DIFFERENTIAL OPERATORS ON HOMOGENEOUS SPACES

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Introduction. Among all linear differential operators in Euclidean space $\mathbf{R}^n$, those that have constant coefficients are characterized by their invariance under the transitive group of all translations. The special role played by Laplace's equation is partly due to its invariance under all rigid motions. Another example of physical importance is the wave equation which can essentially be characterized by its invariance under the Lorentz group. This implicit physical significance of the Lorentz group so far as electromagnetic phenomena is concerned is made explicit in Einstein's special theory of relativity. Here the Lorentz group is given an interpretation in terms of pure mechanics.

In the present paper a study is made of differential operators on a manifold under the assumption that these operators are invariant under a transitive group G of "automorphisms" of this manifold M. Let p be a point of M, H the subgroup of G leaving p fixed and M_p the tangent space to M at p. It is easy to set up a linear correspondence between the set of invariant differential operators on M and the set of all polynomials on M_p that are invariant under the action of the isotropy group H at p. However, the multiplicative properties of this correspondence are complicated and are better understood (at least in case G/H is reductive) by describing the differential operators by means of the Lie algebras of G and H (Theorem 10).

Our purpose is to study various geometrical properties of solutions of differential equations involving these invariant operators. We give now a summary of the different chapters.

Chapter I contains a general discussion of linear differential operators on manifolds. On pseudo-Riemannian manifolds there is always one differential operator, the Laplace-

Beltrami operator, which is invariant under all isometries but under no other diffeomorphisms.

Chapter II. In § 1 we recall some essentially known results on transitive transformation groups and homogeneous spaces. Two-point homogeneous spaces admit essentially only one invariant differential operator, the Laplace-Beltrami operator. Potential theory has a particularly explicit character. In § 3 we prove some properties of these spaces which are used later, e.g., the symmetry of non-compact two-point homogeneous spaces. A fairly direct proof of this fact is possible, but for the compact spaces such a proof seems to be unknown although the symmetry can be verified by means of Wang's classification. In § 4 we investigate in some detail Lorentzian spaces of constant curvature and the behavior of the geodesics on these spaces. For the spaces of negative curvature (simply connected) the timelike geodesics through a given point are infinite and do not intersect each other. The spaces of positive curvature that we consider have infinite cyclic fundamental group. Their timelike geodesics through a given point are all closed and do not intersect each other.

Chapter III. In § 1 we represent the algebra $\mathbf{D}(G/H)$ of invariant differential operators by means of the symmetric invariants of the group $Ad_G(H)$. Thus if H is semi-simple, $\mathbf{D}(G/H)$ has a finite system of generators. If G/K is a Riemannian symmetric space, $\mathbf{D}(G/K)$ is finitely generated and commutative (Gelfand [11], Selberg [36]). For Lorentz spaces of constant curvature (or two-point homogeneous spaces) $\mathbf{D}(G/H)$ is generated by the Laplace-Beltrami operator.

Chapter IV. We consider in § 1 the mean value operators M^x which in a natural way generalize the operation M^r of averaging over spheres in $\mathbf{R}^n$ of fixed radius r. It is well known that M^r is formally a function (Bessel function) of the Laplacian Δ. The analogue holds for the space G/K if K is compact. In fact M^x is formally a function of the generators $D^1, \ldots, D^l$ of $\mathbf{D}(G/K)$. This has some applications, for example a generalization of the mean value theorem of Ásgeirsson. For two-point homogeneous spaces we obtain more explicit results, for example a simple geometric solution of Poisson's equation. In § 4 is given for a Riemannian space of constant curvature a decomposition of a function into integrals over totally geodesic submanifolds. A somewhat analogous problem is treated in § 7 for a Lorentzian space of constant curvature. Here a function is represented by means of its integrals over Lorentzian spheres. We use here methods of analytic continuation introduced by M. Riesz in his treatment of the wave equation. In § 8 we verify that Huygens' principle in Hadamard's formulation is absent for non-flat harmonic Lorentz spaces.(1)

(1) This confirms, in a very special case, a well-known conjecture attributed to Hadamard.

An outline of the results of this paper (with the exception of Ch. II, § 4 and Ch. IV, §§ 5–8) was given [25] at the Scandinavian Mathematical Congress in Helsinki, August 1957. An exposition was given in a course at the University of Chicago, Spring 1958. I am grateful to Professor Ásgeirsson for advice concerning some problems dealt with in Chapter IV, § 3. I am also grateful to Professor Harish-Chandra for interesting conversations about the topic of his paper [21].

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CHAPTER I

Preliminary remarks on differential operators

Let M be a locally connected topological space with the property that each connected component of M is a differentiable manifold of class C^∞ and dimension n. We shall then say that M is a C^∞-manifold of dimension n. We shall only be dealing with separable C^∞-manifolds and will simply refer to them as manifolds. If p is a point on the manifold M, the tangent space to M at p will be denoted by M_p. The set of real valued indefinitely differentiable functions on M constitutes an algebra $C^\infty(M)$ over the real numbers $\mathbf{R}$, the multiplication in $C^\infty(M)$ being given by pointwise multiplication of functions. The functions in $C^\infty(M)$ that have compact support form a subalgebra $C_c^\infty(M)$. We use the topology on $C_c^\infty(M)$ which is familiar from the theory of distributions (L. Schwartz [33] I, p. 67), and is based on uniform convergence of sequences of functions and their derivatives. The derivations $\mathfrak{D}$ of the algebra $C^\infty(M)$ are the C^∞-vector fields on M; each $X \in \mathfrak{D}$ leaves $C_c^\infty(M)$ invariant. An endomorphism of a vector space V is a linear mapping of V into itself. If D is an endomorphism of $C^\infty(M)$ and $f \in C^\infty(M)$ then $[Df](p)$ will always denote the value of Df at $p \in M$. If $X \in \mathfrak{D}$, then the linear functional X_p on $C^\infty(M)$ defined by $X_p(f) = [Xf](p)$ for $f \in C^\infty(M)$ is a tangent vector(1) to M at p, that is $X_p \in M_p$. Let $\mathbf{R}^n$ denote the Euclidean n-space with a fixed coordinate system. If the mapping $\Psi : x \to (x_1, \ldots, x_n) \in \mathbf{R}^n$ is a local coordinate system valid in an open subset $U \subset M$, we shall often write f^* for the composite function $f \circ \Psi^{-1}$ defined on $\Psi(U)$. We also write D_i for the partial differentiation $\partial/\partial x_i$ and if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of indices $\alpha_i \geqslant 0$ we put $D^\alpha = D_n^{\alpha_1} \ldots D_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

DEFINITION. A continuous endomorphism D of $C_c^\infty(M)$ is called a *differential operator* on M if it is of local character. This means that whenever U is an open set in M and $f \in C_c^\infty(M)$ vanishes on U, then Df vanishes on U.

PROPOSITION 1.(2) *Let D be an endomorphism of $C_c^\infty(M)$ which has the following property. For each $p \in M$ and each open connected neighborhood U of p on which the local coordinate system $\Psi : x \to (x_1, \ldots, x_n)$ is valid there exists a finite set of functions a_α of class C^∞ such that for each $f \in C_c^\infty(M)$ with support contained in U*

$$[Df](x) = \sum_\alpha a_\alpha(x) [D^\alpha f^*](x_1, \ldots, x_n) \qquad \textit{for } x \in U$$

$$[Df](x) = 0 \qquad \textit{for } x \notin U.$$

Then D is a differential operator on M and each differential operator on M has the property above.

Proof. Let E be a differential operator, p, U and Ψ as above. Let V be an open subset of U whose closure $\overline{V}$ is compact and contained in U. Let $C_{\overline{V}}(M)$ and $C_V(M)$ denote the the set of functions $f \in C_c^\infty(M)$ with compact support contained in $\overline{V}$ and V respectively. The operator E induces a continuous endomorphism of $C_{\overline{V}}(M)$. This implies that for each $\varepsilon > 0$ there exists an integer m and a real number $\delta > 0$ such that

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(1) We use here and often in the sequel the terminology of Chevalley [7].

(2) This proposition is attributed to L. Schwartz in A. Grothendieck, *Sur les espaces de solutions d'une classe générale d'équations aux dérivées partielles. J. Analyse Math. 2* (1953) 243–280.

$$|Ef(x)| < \varepsilon \qquad \text{for all } x \in \bar{V}$$

whenever $[D^{\alpha}f^*](x_1, \ldots, x_n)$ is in absolute value less than δ for all $(x_1, \ldots, x_n) \in \Psi(\bar{V})$ and all α satisfying $|\alpha| \leqslant m$. For a fixed point $x \in V$ we put $T(f^*) = [Ef](x)$ for all $f \in C_V(M)$. The linear functional $f^* \to T(f^*)$ is then a distribution on $\Psi(V)$ of order $\leqslant m$ in the sense of [33] I, p. 25. From the local character of E it follows that this distribution has support at the point $\Psi(x)$. Due to Schwartz' theorem on distributions with point supports (*loc. cit.* p. 99), $T(f^*) = [Ef](x)$ can be written as a finite sum

$$[Ef](x) = \sum_{|\alpha| \leqslant m} a_{\alpha}(x)\,[D^{\alpha} f^*](x_1, \ldots, x_n) \tag{1.1}$$

where the coefficients $a_{\alpha}(x)$ are certain constants. Each constant $a_{\alpha}(x)$ varies differentiably with x as is easily seen by choosing f such that $D^{\alpha}f^*$ is constant in a neighborhood of $\Psi(x)$. The representation (1.1) holds for all $x \in V$ and all $f \in C_V(M)$. However, since U can be covered by a chain of open sets each of which has compact closure it is easily seen that (1.1) is valid for all $x \in U$ and all $f \in C_c^{\infty}(M)$ with support contained in U.

On the other hand, let D be an endomorphism of $C_c^{\infty}(M)$ with the properties described in the proposition. D is obviously of local character. Also D is continuous on the subspace of functions that have support inside a fixed coordinate neighborhood. Using the well-known technique of partition of unity (see for example [7], p. 163), D is seen to be continuous on the entire $C_c^{\infty}(M)$.

A differential operator on M can be extended to an endomorphism of $C^{\infty}(M)$ such that the condition of local character holds for all $f \in C^{\infty}(M)$. This extension is unique.

Let Φ be a homeomorphism of M onto itself such that Φ and Φ^{-1} are differentiable mappings. The mapping Φ is then called a diffeomorphism of M. If $p \in M$, the differential $d\Phi_p$ maps M_p onto $M_{\Phi(p)}$ in such a way that $d\Phi_p(X_p)f = X_p(f \circ \Phi)$. For each C^{∞} – vector field X on M we obtain a new vector field X^{Φ} by putting $X^{\Phi} f = (X(f \circ \Phi)) \circ \Phi^{-1}$ for $f \in C^{\infty}(M)$. It follows then that $(X^{\Phi})_{\Phi(p)} = d\Phi_p(X_p)$ and we often write $d\Phi \cdot X$ instead of X^{Φ}. If A is an endomorphism of $C^{\infty}(M)$ we define the operator A^{Φ} in accordance with the notation above by $A^{\Phi}(f) = (A(f \circ \Phi)) \circ \Phi^{-1}$. If D is a differential operator on M, then so is the operator D^{Φ}. The transformation Φ is said to leave D *invariant* if $D^{\Phi} = D$. We sometimes write f^{Φ} for the composite function $f \circ \Phi^{-1}$. We have then the convenient rule $A^{\Phi} f = (A f^{\Phi^{-1}})^{\Phi}$.

Let M be a connected manifold. A *linear connection* on M is a rule which assigns to each $X \in \mathfrak{D}$ a linear operator ∇_X on $\mathfrak{D}$ satisfying the following two conditions

(i) $\qquad \nabla_{fX+gY} = f\nabla_X + g\nabla_Y$

(ii) $\qquad \nabla_X(fY) = f\nabla_X(Y) + (Xf)\,Y$

for $f, g \in C^\infty(M)$, $X, Y \in \mathfrak{D}$. The operator ∇_X is called *covariant differentiation* with respect to X. This definition of a linear connection is adopted in K. Nomizu [33], and we refer to this paper for a treatment of concepts in the theory of linear connections such as parallelism, curvature and torsion tensor. A curve in M is called a *path* if it has a parameter representation such that all its tangent vectors are parallel. Let p be a point in M and $X \neq 0$ a vector in M_p. There exists a unique parametrized path $t \to \gamma_X(t)$ such that $\gamma_X(0) = p$ and $\gamma'_X(t) = X$. The parameter t is called the *canonical parameter* with respect to X. We put $\gamma_0(t) = p$. The mapping $X \to \gamma_X(1)$ is a one-to-one C^∞-mapping of a neighborhood of 0 in M_p onto a neighborhood of p in M. This mapping is called Exp (the Exponential mapping at p) and will often be used in the sequel.

A pseudo-Riemannian metric Q on a connected manifold M is a rule which in a differentiable way assigns to each $p \in M$ a non-singular symmetric real bilinear form Q_p on the tangent space M_p. Since M is connected the signature of Q_p is the same for all p. If the signature is $+ + \cdots +$ we call M a Riemannian space; if the signature is $+ - - \cdots -$ we speak of a Lorentzian space, otherwise of a pseudo-Riemannian manifold. On a pseudo-Riemannian manifold there exists one and only one linear connection (the pseudo-Riemannian connection) satisfying the conditions: 1°. The torsion is 0. 2.° The parallel displacement preserves the inner product Q_p on the tangent spaces. In the case of a Riemannian space, arc length can be defined for all differentiable curves. The space can then be metrized by defining the distance between two points as the greatest lower bound of length of curves joining the two points. For a Lorentzian space where this procedure fails we adopt the following terminology from the theory of relativity. The cone C_p in the tangent space M_p given by $Q_p(X, X) = 0$ is called the *null cone* or the *light cone* in M_p with vertex p. A vector $X \in M_p$ is called *timelike, isotropic,* or *spacelike* if $Q_p(X, X)$ is positive, 0, or negative respectively. Similarly we use the terms timelike, isotropic, and spacelike for rays (oriented half lines) or unoriented straight lines in M_p. A timelike curve is a curve each of whose tangent vectors is timelike. Such curves have well-defined arc length. If a path has timelike tangent vector at a point, then all of its tangent vectors are timelike and the path is called a timelike path. A curve is said to have length 0 if all its tangent vectors are isotropic.

A diffeomorphism Φ of a pseudo-Riemannian manifold M is called an *isometry* if

$$Q_p(X, Y) = Q_{\Phi(p)}(d\Phi_p X, d\Phi_p Y)$$

for each $p \in M$ and each $X, Y \in M_p$. The group of all isometries of M will be denoted by $\mathbf{I}(M)$. Let U be an open neighborhood of p on which local coordinates $x \to (x_1, \ldots, x_n)$ are valid. We put

$$q_{ij}(x) = Q_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

and define the functions $q^{jk}(x)$, $q(x)$ on U by

$$\sum_k q^{jk}(x)\, q_{jk}(x) = \delta_i^j, \quad q(x) = |\det(q_{ij}(x))|,$$

δ_i^j being Kronecker deltas. For each function $f \in C_c^\infty(\mathrm{M})$ we set

$$\Delta f = \frac{1}{\sqrt{q(x)}} \sum_k \frac{\partial}{\partial x_k}\left(\sum_i q^{ik}(x) \sqrt{q(x)}\, \frac{\partial f^*}{\partial x_i}\right).$$

The expression on the right is invariant under coordinate changes due to the classical transformation formulas for the functions q_{ij}. It is easily seen that Δ is a differential operator on M. It is called the Laplace-Beltrami operator.

PROPOSITION 2. *Let Φ be a diffeomorphism of M. Then Φ leaves the Laplace-Beltrami operator invariant if and only if it is an isometry.*

Proof. Let $p \in M$ and let U be a neighborhood of p on which local coordinates $x \to (x_1, \ldots, x_n)$ are given. Then $\Phi(U)$ is a neighborhood of the point $q = \Phi(p)$ and $y \to (y_1, \ldots, y_n)$ where $y = \Phi(x)$, $y_i = x_i$ $(i = 1, 2, \ldots, n)$ is a local coordinate system on $\Phi(U)$. We also have

$$d\,\Phi_x\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \qquad (i = 1, 2, \ldots, n).$$

For each function $f \in C_c^\infty(M)$ we have

$$[(\Delta f)\Phi^{-1}](x) = [\Delta f](y) = \frac{1}{\sqrt{q(y)}} \sum_k \frac{\partial}{\partial y_k}\left(\sum_i q^{ik}(y) \sqrt{q(y)}\, \frac{\partial f^*}{\partial y_i}\right) \tag{1.2}$$

$$[\Delta f \Phi^{-1}](x) = \frac{1}{\sqrt{q(x)}} \sum_k \frac{\partial}{\partial x_k}\left(\sum_i q^{ik}(x) \sqrt{q(x)}\, \frac{\partial (f \circ \Phi)^*}{\partial x_i}\right). \tag{1.3}$$

Due to the choice of coordinates we have

$$\frac{\partial f^*}{\partial y_i} = \frac{\partial (f \circ \Phi)^*}{\partial x_i}, \quad \frac{\partial^2 f^*}{\partial y_i \partial y_k} = \frac{\partial^2 (f \circ \Phi)^*}{\partial x_i \partial x_k} \qquad (i, k = 1, 2, \ldots, n).$$

Now, if Φ is an isometry, then $q_{ik}(x) = q_{ik}(y)$ for all i, k so the expressions (1.2) and (1.3) are the same and $\Delta^\Phi = \Delta$. On the other hand, if (1.2) and (1.3) agree we obtain by equating coefficients, $q_{ik}(x) = q_{ik}(y)$ for all i, k, which shows that Φ is an isometry.

For Lorentzian spaces the Laplace-Beltrami operator will always be denoted by $\square$.

CHAPTER II

Homogeneous spaces

1. The analytic structure of a coset space

Let G be a separable Lie group and H a closed subgroup. The identity element of a group will always be denoted by e. Let $L(g)$ and $R(g)$ denote the left and right translations of G onto itself given by $L(g)\cdot x = gx$, $R(g)\cdot x = xg^{-1}$. The system G/H of left cosets gH has a unique topology with the property that the natural projection π of G onto G/H is a continuous and open mapping. This is called the *natural topology* of the coset space G/H. In this topology G/H is a locally connected Hausdorff space. For each $x \in G$, the mapping $\tau(x) : gH \to xgH$ is a homeomorphism of G/H onto itself. The connected components of G/H are all homeomorphic to $G_0/G_0 \cap H$ where G_0 is the identity component of G. The point $\pi(e)$ will usually be denoted by p_0. For later purposes we need Lemma 1 below which gives a special local cross-section in G, considered as a fiber bundle over G/H. The group H is a Lie group, regularly imbedded in G and thus the Lie algebra $\mathfrak{h}$ ($= H_e$) of H can be regarded as a subalgebra of the Lie algebra $\mathfrak{g}$ ($= G_e$) of G. We choose a fixed complementary subspace to $\mathfrak{h}$ in $\mathfrak{g}$ and denote it by $\mathfrak{m}$. Let exp denote the usual exponential mapping of $\mathfrak{g}$ into G and Ψ its restriction to $\mathfrak{m}$.

LEMMA 1. *There exists a neighborhood U of 0 in $\mathfrak{m}$ which is mapped homeomorphically under Ψ and such that π maps $\Psi(U)$ homeomorphically onto a neighborhood of p_0 in G/H.*

Proof. Consider the mapping $\Phi : (X, Y) \to \exp X \cdot \exp Y$ of the product space $\mathfrak{m} \times \mathfrak{h}$ into G. We choose a basis $X_1, \ldots, X_r$ of $\mathfrak{g}$ such that the first n elements form a basis of $\mathfrak{m}$ and the $r-n$ last elements form a basis of $\mathfrak{h}$. Let $x_1, \ldots, x_r$ be a system of canonical coordinates with respect to this basis, valid in a neighborhood V' of e in G. For sufficiently small t_i the element

$$\exp(t_1 X_1 + \cdots + t_n X_n) \exp(t_{n+1} X_{n+1} + \cdots + t_r X_r)$$

belongs to V' and its canonical coordinates are given by $x_j = \xi_i(t_1, \ldots, t_r)$ where ξ_i are analytic functions in a neighborhood of 0 in $\mathfrak{g}$. The Jacobian determinant of the transformation $(t_1, \ldots, t_r) \to (x_1, \ldots, x_r)$ is $\neq 0$ in a neighborhood of 0 in $\mathfrak{g}$. There exists therefore a neighborhood of 0 in $\mathfrak{g}$ of the form $N_1 \times N_2$ where $N_1 \subset \mathfrak{m}$, $N_2 \subset \mathfrak{h}$ which Φ maps homeomorphically onto an open subset V'' of V'. Choose a neighborhood V of e such that $V^{-1}\cdot V \subset V''$. Let U be a compact neighborhood of 0 in $\mathfrak{m}$ contained in $\Phi^{-1}(V) \cap N_1$. Then Ψ maps U homeomorphically onto $\Psi(U)$. Also π maps $\Psi(U)$ in a one-to-one fashion because otherwise there exist $X_1, X_2 \in U$ and $h \in H$, $h \neq e$ such that $\exp X_1 = \exp X_2 h$. It follows that $h \in V''$ and there exists a $Y \in N_2$ such that $h = \exp Y$, $Y \neq 0$. The elements $(X_1, 0)$ and (X_2, Y) belong to $N_1 \times N_2$ and have the same image under Φ which is a contradiction. The set $\Psi(U)$, being compact, is mapped homeomorphically by π and the image $\pi(\Psi(U))$ is a neighborhood of p_0 in G/H because $\pi(\Psi(U)) = \pi(\Psi(U)H)$, $\Psi(U)H$ is a neighborhood of e in G and π is an open mapping.

THEOREM 1. *The coset space G/H has a unique analytic structure with the property that the mapping $(x, gH) \to xgH$ is a differentiable mapping of the product manifold $G \times G/H$ onto G/H. A coset space G/H will always be given this analytic structure.*

Proof. We use the terminology of Lemma 1 and introduce coordinates in G/H as follows. For each $p \in G/H$ we can find a $g \in G$ such that $\pi(g) = p$; let N_p denote the interior of the set $\pi(g\Psi(U))$. Then the mapping

$$\pi(g \exp(t_1 X_1 + \cdots + t_n X_n)) \to (t_1, \ldots, t_n)$$

is a system of coordinates valid on N_p. It is not difficult to show that this procedure defines an analytic structure on G/H with the property that the mapping $(x, gH) \to xgH$ is an analytic mapping of $G \times G/H$ onto G/H (Chevalley [7], p. 111). The uniqueness statement is contained in the following theorem.

THEOREM 2. *Let G be a separable, transitive Lie group of diffeomorphisms of a manifold M. Assume that the mapping $(g, q) \to g \cdot q$ of $G \times M$ onto M is continuous. Let p be a point on M and G_p the subgroup of G that leaves p fixed. Then G_p is closed and the mapping $g \cdot p \to gG_p$ is a diffeomorphism of M onto G/G_p in the analytic structure defined above.*

If M is connected, then G_0, the identity component of G, is transitive on M.

Proof. We first prove (following R. Arens, "Topologies for homeomorphism groups", *Amer. J. Math.* 68 (1946), 593–610) that the coset space G/G_p, in its natural topology, is homeomorphic to M. For this it suffices to prove that the mapping $\Phi: g \to g \cdot p$ of G onto M is an open mapping. Let V be a compact symmetric neighborhood of e in G; then there exists a sequence $(g_n) \in G$ such that $G = \bigcup_n g_n V$. Thus $M = \bigcup_n g_n V \cdot p$ and it follows by a category argument that at least one of the summands has an inner point. Hence $V \cdot p$ has an inner point, say $h \cdot p$ where $h \in V$. Then p is an inner point of $h^{-1} V \cdot p \subset V^2 \cdot p$ so Φ is an open mapping. In particular $\dim G/G_p = \dim M$.

Consider now the interior B of the subset $\Psi(U)$ from Lemma 1. B is a submanifold of G because $(t_1, \ldots, t_n)$ and $(t_1, \ldots, t_r)$ are local coordinates of the points

$$\exp\,(t_1 X_1 + \cdots + t_n X_n)$$

and
$$\exp\,(t_1 X_1 + \cdots + t_n X_n)\ \exp\,(t_{n+1} X_{n+1} + \cdots + t_r X_r)$$

in B and G respectively and thus the injection i of B into G is regular. By the definition of the analytic structure of G/G_p, π is a differentiable transformation of B onto an open subset N of G/G_p. Due to a theorem of S. Bochner and D. Montgomery ("Groups of differentiable and real or complex analytic transformations", *Ann. of Math.* 46 (1945), 685–694), the continuous mapping $(g, q) \to g \cdot q$ is automatically differentiable. The mapping $gG_p \to g \cdot p$ is a homeomorphism of N onto an open set in M and is differentiable since it is of the form $\Phi \circ i \circ \pi^{-1}$. To show that the inverse is differentiable we just have to show that the Jacobian of Φ at $g = e$ has rank equal to dim M. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and G_p respectively. We shall prove that if $X \in \mathfrak{g}$ and $X \notin \mathfrak{h}$ then $(d\Phi)_e X \neq 0$, in other words the Jacobian of Φ at e has rank equal to $\dim \mathfrak{g} - \dim \mathfrak{h} = \dim M$. Suppose to the contrary that $(d\Phi)_e X = 0$; then if $f \in C^\infty(M)$ we have

$$0 = (d\,\Phi)_e\, X \cdot f = X_e\,(f \circ \Phi) = \frac{d}{d\,t} f\,(\exp\ tX \cdot p)\Big\}_{t=0}.$$

If we use this relation on the function $f^*(q) = f\,(\exp\, sX \cdot q)$ we obtain

$$0 = \frac{d}{dt} f^*(\exp tX \cdot p)\Big\}_{t=0} = \frac{d}{d\,s} f\,(\exp\ sX \cdot p)$$

which shows that $f(\exp sX \cdot p)$ is constant in s. Hence $\exp sX \cdot p = p$ and $X \in \mathfrak{h}$. This shows that M is diffeomorphic to G/G_p. For the last statement of the theorem consider a sequence $(x_n) \in G$ such that $G = \bigcup_n G_0 x_n$. Each orbit $G_0 x_n \cdot p$ is an open subset of M; since M is connected we conclude that G_0 is transitive on M.

In general, if G is a group of diffeomorphisms of a manifold M, the *isotropy* group at $p \in M$, G_p, is the subgroup of G which leaves p fixed. The *linear isotropy group* at p is the group of linear transformations of M_p induced by G_p.

Suppose now G is a connected Lie group with Lie algebra $\mathfrak{g}$. Let $g \to \mathrm{Ad}(g)$ denote the adjoint representation of G on $\mathfrak{g}$ and $X \to \mathrm{ad}\,X$ the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$. Then $\mathrm{ad}\,X(Y) = [X, Y]$ and $\mathrm{Ad}\,(\exp X) = e^{\mathrm{ad}\,X}$ for $X, Y \in \mathfrak{g}$. Let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. The coset space G/H is called *reductive* (Nomizu [31]) if there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ complementary to $\mathfrak{h}$ such that $\mathrm{Ad}(h)\,\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$. We shall only be dealing with reductive coset spaces. All spaces G/H where H is compact or connected and semi-simple are reductive. For reductive coset spaces G/H, the mapping $(d\pi)_e$ maps $\mathfrak{m}$ isomorphically onto the tangent space to G/H at p_0 such that the action of $\mathrm{Ad}(h)$ on $\mathfrak{m}$ corresponds to the action of $d\tau(h)$ on the tangent space. It is customary to identify these spaces. If in a reductive coset space the subspace $\mathfrak{m}$ satisfies $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ we say that G/H is *infinitesimally symmetric*. Suppose the group G has an involutive automorphism σ such that H lies between the group H_σ of fixed points of σ and the identity component of H_σ. The space G/H is then called a *symmetric* coset space. Such a space is infinitesimally symmetric as is easily seen by taking $\mathfrak{m}$ as the eigenspace for the eigenvalue -1 of the automorphism $d\sigma$ of $\mathfrak{g}$.

Let G/H be an infinitesimally symmetric coset space. Here one has the relations

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m} \text{ for all } h \in H, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{2.1}$$

On G/H we consider the *canonical linear connection* which is defined in Nomizu [31] and has the following properties. It is torsion free, invariant under G and the paths (that is the autoparallel curves) through p_0 have the form $t \to \exp tX \cdot p_0$ where $X \in \mathfrak{m}$. This last property is usually expressed: paths in G/H are orbits of one-parameter groups of transvections. In terms of the Exponential mapping at p_0 we can express this property by the relation

$$\mathrm{Exp}\, X = \pi \circ \exp X \qquad \text{for } X \in \mathfrak{m}. \tag{2.2}$$

In particular G/H is *complete* in the sense that each path can be extended in both directions to arbitrary large values of the canonical parameter. Now it is known that the differential of the exponential mapping of the manifold $\mathfrak{g}$ into G is given by

$$d \exp_X = d\,L\,(\exp X) \circ \frac{1 - e^{-\mathrm{ad}\, X}}{\mathrm{ad}\, X} \qquad X \in \mathfrak{g}. \tag{2.3}$$

This is essentially equivalent to the formula of Cartan (proved in Chevalley [7]), which expresses the Maurer–Cartan forms in canonical coordinates. A different proof without the use of differential forms is given in Helgason [24]. To derive a similar formula for $d\,\mathrm{Exp}_X$ $(X \in \mathfrak{m})$ we observe, as a consequence of (2.1), that the linear mapping $(\mathrm{ad}\, X)^2$ maps $\mathfrak{m}$ into itself. Let T_X denote the restriction of $(\mathrm{ad}\, X)^2$ to $\mathfrak{m}$. From the relation $\pi \circ L(g) = \tau(g) \circ \pi$ and (2.2) we obtain for $Y \in \mathfrak{m}$

$$\begin{aligned} d\ \mathrm{Exp}_X(Y) &= d\pi \circ d\,\exp_X(Y) = d\pi \circ d\,L\,(\exp X) \circ \frac{1 - e^{-\mathrm{ad}\, X}}{\mathrm{ad}\, X}(Y) \\ &= d\,\tau\,(\exp X) \circ d\pi \sum_0^\infty (-1)^m \frac{(\mathrm{ad}\, X)^m}{(m+1)!}(Y). \end{aligned}$$

From the relations (2.1) it follows that

$$d\pi \circ (\mathrm{ad}\, X)^m (Y) = \begin{cases} (T_X)^n (Y) & \text{if } m = 2n \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

We have then proved the desired formula

$$d \operatorname{Exp}_X = d\tau\, (\exp X) \circ \sum_0^{\infty} \frac{T_X^n}{(2n+1)!} \qquad \text{for } X \in \mathfrak{m} \tag{2.4}$$

which will be used presently.

2. Spherical areas

Let M be a Riemannian manifold such that the group $\mathbf{I}(M)$ of all isometries of M is transitive on M. M is then called a *Riemannian homogeneous space*. The group $\mathbf{I}(M)$, endowed with the compact-open topology, is a Lie group (S. Myers and N. Steenrod [30]). Let p_0 be a point in M and $\tilde{K}$ the subgroup of $\mathbf{I}(M)$ that leaves p_0 fixed. It is well known that $\tilde{K}$ is compact. Now M, and consequently the group $\mathbf{I}(M)$, are separable. By the definition of the topology of $\mathbf{I}(M)$, the mapping $\Phi: (g, p) \to g \cdot p$ of $\mathbf{I}(M) \times M$ onto M is continuous. Theorem 2 then implies that $\mathbf{I}(M)/\tilde{K}$ is homeomorphic to M, in particular connected. The group $\tilde{K}$, being compact, has finitely many components and it follows easily that the same is true of $\mathbf{I}(M)$. Let G denote the identity component of $\mathbf{I}(M)$ and let $K = G \cap \tilde{K}$. Then K is compact and due to Theorem 2 we can state

LEMMA 2. *A Riemannian homogeneous space M can (with respect to the differentiable structure) be identified with the coset space G/K where G is the identity component of $\mathbf{I}'(M)$ and K is compact. Here $\mathbf{I}'(M)$ is any closed subgroup of $\mathbf{I}(M)$, transitive on M.*

On the other hand let G be a connected Lie group and H a closed subgroup. We assume that the group $\operatorname{Ad}_G(H)$ consisting of all the linear transformations $\operatorname{Ad}(h)$, $h \in H$, is compact. Then G/H is reductive and there exists a positive definite quadratic form on $\mathfrak{m}$ invariant under $\operatorname{Ad}_G(H)$. This form gives by translation a positive definite Riemannian metric on G/H which is invariant under the action of G. Such a space we shall call a *Riemannian coset space.*

LEMMA 3. *Let G/H be a symmetric Riemannian coset space which is non-compact, simply connected and irreducible (that is, $\operatorname{Ad}_G(H)$ acts irreducibly on $\mathfrak{m}$). Let $A(r)$ denote the area of a geodesic sphere in G/H of radius r. Then $A(r)$ is an increasing function of r.*

Proof. We can assume that G acts effectively on G/H because if N is a closed normal subgroup of G contained in H then the coset space G^*/H^*, where $G^* = G/N$, $H^* = H/N$ satisfies all the conditions of the lemma. The G-invariant metric on G/H induces the canonical linear connection on G/H (K. Nomizu [31]), and the paths are now geodesics. Since G/H is irreducible and non-compact it has sectional curvature everywhere $\leqslant 0$ due to a theorem of E. Cartan [4]. (Another proof is given in [24]). Furthermore, since G/H is simply connected and has negative curvature, a well-known result of J. Hadamard and

E. Cartan ([5] and [17]) states that the mapping Exp is a one-to-one mapping of $\mathfrak{m}$ onto [1] G/H. Now, each $\tau(x)$, $x \in G$ is an isometry of G/H. From (2.4) it follows therefore that the ratio of the volume elements in G/H and $\mathfrak{m}$ is given by the determinant of the endomorphism

$$A_X = \sum_0^\infty \frac{T_X{}^n}{(2n+1)!}. \tag{2.5}$$

For the volume of a geodesic sphere in G/H with radius r we obtain the expression

$$V(r) = \int_{\|X\| \leqslant r} \det(A_X)\, dX.$$

Here dX and $\| \ \|$ denote the volume element and norm respectively in the space $\mathfrak{m}$. On differentiation with respect to r we obtain

$$A(r) = \int_{\|X\| = r} \det(A_X)\, d\omega_r(X) \tag{2.6}$$

where $d\omega_r$ is the Euclidean surface element of the sphere $\|X\| = r$ in $\mathfrak{m}$. Now it is known that the irreducibility of G/H implies that either $\mathfrak{g}$ is semi-simple or $[\mathfrak{m}, \mathfrak{m}] = 0$. (A proof can be found in K. Nomizu [31] p. 56; observe the slight difference in the definition of irreducibility). In the case $[\mathfrak{m}, \mathfrak{m}] = 0$, Lemma 3 is obvious so we shall now assume $\mathfrak{g}$ semi-simple. In the proof of Theorem 2 in [24] it is shown that the Killing form B is not only non-degenerate on $\mathfrak{g}$ but

$$B(X, X) > 0 \qquad \text{for } X \neq 0 \text{ in } \mathfrak{m} \tag{2.7}$$

$$B(Y, Y) < 0 \qquad \text{for } Y \neq 0 \text{ in } \mathfrak{h}. \tag{2.8}$$

Using the invariance of the Killing form we obtain also

$$B(\{\operatorname{ad} X\}^2 Z_1, Z_2) = -B([X, Z_1], [X, Z_2]) = B(Z_2, \{\operatorname{ad} X\}^2 (Z_1)) \tag{2.9}$$

which shows that for $X \in \mathfrak{m}$, T_X is symmetric with respect to B. Using (2.7), (2.8) and (2.9) for $Z_1 = Z_2$ we see also that the eigenvalues of T_X are all $\geqslant 0$. If we call these $\xi_1(X)$, $\ldots$, $\xi_n(X)$ and throw T_X into diagonal form we obtain the formula

$$\det(A_X) = \prod_{i=1}^n \frac{\sinh(\xi_i(X))^{\frac{1}{2}}}{(\xi_i(X))^{\frac{1}{2}}}. \tag{2.10}$$

The function $\sinh t/t$ is increasing; it follows then from (2.6) that the function $A(r)$ increases with r, in fact faster than r^{n-1}.

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(1) Using the theory of symmetric spaces, the assumption in Lemma 3 could be reduced somewhat. In fact, either G/H is a Euclidean space or G is semi-simple. In the latter case it can be proved directly, without using the simple connectedness (Cartan [3], Mostow [29]) that Exp is a homeomorphism of $\mathfrak{m}$ onto G/H.

The example of a sphere shows that the hypothesis in Lemma 3 that G/H is non-compact cannot be dropped. However, it seems very likely that the conclusion of Lemma 3 holds for every simply connected Riemannian manifold of negative curvature. The proof above shows (after decomposition) that this is the case if the space is symmetric.

3. Two-point homogeneous spaces

DEFINITION. A connected differentiable manifold M with a positive definite Riemannian metric of class C^∞ is called a *two-point homogeneous space* if the group $\mathbf{I}(M)$ is transitive on the set of all equidistant point pairs of M.

We shall now outline a proof of a theorem which will be of use later. This theorem is known through the classification of the two-point homogeneous spaces. We aim at proving the theorem more directly.

THEOREM 3.

(i) *A two-point homogeneous space M is isometric to a symmetric Riemannian coset space G/K where G is the identity component of $\mathbf{I}(M)$ and K is compact.*

(ii) *If M has odd dimension it has constant sectional curvature.*

(iii) *The non-compact spaces M are all simply connected, in fact homeomorphic to a Euclidean space.*

REMARK. Considerably more is known about two-point homogeneous spaces even under less restrictive definition. A complete classification of the compact two-point homogeneous spaces was given by H. Wang [36]. He found that these are the spherical spaces, real elliptic spaces, complex elliptic spaces, quaternian elliptic spaces and the Cayley elliptic plane. The dimensions of these spaces are respectively d, $d+1$, $2d$, $4d$ and 16 ($d = 1, 2, \ldots$). These are known to be symmetric spaces, that is the geodesic symmetry with respect to each point extends to a global isometry of the whole space. We indicate briefly how (i) follows in the compact case.

Choose a fixed point $p_0 \in M$ and let s_0 denote the geodesic symmetry around p_0. In view of Lemma 2 we can identify M and G/K. (Here K is the subgroup of G that leaves p_0 fixed). The mapping $\sigma: g \to s_0 g s_0$ is an automorphism of $\mathbf{I}(M)$ which maps the identity component G into itself. Also $s_0 \cdot k \cdot s_0 = k$ since both sides are isometries which induce the same mapping on M_{p_0}. It follows that the involutive automorphism $(d\sigma)_e$ of $\mathfrak{g}$ is identity on $\mathfrak{k}$, the Lie algebra of K. On the other hand if $(d\sigma)_e X = X$ for some X in $\mathfrak{g}$ then $\sigma \cdot \exp X = \exp X$ and $\exp X \cdot p_0$ is a fixed point under s_0. Hence $X \in \mathfrak{k}$. Thus $\mathfrak{k}$ is the set of fixed points of $(d\sigma)_e$ and it follows immediately that G/K is a symmetric coset space.

The non-compact two-point homogeneous spaces were classified by J. Tits [35]. In the following we shall establish (i) more directly. When this is done Tits' classification could be obtained from Cartan's classification of non-compact symmetric spaces of rank 1 ([4], p. 385). Using Cartan's terminology, the spaces that occur are: A IV (the hermitian hyperbolic spaces), BD II (the real hyperbolic spaces), C II (for $q = 1$) (the quaternian hyperbolic spaces) and F II (the hyperbolic analogue of the Cayley plane).

Suppose now M is a two-point homogeneous space, p_0 a fixed point in M and $\tilde{K}$ the subgroup of $\mathbf{I}(M)$ that leaves p_0 fixed.

LEMMA 4. *Let G be a closed, connected subgroup of $\mathbf{I}(M)$, and assume that G is transitive on equidistant point pairs of M. If G' is a closed connected normal subgroup of G ($G' \neq e$) then G' is transitive on M.*

This lemma is essentially due to Wang and Tits. We give a proof for the reader's convenience. Let $p \in M$ and let H be the subgroup of G leaving p fixed. H is compact. The Lie algebra $\mathfrak{g}$ of G can be written $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is invariant under $\mathrm{Ad}_G(H)$. From Lemma 2 it is clear that M is isometric with the Riemannian coset space G/H and $\mathfrak{m}$ can be identified with the tangent space M_p. Now the group G, being a group of motions, acts effectively on M, so M', the orbit of p under G', does not consist of p alone. Due to S. Myers and N. Steenrod [30], we know that this orbit is a regularly imbedded submanifold of M. We can choose a one-parameter subgroup g_t of G' which does not keep p fixed. Let X be the tangent vector to the curve $g_t \cdot p$ at $t = 0$. Then $X \neq 0$. In fact, assume to the contrary that $X = 0$. Then we have for each $f \in C^\infty(M)$, $Xf = \left.\frac{d}{dt} f(g_t \cdot p)\right\}_{t=0} = 0$. Using this on the function f^* given by $f^*(q) = f(g_u \cdot q)$ we find $\frac{d}{du} f(g_u \cdot p) = 0$ so $g_u \cdot p = p$ which is a contradiction. If $h \in H$ the curve $h g_t h^{-1} \cdot p$ lies in M' and has tangent vector $\mathrm{Ad}(h)X$. But the group $\mathrm{Ad}_G(H)$ acts transitively on the directions in $\mathfrak{m}$. Therefore, if I denotes the imbedding of M' into M, dI_p is an isomorphism of M'_p onto M_p. Consequently some neighborhood of p in M lies in M'. By homogeneity this holds for each $p \in M'$ and M' is open in M. This proves that each orbit in M under G' is open. By the connectedness of M this is impossible unless $M' = M$ and the lemma is proved.

LEMMA 5. *Let G/H be a reductive coset space ($H \neq G$) and let H_0 denote the identity component of H. Let $\mathfrak{m}$ be a subspace of $\mathfrak{g}$ (the Lie algebra of G) such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$ for $h \in H$. Here $\mathfrak{h}$ is the Lie algebra of H.*

(i) *If $\mathrm{Ad}_G(H_0)$ acts irreducibly on $\mathfrak{m}$, then $\mathfrak{h}$ is a maximal proper subalgebra of $\mathfrak{g}$.*

(ii) *If $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ (that is G/H is infinitesimally symmetric), the converse of* (i) *is true.*

This lemma which is undoubtedly known can be proved as follows. Suppose $\mathrm{Ad}_G(H_0)$

acts irreducibly on $\mathfrak{m}$ and that $\mathfrak{h}$ is not maximal. Then there exists a subalgebra $\mathfrak{h}^*$ of $\mathfrak{g}$ such that we have the proper inclusions $\mathfrak{h} \subset \mathfrak{h}^* \subset \mathfrak{g}$. The subspace $\mathfrak{m}^* = \mathfrak{h}^* \cap \mathfrak{m}$ satisfies $[\mathfrak{h}, \mathfrak{m}^*] \subset \mathfrak{h}^* \cap \mathfrak{m} = \mathfrak{m}^*$ so $\mathfrak{m}^*$ is invariant under $\mathrm{Ad}_G(H_0)$. Hence $\mathfrak{m}^* = 0$ or $\mathfrak{m}^* = \mathfrak{m}$. This last possibility is impossible because it implies $\mathfrak{h}^* = \mathfrak{g}$. But the relation $\mathfrak{m}^* = 0$ is also impossible because if X belongs to the complement of $\mathfrak{h}$ in $\mathfrak{h}^*$ we have $X = Y + Z$, where $Y \in \mathfrak{h}$, $Z \in \mathfrak{m}$ and $Z \neq 0$. But $Z = X - Y \in \mathfrak{h}^*$ so $Z \in \mathfrak{m} \cap \mathfrak{h}^* = 0$. This proves (i). In order to prove (ii) assume $\mathfrak{n}$ is a proper subspace of $\mathfrak{m}$, invariant under $\mathrm{Ad}_G(H_0)$. The relation $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ shows that $\mathfrak{h} + \mathfrak{n}$ is a proper subalgebra of $\mathfrak{g}$, which properly contains $\mathfrak{h}$.

We shall now indicate a proof of Theorem 3 in case M is non-compact. Let G be the identity component of $\mathbf{I}(M)$. We know that M is isometric to G/K where $K = G \cap \tilde{K}$. We can assume $\dim M > 1$. Then a small geodesic sphere S_r around p_0 is connected and $\tilde{K}$ acts transitively on S_r. From Theorem 2 we see that K, having the same dimension as $\tilde{K}$, acts transitively on S_r and thus G acts transitively on equidistant point pairs of M. If G is not semi-simple, G contains an abelian connected normal subgroup $\neq e$ which by Lemma 4 acts transitively on M. M is then a vector space for which Theorem 3 is obvious. If on the other hand G is semi-simple, we see from Lemma 5 that $\mathfrak{k}$, the Lie algebra of K, is a maximal proper subalgebra of $\mathfrak{g}$. Since maximal compact subgroups of connected semi-simple groups are connected, we conclude that K is a maximal compact subgroup of G and G/K is a symmetric coset space. Due to a well-known theorem of Cartan on semi-simple groups, G/K is homeomorphic to a Euclidean space. In our special case, this can be established as follows. Clearly G/K has an infinite geodesic and therefore all its geodesics are infinite. The mapping Exp of $\mathfrak{m}$ into G/K has Jacobian determinant at X given by (2.10) (the derivation of (2.10) did not use the simple connectedness of G/H). The expression (2.10) is always $\neq 0$ so Exp is everywhere regular. Since geodesics issuing from p_0 intersect the geodesic spheres around p_0 orthogonally we see that geodesics issuing from p_0 do not intersect again. Thus Exp is one-to-one and hence a homeomorphism.

Part (ii) of Theorem 3 which is due to Wang [36] depends on the fact that if a linear group of motions acts transitively on an even-dimensional sphere then the action is transitive on equidistant point pairs.

4. Harmonic Lorentz spaces

Let M be a Lorentz space with metric tensor Q. Let p_0 be an arbitrary but fixed point of M and let Exp be the Exponential mapping at p_0 which maps a neighborhood U_0 of 0 in M_{p_0} in a one-to-one manner onto a neighborhood U of p_0 in M. Let $X_1, \ldots, X_n$ be any basis of M_{p_0}. If $X = \sum x_i X_i$ and $x = \operatorname{Exp} X$ the mapping $x \to (x_1, \ldots, x_n)$ is a system of coordinates valid on U. Following Hadamard we consider now the distance function $\Gamma(x) = Q_{p_0}(X, X)$ defined for $x = \operatorname{Exp} X$ in U.

DEFINITION. Suppose M and Q are analytic. M is called *harmonic* if for each $p_0 \in M$ there exists a neighborhood of p_0 in which $\square \Gamma$ is a function of Γ only, $\square \Gamma = f(\Gamma)$.

We shall now study in some detail three types of harmonic homogeneous spaces. These are denoted G^0/H, G^-/H and G^+/H below. For each integer $n \geqslant 1$ there is one space of each class with dimension n. If $n = 1$, $G^0/H = G^-/H$. If $n = 2$, G^-/H and G^+/H are diffeomorphic but not isometric. Otherwise the spaces are all different (even topologically). Due to Theorem 9 these spaces exhaust the class of harmonic Lorentz spaces up to local isometry.

G^0/H. *Flat Lorentz spaces.* We consider the Euclidean space $\mathbf{R}^n$ as a manifold in the usual way that the tangent space at each point is identified with $\mathbf{R}^n$ under the usual identification of parallel vectors. We define a Lorentzian metric Q^0 on $\mathbf{R}^n$ by

$$Q_p^0(Y, Y) = y_1^2 - y_2^2 - \cdots - y_n^2$$

if $Y = (y_1, \ldots, y_n)$ is a vector at $p \in \mathbf{R}^n$. We have then obtained a Lorentz space M. Let L_n denote the general Lorentz group, that is the group of all linear homogeneous transformations h of $\mathbf{R}^n$ such that $Q_0^0(h \cdot X, h \cdot X) = Q_0^0(X, X)$ for all $X \in \mathbf{R}^n$. Each isometry $g \in \mathbf{I}(M)$ can be uniquely decomposed $g = th$ where t is a translation and $h \in L_n$. Hence $\mathbf{I}(M) = \mathbf{R}^n \cdot L_n$. $\mathbf{R}^n$ is a normal subgroup of $\mathbf{I}(M)$. If G^0 is the identity component of $\mathbf{I}(M)$ and H is the subgroup of G^0 that leaves 0 fixed, then M is diffeomorphic to G^0/H and H is connected. G^0/H is a symmetric coset space under the mapping $th \to t^{-1}h$, $t \in \mathbf{R}^n$, $h \in H$.

The group L_n acts transitively on the set of timelike rays from 0; L_n also acts transitively on the set of spacelike rays from 0. Furthermore L_n acts transitively on the punctured cone $\mathbf{C}_0 - 0$. Since $\square$ is invariant under L_n, it follows in particular that $\square \Gamma$ is a function of Γ only; G^0/H is harmonic.

G^-/H. *Negatively curved harmonic Lorentz spaces.* We consider now the quadratic form

$$T(Y, Y,) = -y_1^2 + y_2^2 + \cdots + y_{n+1}^2 \qquad Y = (y_1, \ldots, y_{n+1})$$

and let G^- denote the identity component of the group L_{n+1} which leaves the form $T(Y, Y)$ invariant. Let H be the subgroup of G^- that leaves the point $(0, 0, \ldots, 1)$ fixed. If the transformations $g \in L_{n+1}$ are represented in matrix form $g = (g_{ij})$ then $g \in G^-$ if and only if $g_{11} > 0$ and $\det g = 1$. From this well-known fact follows immediately that H is connected and actually the same as the group H above. The coset space G^-/H can be identified with the orbit of the point $(0, 0, \ldots, 1)$ under G^-. This is the hyperboloid $-y_1^2 + y_2^2 + \ldots + y_{n+1}^2 = 1$ which is homeomorphic to $S^{n-1} \times \mathbf{R}$ (S^m denotes the m-dimensional sphere). It is clear that

G^- acts effectively on G^-/H. Let $\mathfrak{g}^-$ and $\mathfrak{h}$ denote the Lie algebras of G^- and H respectively. If J denotes the matrix of the quadratic form T then a matrix A belongs to G^- if and only if ${}^tAJA = J$ (${}^tA$ is the transpose of A). Using this on matrices of the form $A = \exp X$, $X \in \mathfrak{g}^-$ we find that a basis of $\mathfrak{g}^-$ is given by

$$X_i = E_{1i} + E_{i1} \quad (2 \leqslant i \leqslant n+1), \; X_{ij} = E_{ij} - E_{ji} \quad (2 \leqslant i < j \leqslant n+1). \tag{2.11}$$

Here E_{ij} denotes as usual the matrix (a_{km}) where all $a_{km} = 0$ except $a_{ij} = 1$. A basis of $\mathfrak{h}$ is given by

$$Y_i = E_{1i} + E_{i1} \, (2 \leqslant i \leqslant n), \quad Y_{ij} = E_{ij} - E_{ji} \qquad (2 \leqslant i < j \leqslant n).$$

Let $B^-(X, X)$ denote the Killing form $\mathrm{Tr}\,(\mathrm{ad}\, X \,\mathrm{ad}\, X)$ on $\mathfrak{g}^-$.

Lemma 6. *The Killing form* $\mathfrak{g}^-$ *is given by*

$$B^-(X, X) = (n-1)\,\mathrm{Tr}\,(XX) = 2\,(n-1) \Big\{ \sum_{2 \leqslant i \leqslant n+1} x_i^2 - \sum_{2 \leqslant i < j \leqslant n+1} x_{ij}^2 \Big\}$$

if

$$X = \sum_{2 \leqslant i \leqslant n+1} x_i X_i + \sum_{2 \leqslant i < j \leqslant n+1} x_{ij} X_{ij}.$$

Proof. The complexification $\mathfrak{g}^c$ of $\mathfrak{g}^-$ is the Lie algebra of complex linear transformations which leave invariant the form $-z_1^2 + z_2^2 + \ldots + z_{n+1}^2$. However within the complex number field the signature $- + + \ldots +$ is equivalent to the signature $+ + \ldots +$ and thus $\mathfrak{g}^c$ is isomorphic to the Lie algebra $\mathfrak{o}(n+1, C)$ which consists of all skew symmetric complex matrices. The isomorphism $X \to X'$ in question is given by the mapping $X_i \to i\,(E_{i1} - E_{1i})$ and $X_{ij} \to X_{ij}$. Now the Killing form B' on $\mathfrak{o}(n+1, C)$ is well known to be $B'(X', X') = (n-1)\mathrm{Tr}(X'X')$. Since $\mathrm{Tr}(XX) = \mathrm{Tr}(X'X')$ and since Killing forms are preserved by isomorphisms we see that the Killing form B^c on $\mathfrak{g}^c$ i given by $B^c(X, X) = (n-1)\;\mathrm{Tr}(X, X)$. Now the restriction of B^c to $\mathfrak{g}^-$ coincides with B^- and Lemma 6 follows.

Let s_0 be the linear transformation

$$s_0 : (y_1, \ldots, y_{n+1}) \to (-y_1, -y_2, \ldots, -y_n, y_{n+1}).$$

s_0 leaves the form T invariant and the mapping $\sigma : g \to s_0 g s_0$ is an involutive automorphism of G^-. The corresponding automorphism of $\mathfrak{g}$ is $d\sigma : X \to s_0 X s_0$ and it is easy to see that $\mathfrak{h}$ is the set of all fixed points of $d\sigma$. Thus G^-/H is a symmetric coset space. Let $\mathfrak{p}$ be the eigenspace for the eigenvalue -1 of $d\sigma$. $\mathfrak{p}$ is the subspace of $\mathfrak{g}^-$ spanned by the basis vectors X_{n+1} and $X_{i,n+1}$ $(2 \leqslant i \leqslant n)$, and we have the relations

$$\mathfrak{g}^- = \mathfrak{h} + \mathfrak{p}, \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h} \tag{2.12}$$

and since H is connected, $\mathrm{Ad}(h)\,\mathfrak{p} \subset \mathfrak{p}$. As usual we identify $\mathfrak{p}$ with the tangent space to G^-/H at p_0.

Since the Killing form B^- is invariant under all $\mathrm{Ad}(g)$, $g \in G^-$ we see that the quadratic form Q^- on $\mathfrak{p}$ given by

$$Q^- (X, X) = x_{n+1}^2 - \sum_2^n x_{i,n+1}^2, \quad X = x_{n+1} X_{n+1} + \sum_2^n x_{i,n+1} X_{i,n+1} \tag{2.13}$$

is invariant under the action of $\mathrm{Ad}(H)$ on $\mathfrak{p}$. The form Q^- "extends" uniquely to a G^--invariant Lorentzian metric on G^-/H which induces the canonical linear connection on G^-/H (Nomizu [31]). We denote the metric tensor also by Q^-. Consider now the action of the group $\mathbf{I}(G^-/H)$. Let H^* denote the corresponding linear isotropy group at p_0 which consists of certain linear transformations leaving the form Q^- invariant.

LEMMA 7. *H^* acts transitively on* 1°. *The punctured cone* $C_{p_0} - 0$; 2°. *The set of all timelike rays from* 0; 3°. *The set of all spacelike rays from* 0.

Proof. H^* contains the restriction of the group $\mathrm{Ad}_{G^-}(H)$ to $\mathfrak{p}$ which is isomorphic to H. H^* contains also the symmetry $X \to -X$. As remarked earlier L_n acts transitively on the set

$$M_c = \{ X \in \mathfrak{p} \mid X \neq 0, \ Q^- (X, X) = c \}.$$

Here c is any real number. Due to Theorem 2, H acts transitively on each component of M_c. If $(n, c) \neq (2, 0)$, M_c consists of one or two components, symmetric with respect to 0. If we exclude for a moment the case $(n, c) = (2, 0)$, H^* acts transitively on M_c, as stated in the lemma. If $n = 2$, M_0 consists of four components which are the rays

$$t\,(X_3 + X_{23}), \ t\,(X_3 - X_{23}), \ t\,(-X_3 + X_{23}), \ t\,(-X_3 - X_{23})$$

where $0 < t < \infty$. H^* will clearly be transitive on M_0 if we can prove that the mapping

$$A: \quad x_3 X_3 + x_{23} X_{23} \to -x_3 X_3 + x_{23} X_{23}$$

belongs to H^*. The Killing form on $\mathfrak{g}^-$ is

$$B^- (X, X) = 2\,(x_2^2 + x_3^2 - x_{23}^2), \quad X = x_2 X_2 + x_3 X_3 + x_{23} X_{23}$$

G^- is the group leaving B^- invariant and H is the subgroup of G^- which leaves the point $(1, 0, 0)$ fixed. G^-/H can thus be identified with the hyperboloid $B^-(X, X) = 2$. Hence G^-/H is isometrically imbedded in the flat Lorentz space $\mathfrak{g}^-$ with metric B^-. Now the transformation $(x_2, x_3, x_{23}) \to (x_2, -x_3, x_{23})$ is an isometry of $\mathfrak{g}^-$ which maps the hyperboloid onto itself and leaves the point $(1, 0, 0)$ fixed. Hence A belongs to H^*, as we wanted to prove.

COROLLARY. *G^-/H is harmonic.*

In fact $\square\Gamma$ is invariant under the isotropy subgroup of $\mathbf{I}(G^-/H)$. Due to Lemma 7 $\square\Gamma$ is a function of Γ only.

LEMMA 8. *The timelike paths in G^-/H are infinite and have no double points.*

Proof. Consider the vector $X_{n+1} \in \mathfrak{p}$ which lies inside the cone $Q^-(X, X) = 0$. The path with tangent vector X_{n+1} has the form $\pi \circ \exp tX_{n+1}$, $(t \in \mathbf{R})$. If we use the matrix representation (2.11) we get

$$\exp tX_{n+1} = I + (\cosh t - 1)(E_{11} + E_{n+1, n+1}) + (\sinh t) X_{n+1}$$

and this one-parameter subgroup intersects H only for $t = 0$. It follows easily that the path in question has no double points and since $\mathbf{I}(G^-/H)$ is transitive on the timelike paths the lemma follows.

As before, let Exp denote the Exponential mapping of $\mathfrak{p}$ into G^-/H and A_X the linear transformation (2.5).

LEMMA 9.

$$\det A_X = \left\{\frac{\sinh (Q^-(X, X))^{\frac{1}{2}}}{(Q^- X, X))^{\frac{1}{2}}}\right\}^{n-1}$$

if $Q^-(X, X) > 0$. *In particular,* Exp *is regular in the cone* $Q^-(X, X) > 0$.

Proof. Let as before T_X be the restriction of $(\operatorname{ad} X)^2$ to $\mathfrak{p}$. If $n = 1$, $T_X = 0$ and $G^-/H = \mathbf{R}$; hence we assume $n > 1$. Suppose now $Q^-(X, X) > 0$ and that $Y \neq 0$ is an eigenvector of T_X with eigenvalue ξ. There exists an element $h \in H$ such that $\operatorname{Ad}(h) X = cX_{n+1}$ where $c^2 = Q^-(X, X)$. The relation $T_X \cdot Y = \xi Y$ implies

$$T_{X_{n+1}} Y^* = \frac{\xi}{c^2} Y^* \qquad \text{where } Y^* = \operatorname{Ad}(h)\, Y. \tag{2.14}$$

Writing $Y^* = y_{n+1} X_{n+1} + \sum_2^n y_{i, n+1} X_{i, n+1}$ we find easily $[X_{n+1}, Y^*] = -\sum_2^n y_{i, n+1} X_i$ and

$$T_{X_{n+1}} \cdot Y^* = \sum_2^n y_{i, n+1} X_i. \tag{2.15}$$

From (2.14) and (2.15) we obtain

$$\xi\,(y_{n+1} X_{n+1} + \sum_2^n y_{i, n+1} X_{i, n+1}) = c^2 (\sum_2^n y_{i, n+1} X_{i, n+1}).$$

This shows that either $\xi = 0$ (in which case Y is a non-zero multiple of X) or $\xi = c^2$ (in which case $y_{n+1} = 0$, $y_{i, n+1}$ arbitrary). This shows that the eigenvalues of T_X are 0 and $Q^-(X, X)$; the latter is an $(n-1)$-tuple eigenvalue. The lemma now follows from the relation (2.10).

Suppose now M is an arbitrary complete Lorentz space with metric tensor Q. For a given point $p \in M$ let $\mathrm{S}_r(p)$ be a "sphere" in M_p of radius r and center p; that is $\mathrm{S}_r(p)$ is one of the two components of the set of vectors $\{X \mid X \in M_p,\ Q_p(X, X) = r^2\}$. If Exp is the Exponential mapping at p we put $\mathrm{S}_r(p) = \mathrm{Exp}\, S_r(p)$. For the present considerations it is convenient not to specify which of the two components is chosen. In Chapter IV, § 5 we shall (for the special cases treated there) make such a choice in a continuous manner over the entire manifold.

LEMMA 10. *The timelike paths in G^-/H issuing from p_0 intersect the manifold $\mathrm{S}_r(p_0)$ at a right angle (in the Lorentzian sense).*

Proof. $\mathrm{S}_r(p_0)$ is a manifold since Exp is regular in an open set containing $S_r(p_0)$. Let p be a point on $S_r(p_0)$, X the vector $\overrightarrow{p_0 p}$ and Y a tangent vector to $S_r(p_0)$ at p. Clearly $Q_p^-(X, Y) = 0$. To prove the lemma we have to prove

$$Q_q^-(d\,\mathrm{Exp}_X(X),\ d\,\mathrm{Exp}_X(Y)) = 0 \qquad (q = \mathrm{Exp}\,X). \tag{2.16}$$

(Here we have considered X as a tangent vector to $\mathfrak{p}$ at p, parallel to $\overrightarrow{p_0 p}$.) Using (2.4) and the fact that $\tau(g)$, $g \in G$ is an isometry of G^-/H we see that (2.16) amounts to

$$B^-(A_X(X),\ A_X(Y)) = 0.$$

This relation, however, is immediate from the invariance of B^-.

It is possible to extend Lemma 9 to arbitrary Lorentz spaces by using the structural equations for pseudo-Riemannian connections. We do not do this here since the proof in the special case above is much simpler.

LEMMA 11. *Let Z be a non-vanishing tangent vector to $\mathrm{S}_r(p_0)$ at q. Then $Q_q^-(Z, Z) < 0$.*

Proof. It suffices to prove this when $q = \mathrm{Exp}\, X_{n+1}$ in which case

$$Z = d\,\mathrm{Exp}_{X_{n+1}}(Y) \ \text{ with } \ Y = \sum_2^n y_{i,n+1} X_{i,n+1}$$

To prove $Q_q^-(Z, Z) < 0$ we just have to prove

$$Q^-(A_{X_{n+1}}(Y),\ A_{X_{n+1}}(Y)) < 0. \tag{2.17}$$

This however is obvious since $T_{X_{n+1}} \cdot Y = Y$ and $Q^-(Y, Y) < 0$.

From Lemma 11 it follows that $\mathrm{S}_r(p_0)$ has at each point a unique Lorentzian normal direction. Combining this with Lemmas 8, 9, and 10 we obtain

THEOREM 4. *The Exponential mapping at p_0 which maps $\mathfrak{p}$ into G^-/H is a diffeomorphism of the interior of C_{p_0} into G^-/H. (By the interior of C_{p_0} we mean the set of points $p \in M_{p_0}$ such that $\overrightarrow{p_0 p}$ is timelike).*

On the manifold $\mathrm{S}_r(p_0)$ the tensor $-Q^-$ induces a positive definite Riemannian metric. The same applies clearly to $\mathrm{S}_r(0)$ in the flat Lorentz space $\mathbf{R}^n$.

THEOREM 5. *Suppose the space G^-/H has dimension $n > 2$. With the metric induced by $-Q^-$, $\mathrm{S}_r(p_0)$ is a Riemannian manifold of constant negative curvature. The same statement holds for $\mathrm{S}_r(0)$ in the flat Lorentz space $\mathbf{R}^n$ $(n > 2)$.*

Proof. Let $q_0 = \operatorname{Exp}(rX_{n+1})$. The group H acts transitively on $\mathrm{S}_r(p_0)$ and leaves invariant the positive definite metric on $\mathrm{S}_r(p_0)$. Let H_1 be the subgroup of H leaving q_0 fixed. H_1 is connected since $\mathrm{S}_r(p_0)$ is simply connected. The group $\mathrm{Ad}_H(H_1)$ is the group of all proper rotations in the tangent space to $\mathrm{S}_r(p_0)$ at q_0. In particular, $\mathrm{Ad}_H(H_1)$ acts transitively on the set of two-dimensional subspaces through q_0. Thus $\mathrm{S}_r(p_0)$ has constant sectional curvature at q_0 and, due to the homogeneity, at all points. Since $\mathrm{S}_r(p_0)$ is noncompact the curvature is non-positive. If $n = 2$, $\mathrm{S}_r(p_0)$ is flat, but for $n > 2$ we see from Lemma 6 that H is semi-simple (actually simple), and $\mathrm{S}_r(p_0)$ cannot be flat.

Let M be a connected manifold with a linear connection $X \to \nabla_X$. The curvature tensor R of this connection is a mapping of $\mathfrak{D} \times \mathfrak{D}$ into the space of linear mappings of $\mathfrak{D}$ into itself given by $(X, Y) \to R(X, Y)$ where

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

Here $[X, Y]$ is the usual Poisson bracket of vector fields. If $x \to (x_1, \ldots, x_n)$ is a system of coordinates valid in an open subset of M the coefficients $R_l{}^i{}_{jk}$ of R are defined by

$$R\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) \cdot \frac{\partial}{\partial x_l} = \sum_i R_l{}^i{}_{jk} \frac{\partial}{\partial x_i}.$$

Suppose the connection $X \to \nabla_X$ is the connection induced by a pseudo-Riemannian metric Q on M. If q_{ij} is defined as in Chapter I, the coefficients $R_l{}^i{}_{jk}$ are given by

$$R_l{}^i{}_{jk} = \frac{\partial \Gamma_k{}^i{}_l}{\partial x_j} - \frac{\partial \Gamma_j{}^i{}_l}{\partial x_k} + \sum_m \left(\Gamma_j{}^i{}_m \Gamma_k{}^m{}_l - \Gamma_k{}^i{}_m \Gamma_j{}^m{}_l\right)$$

where Γ^j_{ik} are the Christoffel symbols

$$\Gamma_i{}^j{}_k = \tfrac{1}{2} \sum_l q^{jl} \left(\frac{\partial q_{il}}{\partial x_k} + \frac{\partial q_{lk}}{\partial x_i} - \frac{\partial q_{ik}}{\partial x_l}\right).$$

As usual, we put

$$R_{ijkl} = \sum_m q_{jm} R_i{}^m{}_{kl}.$$

The pseudo-Riemannian manifold is said to have *constant curvature* $\varkappa$ if the relation

$$R_{ijkl} = \varkappa\,(q_{jk} q_{il} - q_{ik} q_{jl}) \tag{2.18}$$

holds on M. For a Riemannian manifold (with positive definite metric) the relation (2.18) is a necessary and sufficient condition for the manifold to have constant sectional curvature in the ordinary sense.

THEOREM 6. *The space G^-/H $(n > 1)$ has constant curvature $\varkappa = -1$.*

Proof. The G^--invariant Lorentzian metric induces the canonical linear connection on the symmetric space G^-/H. The curvature tensor at p_0 is given by $R(X, Y)\cdot Z = -[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{p}$; see e.g. K. Nomizu [31]. We choose coordinates $x_1, \ldots, x_n$ in a neighborhood of p_0 such that

$$\left(\frac{\partial}{\partial x_1}\right)_{p_0} = X_{n+1}, \quad \left(\frac{\partial}{\partial x_i}\right)_{p_0} = X_{i,n+1} \qquad (2 \leqslant i \leqslant n).$$

At p_0 we have $q_{11} = 1$, $q_{22} = \cdots = q_{nn} = -1$. The coefficients of the curvature tensor at p_0 can be found by routine computation. The result is

$$\left.\begin{aligned} R_{ijkl} &= \delta_i^k \delta_l^j = \delta_k^j \delta_l^i \\ R_{1i1j} &= -R_{i11j} = R_{i1j1} = -R_{1ij1} = -\delta_j^i \end{aligned}\right\} \quad (2 \leqslant i, j, k, l \leqslant n).$$

All other coefficients vanish. It is immediate to verify that (2.18) holds with $\varkappa = -1$. Since the validity of (2.18) is independent of the choice of coordinates and since G^-/H is homogeneous, the theorem follows.

G^+/H. *Positively curved harmonic Lorentz space.* Still maintaining the notation from above, we consider the complexification $\mathfrak{g}^c$ of the Lie algebra $\mathfrak{g}^-$. If we consider $\mathfrak{g}^c$ as a real Lie algebra, it is clear that $\mathfrak{g}^+ = \mathfrak{h} + i\,\mathfrak{p}$ is a real subspace, and in fact a real subalgebra due to the relations (2.12). Let G^+ denote the corresponding real analytic subgroup of the general linear group $\mathbf{GL}(n+1, C)$, considered as a real group. H is then a closed subgroup of G^+ and we shall now investigate the space G^+/H of left cosets gH. A basis for $\mathfrak{g}^+$ is given by

$$X_i \;\; (2 \leqslant i \leqslant n), \;\; X_{ij} \;\; (2 \leqslant i < j \leqslant n), \;\; i\,X_{n+1}, \;\; i\,X_{j,n+1} \qquad (2 \leqslant j \leqslant n).$$

and the bracket operation in $\mathfrak{g}^+$ is the ordinary matrix bracket $[A, B] = AB - BA$. The relations

$$\mathfrak{g}^+ = \mathfrak{h} + i\,\mathfrak{p}, \;\; [\mathfrak{h}, i\,\mathfrak{p}] \subset i\,\mathfrak{p}, \;\; [i\,\mathfrak{p}, i\,\mathfrak{p}] \subset \mathfrak{h} \tag{2.19}$$

are obvious from (2.12) and, since H is connected, $\mathrm{Ad}_{G^+}(h)\,i\,\mathfrak{p} \subset i\,\mathfrak{p}$ for each $h \in H$. Thus G^+/H is an infinitesimally symmetric coset space. To see that G^+/H is a symmetric coset space, let s_0 denote the linear transformation

$$s_0 : (y_1, \ldots, y_n, y_{n+1}) \to (-y_1, -y_2, \ldots, -y_n, y_{n+1}).$$

It is easy to see that the mapping $\sigma : g \to s_0 g s_0$ is an involutive automorphism of G^+ and $\mathfrak{h}$ is the set of fixed points of $d\sigma$.

LEMMA 12. *The Killing form on* $\mathfrak{g}^+$ *is given by*

$$B^+(X,X) = (n-1)\operatorname{Tr}(XX) = 2(n-1)\left\{\sum_{2 \leq i \leq n} x_i^2 - \sum_{2 \leq i < j \leq n} x_{ij}^2 - x_{n+1}^2 + \sum_{2 \leq j \leq n} x_{j,n+1}^2\right\}$$

in terms of the basis above.

Proof. Let B^c denote the Killing form on the complex Lie algebra $\mathfrak{g}^c$. The forms B^+ and B^- are the restrictions of B^c to $\mathfrak{g}^+$ and $\mathfrak{g}^-$ respectively. If we write $X = Y + Z$, $Y \in \mathfrak{h}$, $Z \in \mathfrak{p}$ we have

$$B^+(X,X) = B^c(X,X) = 2\,i\,B^-(Y,Z) + B^-(Y,Y) - B^-(Z,Z)$$
$$= (n-1)\operatorname{Tr}(Y + iZ)(Y + iZ)$$

Due to the invariance of the Killing form the quadratic form on $i\mathfrak{p}$ given by

$$Q^+(X,X) = x_{n+1}^2 - \sum_2^n x_{j,n+1}^2, \quad X = x_{n+1}(i X_{n+1}) + \sum_2^n x_{j,n+1}(i X_{j,n+1})$$

is invariant under the action of $\mathrm{Ad}_{G^+}(H)$ on $i\mathfrak{p}$. The tangent space to G^+/H at p_0 can be identified with the subspace $i\mathfrak{p}$ of $\mathfrak{g}^+$. As before Q^+ extends to a G^+-invariant Lorentzian metric on G^+/H. If $n = 1$, G^+/H can be identified with $\mathbf{S}^1$. If $n > 1$, G^+ is semi-simple and from the signature of B^+, ($\frac{1}{2}(n^2 - 3n + 4)$ minus signs), one knows that G^+ has a maximal compact subgroup of dimension $\frac{1}{2}(n^2 - 3n + 4)$. This group is generated by X_{ij} $(2 \leq i < j \leq n)$ and iX_{n+1}. The vectors X_{ij} $(2 \leq i < j \leq n)$ generate a maximal compact subgroup of H. From this it can be concluded that G^+/H is homeomorphic to $\mathbf{S}^1 \times \mathbf{R}^{n-1}$ (also for $n = 1$) but we shall not need this fact. Lemma 7 extends easily to the space G^+/H, and G^+/H is a harmonic Lorentz space. Note that for $n = 2$, G^+/H and G^-/H are diffeomorphic to a hyperboloid F: $-y_1^2 + y_2^2 + y_3^2 = 1$ such that $Q_{p_1}^- = -Q_{p_2}^-$ if p_1 and p_2 correspond to the same $p \in F$.

LEMMA 13. *All the timelike paths issuing from* p_0 *are closed and have length* 2π.

Proof. We consider the one parameter subgroup of G^+ generated by the timelike vector iX_{n+1}. We find

$$\exp\, t\,i\,X_{n+1} = I + (\cos t - 1)(E_{11} + E_{n+1,\,n+1}) + (\sin t)(i\,X_{n+1}).$$

The path in G^+/H with tangent vector iX_{n+1} at p_0 has the form $\pi \circ \exp t i X_{n+1}$ and this is clearly a closed path of length 2π. (The matrix $I - 2(E_{11} + E_{n+1,n+1})$ does not belong to H). Since $\mathrm{Ad}_{G^+}(H)$ acts transitively on the set of timelike lines through p_0, the lemma follows.

LEMMA 14.
$$\det(A_Z) = \left\{ \frac{\sin(Q^+(Z,Z))^{\frac{1}{2}}}{(Q^+(Z,Z))^{\frac{1}{2}}} \right\}^{n-1}$$

for all $Z \in i\mathfrak{p}$ that lie in the cone $Q^+(Z,Z) > 0$. From (6) *it follows that* Exp *is regular in the set $0 < Q^+(Z,Z) < \pi^2$.*

The proof is entirely analogous to that of Lemma 8 and will be omitted. Just as before it can also be proved that $Q_q^+(Z,Z) < 0$ if Z is a non-vanishing tangent vector at q to $S_r(p_0)$, $(r < \pi)$, and Lemma 10 remains valid here if $r < \pi$. Combining these results we have

THEOREM 7. *The Exponential mapping at p_0 which maps $i\mathfrak{p}$ into G^+/H is a diffeomorphism of the open set $0 < Q^+(Z,Z) < \pi^2$ into G^+/H.*

The situation is thus somewhat analogous to the sphere in Euclidean space. The following question arises. Do the timelike paths issuing from p_0 all meet at the point

$$p^* = \pi(I - 2(E_{11} + E_{n+1,\,n+1}))$$

in G^+/H which corresponds to the antipodal point on the sphere? The answer is no and the timelike paths behave more like geodesics in a real elliptic space.

LEMMA 15. *Two different timelike paths issuing from p_0 have no other point in common.*

Proof. We can assume that one of the paths is $t \to \pi(\exp t i X_{n+1})$. The other then has the form $t \to \pi(\exp t \operatorname{Ad}(h)\, i X_{n+1})$ with $h \in H$. By Theorem 7 it is clear that the only possible point of intersection other than p_0 would be the point p^* above, occurring for $t = \pi$. Then there exists $h_1 \in H$ such that

$$(E_{11} + E_{n+1,\,n+1})\, h_1 = h\,(E_{11} + E_{n+1,\,n+1})\, h^{-1}.$$

We can represent h_1, h and h^{-1} in the form

$$h_1 = E_{n+1,\,n+1} + \sum_{i,j=1}^{n} a_{ij} E_{ij}$$

$$h = E_{n+1,\,n+1} + \sum_{i,j=1}^{n} b_{ij} E_{ij}$$

$$h^{-1} = E_{n+1,\,n+1} + \sum_{i,j=1}^{n} c_{ij} E_{ij}.$$

Then the relation above implies

$$b_{11} c_{1j} = a_{1j} \qquad (1 \leqslant j \leqslant n)$$
$$b_{i1} c_{1j} = 0 \qquad (1 < i \leqslant n,\ 1 \leqslant j \leqslant n).$$

Also $c_{11}^2 - \sum_1^n c_{i1}^2 = 1$ so $c_{11} \neq 0$ and therefore $b_{i1} = 0$ for $1 < i \leq n$. On the other hand,

$$b_{11} b_{1j} - \sum_{i=2}^{n} b_{i1} b_{ij} = 0 \qquad (1 < j)$$

so $b_{1j} = 0$. Hence

$$h = E_{11} + E_{n+1,\,n+1} + \sum_{i,j=2} b_{ij} E_{ij}$$

which obviously commutes with $i X_{n+1}$; this implies that the paths coincide, contrary to assumption.

THEOREM 8. *The space G^+/H has constant curvature $\varkappa = +1$.*

The proof is entirely analogous to that of Theorem 6 and will be omitted.

Now let M be an arbitrary harmonic Lorentz space. An important theorem of A. Lichnerowicz and A. G. Walker [28] states that such a space has constant curvature in the sense of the relation (2.18). Using a similarity transformation (i.e. a multiplication of Q by a positive constant) we can assume that the curvature $\varkappa$ is 0, 1 or -1. In particular, the covariant derivatives of the curvature tensor all vanish, $\nabla_X R = 0$ for all $X \in \mathfrak{D}$. A torsion-free linear connection with this last property is uniquely determined in a suitable neighborhood U_p of a given point p, by the value R_p (see e.g. [31]). Furthermore, a diffeomorphism Φ leaving invariant a pseudo-Riemannian connection is an isometry if $(d\Phi)_p$ is an isometry for some point p. From the quoted result of Lichnerowicz and Walker follows

THEOREM 9. *The spaces G^0/H, G^-/H* and *G^+/H exhaust the class of harmonic Lorentz spaces up to local isometry.*

It is customary to denote by $\mathbf{S\,O}^h(n)$ the identity component of the group of non-singular real $n \times n$ matrices that leave invariant the quadratic form $-\sum_1^h x_i^2 + \sum_{h+1}^n x_i^2$. $\mathbf{S\,O}^0(n)$ is the usual rotation group $\mathbf{S\,O}(n)$. In this terminology we have

$$G^0/H = R^n \cdot \mathbf{S\,O}^1(n)/\mathbf{S\,O}^1(n), \quad G^-/H = \mathbf{S\,O}^1(n+1)/\mathbf{S\,O}^1(n).$$

CHAPTER III

Invariant differential operators

1. A general representation theorem

To begin with we introduce some notation which will be used in the rest of the paper. Let G/H be a reductive coset space with a fixed decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathrm{Ad}(h)\,\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$. We shall in this chapter study the set $\mathbf{D}(G/H)$ of differential operators on

G/H that are invariant under the action of G; a differential operator D on G/H belongs to $\mathbf{D}(G/H)$ if and only if $D^{\tau(g)} = D$ for all $g \in G$. We shall write $\mathbf{D}(G)$ instead of $\mathbf{D}(G/e)$. Let $L(g)$ and $R(g)$ denote the left and right translations of G onto itself given by $L(g) \cdot x = gx$, $R(g) \cdot x = xg^{-1}$. For each $f \in C^\infty(G/H)$ we put $\tilde{f} = f \circ \pi$. Then $\tilde{f} \in C^\infty(G)$ and $\tilde{f}$ is constant on each coset gH. The set of all such functions will be denoted by $C_0^\infty(G)$. Finally let $\mathbf{D}_0(G)$ denote the subset of $\mathbf{D}(G)$ consisting of operators that are invariant under right translations by H, that is $D \in \mathbf{D}_0(G)$ if and only if $D^{L(g)} = D$ and $D^{R(h)} = D$ for all $g \in G$ and all $h \in H$. Each $D \in \mathbf{D}_0(G)$ leaves the space $C_0^\infty(G)$ invariant.

LEMMA 16. *The algebra $\mathbf{D}(G/H)$ is isomorphic with the algebra of restrictions of $\mathbf{D}_0(G)$ to $C_0^\infty(G)$.*

Proof. The mapping $f \to f \circ \pi$ is an isomorphism of $C^\infty(G/H)$ onto $C_0^\infty(G)$. Let $D_0 \in \mathbf{D}_0(G)$; we define $D \in \mathbf{D}(G/H)$ by the requirement $(Df)^\sim = D_0 \tilde{f}$ for all $f \in C^\infty G/H)$. This gives a mapping Ψ: $D_0 \to D$ of the algebra of restriction of $\mathbf{D}_0(G)$ to $C_0^\infty(G)$ into $\mathbf{D}(G/H)$. It is easy to see that Ψ is one-to-one, linear and preserves multiplication. To see that the image of Ψ is all of $\mathbf{D}(G/H)$, let $D' \in \mathbf{D}(G/H)$. We choose a basis $X_1, \dots, X_n$ of $\mathfrak{m}$; Lemma 1 shows that for small t, $\exp(t_1 X_1 + \cdots + t_n X_n)$ is a local cross section in G over a neighborhood N of p_0 in G/H and the mapping

$$\pi(\exp(t_1 X_1 + \cdots + t_n X_n)) \to (t_1, \dots, t_n)$$

defines a local coordinate system on G/H valid in N. There exists by Proposition 1 a polynomial P in n variables such that

$$[D' f](p_0) = \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) \tilde{f}(\exp(t_1 X_1 + \cdots + t_n X_n))\right]_{t=0} \tag{3.1}$$

for $f \in C^\infty(G/H)$. Using $(D')^{\tau(g)} = D'$ we find easily that if $g \cdot p_0 = p$

$$[D' f](p) = \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) \tilde{f}(g \exp(t_1 X_1 + \cdots + t_n X_n))\right]_{t=0}. \tag{3.2}$$

If $X_{n+1}, \dots, X_r$ is a basis of $\mathfrak{h}$, the mapping

$$g \exp(t_1 X_1 + \cdots + t_r X_r) \to (t_1, \dots, t_r)$$

is a coordinate system valid in a neighborhood of $g \in G$ and the operator D_0' defined by

$$[D_0' F](g) = \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) F(g \exp(t_1 X_1 + \cdots + t_r X_r))\right]_{t=0} \tag{3.3}$$

for $F \in C^\infty(G)$, is a differential operator on G. Now if $h \in H$ we know $(D')^{\tau(h)} = D'$ so for $f \in C^\infty(G/H)$

$$[D' f](p_0) = [D' f^{\tau(h^{-1})}](p_0)$$
$$= \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) \tilde{f}(h \exp(t_1 X_1 + \dots + t_n X_n) h^{-1})\right]_{t=0}$$
$$= \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) \tilde{f}(\exp \operatorname{Ad}(h)(t_1 X_1 + \dots + t_n X_n))\right]_{t=0}$$

which in view of (3.1) implies

$$P(X_1, \dots, X_n) = P(\operatorname{Ad}(h)\cdot X_1, \dots, \operatorname{Ad}(h)\cdot X_n) \text{ for all } h \in H,$$

that is, P is invariant under H. It follows quickly that D_0' is invariant under all $R(h)$, $h \in H$. Similarly, if $x \in G$

$$[(D_0')^{L(x)} F](g) = [D_0' F^{L(x^{-1})}](x^{-1} g)$$
$$= \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) F^{L(x^{-1})}(x^{-1} g \exp(t_1 X_1 + \dots + t_r X_r))\right]_{t=0} = [D_0' F](g)$$

so $D_0' \in \mathbf{D}_0(G)$. The relations (3.2) and (3.3) imply that

$$(D' f)^{\sim} = D_0' \tilde{f} \qquad \text{for } f \in C^\infty(G/H)$$

so the image of Ψ' is all of $\mathbf{D}(G/H)$.

Now each $X \in \mathfrak{g}$ defines uniquely a left invariant vector field on G. This vector field is a differential operator on G (again denoted X) satisfying $X^{L(g)} = X$ for all $g \in G$. It follows easily that

$$[X f](g) = \left[\frac{d}{dt} f(g \exp tX)\right]_{t=0} \qquad \text{for } f \in C^\infty(G). \tag{3.4}$$

This mapping of $\mathfrak{g}$ into $\mathbf{D}(G)$ sends the Lie algebra element $[X, Y]$ in the operator $X \cdot Y - Y \cdot X$ and extends uniquely to a homomorphism ξ of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, into $\mathbf{D}(G)$. More crucially, ξ is an isomorphism of $U(\mathfrak{g})$ onto $\mathbf{D}(G)$. (See Harish-Chandra [22]). On the other hand, let $X_1, \dots, X_r$ be a basis of $\mathfrak{g}$ and $S(\mathfrak{g})$ the symmetric algebra over $\mathfrak{g}$, that is the set of polynomials over $\mathbf{R}$ in the letters $X_1, \dots, X_r$. Harish-Chandra's version [19] of the Poincaré-Birkhoff-Witt theorem gives a one-to-one linear mapping λ of $S(\mathfrak{g})$ onto $\mathbf{D}(G)$ with the property that for arbitrary elements $Y_1, \dots Y_p$

$$\lambda(Y_1 Y_2 \dots Y_p) = \frac{1}{p!} \sum_\sigma Y_{\sigma(1)} \cdot Y_{\sigma(2)} \dots Y_{\sigma(p)} \tag{3.5}$$

where σ runs over the symmetric group on p letters. (Note the difference in the notation for multiplication in $S(\mathfrak{g})$ and $\mathbf{D}(G)$.) We shall refer to a mapping with the property (3.5) as "symmetrization".

For each $g \in G$, $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$ and extends uniquely to an automorphism of $U(\mathfrak{g})$ which combined with ξ gives an automorphism of $\mathbf{D}(G)$. Denoting this automorphism again by $\mathrm{Ad}(g)$ we have

$$\mathrm{Ad}\,(g) \cdot D = D^{R(g)} \qquad \text{for } D \in \mathbf{D}\,(G). \tag{3.6}$$

In fact, since $D \rightarrow D^{R(g)}$ is an automorphism of $\mathbf{D}(G)$ it suffices, due to the uniqueness mentioned, to prove (3.6) when D is a vector field X. But $\mathrm{Ad}(g) \cdot X = X(X^{L(g)})^{R(g)} = X^{R(g)}$. Now if f is analytic in a neighborhood of $g \in G$, (3.4) implies that

$$f\,(g \exp tX) = \sum_0^\infty \frac{t^n}{n\,!}\,[X^n f]\,(g) \tag{3.7}$$

for sufficiently small t. Using the fact that $D \in \mathbf{D}\,(G)$ has analytic coefficients we obtain from (3.4) and (3.7)

$$D \cdot X = X \cdot D \text{ if and only if } D^{R\,(\exp tX)} = D \text{ for all } t. \tag{3.8}$$

Let $\mathbf{Z}(G)$ denote the center of $\mathbf{D}(G)$; from (3.6) and (3.8) we see [1] that $D \in \mathbf{Z}(G)$ if and only if $\mathrm{Ad}(g) \cdot D = D$ for all $g \in G$.

If V is a finite dimensional vector space over $\mathbf{R}$, $X_1, \ldots, X_l$ a basis of V, $S(V)$ shall denote the symmetric algebra over V, that is the algebra of polynomials over $\mathbf{R}$ in the letters $X_1, \ldots, X_l$. Let A be an endomorphism of V. A induces a homomorphism of $S(V)$, say $P \rightarrow A \cdot P$ where $(A \cdot P)(X_1, X_2, \ldots, X_l) = P(AX_1, AX_2, \ldots, AX_l)$. Using (3.5) it follows that $\lambda^{-1}(\mathbf{Z}(G))$ is the subset $I(\mathfrak{g})$ of $S(\mathfrak{g})$ consisting of all polynomials that are invariant under $\mathrm{Ad}(G)$. In the same manner we obtain

LEMMA 17. *$\lambda^{-1}(\mathbf{D}_0(G))$ is the set of polynomials $P \in S(\mathfrak{g})$ such that $\mathrm{Ad}(h)P = P$ for all $h \in H$.*

LEMMA 18. *$\mathbf{D}(G) = \mathbf{D}(G)\mathfrak{h} + \lambda(S(\mathfrak{m}))$ where the sum is a direct sum of vector spaces. (Here $\mathbf{D}\,(G)\,\mathfrak{h}$ denotes the left ideal in $\mathbf{D}\,(G)$ generated by $\mathfrak{h}$).*

Proof. To begin with we shall prove by induction that for each $P \in S\,(\mathfrak{g})$ there exists $Q \in S\,(\mathfrak{m})$ such that $\lambda\,(P - Q) \in \mathbf{D}\,(G)\,\mathfrak{h}$. This is obvious if P has degree 1 and we assume it true for all $P \in S\,(\mathfrak{g})$ of degree $< d$. To prove it for P of degree d we can assume P has the form $X_1^{e_1} \ldots X_r^{e_r}$, $e_1 + \cdots + e_r = d$ where $X_1, \ldots, X_r$ is a basis of $\mathfrak{g}$ such that $X_i \in \mathfrak{m}$ for $1 \leqslant i \leqslant n$ and $X_j \in \mathfrak{h}$ for $n+1 \leqslant j \leqslant r$. If $e_{n+1} = \cdots = e_r = 0$ there is nothing to prove; otherwise $\lambda\,(P)$ is a linear combination of terms of the form

[1] Equivalent result is given in Harish-Chandra [20] Cor. of Lemma 11 and in I. Gelfand [12].

$X_{\alpha_1} \cdot X_{\alpha_2} \dots X_{\alpha_d}$ where for some i, $X_{\alpha_i} \in \mathfrak{h}$. Let $S_e(\mathfrak{g})$ denote the set of homogeneous polynomials in $S(\mathfrak{g})$ of degree e and put $\mathbf{D}_d(G) = \lambda(\sum_0^d S_e(\mathfrak{g}))$. Then

$$(X_{\alpha_1} \cdot X_{\alpha_2} \dots \cdot X_{\alpha_d}) - (X_{\alpha_1} \dots \cdot X_{\alpha_{i-1}} \cdot X_{\alpha_{i+1}} \dots X_{\alpha_d} X_{\alpha_i}) \in \mathbf{D}_{d-1}(G).$$

Therefore, there is an element $D \in \mathbf{D}_{d-1}(G)$ such that

$$\lambda(P) \equiv D \mod (\mathbf{D}(G)\mathfrak{h}).$$

Using the inductive assumption we obtain a $Q \in S(\mathfrak{m})$ such that $\lambda(P - Q) \in \mathbf{D}(G)\mathfrak{h}$ as desired. To prove the uniqueness we note first that if $P \in S(\mathfrak{g})$, $f \in C^\infty(G)$

$$[\lambda(P) f](e) = \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}\right) f(\exp(t_1 X_1 + \dots + t_r X_r)) \right]_{t=0}. \tag{3.9}$$

In fact, if f is analytic on a neighborhood of e in G, (3.7) shows that for sufficiently small t_i

$$\begin{aligned} f(\exp(t_1 X_1 + \dots + t_r X_r)) &= \sum_0^\infty \frac{1}{m!} [(t_1 X_1 + \dots + t_r X_r)^m f](e) \\ &= \sum_{M=0}^\infty \frac{1}{M!} \sum_{m_1 + \dots + m_r = m} t_1^{m_1} \dots t_r^{m_r} \frac{M!}{m_1! \dots m_r!} [\lambda(X_1^{m_1} \dots X_r^{m_r}) f](e). \end{aligned}$$

Comparison with the usual Taylor formula yields (3.9). Now, by Lemma 1, exp $(t_1 X_1 + \dots + t_n X_n)$ defines for small t_i a local cross section in G over a neighborhood N of p_0 and $(t_1, \dots, t_n)$ are local coordinates on N. If $P \in S(\mathfrak{m})$, $P \neq 0$ we can choose $f^* = f^*(t_1, \dots, t_n)$ of class C^∞ such that

$$\left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) f^* \right]_{t=0} \neq 0$$

and there exists a function $f \in C_0^\infty(G)$ such that

$$f(\exp(t_1 X_1 + \dots + t_n X_n)) = f^*(t_1, \dots, t_n)$$

for sufficiently small t_i. From (3.9) we have

$$\begin{aligned} [\lambda(P) f](e) &= \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) f(\exp(t_1 X_1 + \dots + t_r X_r)) \right]_{t=0} \\ &= \left[P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) f(\exp(t_1 X_1 + \dots + t_n X_n)) \right]_{t=0} \neq 0. \end{aligned}$$

Since each operator in $\mathbf{D}(G)\mathfrak{h}$ annihilates all of $C_0^\infty(G)$ we see that $\lambda(S(\mathfrak{m})) \cap \mathbf{D}(G)\mathfrak{h} = 0$. This proves Lemma 18.

LEMMA 19. *Each $P \in S(\mathfrak{g})$ which is invariant under* $\mathrm{Ad}_G(H)$ *is congruent* mod $\lambda^{-1}(\mathbf{D}(G)\mathfrak{h})$ *to a polynomial in $S(\mathfrak{m})$ invariant under* $\mathrm{Ad}_G(H)$.

Proof. By Lemma 18, $P = Q + Q_0$ where $Q \in S(\mathfrak{m})$ and $\lambda(Q_0) \in \mathbf{D}(G)\mathfrak{h}$. For each $h \in H$ we obtain $P = \mathrm{Ad}(h)\cdot Q + \mathrm{Ad}(h)\cdot Q_0$ and by (3.5) and (3.6), $\lambda(\mathrm{Ad}(h)\cdot Q_0) = \lambda(Q_0)^{R(h)}$. Now the mapping $D \to D^{R(h)}$ is an automorphism of $\mathbf{D}(G)$ leaving $\mathfrak{h}$ invariant. Hence it leaves $\mathbf{D}(G)\mathfrak{h}$ invariant and $\lambda(\mathrm{Ad}(h)\cdot Q_0) \in \mathbf{D}(G)\mathfrak{h}$. On the other hand $\mathrm{Ad}(h)Q_0 \in S(\mathfrak{m})$ and Lemma 19 follows from the uniqueness statement in Lemma 18.

Let $I(\mathfrak{g}/\mathfrak{h})$ denote the set of polynomials in $S(\mathfrak{m})$ that are invariant under $\mathrm{Ad}_G(H)$. We define a mapping of $I(\mathfrak{g}/\mathfrak{h})$ into $\mathbf{D}(G/H)$ as follows. If $P \in I(\mathfrak{g}/\mathfrak{h})$, then $\lambda(P) \in \mathbf{D}_0(G)$ and the restriction of $\lambda(P)$ to $C_0^\infty(G)$ gives by Lemma 16 rise to a well-defined operator $D_P \in \mathbf{D}(G/H)$. This mapping $P \to D_P$ is linear. It maps $I(\mathfrak{g}/\mathfrak{h})$ onto $\mathbf{D}(G/H)$ because Lemma 19 shows that if $P \in S(\mathfrak{g})$ is invariant under $\mathrm{Ad}_G(H)$ there exists a $Q \in I(\mathfrak{g}/\mathfrak{h})$ such that $\lambda(P)$ and $\lambda(Q)$ have the same restrictions to $C_0^\infty(G)$. Finally the mapping $P \to D_P$ is one-to-one. In fact, let $P \in I(\mathfrak{g}/\mathfrak{h})$, $P \neq 0$. As shown in the proof of Lemma 18 there exists a function $f \in C_0^\infty(G)$ such that $[\lambda(P)f](e) \neq 0$. The following theorem gives the desired representation of $\mathbf{D}(G/H)$.

THEOREM 10. *Let G/H be a reductive coset space, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, $\mathrm{Ad}(h)(\mathfrak{m} \subset \mathfrak{m}$ for $h \in H$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{m}$, and let $\tilde{f} = f \circ \pi$ for $f \in C^\infty(G/H)$. There is a one-to-one linear correspondence $Q \to D_Q$ between $I(\mathfrak{g}/\mathfrak{h})$ and $\mathbf{D}(G/H)$ such that*

$$[D_Q f](p) = \left[Q\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}\right)\tilde{f}(g \exp(t_1 X_1 + \cdots + t_n X_n))\right]_{t=0}$$

where $p = \pi(g)$. D_Q is obtained from $Q(X_1, \ldots, X_n)$ by symmetrization (followed by the mapping Ψ from Lemma 16).

REMARK. If $P = X_1^{e_1} \ldots \cdot X_r^{e_r}$ then (3.5) shows easily that

$$\lambda(P) = X_1^{e_1} X_2^{e_2} \ldots \cdot X_r^{e_r} + \lambda(Q)$$

where Q is of lower degree than P. It follows that if $P_1, P_2 \in I(\mathfrak{g}/\mathfrak{h})$ then $D_{P_1 P_2} = D_{P_1} D_{P_2} + D$ where the order of D is less than the sum of the degrees of P_1 and P_2.

COROLLARY *If $I(\mathfrak{g}/\mathfrak{h})$ has a finite system of generators, say $P_1, \ldots, P_l$, and we put $D_i = D_{P_i}$, then each D can be written*

$$D = \sum \alpha_{n_1 \ldots n_l} D_1^{n_1} \ldots D_l^{n_l} \qquad \textit{where } \alpha_{n_1 \ldots n_l} \in \mathbf{R}.$$

In fact, suppose $D = D_P$ where $P \in I(\mathfrak{g}/\mathfrak{h})$. Then P can be written

$$P = \sum \beta_{n_1 \ldots n_l} P_1^{n_1} \ldots P_l^{n_l}, \quad \beta_{n_1 \ldots n_l} \in \mathbf{R}.$$

If $\beta P_1^{N_1} \dots P_l^{N_l}$ is the term of highest degree, the preceding remark shows that

$$D - \beta_1^{N_1} \dots \cdot D_l^{N_l}$$

is of lower order than D and the corollary follows by induction.

2. Invariant differential operators on two-point homogeneous spaces and on harmonic Lorentz spaces

THEOREM 11. *Let M be a two-point homogeneous space. The only differential operators on M that are invariant under all isometries of M are the polynomials in the Laplace-Beltrami operator Δ.*

Proof. If $\dim M = 1$, M is isometric to the real line or to a circle and in both cases Theorem 11 is obvious. We can therefore assume that $\dim M > 1$. From Chapter II, § 3 we know that M is isometric to a homogeneous space G/K where K is compact, G is a connected Lie group of isometries which is pairwise transitive on G/K. The Lie algebra $\mathfrak{g}$ of G can be written $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ where $\mathfrak{k}$ is the Lie algebra of K, the group $\mathrm{Ad}_G(K)$ leaves $\mathfrak{m}$ invariant and acts transitively on the directions in $\mathfrak{m}$. $\mathrm{Ad}_G(K)$ leaves invariant a positive definite inner product on $\mathfrak{m}$; let $X_1, \dots, X_n$ be an orthonormal basis with respect to this inner product. Each $D \in \mathbf{D}(G/K)$ has by Theorem 10 the form D_P where $P \in I(\mathfrak{g}/\mathfrak{k})$. Explicitly, we write $P = \sum a_{r_1 \dots r_n} X_1^{r_1} \dots . X_n^{r_n}$ and consider the corresponding polynomial function P^* on $\mathfrak{m}$ given by $P^*(X) = \sum a_{r_1 \dots r_n} \cdot x_1^{r_1} \dots . x_n^{r_n}$ if $X = \sum x_i X_i$. Since $P \in I(\mathfrak{g}/\mathfrak{k})$ we have $P^*(\mathrm{Ad}(k) X) = P^*(X)$ for all $k \in K$, and it follows that P^* is constant on each sphere around the origin in $\mathfrak{m}$. Thus P^* can be written

$$P^*(X) = \sum_1^N a_k (x_1^2 + \cdots + x_n^2)^k \qquad \text{where } a_k \in \mathbf{R}$$

and
$$P = \sum_1^N a_k (X_1^2 + \cdots + X_n^2)^k.$$

Let Δ denote the member of $\mathbf{D}(G/K)$ that corresponds to the invariant polynomial $X_1^2 + \cdots + X_n^2$. From the remark following Theorem 10 we know that $D_P - \sum_1^N a_k \Delta^k = D_Q$ where Q belongs to $I(\mathfrak{g}/k)$ and has degree lower than P. Theorem 11 now follows by a simple induction.

It is to be expected in view of Theorem 11 that potential theory on two-point homogeneous spaces parallels potential theory in Euclidean spaces very closely. This agrees also with the fact that two-point homogeneous spaces are harmonic spaces and as Willmore

[37] has shown, harmonic spaces can be characterized by the fact that the usual mean value theorem for solutions of Laplace's equation $\Delta u = 0$ remains valid.

We shall next consider the case of a harmonic Lorentz space M with metric tensor Q.

THEOREM 12. *The algebras* $\mathbf{D}(G^0/H)$, $\mathbf{D}(G^-/H$ *and* $\mathbf{D}(G^+/H)$ *consist of all polynomials in the Laplace-Beltrami operator* $\square$.

It is easy to adapt the proof of Theorem 11 to the present case. The essential point is that $\mathrm{Ad}_G(H)$, $(G = G^0,\ G^-$ or $G^+)$, acts transitively on each component of the set $\{X \in \mathfrak{m} \mid X \neq 0,\ Q(X, X) = c\}$. Here Q is the quadratic form on $\mathfrak{m}$ invariant under $\mathrm{Ad}_G(H)$.

3. The case of a symmetric coset space

The assumption that M is symmetric also has important consequences as Theorem 13 shows. This theorem is essentially known from Gelfand's paper [11], and in [34] A. Selberg gave a very direct and transparent proof.

THEOREM 13. *Let* G/K *be a symmetric coset space,* K *compact. Then* $\mathbf{D}(G/K)$ *is commutative.*

In the special case when G is a complex semi-simple Lie group and K is a maximal compact subgroup, the algebra $\mathbf{D}(G/K)$ can be described more explicitly. It is known that K is connected and the Lie algebra $\mathfrak{g}$ of G is the complexification of $\mathfrak{k}$, the Lie algebra of K. We express this by the relation $\mathfrak{g} = \mathfrak{k} + i\,\mathfrak{k}$ where $\mathfrak{g}$ and $\mathfrak{k}$ are considered as Lie algebras over $\mathbf{R}$. As is well known G/K is a symmetric coset space and thus $\mathbf{D}(G/K)$ is commutative. Let $I(\mathfrak{k})$ denote the set of polynomials in $S(\mathfrak{k})$ that are invariant under the adjoint group of K. Then it is easy to see that the mapping $iX \rightarrow X$ of $i\,\mathfrak{k}$ onto $\mathfrak{k}$ induces an isomorphism of $I(\mathfrak{g}\,\mathfrak{k}/)$ onto $I(\mathfrak{k})$. The algebra $I(\mathfrak{k})$ has significance in topological study of the group K (see e.g. C. Chevalley [8]) during which the following results have been proved. Let 1 be the rank of K (dimension of the maximal tori) and let p_i be the indices occurring in the Hopf-splitting of the Poincaré polynomial of K

$$\sum_p B_p\, t^p = \prod_{i=1}^{l} (1 + t^{p_i}).$$

Then $I(\mathfrak{k})$ is generated by l algebraically independent polynomials $P_1, \ldots, P_l$ of degrees $\frac{1}{2}(p_i + 1)$, $i = 1, \ldots, l$. The corresponding operators $D_{P_1}, \ldots, D_{P_l}$ form a system of generators of $\mathbf{D}(G/K)$.

Chapter IV

Mean value theorems

1. The mean value operator

Suppose now that G is a connected Lie group and K a compact subgroup. We fix a G-invariant Riemannian metric tensor Q on G/K, and denote the distance function by d. There exists in this case a finite system $D^1, \ldots, D^l$ of generators(1) for $\mathbf{D}(G/K)$. Let dk denote a normalized invariant measure on K. If π is the natural projection of G onto G/K we put as before $\tilde{f} = f \circ \pi$ for each $f \in C^\infty(G/K)$. Let x be a fixed element of G. The function

$$g \to \int_K \tilde{f}(gkx)\, dk$$

is constant on each coset gK and determines a C^∞-function on G/K which we call $M^x f$. M^x is therefore the linear operator on $C^\infty(G/K)$ given by

$$[M^x f](p) = \int_K \tilde{f}(gkx)\, dk \qquad \text{if } \pi(g) = p.$$

The set $\{\pi(gkx) \mid k \in K\}$ is the orbit of the point $\pi(gx)$ under the group gKg^{-1} and lies on a sphere in G/K with center $\pi(g)$. $[M^x f](p)$ is the average of the values of f on this orbit. In the case that G is pairwise transitive on G/K, M^x is the operation of averaging over a sphere of fixed radius equal to $d(\pi(e), \pi(x))$. Next theorem shows that M^x can be represented as a function of the operators $D^1, \ldots, D^l$. This was proved by Berezin and Gelfand in [2] for the case when G/K is symmetric. Their proof, which does not seem to generalize to the non-symmetric case, is different from ours, which was found independently.

Theorem 14. *Let $p \in G/K$ and let U be a neighborhood of p. Suppose $X \in \mathfrak{g}$ is so small that U contains the sphere with center p and radius $d(\pi(e), \pi(\exp X))$. Then there exists a neighborhood V of p, $V \subset U$, and certain polynomials without constant term, say p_n, such that*

$$[M^{\exp X} f](q) = f(q) + \sum_n [p_n(D^1, \ldots, D^l) f](q)$$

for each f analytic on U and each $q \in V$.

Proof. Choose $g_0 \in G$ such that $\pi(g_0) = p$, and let $x = \exp X$. Then $\pi(g_0 kx) \in U$ for all $k \in K$, and there exists a neighborhood U^* of g_0 in G such that $\pi(gkx) \in U$ for all $g \in U^*$ and all $k \in K$. Put $V = \pi(U^*)$. Now suppose f is analytic in U and $q \in V$. Select $g \in G$ such that $\pi(g) = q$. Then

(1) This is also proved in [34].

$$[M^x f](q) = \int_K \tilde{f}(g k x k^{-1}) \, dk = \int_K \tilde{f}(g \exp Ad(k) X) \, dk$$

which by (3.4) is equal to

$$\int_K \sum_0^\infty \frac{1}{m!} [(Ad(k) X)^m \tilde{f}](g) \, dk \tag{4.1}$$

Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{M}$ and write $Ad(k) = r \sum_1^n k_i X_i$ where $r^2 = Q_{p_0}(X, X)$ and $\sum k_i^2 = 1$. Put

$$f^*(t_1, \ldots t_n) = \tilde{f}(g \exp (t_1 X_1 + \cdots + t_n X_n).$$

Then by (3.9)

$$[(Ad(k) X)^m \tilde{f}](g) = \left[\left(r k_1 \frac{\partial}{\partial t_1} + \cdots + r k_n \frac{\partial}{\partial t_n} \right)^m f^*(t_1, \ldots, t_n) \right]_{t=0}$$

and (4.1) is just the ordinary Taylor series for $f^*(r k_1 \ldots r k_n)$. Thus the series (4.1) converges uniformly in k so the summation and integration can be interchanged; also $(\mathrm{Ad}(k) X)^m = \mathrm{Ad}(k) \cdot X^m$ and the operator $\int \mathrm{Ad}(k) \cdot X^m \, dk$ belongs to $\mathbf{D}_0(G)$. By Lemma 16 this corresponds to an operator $D^m \in \mathbf{D}(G/K)$ which can be written $p^m(D^1, \ldots, D^l)$ as we have seen, and the theorem follows.

We shall now generalize the well-known mean value theorem of Ásgeirsson [1] for solutions of the ultrahyperbolic equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \frac{\partial^2 u}{\partial y_1^2} + \cdots + \frac{\partial^2 u}{\partial y_n^2}$$

which states that each solution $u(x_1, \ldots, x_n; y_1, \ldots, y_n) = u(X, Y)$ satisfies the relation

$$\int_{\mathbf{S}_r(X_0)} u(X, Y_0) \, d\boldsymbol{\omega}_r(X) = \int_{\mathbf{S}_r(Y_0)} u(X_0, Y) \, d\boldsymbol{\omega}_r(Y)$$

for every $X_0, Y_0 \in \mathbf{R}^n$. Here $d\boldsymbol{\omega}_r$ stands for the Euclidean area element of the sphere $\mathbf{S}_r$.

DEFINITION. Let u be a function in $C^\infty(G/K \times G/K)$. We say u is of *slow growth* if $D_1 u$ and $D_2 u$ are bounded for each $D \in \mathbf{D}(G/K)$.

THEOREM 15. *Let u be a function on $G/K \times G/K$ which is either of slow growth or analytic. Suppose u satisfies the differential equations*

$$D_1 u = D_2 u \quad \text{for all} \quad D \in \mathbf{D}(G/K) \tag{4.2}$$

Then

$$M_1^x u = M_2^x u \quad \text{for all} \quad x \in G. \tag{4.3}$$

(*Here the subscripts* 1, 2 *on an operator indicate that it operates on the first and second variable respectively.*) *Conversely, if* (4.3) *holds for a function of class C^∞, then* (4.2) *follows.*

Proof. We first prove the theorem under the assumption that u is analytic (but not necessarily bounded). If (4.2) holds, it follows from Theorem 14 that (4.3) is valid at least if x lies in a suitable neighborhood of e in G. Since however both sides of (4.3) are analytic in x, (4.3) holds for all x. On the other hand, if u belongs to $C^\infty(G/K \times G/K)$ and $\tilde{u}$ is defined by $\tilde{u}(g, g') = u(\pi(g), \pi(g'))$ then $\tilde{u} \in C^\infty(G \times G)$ and the relation (4.3) can be written

$$\int_K \tilde{u}(gkx, g')\,dk = \int_K \tilde{u}(g, g'kx)\,dk. \tag{4.4}$$

We take now an operator $T \in \mathbf{D}_0(g)$ and apply to both sides of (4.4) considered as functions of x, and put $x = e$. It follows that

$$[T_1\tilde{u}](g, g') = [T_2\tilde{u}](g, g')$$

which is equivalent to (4.2).

Let us consider the case when u is constant in the second argument, i.e., $\tilde{u}(g, g') = \tilde{u}(g, e)$ and put $\tilde{v}(g) = \tilde{u}(g, e)$. The algebra $\mathbf{D}(G/K)$ always contains an elliptic operator, e.g., the Laplace-Beltrami operator with respect to the G-invariant metric. By S. Bernstein's theorem, a function v that satisfies the equation

$$Dv = 0 \tag{4.5}$$

for all D that annihilate constants, is automatically analytic. Using (4.3) we see that each solution of (4.5) is characterized by the mean value relation

$$M^x v = v \text{ for all } x \in G.$$

This result was proved somewhat differently by Godement [15]. It generalizes the mean value theorem for harmonic functions in $\mathbf{R}^n$. Earlier Feller [10] had extended this theorem to certain non-Euclidean spaces in connection with mean value theorems for more general elliptic equations. Whereas the assumption of analyticity is no restriction in Godement's theorem, this is not so in Theorem 15 where the most interesting solutions are the non-analytic ones.

Let dg denote a left invariant Haar measure on G. The convolution $f_1 * f_2$ of two functions f_1 and f_2 on G is defined by

$$f_1 * f_2(x) = \int_G f_1(y) f_2(y^{-1}x)\,dy$$

whenever this integral exists. We shall use the following lemma to prove Theorem 15 in full generality.

LEMMA 20. *Let f be a bounded continuous function on G, ε a number > 0 and C a compact subset of G. Then there exists a function φ on G such that*

$$\varphi * f \quad \textit{is analytic} \tag{4.6}$$

$$|(\varphi * f)(x) - f(x)| < \varepsilon \quad \textit{for all } x \in C. \tag{4.7}$$

Proof. If G is compact the lemma is an easy consequence of the Peter-Weyl theory and it can also be proved directly for a commutative Lie group. The general case is handled by using the fact that as a manifold G is analytically isomorphic to a product manifold $K \times N$ where K is a compact subgroup of G and N is a submanifold of G analytically isomorphic to a Euclidean space.

An analogous procedure is followed in Harish-Chandra's theory of well-behaved vectors (see Harish-Chandra [20] and the generalization given by Cartier-Dixmier [6]). As we shall indicate, Lemma 20 is essentially contained in the theorem which states that the well-behaved vectors are dense in the representation space.

Let π denote the left regular representation of G on the Banach space $L^1(G)$ of functions on G that are integrable with respect to left invariant Haar measure, that is $[\pi(x)h](y) = h(x^{-1}y)$ for $h \in L^1(G)$. If h is a well-behaved vector in $L^1(G)$ then so is $\pi(x)h$ and from Lemma 18 in [20] it follows that if f is bounded and continuous on G, the function

$$x \to \int_G f(y)\,[\pi(x)h](y)\,dy$$

is analytic on G and the function $h * f$ likewise. Now to prove Lemma 20 we select a continuous function γ on G of compact support such that

$$|\gamma * f(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for } x \in C;$$

next we select a sequence (φ_n) of well-behaved vectors converging to γ. Then the sequence $(\varphi_n * f)(g)$ converges to $(\gamma * f)(g)$ uniformly on G and a suitable φ_N satisfies (4.6) and (4.7).

Now we can finish the proof of Theorem 15. Let u be a solution of (4.2) of slow growth. The function $\tilde{u}$ on $G \times G$ introduced earlier satisfies

$$T_1 \tilde{u} = T_2 \tilde{u} \quad \text{for each } T \in \mathbf{D}_0(G).$$

If φ belongs to $C^\infty(G \times G)$ and $L^1(G \times G)$ the convolution

$$(\varphi * u)(x_1, x_2) = \int_{G \times G} \varphi(y_1, y_2)\,\tilde{u}(y_1^{-1}x_1, y_2^{-1}x_2)\,dy_1\,dy_2$$

exists, and since u is of slow growth

$$T_1(\varphi * \tilde{u}) = \varphi * T_1 \tilde{u}$$
$$T_2(\varphi * \tilde{u}) = \varphi * T_2 \tilde{u}$$

and therefore

$$T_1(\varphi * \tilde{u}) = T_2(\varphi * \tilde{u}) \quad \text{for all } T \in \mathbf{D}_0(G). \tag{4.8}$$

The function $\varphi * \tilde{u}$, being constant on left cosets mod $(K \times K)$ determines a function $v \in C^\infty(G/K \times G/K)$ such that

$$v(\pi(g), \pi(g')) = (\varphi * \tilde{u})(g, g')$$

and

$$D_1 v = D_2 v \quad \text{for all } D \in \mathbf{D}(G/K).$$

If we choose φ in accordance with Lemma 20, v is an analytic solution of (4.2), and a suitable sequence of such solutions approximates u uniformly on compact subsets of G/K and (4.3) follows.

Remark. The relation $M^x v = v$ which characterizes the solutions of (4.5) can be written differently. Let h have compact support on G and satisfy the conditions:

(i) $h(xk) = h(x)$ for all $x \in G$ and all $k \in K$

(ii) $\int_G h(x^{-1})\, dx = 1.$

The relation $M^x v = v$ for all x is then equivalent to

$$\tilde{v} * h = \tilde{v}$$

for every h with the properties (i) and (ii). This is easily proved by using the integration theory on homogeneous spaces and shows how the operators in $\mathbf{D}(G/K)$ appear as infinitesimal generators for the convolution operators $\tilde{f} \to \tilde{f} * h$ considered as operators on $C^\infty(G/K)$.

2. The Darboux equation in a symmetric space

We shall now suppose G/K is a symmetric coset space and K compact. Here the algebra $\mathbf{D}(G/K)$ is commutative; we shall give certain consequences of this fact.

Theorem 16. *For each $x \in G$, M^x commutes with all the operators in $\mathbf{D}(G/K)$.*

Proof. It is clear from Theorems 13 and 14 that if f is analytic on G/K and $D \in \mathbf{D}(G/K)$, then

$$D M^x f = M^x D f \tag{4.9}$$

if x is sufficiently close to e in G. However f and Df are analytic so (4.9) holds for all $x \in G$. Let T be the operator in $\mathbf{D}_0(G)$ that corresponds to D according to Lemma 16, and N^x the operator on $C^\infty(G)$ given by

$$[N^x F](g) = \int_K F(gkx)\, dk.$$

We shall now prove (4.9) for $f \in C_c^\infty(G/K)$ and fixed $x \in G$. The arguments used in the proof of Lemma 20 show that there exists a sequence φ_n of functions on G such that $\varphi_n * \tilde{f}$, $\varphi_n * TN^x \tilde{f}$ and $\varphi_n * T\tilde{f}$ are all analytic functions on G and the sequences $(\varphi_n * \tilde{f})$, $(\varphi_n * TN^x \tilde{f})$, $(\varphi_n * T\tilde{f})$ converge to the functions $\tilde{f}$, $TN^x \tilde{f}$ and $T\tilde{f}$ uniformly on G. Using the obvious relations

$$\varphi_n * N^x \tilde{f} = N^x (\varphi_n * \tilde{f})$$

$$\varphi_n * T \tilde{f} = T (\varphi_n * \tilde{f}),$$

(4.9) follows easily for each $f \in C_c^\infty(G/K)$. Finally, to prove (4.9) for each $f \in C^\infty(G/K)$, one just has to observe that for each compact subset M of G/K there exists a function $f_M \in C_c^\infty(G/K)$ which agrees with f on an open set containing M.

The following corollary is proved in [2] in a different way.

Corollary. *Let $f \in C^\infty(G/K)$ and put*

$$V(x, g) = \int_K \tilde{f}(gkx)\, dk.$$

Then V satisfies the "Darboux Equation"

$$T_1 V = T_2 V \quad \text{for each } T \in \mathbf{D}_0(G).$$

In fact, write $T\tilde{f} = F$. Then

$$[T_1 V](x, g) = \int_K F(gkx)\, dk = [N^x T \tilde{f}](g) = [T N^x \tilde{f}](g) = [T_2 V](x, g).$$

The zonal spherical functions φ on G/K introduced by E. Cartan and I. Gelfand are by definition the (analytic) eigenfunctions of all $D \in \mathbf{D}(G/K)$ which are invariant under K, that is $\varphi^{\tau(k)} = \varphi$ for all $k \in K$. Since M^x (for x near e in G) is a power series in the generators $D^1, \ldots, D^l$, $M^x = P(D^1, \ldots, D^l)$, it is clear that φ is an eigenfunction of M^x, $M^x \varphi = \lambda \varphi$. It follows that if φ is not identically 0, then $\varphi(\pi(e)) \neq 0$ so we assume the zonal spherical functions normalized by $\varphi(\pi(e)) = 1$. These functions then satisfy the functional equation

$$M^x \varphi = \varphi(\pi(x))\, \varphi.$$

On the other hand, there exist constants $\lambda^1, \ldots, \lambda^l$ such that $D^i \varphi = \lambda^i \varphi$. Hence $M^x \varphi = P(\lambda^1, \ldots, \lambda^l) \varphi$ so

$$\varphi(\pi(x)) = P(\lambda^1, \ldots, \lambda^l).$$

This shows that φ is determined by the ordered system $(\lambda^1, \ldots, \lambda^l)$ of eigenvalues. Formally M^x is a zonal spherical function of the operators $D^1, \ldots, D^l$.

3. Invariant differential equations on two-point homogeneous spaces

We shall now combine the previous group theoretic methods with special geometric properties of two-point homogeneous spaces. This leads naturally to more explicit results.

We shall now assume that M is a two-point homogeneous space, and we exclude in advance the trivial case when M has dimension 1. Let G be the connected component of e in the group of all isometries of M. Then M can be represented G/K where K is compact and G is pairwise transitive on M. $\mathbf{D}(G/K)$ consists of all polynomials in the Laplace-Beltrami operator Δ. We see also that the mean value operators M^x and M^y are the same if $d(\pi(e), \pi(x)) = d(\pi(e), \pi(y))$ and consequently we write M^r instead of M^x if $r = d(\pi(e), \pi(x))$. Let p be a point in M, $\mathrm{S}_r(p)$ the geodesic sphere around p with radius r, $d\omega_r$ the volume element on $\mathrm{S}_r(p)$ and $A(r)$ the area of $\mathrm{S}_r(p)$.

LEMMA 21. *In geodesic polar coordinates around p, Δ has the form*

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)} \frac{dA}{dr} \frac{\partial}{\partial r} + \Delta'$$

where Δ' is the Laplace-Beltrami operator on $\mathrm{S}_r(p)$.

Proof. Let the geodesic polar coordinates be denoted by $r, \theta_1, \ldots, \theta_{n-1}$. Due to the fact that the geodesics emanating from p are perpendicular to $\mathrm{S}_r(p)$ the metric tensor must have the form

$$ds^2 = dr^2 + \sum_{i,j=1}^{n-1} g_{ij}\, d\theta_i\, d\theta_j$$

and the Laplace-Beltrami operator is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r} \frac{\partial}{\partial r} + \frac{1}{\sqrt{g}} \sum_k \frac{\partial}{\partial \theta_k} \left(\sum_i g^{ik} \sqrt{g} \frac{\partial}{\partial \theta_i} \right).$$

Since r and Δ are invariant under the subgroup of G that leaves p fixed, Δr is also invariant under this subgroup which acts transitively on the geodesics emanating from p. Hence

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r}$$

is a function of r alone so

$$\log \sqrt{g} = \alpha(r) + \beta(\theta_1, \ldots, \theta_{n-1})$$

and

$$\sqrt{g} = e^{\alpha(r)} e^{\beta(\theta_1, \ldots, \theta_{n-1})}.$$

On the other hand, the volume of $\mathrm{S}_r(p)$ is given by

$$V(r) = \int \sqrt{g}\, dr\, d\theta_1, \ldots, d\theta_{n-1}$$

and thus we find for $A(r) = dV/dr$ the formula

$$A(r) = \int \sqrt{g}\, d\theta_1, \ldots, d\theta_{n-1} = C e^{\alpha(r)} \quad (C = \text{constant})$$

and

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r} = \frac{1}{A(r)} \frac{dA}{dr}.$$

The lemma now follows by observing that the induced metric on $S_r(p)$ is given by

$$ds_1^2 = \sum_{i,j=1}^{n-1} g_{ij}\, d\theta_i\, d\theta_j.$$

The next lemma, which also is proved by Günther [16], is just a special case of the corollary of Theorem 16.

LEMMA 22. *Let $f \in C^\infty(M)$ and put $F(p, q) = [M^r f](p)$ if $p, q \in M$, $d(p, q) = r$. Then $\Delta_1 F = \Delta_2 F$.*

We shall now state and give a different proof for the extension of Ásgeirsson's theorem to two-point homogeneous spaces. The proof is based on an ingenious method used in Ásgeirsson's original proof ([1], p. 334).

THEOREM 17. *Let M be a two-point homogeneous space and let u be a twice continuously differentiable function on $M \times M$ which satisfies the equation*

$$\Delta_1 u = \Delta_2 u \tag{4.10}$$

Then for each $(x_0, y_0) \in M \times M$

$$\int_{S_r(x_0)} u(x, y_0)\, d\omega_r(x) = \int_{S_r(y_0)} u(x_0, y)\, d\omega_r(y). \tag{4.11}$$

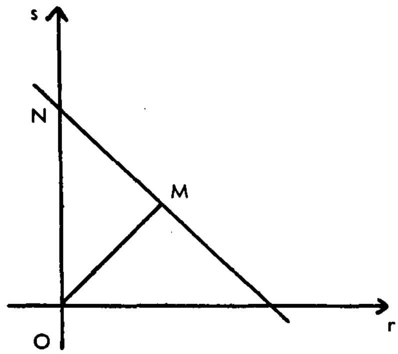
Proof. We assume first M is non-compact. From Theorem 3 we know that M is isometric to a symmetric Riemannian space G/K. $\mathrm{Ad}_G(K)$ is transitive on the directions in the tangent space to G/K at $\pi(e)$, in particular G/K is irreducible. As we saw at the end of the proof of Theorem 3, geodesic polar coordinates with origin at a point $p \in M$ are valid on the entire M.

Now, suppose the function u satisfies (4.10) and let (x_0, y_0) be an arbitrary point in $M \times M$. Consider the function U defined by

$$U(r, s) = [M_1^r M_2^s u](x_0, y_0) \quad \text{for } r, s \geqslant 0$$

We view U as a function on $M \times M$ by giving it the value $U(r, s)$ on the set $S_r(x_0) \times S_s(y_0)$. Since Δ commutes (1) with M^r we obtain from (4.10) and Lemma 21

(1) Theorem 16 shows that Δ and M^r commute when applied to C^∞-functions. In the same way it can be shown that they commute when applied to C^2-functions.



$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{A(r)}\frac{dA}{dr}\frac{\partial U}{\partial r}=\frac{\partial^2 U}{\partial s^2}+\frac{1}{A(s)}\frac{dA}{ds}\frac{\partial U}{\partial s}\,.$$

If we put $F(r, s)=U(r, s)-U(s, r)$ we obtain the relations

$$\frac{\partial^2 F}{\partial r^2}+\frac{1}{A(r)}\frac{dA}{dr}\frac{\partial F}{\partial r}-\frac{\partial^2 F}{\partial s^2}-\frac{1}{A(s)}\frac{dA}{ds}\frac{\partial F}{\partial s}=0 \tag{4.12}$$

$$F(r, s)=-F(s, r).$$

After multiplication of (4.12) by $2\,A(r)\partial F/\partial s$ and some manipulation we obtain

$$-A(r)\frac{\partial}{\partial s}\left[\left(\frac{\partial F}{\partial r}\right)^2+\left(\frac{\partial F}{\partial s}\right)^2\right]+2\frac{\partial}{\partial r}\left(A(r)\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\right)-\frac{2A(r)}{A(s)}\frac{dA}{ds}\left(\frac{\partial F}{\partial s}\right)^2=0. \tag{4.13}$$

Now consider the line MN with equation $r+s=$ constant in the (r, s)-plane and form the plane integral of (4.12) over the triangle OMN, (see figure), and use Green's formula. If $\partial/\partial n$ denotes derivation in the direction of the outgoing normal and dl is the element of arc length, we obtain

$$\oint_{OMN}\left\{-A(r)\left[\left(\frac{\partial F}{\partial r}\right)^2+\left(\frac{\partial F}{\partial s}\right)^2\right]\frac{\partial s}{\partial n}+2A(r)\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\frac{\partial r}{\partial n}\right\}dl-\iint_{OMN}\frac{2A(r)}{A(s)}\frac{dA}{ds}\left(\frac{\partial F}{\partial r}\right)^2 dr\,ds=0. \tag{4.14}$$

On OM: $\left(\frac{\partial r}{\partial n}, \frac{\partial s}{\partial n}\right)=\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $F(r, r)=0$ so $\frac{\partial F}{\partial r}+\frac{\partial F}{\partial s}=0$.

On MN: $\left(\frac{\partial r}{\partial n}, \frac{\partial s}{\partial n}\right)=\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

On ON: $A(r)=0$.

From (4.14) follows the relation

$$\frac{1}{\sqrt{2}}\int_{MN}A(r)\left(\frac{\partial F}{\partial r}-\frac{\partial F}{\partial s}\right)^2 dl+\iint_{OMN}\frac{2A(r)}{A(s)}\frac{dA}{ds}\left(\frac{\partial F}{\partial r}\right)^2 dr\,ds=0. \tag{4.15}$$

Now Lemma 3 shows that $dA/ds \geqq 0$ for all s and (4.15) shows therefore that

$$\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s},$$

which is the directional derivative of F in the direction MN, vanishes on MN. Consequently $F \equiv 0$ so U is symmetric. In particular $U(r, 0) = U(0, r)$ and this is (4.11). If M is compact the proof above fails since $A(r)$ is no longer an increasing function of r. To show that (4.11) is valid even if the solution u is not analytic we resort again to Lemma 20 to approximate u by analytic solutions. Since M is compact this requires only the Peter-Weyl theory and not the theory of well-behaved vectors.

We recall now some facts from [4] about the behaviour of the geodesics on $M = G/K$. Let $p_0 = \pi(e)$ and dim $M = n$. If M is non-compact Exp maps $M_{p_0} (= \mathfrak{m})$ homeomorphically onto M. If M is compact all geodesics are closed and have the same length 2σ. The mapping Exp maps the ball $0 \leqslant Q_{p_0}(X, X) \leqslant \sigma^2$ onto M and is one-to-one on the open ball $0 \leqslant Q_{p_0}(X, X) < \sigma^2$. Except for the real elliptic spaces, Exp becomes singular on the sphere $Q_{p_0}(X, X) = \sigma^2$, and thus the set $S_\sigma(p_0)$, which Cartan calls the *antipodal variety associated to* p_0, will in general have dimension inferior to $n-1$. For the various n-dimensional two-point homogeneous spaces the dimension of $S_\sigma(p_0)$ is given in [4] as 0, $n-1$, $n-2$, $n-4$ for the spheres, real elliptic spaces, hermitian elliptic spaces and quaternian elliptic spaces respectively. For the Cayley elliptic plane $S_\sigma(p_0)$ has dimension 8.

The following theorem gives a generalization of the Poisson equation to two-point homogeneous spaces. Consider the function

$$\varphi(r) = \int_a^r \frac{1}{A(t)}\, dt \quad \text{where} \quad \begin{cases} a > 0 & \text{if } M \text{ is non-compact} \\ a = \sigma & \text{if } M \text{ is real elliptic} \\ 0 < a < \sigma & \text{otherwise} \end{cases}$$

We define the function Ψ by

$$\Psi(p, q) = \varphi(r) \text{ if } d(p, q) = r.$$

In view of Lemma 21, Ψ satisfies the equations $\Delta_1 \Psi = \Delta_2 \Psi = 0$ and as the following theorem shows Ψ can be regarded as a fundamental solution.

THEOREM 18. *Let f be a twice continuously differentiable function on M with compact support. Then the function u given by (dq is the volume element on M)*

$$u(p) = \int_M f(q) \Psi(p, q)\, dq$$

satisfies the "Poisson equation"

$$\Delta u = f \quad \textit{if } M \textit{ is non-compact} \tag{4.16}$$

$$\Delta u = f - M^\sigma f \quad \textit{if } M \textit{ is compact.} \tag{4.17}$$

In the compact case the compensating term $[M^\sigma f](p)$ is the average of f on the antipodal variety associated to p. In the case when M is a sphere, $M^\sigma f = f \circ A$ where A is the antipodal mapping.

We first prove (4.16). Since $\varphi(r) = O(r^{2-n})$ as $r \to 0$ the integral $\int f(q)\psi(p, q)\,dq$ is convergent and

$$u(p) = \int_M f(q)\Psi(p, q)\,dq = \int_0^\infty dr \int_{S_r(p)} f(q)\Psi(p, q)\,d\boldsymbol{\omega}_r(q)$$

$$= \int_0^\infty A(r)\varphi(r)[M^r f](p)\,dr.$$

We apply Δ to this relation and make use of Lemma 22. Then we obtain

$$[\Delta u](p) = \int_0^\infty \varphi(r)A(r)[\Delta M^r f](p)\,dr = \int_0^\infty \varphi(r)A(r)\Delta_r([M^r f](p))\,dr.$$

Now we keep p fixed (and omit writing it in the formulas below) and use Lemma 21. Then

$$\Delta u = \int_0^\infty \varphi(r)A(r)\left\{\frac{\partial^2 M^r f}{\partial r^2} + \frac{1}{A(r)}\frac{dA}{dr}\frac{\partial M^r f}{\partial r}\right\}dr$$

$$= \lim_{\varepsilon\to 0}\left[\varphi(r)A(r)\frac{\partial}{\partial r}M^r f\right]_\varepsilon^\infty - \lim_{\varepsilon\to 0}\int_\varepsilon^\infty \varphi'(r)A(r)\frac{\partial}{\partial r}M^r f\,dr.$$

Since $\lim_{\varepsilon\to 0}\varphi(\varepsilon)A(\varepsilon) = 0$ and $\varphi'(r)A(r) = 1$, the relation (4.16) follows.

We next consider the case when M is compact. Here $\varphi(r)\to\infty$ as $r\to\sigma$ (except for the real elliptic space). Nevertheless $A(r)\varphi(r)$ is bounded as $r\to\sigma$ and the integral $\int f(q)\Psi(p, q)\,dq$ exists. As before we obtain

$$u(p) = \int_0^\sigma A(r)\varphi(r)[M^r f](p)\,dr.$$

Using Lemma 21 and 22 it follows that

$$\Delta u = \int_0^\sigma \varphi(r)A(r)\left\{\frac{\partial^2 M^r f}{\partial r^2} + \frac{1}{A(r)}\frac{dA}{dr}\frac{\partial M^r f}{\partial r}\right\}dr$$

$$= \lim_{\varepsilon_1,\varepsilon_2\to 0}\left[\varphi(r)A(r)\frac{\partial M^r f}{\partial r}\right]_{\varepsilon_1}^{\sigma-\varepsilon_2} - \lim_{\varepsilon_1,\varepsilon_2\to 0}\int_{\varepsilon_1}^{\sigma-\varepsilon_2}\varphi'(r)A(r)\frac{\partial M^r f}{\partial r}\,dr.$$

If M is real elliptic,

$$\lim_{r\to\sigma} A(r) \neq 0 \text{ and } \lim_{r\to\sigma}\varphi(r) = 0$$

due to the choice of a. If, on the other hand, M is not a real elliptic space

$$\lim_{r\to\sigma} A(r)=0 \text{ and } \lim_{r\to\sigma} \varphi(r)=\infty.$$

Now $\varphi'(r)A(r)=1$ and $\lim \varphi(\varepsilon_1)A(\varepsilon_1)=0$ as ε_1 tends to 0. To determine $\lim_{r\to\sigma}\varphi(r)A(r)$ as r converges to σ we observe that $A(r)$ is given by the formula

$$A(r)=\int_{\|X\|=r} \det(A_X)\,d\omega_r(X) \quad 0\leq r<\sigma, \tag{4.18}$$

where A_X is the linear transformation (2.5). Since $\det(A_X)$ is invariant under the group $Ad_G(K)$ it is a function of r only and

$$A(r)=\det(A_X)\,r^{n-1}\Omega_n$$

where Ω_n is the surface area of the unit sphere in $\mathbf{R}^n$. We can use this last formula to continue $A(r)$ to an analytic function in an open interval containing $r=\sigma$. Consequently $A(r)$ has the form $A(r)=(r-\sigma)^m h(r)$ in such an interval. Here m is an integer and $h(r)$ is an analytic function, $h(\sigma)\neq 0$. This being established, the relation

$$\lim_{r\to\sigma}\varphi(r)A(r)=0$$

follows easily. We find therefore, whether M is real elliptic or not,

$$\Delta u=-M^\sigma f+M^0 f=f-M^\sigma f.$$

4. Decomposition of a function into integrals over totally geodesic submanifolds

The formula of J. Radon determining a function on $\mathbf{R}^n$ by means of its integrals over hyperplanes has had considerable importance for partial differential equations, particularly in G. Herglotz' treatment of hyperbolic equations with constant coefficients (G. Herglotz [26], F. John [27]). We give below an extension of Radon's formula to spherical and hyperbolic spaces. The proof seems to be new in the Euclidean case.

DEFINITION. Let S be a connected submanifold of a Riemannian manifold M. S is called *totally geodesic* if each geodesic in M which touches S lies entirely in S.

Let M be a simply connected Riemannian manifold of constant curvature $\varkappa$ and dimension $n>1$. Such a space is either a hyperbolic, Euclidean or a spherical space. It is well known that for each integer d, $0<d<n$ there exist totally geodesic submanifolds of M of dimension d. Using the notation from the end of Chapter II, M can be written

$$\mathbf{SO}(n+1)/\mathbf{SO}(n),\ \ \mathbf{R}^n\cdot\mathbf{SO}(n)/\mathbf{SO}(n),\ \ \mathbf{SO}^1(n+1)/\mathbf{SO}(n) \tag{4.19}$$

according as $\varkappa$ is positive, 0 or negative. Let $\mathbf{M}_{n,d}(p)$ denote the set of d-dimensional totally geodesic submanifolds of M passing through some fixed point p. Since $\mathbf{O}(n)$ acts transitively on the set of d-dimensional subspaces of $\mathbf{R}^n$ we see that $\mathbf{M}_{n,d}(p)$ can be identified with the coset space $\mathbf{O}(n)/\mathbf{O}(d)\times\mathbf{O}(n-d)$. In particular $\mathbf{M}_{n,d}(p)$ has a unique normalized measure invariant under the action of $\mathbf{O}(n)$.

THEOREM 19. *Let M be a simply connected Riemannian manifold of constant curvature $\varkappa$ and dimension $n>1$. For d even, $0<d<n$, let $Q_d(x)$ denote the polynomial*

$$Q_d(x)=[x-\varkappa(d-1)(n-d)][x-\varkappa(d-3)(n-d+2)]\ \dots\ .[x-\varkappa\cdot 1\,(n-2)]$$

of degree $d/2$. For each function $f\in C_c^\infty(M)$, let $[I_d f](p)$ denote the average of the values of the integrals of f over all d-dimensional totally geodesic submanifolds through p. Then

$$Q_d(\Delta)\,I_d f=\gamma f \qquad \text{if } M \text{ is non-compact}$$

$$Q_d(\Delta)\,I_d f=\gamma(f+f\circ A) \qquad \text{if } M \text{ is compact.}$$

In the latter case $M=\mathbf{S}^n$ and A denotes the antipodal mapping. The constant γ equals

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-d}{2}\right)}(-4\pi)^{\frac{1}{2}d}.$$

Proof. We consider first the non-compact case, $M=G/K$, $K=\mathbf{SO}(n)$. In geodesic polar coordinates which are valid on the entire M the metric is given by

$$d s^2=d r^2+\frac{\sinh^2(r\sqrt{-\varkappa})}{(r\sqrt{-\varkappa})^2}\,r^2\,d\sigma^2$$

where $d\sigma^2$ is the fundamental metric form on the unit sphere in $\mathbf{R}^n$. Let $p_0=\pi(e)$ and choose g such that $g\cdot p_0=p$. If E is a fixed element in $\mathbf{M}_{n,d}(p_0)$ we consider the integral

$$F(k)=\int_E f(g\,k\cdot q)\,d q \qquad k\in K$$

where dq denotes the volume element in E. If K_0 is the subgroup of K that transforms E into itself then $F(kk_0)=F(k)$ for $k_0\in K_0$; consequently the average $[I_d f](p)=\int_K F(k)\,dk$ where dk is the normalized Haar measure on K.

$$\begin{aligned}[I_d f](p)&=\int_K d k\int_E f(g\,k\cdot q)\,d q\\ &=\int_E d q\int_K f(g\,k\cdot q)\,d k=\int_E [M^r f](p)\,d q\end{aligned}$$

where $r = d(p_0, q)$. Now we make use of the fact that E is totally geodesic. Let γ be an E-geodesic in E; let Γ be an M-geodesic touching γ at p. Then $\Gamma \subset E$ and due to the local minimizing property of geodesics, $\Gamma = \gamma$. It follows immediately that E is complete and thus two arbitrary points $q_1, q_2 \in E$ can be joined by a minimizing E-geodesic arc $\gamma_{q_1 q_2}$. Let Γ_{q_1} be an M-geodesic touching $\gamma_{q_1 q_2}$ at q_1. Then by the previous remark $\gamma_{q_1 q_2} \subset \Gamma_{q_1}$. Since two arbitrary points in M can be joined by exactly one geodesic the same is true of E and the distance between q_1 and q_2 is the same whether it is measured in the E-metric or the M-metric. In particular E and M have the same constant sectional curvature $\varkappa$. Let S_r^{d-1} and S_r^{n-1} be geodesic spheres in E and M respectively with radius r. Their areas are

$$A_d(r) = \left[\frac{\sinh(r\sqrt{-\varkappa})}{\sqrt{-\varkappa}}\right]^{d-1} \Omega_d$$

$$A(r) = \left[\frac{\sinh(r\sqrt{-\varkappa})}{\sqrt{-\varkappa}}\right]^{n-1} \Omega_n.$$

From this we find
$$[I_d f](p) = \int_0^\infty A_d(r)\,[M^r f](p)\,dr \tag{4.20}$$

Now we apply Δ to both sides of (4.20) and make use of Lemma 22;

$$[\Delta I_d f](p) = \int_0^\infty A_d(r)\,[\Delta M^r f](p)\,dr = \int_0^\infty A_d(r)\,\Delta_r([M^r f](p))\,dr.$$

We shall now keep p fixed and write $F(r) = [M^r f](p)$.

LEMMA 23. *Let m be an integer, $0 < m < n = \dim M$. Put $\lambda = \sqrt{-\varkappa}$. Then*

$$\int_0^\infty \sinh^m \lambda r\, \Delta_r F\, dr = (-\lambda^2)(n-m-1)\left[m\int_0^\infty \sinh^m \lambda r\, F(r)\,dr + (m-1)\int_0^\infty \sinh^{m-2}\lambda r\, F(r)\,dr\right].$$

If $m = 1$ the term $(m-1)\int_0^\infty \sinh^{m-2}\lambda r\, F(r)\,dr$ should be replaced by $\frac{1}{\lambda}F(0)$.

Proof. Using Lemma 21 we have

$$\int_0^\infty \sinh^m \lambda r\, \Delta_r F\, dr = \int_0^\infty \sinh^m \lambda r \left(\frac{d^2 F}{dr^2} + \frac{1}{A(r)}\frac{dA}{dr}\frac{dF}{dr}\right) dr$$

and the result follows after repeated integration by parts. From Lemma 23 we see that

$$[\Delta_p + \lambda^2 m (n-m-1)] \int_0^\infty \sinh^m \lambda r \, [M^r f](p) \, dr$$
$$= (-\lambda^2)(n-m-1)(m-1) \int_0^\infty \sinh^{m-2} \lambda r \, [M^r f](p) \, dr.$$

Applying this repeatedly to (4.20) the first relation of Theorem 19 follows.

If M is compact it is a sphere and we can proceed in a similar way as in the non-compact case, but here we have to observe that the geodesics emanating from p all intersect at the antipodal point $A(p)$. In geodesic polar coordinates the metric on M is given by

$$d s^2 = d r^2 + \frac{\sin^2 (r \sqrt{\varkappa})}{(r \sqrt{\varkappa})^2} r^2 d \sigma^2$$

where $d\sigma^2$ is the fundamental metric form on the unit sphere in $\mathbf{R}^n$. As in the non-compact case we prove the formula

$$[I_d f](p) = \int_0^{\frac{\pi}{\sqrt{\varkappa}}} A_d(r) \, [M^r f](p) \, dr \tag{4.21}$$

where
$$A_d(r) = \left[\frac{\sin (r \sqrt{\varkappa})}{\sqrt{\varkappa}} \right]^{d-1} \Omega_d.$$

For a fixed p, we put $F(r) = [M^r f](p)$. The analogue of Lemma 23 is here

LEMMA 24. *Let m be an integer satisfying $0 < m < n = \dim M$. We put $\lambda = \sqrt{\varkappa}$. Then*

$$\int_0^{\frac{\pi}{\lambda}} \sin^m \lambda r \, \Delta_r F \, dr = \lambda^2 (n-m-1) \left[m \int_0^{\frac{\pi}{\lambda}} \sin^m \lambda r \, F(r) \, dr - (m-1) \int_0^{\frac{\pi}{\lambda}} \sin^{m-2} \lambda r \, F(r) \, dr \right].$$

If $m = 1$, the term $(m-1) \int_0^{\frac{\pi}{\varkappa}} \sin^{m-2} \lambda r \, F(r) \, dr$ should be replaced by $\frac{1}{\lambda} \left\{ f(p) + f(A(p)) \right\}$.

This is easily verified by using the formula

$$A(r) = \left[\frac{\sin (r \sqrt{\varkappa})}{\sqrt{\varkappa}} \right]^{n-1} \Omega_n.$$

Lemma 24 can be rewritten by using Lemma 22 and we obtain

$$[\Delta_p - m \lambda^2 (n-m-1)] \int_0^{\frac{\pi}{\lambda}} \sin^m \lambda r \, [M^r f](p) \, dr$$
$$= (-\lambda^2)(n-m-1)(m-1) \int_0^{\frac{\pi}{\lambda}} \sin^{m-2} \lambda r \, [M^r f](p) \, dr.$$

If we apply this repeatedly to (4.21) the latter part of Theorem 19 follows.

5. Wave equations on harmonic Lorentz spaces

In the following sections we shall show that certain mean value theorems connected with the Laplace operator are not restricted to a positive definite metric as given in ordinary potential theory. We extend the definition of the mean value operator M^r to harmonic Lorentz spaces and establish various relations between $\square$ and M^r. The situation changes considerably as we pass to Lorentzian metric. "Spheres" are no longer compact and a family of concentric spheres does not shrink to a point as the radius converges to 0. Also the analyticity of the solution of Laplace's equation is lost.

We consider the Lorentz spaces of constant curvature studied in Chapter II, § 4, where the wave operator has a simple characterization (Theorem 12). Let $M = G/H$ be such a Lorentz space of dimension $n > 1$, carrying the metric tensor Q. Here $H = \mathbf{S}\,\mathbf{O}^1(n)$ and G is either $G^0 = \mathbf{R}^n \cdot \mathbf{S}\mathbf{O}^1(n)$, $G^- = \mathbf{S}\,\mathbf{O}^1(n+1)$ or G^+ as defined in Chapter II, § 4. Let s_0 be the geodesic symmetry of G/H with respect to the point p_0. Then s_0 extends to an isometry of G/H as we have seen in Chapter II. The mapping $\sigma : g \rightarrow s_0 g s_0$ is an involutive automorphism of G which is identity on H. Let $\mathfrak{m}$ be the eigenspace for the eigenvalue -1 of the automorphism $d\sigma$ of the Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ denotes as before the Lie algebra of H we have

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \tag{4.22}$$

As before we identify $\mathfrak{m}$ with M_{p_0} and denote by C_{p_0} the light cone in M_{p_0} at p_0. The interior of the cone C_{p_0} has two components; the component that contains the timelike vectors $(-1, 0, \ldots, 0)$, $-X_{n+1}$, $-iX_{n+1}$ in the cases G^0/H, G^-/H, G^+/H respectively we call the *retrograde cone* in $\mathfrak{m}$ at p_0. It will be denoted by D_{p_0}. The component of the hyperboloid $Q_{p_0}(X, X) = r^2$ that lies in D_{p_0} will be denoted by $S_r(p_0)$ in agreement with previous terminology. If p is any other point of M, we define the light cone C_p in M_p at p, and the retrograde cone D_p in M_p at p as follows. We choose $g \in G$ such that $\tau(g) \cdot p_0 = p$ and put $C_p = d\tau(g) \cdot C_{p_0}$, $D_p = d\tau(g) \cdot D_{p_0}$. Due to the connectedness of H this is a valid definition. Similarly the "sphere" $S_r(p)$ (the ball $B_r(p)$) is the component of the hyperboloid $Q_p(X, X) = r^2$ $(0 < Q_p(X, X) < r^2)$ which lies in D_p. Finally, if Exp is the Exponential mapping of M_p into M we put

$$\mathbf{D}_p = \operatorname{Exp} D_p \qquad \mathbf{C}_p = \operatorname{Exp} C_p$$

$$\mathbf{S}_r(p) = \operatorname{Exp} S_r(p) \qquad \mathbf{B}_r(p) = \operatorname{Exp} B_r(p)$$

$\mathbf{C}_p$ and $\mathbf{D}_p$ are called the light cone in M with vertex p and the retrograde cone in M with vertex p. For the spaces G^+/H we tacitly assume $r < \pi$ in order that Exp will be one-to-one.

We wish now to study solutions of various equations involving $\square$ inside the retrograde cone $\mathbf{D}_p$ for $p \in M$. This emphasis on $\mathbf{D}_p$ is in agreement with the physical and geometric situation occurring in relativity theory and in Hadamard's theory of hyperbolic equations.

Let dh denote a two-sided invariant measure on the unimodular group H. Let p be a point in M, and u a function defined in the retrograde cone $\mathbf{D}_p$. Let $q \in \mathbf{S}_r(p)$ $(r > 0)$ and consider the integral

$$\int_H u(g h g^{-1} \cdot q)\, dh$$

where g is an arbitrary element in G such that $\pi(g) = p$. The choice of g in the coset gH and of $q \in \mathbf{S}_r(p)$ is immaterial due to the invariance of dh. The integral is thus an invariant integral of u over $\mathbf{S}_r(p)$ and in analogy with the previous mean value we write

$$[M^r u](p) = \int_H u(g h g^{-1} \cdot q)\, dh$$

Now $\mathbf{S}_r(p)$ has a positive definite Riemannian metric induced by the Lorentzian metric on M. Let $d\boldsymbol{\omega}_r$ denote the volume element on $\mathbf{S}_r(p)$. Then if K denotes the (compact) subgroup of gHg^{-1} which leaves the point q fixed, $\mathbf{S}_r(p)$ can be identified with coset space gHg^{-1}/K and

$$\int_H u(g h g^{-1} \cdot q)\, dh = \frac{1}{A(r)} \int_{\mathbf{S}_r(p)} u(q)\, d\boldsymbol{\omega}_r(q)$$

where $A(r)$ is a positive scalar depending on r only. We have thus $dh = d\boldsymbol{\omega}_r\, dk / A(r)$ where dk is the normalized Haar measure on K. Now the Exponential mapping at p which maps D_p onto $\mathbf{D}_p$ is length preserving on the geodesics through p and maps $S_r(p)$ onto $\mathbf{S}_r(p)$. Consequently, if $s \in S_r(p)$ and X denotes the vector $\overrightarrow{ps}$ in M_p, the ratio of the volume elements of $\mathbf{S}_r(p)$ and $S_r(p)$ at s is given by $\det(d\,\mathrm{Exp}_X)$. By Lemma 8 and 13 this equals 1, $(\sinh r/r)^{n-1}$, $(\sin r/r)^{n-1}$ in the flat, negatively curved and positively curved case respectively. It follows that $A(r) = cr^{n-1}$, $c(\sinh r)^{n-1}$, $c(\sin r)^{n-1}$ in the three cases. Here c is a constant which depends on the choice of dh. We normalize dh in such a way that $c = 1$ and have then the relation

$$[M^r u](p) = \int_H u(g h g^{-1} \cdot q)\, dh = \int_{\mathbf{S}_r(p)} u(q)\, d\sigma(q) \tag{4.23}$$

where $d\sigma = 1/A(r)\, d\boldsymbol{\omega}_r$. Suppose now $x_1, \ldots, x_n$ are coordinates in M_p such that the cone C_p has equation $x_1^2 - x_2^2 - \cdots - x_n^2 = 0$ and the axis in the retrograde cone D_p is the negative x_1-axis. If $\theta_1, \ldots, \theta_{n-2}$ are geodesic polars on the unit sphere in $\mathbf{R}^{n-1}$ we obtain coordinates in D_p by

$$x_1 = -r\cosh\zeta \qquad\qquad 0 \leqq \zeta < \infty,\ 0 < r < \infty$$
$$x_2 = r\sinh\zeta\cos\theta_1$$
$$\dots\dots\dots\dots\dots$$
$$x_n = r\sinh\zeta\sin\theta_1\sin\theta_2, \dots, \sin\theta_{n-2}.$$

The volume element on $S_r(p)$ is then given by

$$d\omega_r = r^{n-1}\sinh^{n-2}\zeta\, d\zeta\, d\omega^{n-2}$$

where $d\omega^{n-2}$ is the volume element on the unit sphere in $\mathbf{R}^{n-1}$. Using the Exponential mapping at p we can consider $(r, \zeta, \theta_1, \dots, \theta_{n-2})$ as coordinates(1) on $\mathbf{D}_p$. Let u be a function defined in $\mathbf{B}_{r_0}(p_0)$. We shall say u has *order* a if there exists a continuous (not necessarily bounded) function $C(r)$, $(0 < r < r_0)$ such that

$$|(u \circ \operatorname{Exp})(q)| \leqslant C(r)\, e^{-a\zeta} \qquad \text{for } q \in B_{r_0}(p_0) \tag{4.24}$$

in terms of the coordinates above.

For R^2 the following result has also been noted by Ásgeirsson (letter to the author).

THEOREM 20. *Suppose u satisfies the equation $\square u = 0$ in $\mathbf{B}_{r_0}(p_0)$. We assume that u and its first and second order partial derivatives have order $a > n-2$. Then*

$$[M^r u](p_0) = \alpha \int_r^\beta \frac{1}{A(r)}\, dr$$

where α and β are constants.

REMARK. If u converges to 0 fast enough in an immediate neighborhood of the cone $\mathbf{C}_{r_0}$ so that

$$[M^r u](p_0) = O\left(\left(\frac{|\log r|}{r}\right)^{n-2}\right)$$

then $[M^r u](p_0)$ is constant. We get thus an analogue of the mean value theorem for harmonic functions.

To prove the relation above we consider the integral

$$F(q) = \int_H u(h \cdot q)\, dh.$$

The measure dh has been normalized such that

$$dh = \sinh^{n-2}\zeta\, d\zeta\, d\omega^{n-2}\, dk.$$

(1) We call these the geodesic polar coordinates on $\mathbf{D}_p$.

Due to the growth condition on u it is clear that the integral is convergent and the operator $\square_q$ can be applied to the integral by differentiating under the integral sign. Since $\square$ is invariant under H we obtain $\square F = 0$. We now need a lemma whose statement and proof are entirely analogous to that of Lemma 21.

LEMMA 25. *In geodesic polarcoordinates on* $\mathbf{D}_p$, $\square$ *can be expressed*

$$\square = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)}\frac{dA}{dr}\frac{\partial}{\partial r} - \Delta'$$

where Δ' *is the Laplace-Beltrami operator on* $\mathbf{S}_r(p)$.

The minus sign is due to the circumstance that Q induces a negative definite metric on $\mathbf{S}_r(p)$ whereas Δ' is taken with respect to the positive definite metric.

The function $F(q)$ is constant on each sphere $\mathbf{S}_r(p_0)$. Due to Lemma 25, $F(q) = [M^r u](p_0)$ is a solution of the differential equation

$$\frac{d^2 v}{d r^2} + \frac{1}{A(r)}\frac{dA}{dr}\frac{dv}{dr} = 0$$

and can therefore be written

$$[M^r u](p_0) = \alpha \int_r^\beta \frac{1}{A(r)} dr$$

where α and β are constants.

6. Generalized Riesz potentials

For two-point homogeneous spaces M^r can be expressed as a power series in Δ when applied to analytic functions. This does not hold for the operators M^r and $\square$ in a harmonic Lorentz space; nevertheless we shall now establish various relations between M^r and $\square$. For this purpose it is convenient to generalize certain facts concerning Riesz potentials (M. Riesz [32]) to harmonic Lorentz spaces. These potentials, defined below, do not however coincide with the generalization to arbitrary Lorentzian spaces given by Riesz himself in [32].

We consider first the case $M = G^-/H$. Let $f \in C_c^\infty(M)$. The integral

$$\int_{\mathbf{D}_p} f(q) \sinh^{\lambda-n} r_{pq}\, dq \qquad dq = dr\, d\omega_r$$

converges absolutely if the complex number λ has real part $\geqslant n$.

We define
$$[I_-^\lambda f](p) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_p} f(q) \sinh^{\lambda-n} r_{pq}\, dq. \tag{4.25}$$

Here
$$H_n(\lambda) = \pi^{\frac{1}{2}(n-2)} 2^{\lambda-1} \Gamma\left(\frac{\lambda}{2}\right) \Gamma\left(\frac{\lambda+2-n}{2}\right)$$

just as for the ordinary Riesz potentials. The right-hand side of (4.25) can be written
$$\frac{1}{H_n(\lambda)} \cdot \int_{D_p} (f \circ \mathrm{Exp})(Q) \frac{\sinh^{\lambda-1} r_{0Q}}{r_{0Q}^{\lambda-1}} r_{0Q}^{\lambda-n}\, dQ$$

which is of the form
$$\frac{1}{H_n(\lambda)} \cdot \int_{D_p} h(Q, \lambda)\, r_{0Q}^{\lambda-n}\, dQ \tag{4.26}$$

where $h(Q, \lambda)$ as well as all its partial derivatives with respect to the first argument are holomorphic in λ and $h(Q, \lambda) \in C_c^\infty(M_p)$ for each λ. The methods of M. Riesz ([32], Ch. III, III) can be applied to such integrals. We find in particular that (4.26), which by its definition is holomorphic in the half plane $\Re \lambda > n$, admits an analytic continuation in the entire plane and the value for $\lambda = 0$ of this entire function is $h(0, 0) = f(p)$. We denote the analytic continuation of (4.25) by $[I^\lambda f](p)$ and have then
$$I_-^0 f = f \tag{4.27}$$

We can differentiate (4.25) with respect to p and carry out the differentiation under the integral sign (for large λ), treating $\mathbf{D}_p$ as a region independent of p. This can be seen ([32] p. 68) by writing the integral (4.25) as $\int_F f(q)\, K(p, q) dq$ over a region F which properly contains the intersection of the support of f and the closure of $\mathbf{D}_p$. $K(p, q)$ is defined as $\sinh^{\lambda-n} r_{pq}$ if $q \in \mathbf{D}_p$, otherwise 0. We obtain thus
$$[\Box I_-^\lambda f](p) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_p} f(q)\, \Box_p \sinh^{\lambda-n} r_{pq}\, dq.$$

Using Lemma 25 and the relation
$$\frac{1}{A(r)} \frac{dA}{dr} = \frac{(n-1)\cosh r}{\sinh r}$$

we find that
$$\Box_p \sinh^{\lambda-n} r_{pq} = \Box_q \sinh^{\lambda-n} r_{pq} = (\lambda-n)(\lambda-1) \sinh^{\lambda-n} r_{pq}$$
$$+ (\lambda-n)(\lambda-2) \sinh^{\lambda-n-2} r_{pq}.$$

We also have $H_n(\lambda) = (\lambda-2)(\lambda-n) H_n(\lambda-2)$ and therefore
$$\Box I_-^\lambda f = (\lambda-n)(\lambda-1) I_-^\lambda f + I_-^{\lambda-2} f.$$

On the other hand, we can use Green's formula to express,
$$\int_{\mathbf{D}_p} (f(q)\, \Box_q (\sinh^{\lambda-n} r_{pq}) - \sinh^{\lambda-n} r_{pq} [\Box f](q))\, dq$$

as a surface integral stretching over a part of $\mathbf{C}_p$ and a surface inside $\mathbf{D}_p$ on which f and its derivatives vanish. It is obvious that these surface integrals vanish (for large λ). This proves the relations

$$\Box I_-^\lambda f = I_-^\lambda \Box f = (\lambda - n)(\lambda - 1) I_-^\lambda f + I_-^{\lambda-2} f \tag{4.28}$$

for all complex λ with sufficiently large real part; due to the uniqueness of the analytic continuation, (4.28) holds for all λ. In particular we have $I_-^{-2} f = \Box f - n f$. Thus our definition (4.25) differs from Riesz' own generalized potential ([32], p. 190) which is suited to obey the law $I^{-2} f = \Box f$.

We consider next the case $M = G^+/H$ and define for $f \in C_c^\infty(M)$

$$[I_+^\lambda f](p) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_p} f(q) \sin^{\lambda - n} r_{pq} \, dq,$$

where $dq = dr\, d\boldsymbol{\omega}_r$. In order to bypass the difficulties caused by the fact that the function $q \to \sin r_{pq}$ vanishes on the antipodal variety $\mathbf{S}_\pi(p)$, we assume that the support of f is disjoint from the antipodal variety $\mathbf{S}_\pi(p)$; this suffices for the present applications. We can then prove just as before

$$[I_+^0 f](p) = f(p)$$

$$[\Box I_+^\lambda f](p) = [I_+^\lambda \Box f](p) = -(\lambda - n)(\lambda - 1)[I_+^\lambda f](p) + [I_+^{\lambda-2}](p). \tag{4.29}$$

In the flat case $M = G^0/H$ we define

$$[I_0^\lambda f](p) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_p} f(q) r_{pq}^{\lambda - n} \, dq, \qquad f \in C_c^\infty(M).$$

Then, as proved by M. Riesz,

$$\Box I_0^\lambda f = I_0^\lambda \Box f = I_0^{\lambda-2} f, \qquad I_0^0 f = f. \tag{4.30}$$

THEOREM 21. *For each of the spaces G^0/H, G^-/H and G^+/H $\Box$ and M^r commute*, i.e.

$$\Box M^r u = M^r \Box u \qquad \textit{for } u \in C_c^\infty(M)$$

(for G^+/H we assume $r < \pi$).

Proof. We restrict ourselves to the case G^+/H. When proving the relation $[\Box M^r u](p^*) = [M^r \Box u](p^*)$ for $r < \pi$ we can assume without loss of generality that the support of u is disjoint from the antipodal variety $\mathbf{S}_\pi(p^*)$. Now we have for $\Re \lambda > n$

$$\int_{\mathbf{D}_p} u(q) \sin^{\lambda - n} r_{pq} \, dq = \int_0^a [M^r u](p) \sin^{\lambda - 1} r \, dr$$

where a is a constant as p varies in some neighborhood of p^*. We now apply $\square$ and make use of (4.29). Then we obtain

$$\int_0^a [M^r \square u](p) \sin^{\lambda-1} r\, dr = \int_0^a [\square M^r u](p) \sin^{\lambda-1} r\, dr.$$

In the same way one can prove

$$\int_0^a [M^r \square u](p) \sin^{\lambda-1} r\, \varphi(r)\, dr = \int_0^a [\square M^r u](p) \sin^{\lambda-1} r\, \varphi(r)\, dr$$

where $\varphi(r)$ is an arbitrary continuous function. It follows that $[\square M^r u](p) = [M^r \square u](p)$.

The following Corollary is obtained just as the Corollary of Theorem 16.

COROLLARY (*The Darboux equation*). *Let $f \in C_c^\infty(M)$ and put $F(p, q) = [M^r f](p)$ if $q \in S_r(p)$. Then*

$$\square_1 F = \square_2 F.$$

7. Determination of a function in terms of its integrals over Lorentzian spheres

In a Riemannian manifold a function is determined in terms of its spherical mean values by the simple relation $u = \lim_{r\to 0} M^r u$. We shall now consider the problem of expressing a function u in a harmonic Lorentz space by means of its mean values $M^r u$ over Lorentzian spheres. Here the situation is naturally quite different because the "spheres" S_r do not shrink to a point as $r \to 0$. For this purpose we use the potentials I_-, I_+ and I_0 defined above; a similar method was used by I. Gelfand and M. Graev [13] in determining a function on a complex classical group by means of the family of integrals I_δ over the conjugacy class given by the diagonal matrix δ. Here I_δ is *bounded* as $\delta \to e$ whereas $M^r u$ is in general unbounded as $r \to 0$. For another related problem see Harish-Chandra's paper [23].

We consider first the negatively curved space $M = G^-/H$ and assume that $n = \dim M$ is even. Let $f \in C_c^\infty(M)$. The potential $I_-^\lambda f(p)$ can be expressed

$$[I_-^\lambda f](p) = \frac{1}{H_n(\lambda)} \int_{D_p} \sinh^{\lambda-1} r\, F(r)\, dr \tag{4.31}$$

where $F(r) = [M^r f](p)$. We use now the coordinates $x_1, \ldots, x_n$ from Chapter IV, § 5. Let R be such that $f \circ \mathrm{Exp}$ vanishes outside the surface $x_1^2 + x_2^2 + \ldots + x_n^2 = R^2$ in M_p. It is easy to see that in the integral

$$F(r) = \iint (f \circ \mathrm{Exp})(-r \cosh \zeta,\ r \sinh \zeta \cos \theta_1, \ldots,$$
$$r \sinh \zeta \sin \theta_1 \ldots \sin \theta_{n-2}) \sinh^{n-2} \zeta\, d\zeta\, d\omega^{n-2}$$

the range of ζ is contained in the interval $(0, \zeta_0)$ where $r^2\cosh^2\zeta_0 + r^2\sinh^2\zeta_0 = R^2$. If $n \neq 2$ we see by the substitution $y = r\sinh\zeta$ that the integral expression for $F(r)$ behaves for small r like

$$\int_0^K \varphi(y)\left(\frac{y}{r}\right)^{n-2}\frac{1}{(r^2+y^2)^{\frac{1}{2}}}\,dy$$

where φ is bounded. If $n=2$ we see in the same way that $F'(r)$ behaves for small r like

$$\int_0^K \varphi(y)\cdot\frac{y}{r}\,\frac{1}{(r^2+y^2)^{\frac{1}{2}}}\,dy.$$

Therefore, the limits

$$a = \lim_{r\to 0}\,(\sinh^{n-2} r)\,F(r) \qquad (n \neq 2) \tag{4.32}$$

$$b = \lim_{r\to 0}\,(\sinh r)\,F'(r) \qquad (n = 2) \tag{4.33}$$

do exist. Consider now the first case $n \neq 2$. We can rewrite (4.31) as

$$[I_-^\lambda f](p) = \frac{1}{H_n(\lambda)}\int_0^R \sinh^{n-2} r\,F(r)\,\sinh^{\lambda-n+1} r\,dr$$

where $F(R)=0$. We now evaluate both sides for $\lambda = n-2$. Since $H_n(\lambda)$ has a simple pole for $\lambda = n-2$ the same is true of the integral and the residue is

$$\lim_{\lambda\to n-2}\int_0^R \sinh^{n-2} r\,F(r)\,(\lambda-n+2)\,\sinh^{\lambda-n+1} r\,dr.$$

Here λ can be restricted to be real and $> n-2$ which is convenient since the integral above is then absolutely convergent and we do not have to think of it as an implicitly given holomorphic extension. We split the integral into two parts

$$\int_0^R (\sinh^{n-2} r\,F(r) - a)\,(\lambda-n+2)\,\sinh^{\lambda-n+1} r\,dr + a\int_0^R (\lambda-n+2)\,\sinh^{\lambda-n+1} r\,dr.$$

Concerning the last term we note that

$$\lim_{\mu\to 0+}\mu\int_0^R \sinh^{\mu-1} r\,dr = \lim_{\mu\to 0+}\mu\int_0^R r^{\mu-1}\,dr = 1.$$

As for the first term, we can for each $\varepsilon > 0$ find a $\delta > 0$ such that

$$|(\sinh^{n-2} r)\,F(r) - a| < \varepsilon \ \text{ for } \ 0 < r < \delta$$

If $N = \max |(\sinh^{n-2} r) F(r)|$ we have for $n-2 < \lambda < n-1$ the estimates

$$\left|\int_0^R (\sinh^{n-2} r\, F(r) - a)(\lambda - n + 2) \sinh^{\lambda-n+1} r\, dr\right| \leqq 2(\lambda - n + 2) N (R - \delta) \sinh^{-1} \delta$$

$$\left|\int_0^\delta (\sinh^{n-2} r\, F(r) - a)(\lambda - n + 2) \sinh^{\lambda-n+1} r\, dr\right| \leqq \varepsilon (\lambda - n + 2) \int_0^\delta r^{\lambda-n+1} dr.$$

We conclude easily that

$$\lim_{\lambda \to (n-2)} \int_0^\infty \sinh^{\lambda-1} r\, F(r)(\lambda - n + 2)\, dr = \lim_{x \to 0} \sinh^{n-2} r\, F(r).$$

Taking into account the formula for $H_n(\lambda)$ we obtain

$$I_-^{n-2} f = (4\pi)^{\frac{1}{2}(2-n)} \frac{1}{\Gamma(\frac{1}{2}(n-2))} \lim_{r \to 0} \sinh^{n-2} r\, M^r f. \tag{4.34}$$

On the other hand, if we use the formula (4.28) recursively, we obtain for arbitrary $u \in C_c^\infty(M)$

$$I^{n-2}(Q(\square)u) = u$$

where

$$Q(\square) = (\square + (n-3)2)(\square + (n-5)4) \ldots (\square + 1(n-2)).$$

We combine this with (4.34) and use on the right-hand side the commutativity of $\square$ and M^r. This yields the desired formula

$$u = (4\pi)^{\frac{1}{2}(2-n)} \frac{1}{\Gamma(\frac{1}{2}(n-2))} \lim_{r \to 0} \sinh^{n-2} r\, Q(\square_r)(M^r u)$$

where

$$\square_r = \frac{d^2}{dr^2} + (n-1) \frac{\cosh r}{\sinh r} \frac{d}{dr}.$$

It remains to consider the case $n = 2$. Here we have by (4.31)

$$I_-^2 f = \frac{1}{H_2(2)} \int_0^\infty \sinh r\, F(r)\, dr \quad f \in C_c^\infty(M)$$

where the integral converges absolutely. In fact $F(r) \leqq C|\log r|$ for small r. We apply this relation to the function $f = \square u$ where u is an arbitrary function in $C_c^\infty(M)$. We also make use of (4.28) and Theorem 21. It follows that

$$I_-^2 \square u = u = \tfrac{1}{2} \int_0^\infty \sinh r\, M^r \square u\, dr = \tfrac{1}{2} \int_0^\infty \sinh r \left(\frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r} \frac{d}{dr} M^r u\right) dr$$

$$= \tfrac{1}{2} \int_0^\infty \frac{\partial}{\partial r} \left(\sinh r \frac{d M^r u}{dr}\right) dr = -\tfrac{1}{2} \lim_{r \to 0} \left(\sinh r \frac{d M^r u}{dr}\right)$$

The spaces G^+/H and G^0/H can be treated in the same manner. The combined result is as follows.

THEOREM 22. *Let M be one of the spaces G^0/H, G^-/H, G^+/H. Let $\varkappa$ denote the curvature of M ($\varkappa = 0, -1, +1$) and assume $n = \dim M$ is even. We also put*

$$Q(x) = (x - \varkappa(n-3)2)(x - \varkappa(n-5)4)\dots\quad (x - \varkappa 1(n-2)).$$

Then if $u \in C_c^\infty(M)$

$$u = (4\pi)^{\frac{1}{2}(2-n)} \frac{1}{\Gamma(\frac{1}{2}(n-2))} \lim_{r\to 0} r^{n-2} Q(\Box_r)(M^r u) \qquad (n \neq 2)$$

and

$$u = -\tfrac{1}{2} \lim_{r\to 0} r \frac{d}{dr} M^r u. \qquad (n = 2)$$

8. Huygens' principle

We consider now an arbitrary Lorentzian space M with metric tensor Q and dimension n. Let U be an open subset of M with the property that arbitrary two points $p, q \in U$ can be joined by exactly one path segment contained in U. All considerations will now take place inside U. The paths of zero length through a point $p \in U$ generate the light cone $\mathbf{C}_p$ in U with vertex p. A submanifold S of U is called spacelike if each tangent vector to S is spacelike. Suppose now that a Cauchy problem is posed for the wave equation $\Box u = 0$ with initial data on a spacelike hypersurface $S \subset U$. From Hadamard's theory it is known that the value $u(p)$ of the solution at $p \in U$ only depends on the initial data on the piece $S^* \subset S$ that lies inside the light cone $\mathbf{C}_p$. Huygens' principle is said to hold for $\Box u = 0$ if the value $u(p)$ only depends on the initial data in an arbitrary small neighborhood of the edge s of S^*, $s = \mathbf{C}_p \cap S$. Hadamard has shown that Huygens' principle can never hold if n is odd. On the other hand the wave equation $\Box u = 0$ in $\mathbf{R}^n$ (n even > 2) is of Huygens' type. A long-standing conjecture, attributed(1) to Hadamard, states that these are essentially the only hyperbolic equations of Huygens' type. A counterexample of the form $\Box u + cu = 0$ was given by K. Stellmacher (Ein Beispiel einer Huygenschen Differentialgleichung, *Nachr. Akad. Wiss. Göttingen* 1953) but for the pure equation $\Box u = 0$ the problem is, to my knowledge, unsettled. For harmonic Lorentz spaces the problem is easily answered by using properties of these spaces obtained in Chapter II.

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(1) Courant-Hilbert, *Methoden der mathematischen Physik*, Vol. II, p. 438. An interesting discussion and results concerning this problem are given in L. Ásgeirsson, Some hints on Huygens' principle and Hadamard's conjecture. *Comm. Pure Appl. Math. IX* (1956), 307–326.

THEOREM 23. *The wave equation $\square u = 0$ in a harmonic Lorentz space M satisfies Huygens' principle if and only if M is flat and has even dimension > 2.*

Proof. Since Huygens' principle is a local property, we can, due to Theorem 9, assume $M = G^-/H$ or $M = G^+/H$. In either case we can find a solution of $\square u = 0$, valid in $\mathbf{D}_{p_0}$, by solving the equation

$$\frac{d^2 v}{d r^2} + \frac{1}{A(r)} \frac{dA}{dr} \frac{dv}{dr} = 0$$

and putting $u(p) = v(r_{p_0 p})$. We find immediately a solution of the form

$$v(r) = \int_a^r \frac{1}{\sinh^{n-1} r} dr \qquad \text{if } M = G^-/H$$

$$v(r) = \int_a^r \frac{1}{\sin^{n-1} r} dr, \qquad \text{if } M = G^+/H.$$

Due to Hadamard's result already quoted we can assume n to be even. Under this assumption it follows by easy computation that v can be written

$$v(r) = \frac{P(r)}{r^{n-2}} + Q(r) \log r, \qquad Q(0) \neq 0$$

where P and Q are regular functions. u is thus an elementary solution and since it contains a non-vanishing logarithmic term, Huygens' principle is absent (Hadamard [18] p. 236, Courant-Hilbert, *loc. cit.*, p. 438).

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