AC_{\mathrm{loc}}[0,1);f,M_k[f]\in L^2[0,1)\Big\},$$

we require that the constant c_0 in (4.3) and the constants e_0, e_1 in (4.4) be zero. This follows since $(1-x)^{\sigma}\ln(1-x)\in\Delta_k[0,1)$ only when $\sigma\geq 2$. A similar simplication can be made for the solutions ψ_1, ψ_0 at the point $x=-1$.

Define $h_{\pm}\in C^4[-1,1]$ as follows: Let h_+ (respectively, h_-) take the value $+1$ in a neighbourhood of 1 (neighbourhood of -1) and the value 0 in a neighbourhood of -1 (neighbourhood of 1). Together, h_+ and h_- are linearly independent modulo the minimal domain of $M_k[\cdot]$ in $L^2(-1,1)$ and they satisfy the symmetry $[h_{\pm},h_{\pm}](\pm 1)=0$ for any choice of $\pm$ signs, where $[\cdot,\cdot]$ is the bilinear form defined by (2.3). We now define the operator A_k in $L^2(-1,1)$ by:

$$\begin{aligned} A_k[f]&:=M_k[f]\\ D(A_k)&:=\big\{f\in\Delta_k\,|\,[f,h_-](-1)=[f,h_+](1)=0\big\}. \end{aligned}$$

Theorem 4.1. *A_k in $L^2(-1,1)$ is self-adjoint. Furthermore, A_k is bounded below in $L^2(-1,1)$ by kI, where I is the identity operator on $L^2(-1,1)$.*

Proof: Because $M_k[\cdot]$ is limit-3 at $x=\pm 1$, the Naimark theory [11, §18] for the construction of self-adjoint operators requires, for the separated case, one boundary condition at each singular endpoint ± 1. The operator A_k satisfies these conditions so A_k is self-adjoint. Notice that the boundary conditions $[f,h_+](1) = [f,h_-](-1) = 0$ imply that

$$\lim_{x\to\pm 1}\left\{\left((1-x^2)^2 f''(x)\right)' - \left(8+4\alpha(1-x^2)\right)f'(x)\right\} = 0, \quad f \in D(A_k).$$

Since $D(A_k) \subset D(T_k)$, it follows from (2.4) and Theorem 2.1 that A_k is Dirchlet and, in particular, from (2.5),

$$\begin{aligned}(A_k f, f) &= \int_{-1}^{1} M_k[f](t)\bar{f}(t)\,dt \\ &= \int_{-1}^{1}\left\{(1-t^2)^2|f''(t)|^2 + (8+4\alpha(1-t^2))|f'(t)|^2 + k|f(t)|^2\right\}dt \\ &\geq k(f,f), \qquad f \in D(A_k).\end{aligned}$$

Hence $A_k[\cdot]$ is bounded below by kI, where I is the identity operator in $L^2(-1,1)$. As with $T_k[\cdot]$, this estimate is best possible. ∎

In order to discuss more properties of $A_k[\cdot]$, we first must return to our earlier study of the solutions of (4.1). Since the solutions ϕ_r, $r = 0,1,2$, are linearly dependent on the four independent solutions ψ_{-1}, ψ_0, ψ_1, ψ_2, we can find constants $\alpha_{rs} \in \mathbf{C}$ such that, for $-1 < x < 1$,

$$\phi_r(x) = \sum_{s=-1}^{2} \alpha_{rs}\psi_s(x).$$

By elimination of ψ_{-1}, if necessary, we can find two linearly independent solutions φ_1 and φ_2 of (4.1) with representations

$$\varphi_r = \sum_{s=0}^{2}\beta_{rs}\phi_s = \sum_{s=0}^{2}\gamma_{rs}\psi_s\,,$$

satisfying $\varphi_r \in \Delta_k$, $r = 1,2$.

By taking $\varphi_+ = \alpha_1\varphi_1 + \alpha_2\varphi_2$, we can choose α_1 and α_2, not both zero, so that $[\varphi_+,1](-1) = 0$. It follows then that $[\varphi_+,1](1) \neq 0$; otherwise, $\varphi_+ \in D(A_k)$. This would imply, however, that φ_+ is an eigenfunction of A_k with eigenvalue 0, contradicting the fact that A_k is bounded below by kI, $k > 0$. By scalar multiplication, we can take $[\varphi_+,1](1) = -1$.

There is a similar construction for a solution φ_- such that $[\varphi_-,1](-1) = 1$ and $[\varphi_-,1](1) = 0$. Note that φ_+ and φ_- are linearly independent; for if $\alpha\varphi_+ + \beta\varphi_- = 0$, then $[\alpha\varphi_+ + \beta\varphi_-,1](1) = 0$. i.e., $\alpha[\varphi_+,1](1) = 0$ and therefore $\alpha = 0$. Similarly, $\beta = 0$. We also note that φ_+ and φ_- are unique. Indeed, if $\varphi_{+,1}$ also satisfies the conditions $[\varphi_{+,1},1](1) = -1$ and $[\varphi_{+,1},1](-1) = 0$, then $\varphi_+ - \varphi_{+,1} \in D(A_k)$, again contradicting the fact that A_k is bounded below by kI.

We pause to state that, since A_k is bounded below by kI, then $0 \in \rho(A_k)$ the resolvent of A_k. Consequently, the resolvent operator $R_0(A_k) := A_k^{-1}$ exists and is a bounded operator from $L^2(-1,1)$ onto $D(A_k)$. This fact will be used when the operator T_k', defined below, is shown to be self-adjoint.

We define the operator T_k' in $L^2_\mu[-1,1]$ as follows:

$$T_k'[f](x) := \begin{cases} \alpha[f,1](-1) & \text{if } x=-1 \\ M_k[f](x) & \text{if } -1<x<1 \\ -\alpha[f,1](1) & \text{if } x=1 \end{cases}$$
$$D(T_k') := \Delta_k$$

Since the calculations made in Theorem 3.1 to show that T_k is symmetric in $L^2_\mu[-1,1]$ are not affected by the terms $kf(\pm 1)$, we see immediately that T_k' is symmetric in $L^2_\mu[-1,1]$. To show that T_k' is self-adjoint in $L^2_\mu[-1,1]$, we appeal to the following theorem [1, I, §41]:

Theorem 4.2. *Suppose A is a symmetric operator in a Hilbert space H. If the range of A is all of H, then A is self-adjoint.*

Theorem 4.3. *T_k' is self-adjoint in $L^2_\mu[-1,1]$.*

Proof: Let $f \in L^2_\mu[-1,1]$. We claim that

$$g(x) := \frac{f(-1)}{\alpha}\varphi_-(x) + (R_0(A_k)f)(x) + \frac{f(1)}{\alpha}\varphi_+(x)$$

is in $D(T_k')$ and satisfies $T_k'[g]=f$. Since $\varphi_\pm \in \Delta_k = D(T_k')$ and $R_0(A_k)f \in D(A_k) \subset D(T_k')$, we have $g \in D(T_k')$. By definition of $\varphi_\pm$ and the fact that $R_0(A_k)f \in D(A_k)$, we see that

$$[g,1](-1) = \frac{f(-1)}{\alpha} \quad \text{and} \quad [g,1](1) = -\frac{f(1)}{\alpha};$$

consequently, we see that $(T_k'g)(-1) = f(-1)$ and $(T_k'g)(1) = f(1)$. Also, since $\varphi_\pm$ are solutions of the homogeneous equation (4.1) and $R_0(A_k)$ is the inverse of $M_k[\cdot]$ on $(-1,1)$, we see that, for $-1<x<1$,

$$\begin{aligned}(T_k'g)(x) &= M_k\Big[\frac{f(-1)}{\alpha}\varphi_-(x) + (R_0(A_k)f)(x) + \frac{f(1)}{\alpha}\varphi_+(x)\Big] \\ &= f(x). \qquad \blacksquare\end{aligned}$$

We now define the operator T_k'' in $L^2_\mu[-1,1]$ by:

$$(T_k''f)(x) := \begin{cases} kf(-1) & \text{if } x=-1 \\ 0 & \text{if } -1<x<1 \\ kf(1) & \text{if } x=1 \end{cases}$$
$$D(T_k'') := L^2_\mu[-1,1].$$

Theorem 4.4. *T_k'' in $L^2_\mu[-1,1]$ is self-adjoint.*

Proof: It is easy to see that T_k'' is symmetric in $L^2_\mu[-1,1]$. Since the domain of T_k'' is all of $L^2_\mu[-1,1]$, it follows from a well-known result [1, I, §41] that T_k'' is self-adjoint. ∎

5. The self-adjointness of $\mathbf{T}_k$. We shall need the following theorems to prove that T_k is self-adjoint in $L^2_\mu[-1,1]$. In the statements of these theorems, H will denote a Hilbert space.

Theorem 5.1. *[11, §14.7]. Suppose A is a closed symmetric operator in H and B is a bounded, Hermitian operator in H. Then A and $A+B$ have the same deficiency indices.*

Theorem 5.2. *[11, §14.4]. A closed symmetric operator in H is self-adjoint if and only if its deficiency indices are both zero.*

Theorem 5.3. *[11, §11.4]. Suppose A is a self-adjoint operator in H. Then A is closed.*

Theorem 5.4. *Suppose A and B are closed operators in H. If B is a bounded operator, then $A+B$ is closed in H.*

Proof: Suppose $x_n \in D(A+B) = D(A) \cap D(B)$ with $x_n \to x$ and $(A+B)x_n \to y$. We are to show that $x \in D(A+B)$ and $(A+B)x = y$. Since $x_n \in D(A)$ and A is closed, we see that $x \in D(A)$; similarly $x \in D(B)$. Hence $x \in D(A+B)$. Since B is bounded,

$$\|Bx_n - Bx_m\| \leq \|B\| \, \|x_n - x_m\| \to 0 \quad \text{as } n, m \to \infty,$$

which implies that $\{Bx_n\}$ converges to, say, $w \in H$. Writing $Ax_n = (A+B)x_n - Bx_n$, we see that $\{Ax_n\}$ also converges to, say, $z \in H$. Since A and B are both closed in H, we see that $Ax = z$ and $Bx = w$. Define $u := w + z$ so $(A+B)x = u$. Since

$$\begin{aligned} \|y - u\| &= \|y - w - z\| \\ &\leq \|y - (A+B)x_n\| + \|Ax_n - z\| + \|Bx_n - w\| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

we see that $y = u$ and we have proved the theorem. ∎

Theorem 5.5. *T_k is self-adjoint in $L^2_\mu[-1,1]$.*

Proof: Observe that $T_k = T_k' + T_k''$, where we recall that T_k' is self-adjoint in $L^2_\mu[-1,1]$ and T_k'' is a bounded self-adjoint operator in $L^2_\mu[-1,1]$ (in fact, $\|T_k''\| = k$). By Theorem 5.3, both T_k' and T_k'' are closed and, from Theorems 5.4 and 3.1, T_k is therefore a closed, symmetric operator in $L^2_\mu[-1,1]$. Since T_k' is self-adjoint, its deficiency indices are both zero and hence, by Theorem 5.1, the deficiency indices of T_k are also 0. By Theorem 5.2, it now follows that T_k is self-adjoint in $L^2_\mu[-1,1]$. ∎

We note that the resolvent operator $R_0(T_k) := T_k^{-1}$ exists for $k > 0$ and can be represented in the form

$$(R_0(T_k)f)(x) = \beta_-\varphi_-(x) + (R_0(A_k)f)(x) + \beta_+\varphi_+(x) \tag{5.1}$$

for all $x \in [-1,1]$ and all $f \in L^2_\mu[-1,1]$ where $\beta_\pm$ are determined uniquely by $\beta_\pm = [R_0(T_k)f, 1](\pm 1)$.

Recall that the Legendre type polynomials (1.2) satisfy:

$$T_k[P_n^\alpha] = (\lambda_n + k)P_n^\alpha$$

i.e., $P_n^\alpha(x)$ is an eigenfunction of $T_k[\cdot]$. From a general theorem ([14, §3.1]) in Szegö, we know that the Legendre type polynomials are complete in $L^2_\mu[-1,1]$. Consequently, the self-adjointness of T_k and the completeness of these polynomials in $L^2_\mu[-1,1]$ imply that the spectrum of T_k is given by

$$\sigma(T_k) = \{\lambda_n + k \mid n = 0, 1, 2, \dots\};$$

i.e., $\sigma(T_k)$ is discrete and bounded below and all eigenvalues are simple.

Since $T_k[P_n^\alpha] = (\lambda_n + k)P_n^\alpha$, we see that

$$\begin{aligned}(\lambda_n + k)P_n^\alpha(\pm 1) &= (T_k[P_n^\alpha])(\pm 1) \\ &= \mp\alpha[P_n^\alpha, 1](\pm 1) + kP_n^\alpha(\pm 1).\end{aligned} \tag{5.2}$$

But since $P_n^\alpha(x)$ is a polynomial, it follows from (2.7) that

$$[P_n^\alpha, 1](\pm 1) = -8\frac{d(P_n^\alpha(x))}{dx}\bigg|_{x=\pm 1}. \tag{5.3}$$

From (5.2) and (5.3), we see that:

$$\begin{aligned}\lambda_n P_n^\alpha(-1) &= -8\alpha\frac{d(P_n^\alpha(x))}{dx}\bigg|_{x=-1} \\ \lambda_n P_n^\alpha(1) &= 8\alpha\frac{d(P_n^\alpha(x))}{dx}\bigg|_{x=1}.\end{aligned} \tag{5.4}$$

These are the λ-dependent boundary conditions discussed in [7] but they are satisfied only by the eigenfunctions of T_k and not, in general, by the elements of $D(T_k)$. In this sense, the equations in (5.4) are to be seen as a property of the Legendre type polynomials but not as an essential element in the definition of T_k.

Remark. If we define the operator T in $L_\mu^2[-1,1]$ by:

$$\begin{aligned}T[f] &:= (T_k - kI)[f] \\ D(T) &:= D(T_k),\end{aligned}$$

where I is the identity operator on $L_\mu^2[-1,1]$, then T is self-adjoint in $L_\mu^2[-1,1]$ and its spectrum is $\sigma(T) = \{\lambda_n \mid n = 0, 1, 2, \ldots\}$. However note that T, which is the operator A that Krall defined in [7], does not have an inverse and $R_0(T)$ is not defined on the whole of $L_\mu^2[-1,1]$. Indeed, in this case, 0 is an eigenvalue of T.

6. The left-definite problem. In this section, we shall assume that $k > 0$. Define the space H by

$$H := \Big\{f : [-1,1] \to \mathbf{C} \mid f \in AC[-1,1], f' \in AC_{\text{loc}}(-1,1),\ f', (1-x^2)f'' \in L^2(-1,1)\Big\}.$$

Then H is a Hilbert space with inner product given by (1.7); i.e.,

$$(f,g)_H := \frac{\alpha}{2}\int_{-1}^{1}\Big\{(1-t^2)^2 f''(t)\bar{g}''(t) + \big(8 + 4\alpha(1-t^2)\big)f'(t)\bar{g}'(t)\Big\}dt + k(f,g)_\mu,\ \ f,g \in H.$$

The completeness of H can be shown by classical arguments; see [12], where similar analysis is used with the left-definite spaces associated with the classical orthogonal polynomials. We emphasize that it is essential for k to be positive; otherwise, if $k = 0$ the equation $(f,f)_H = 0$ implies that f is only constant almost everywhere and not necessarily the zero function in H. Observe that from Theorem 2.1, $\Delta_k = D(T_k)$ is a linear manifold of H. Furthermore, it is easy to see that

$$H \subset L^2_\mu[-1,1] \quad \text{and} \quad k\|f\|^2_\mu \leq \|f\|^2_H, \ f \in H. \tag{6.1}$$

From (3.2), we see that

$$\big(T_k[f], g\big)_\mu = (f,g)_H = \big(f, T_k[g]\big)_\mu \quad \text{for all } \ f, g \in D(T_k).$$

By mimicking the arguments in Theorem 2.1 we can, in fact, show that:

$$\big(T_k[f], g\big)_\mu = (f,g)_H \quad \text{for all } f \in D(T_k), \ \ g \in H \tag{6.2}$$

$$\big(f, T_k[g]\big)_\mu = (f,g)_H \quad \text{for all } f \in H, \ \ g \in D(T_k) \tag{6.3}$$

Define the operator $B_k : H \to H$ by

$$\begin{aligned} B_k[f] &:= R_0(T_k)f \\ D(B_k) &:= H, \end{aligned}$$

where $R_0(T_k)$ is the resolvent operator of T_k, defined by (5.1), at the point $0 \in \rho(T_k)$. By (6.1), we see that $B_k[f]$ is well-defined; furthermore, since $R_0(T_k)f \in D(T_k) = \Delta_k \subset H$, we see that B_k does indeed map H into H.

Now, from (6.2) and since $T_k^{-1} = R_0(T_k)$, we have for all $f, g \in H$,

$$\big(B_k[f], g\big)_H = \big(R_0(T_k)f, g\big)_H = \big(T_k[R_0(T_k)f], g\big)_\mu = (f,g)_\mu \tag{6.4}$$

Similarly, from (6.3), we see that for all $f, g \in H$,

$$\big(f, B_k[g]\big)_H = (f,g)_\mu\,. \tag{6.5}$$

Hence equations (6.4) and (6.5) imply that B_k is symmetric in H and since the domain of B_k is all of H, it follows [1, I, §41] that B_k is self-adjoint and bounded in H. Furthermore, suppose $B_k[f] = 0$, for some $f \in H$. Apply T_k to this equation to get $f \equiv 0$ in H. Hence, $S_k := B_k^{-1}$ exists and, from [1, I, §41], S_k is a self-adjoint operator in H. Noting that $P_n^\alpha \in D(T_k)$, we have that $P_n^\alpha \in H$ and from the fact that $T_k[P_n^\alpha] = (\lambda_n + k)P_n^\alpha$, we obtain:

$$P_n^\alpha = R_0(T_k)\big(T_k[P_n^\alpha]\big) = (\lambda_n + k)R_0(T_k)P_n^\alpha = (\lambda_n + k)B_k[P_n^\alpha].$$

Now apply S_k to obtain

$$S_k[P_n^\alpha] = (\lambda_n + k)P_n^\alpha, \quad n = 0, 1, 2, \ldots$$

Thus, in H, the Legendre type polynomials $\{P_n^\alpha(x)\}$ are eigenfunctions of S_k with eigenvalues $\lambda_n + k$, $n = 0, 1, 2, \ldots$. From this result, we find that S_k is unbounded since $\lim_{n\to\infty}(\lambda_n + k) = \infty$. Also, the Legendre type polynomials are orthogonal in H; we established this in §3, although the orthogonality would follow from the facts that S_k is self-adjoint and the Legendre type polynomials are eigenfunctions of S_k. Equations (1.8) and (3.4) show explicitly this orthogonality relationship. The authors believe that the Legendre type polynomials are complete in H; this will be looked at in the future. If this is true, then the spectrum of S_k is

$$\sigma(S_k) = \{\lambda_n + k \mid n = 0, 1, 2, \dots\}.$$

Unfortunately, as with the case of the left-definite Legendre polynomial problem [4], there is no explicit representation for S_k; it does not seem possible to characterize the operator S_k in terms of the fourth-order differential expression $M_k[\cdot]$. The reason for this is that B_k^{-1} must be computed in H and not in $L^2_\mu[-1,1]$. Also in this case, the operator B_k is less explicit since the formula (5.1) depends on the resolvent $R_0(A_k)$ for which we have no explicit formula.

In one sense we have used some sophisticated (although well-known) operator theorems in Hilbert space and it is not surprising that we have to lose some measure of explicit representation.

7. Half-range Legendre type series. We now discuss some half-range expansions of the Legendre type polynomials. This is similar to the discussion in [4] of half-range Legendre expansions.

For example, we work in the space $L^2_\mu[0,1]$ and define

$$\Delta_k[0,1) = \left\{f : [0,1) \to \mathbf{C} \mid f, f', f'', f^{(3)} \in AC_{\text{loc}}[0,1); f, M_k[f] \in L^2[0,1)\right\}$$

In this case, Theorem 2.1 applies to $\Delta_k[0,1)$, specifically at $x = +1$. The inner product in $L^2_\mu[0,1)$ is

$$(f,g)^+_\mu = \frac{\alpha}{2}\int_0^1 f(t)\bar{g}(t)\,dt + \frac{f(1)\bar{g}(1)}{2}.$$

We define the operators $T_{k,E}$ and $T_{k,0}$ as follows:

$$T_{k,E}[f](x) = T_{k,0}[f](x) := \begin{cases} -\alpha[f,1](1) + kf(1) & \text{if } x = 1 \\ M_k[f](x) & \text{if } 0 \le x < 1 \end{cases}$$

$$D(T_{k,E}) := \{f \in \Delta_k[0,1) \mid f(0) = f''(0) = 0\}$$
$$D(T_{k,0}) := \{f \in \Delta_k[0,1) \mid f'(0) = f^{(3)}(0) = 0\}$$

We introduce the Naimark operators $A^+_{k,E}$ and $A^+_{k,0}$, as defined in §4 :

$$A^+_{k,E}[f] = A^+_{k,0}[f](x) = M_k[f](x) \quad x \in [0,1)$$
$$D(A^+_{k,E}) := \{f \in \Delta_k[0,1) \mid [f,1](1) = 0,\ f(0) = f''(0) = 0\}$$
$$D(A^+_{k,0}) := \{f \in \Delta_k[0,1) \mid [f,1](1) = 0,\ f'(0) = f^{(3)}(0) = 0\}.$$

Then, as by Theorem 4.1, $A^+_{k,E}$ and $A^+_{k,0}$ are self-adjoint in $L^2[0,1)$. By entirely the same procedure as in Sections four and five, the self-adjointness of $T_{k,E}$ and $T_{k,0}$ can be established. The eigenfunctions of $T_{k,E}$ are the even Legendre type polynomials $\{P^\alpha_{2n}(x)\}$, and the eigenfunctions of $T_{k,0}$ are the odd Legendre type polynomials $\{P^{\alpha}_{2n+1}(x)\}$; both sets of eigenfunctions are orthogonal and complete in $L^2_\mu[0,1)$. Similar arguments can be made in the left-definite half-range cases.

Acknowledgement. The second author (L.L.L.) thanks the first author, Norrie Everitt, for the opportunity, financial support and kind hospitality extended to him during a visit to The University of Birmingham, England in the Spring of 1987. Both authors are grateful to Dr. J. Gunson, of the Department of Mathematics in Birmingham, for discussion on the use of the Frobenius method for solutions of linear differential equations.

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