DIFFERENTIAL OPERATORS ON NONCOMMUTATIVE RINGS

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0. Introduction

Let k be a commutative ring and R an associative k-algebra. In this paper we define the ring $D(R)$ of k-linear differential operators on R. In case R is commutative this, of course, coincides with Grothendieck's definition ([G]).

Given left R-modules L, N we define the differential operators $\text{Diff}(L, N)$ from L to N as the differential part of the R-bimodule $Hom_k(L, N)$. Thus our main definition is the differential part M_{diff} of an arbitrary R-bimodule M. This is an R-subbimodule of M. In case R is commutative, M_{diff} is the part of M supported on the diagonal of $\text{Spec}R \times \text{Spec}R$. Building on this analogy for general R, we can think of the R-bimodule R as the "structure sheaf of the diagonal in $\text{Spec}R \times \text{Spec}R$". The differential part M_{diff} is defined by using the bimodule R.

The main property of classical differential operators is their compatibility with localizations of R. We prove that if R is a domain and $R \to R'$ is an Ore localization, then there is a natural ring homomorphism $D(R) \to D(R')$ (Theorem 1.2.1). We show that if the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ acts on R as a Hopf algebra, then it acts by differential operators (1.3). This implies (in case R is a domain) that such an action extends to Ore localizations of R.

This work arose as part of our project to find a localization construction for quantum enveloping algebras (quantum groups), which would be the quantized analogue of the Beilinson-Bernstein localization for reductive Lie algebras. Our localization construction is described in [LR1]. It was one of our first conclusions that quantum groups live naturally in the universe of Q-graded objects (Q is the root lattice). This led us to the notion of quantum (or q-) differential operators $D_q(R)$ when the ring R is graded by an abelian group Γ. The ring $D_q(R)$ includes the "grading" action of Γ. We show the compatibility of $D_q(R)$ with localizations (Theorem 3.3.2) and prove that if a quantum group acts on R as a Hopf algebra, then it acts by quantum differential operators (3.3). In particular, such an action of a quantum group extends to localizations of R (theorem 3.4.1).

The intermediate notion between $D(R)$ and $D_q(R)$ is that of graded (or β-) differential operators $D_\beta(R)$, where $\beta : \Gamma \times \Gamma \to k^*$ is the fixed bicharacter. Recall that the ring $D(R)$ is defined using the R-bimodule R. Similarly, the ring $D_\beta(R)$ (resp. $D_q(R)$) is defined by using the (bigger) bimodule R_Γ (resp. still bigger bimodule R_Γ^q).

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The paper consists of three parts, very similar in structure. The first part concerns with $D(R)$, the second, with $D_\beta(R)$, the third, with $D_q(R)$. We compute the algebra $D_\beta(R)$ in case R is a skew polynomial ring (2.3). Holonomic modules over $D_q(R)$ are discussed in 3.5. In section 4 we discuss differential operators in the category language.

This paper is written in the language of rings and modules and therefore deals with the "affine case". We treat the general case of abelian categories in [LR3], where the picture is in some sense more natural. The β- and q-differential operators appear naturally in the context of braided monoidal categories ([LR4]).

1. Differential operators on noncommutative rings

1.0. Differential calculus on commutative rings

First we recall shortly the differential calculus on commutative rings and schemes following [BB].

Fix a commutative ring k. If not specified otherwise, $\otimes$ means $\otimes_k$. Let R be a commutative k-algebra; and let M be an R-bimodule such that, for any $z \in M$ and any $\lambda \in k$, $\lambda \cdot z = z \cdot \lambda$. The bimodule M can be regarded as an $R \otimes R$-module. For any $r \in R$, define the endomorphism ad_r of the bimodule M (– the adjoint action of r) by $ad_r(z) = r \cdot z - z \cdot r$ for all $z \in M$.

An increasing filtration $\{M_i \mid i \geq -1\}$ on M is called a *D-filtration* if $M_{-1} = 0$ and $ad_r(M_i) \subseteq M_{i-1}$ for all $r \in R$ and $i \geq 0$. There is the largest (with respect to the inclusion) D-filtration $M^\dagger$ on M defined by $M_i^\dagger := \{z \in M \mid ad_r(z) \in M_{i-1}^\dagger$ for all $r \in R\}$, $i \geq 0$. The subbimodule $M^\dagger := \bigcup_{i \geq 0} M_i^\dagger$ is called *the differential part of M*. And M is called *a differential bimodule* if $M^\dagger$ coincides with M.

This allows us to define k-linear differential operators $\mathrm{Diff}(L, N)$ between left R-modules L, N. Namely, note that $Hom_k(L, N)$ is naturally an R-bimodule and put $\mathrm{Diff}(L, N) := Hom_k(L, N)^\dagger$. Thus differential operators $\mathrm{Diff}(L, N)$ have a natural filtration by the degree $\mathrm{Diff}(L, N) = \bigcup_{i \geq 0} \mathrm{Diff}(L, N)_i$, where $f \in \mathrm{Diff}(L, N)_i$ if $[r_{i+1}, [r_i, ...[r_1, f]...]] = 0$ for any $r_1, ..., r_{i+1} \in R$ considered as operators on L and N. We denote $D(R) := \mathrm{Diff}(R, R)$, $D^i(R) := \mathrm{Diff}(R, R)_i$. Then $D^i(R)D^j(R) \subset D^{i+j}(R)$, hence $D(R)$ is a filtered ring. We have the canonical ring homomorphism $R \to D(R)$.

Let A be a k-algebra equipped with an k-algebra homomorphism $i : R \longrightarrow A$. An increasing ring filtration $A. = \{A_i \mid i \geq -1\}$ is called a *D-ring filtration* if it is a D-filtration of the R-bimodule A such that $i(R) \subseteq A_0$ and $i(R)$ lies in the center of the associated graded algebra. One can observe that the largest D-filtration $A^\dagger$ on A is a D-ring filtration. And $i : R \longrightarrow A$ is called *an R-differential algebra* if $A = A^\dagger$, i.e. when A is a differential R-bimodule. An example of an R-differential algebra is the ring $D(R)$.

Note that, if we regard a bimodule M as an $R \otimes R$-module, $M_i^\dagger := \{z \in M \mid I^{i+1}z = 0\}$, where I is the kernel of the multiplication $m : R \otimes R \longrightarrow R$. It follows from this description that the canonical D-filtration is compatible with localizations: for any $t \in R$, there are natural isomorphisms $(M_i^\dagger)_t \simeq (M_t)_i^\dagger \simeq R_t \otimes_R M_i^\dagger \simeq M_i^\dagger \otimes_R R_t$. Here M_t denotes the localization of M at t, i.e. $M_t := R_t \otimes_R M \otimes_R R_t$. This compatibility with localizations allows us to globalize the notion of a differential

bimodule. Namely, we can repeat above definitions with the ring R replaced by the structure sheaf $\mathcal{O}_X$ of a scheme X, and L, N being quasicoherent $\mathcal{O}_X$-modules.

1.1. Differential calculus on noncommutative rings

1.1.1 Reformulation. If we consider the same setup with a noncommutative k-algebra R, the above definition is no good. For example, in $M = N = R$, then operators of left multiplication by elements of R will not necessarily be differential operators. Let us propose a reformulation which carries over to the noncommutative case. Namely, in the notation of 1.0, define $M_i^\dagger$ by induction on i as the largest subbimodule of M, such that the bimodule $M_i^\dagger / M_{i-1}^\dagger$ is a quotient of a direct sum of copies of the bimodule R. It turns out that this reformulation gives a good theory in the noncommutative case.

1.1.2 Fix an associative unital algebra R over a commutative ring k. If not specified otherwise, $\otimes$ means $\otimes_k$. We identify R-bimodules with $R^\mathfrak{e} := R \otimes R^0$-modules.

1.1.2.1. Definition. An $R^\mathfrak{e}$-module M is *differential* iff it has an increasing filtration by submodules

$$0 = M_{-1} \subset M_0 \subset M_1 \subset ..., \quad \bigcup M_i = M,$$

such that M_{i+1}/M_i is a quotient of a direct sum of copies of the $R^\mathfrak{e}$-module R.

1.1.2.2 Definition. Given an R-bimodule M, its *center* is the k-submodule $\mathfrak{z}(M) := \{z \in M | rz = zr \text{ for all } r \in R\}$. We call M *central* if $M = R^\mathfrak{e}\mathfrak{z}(M)$.

Notice that for a central $R^\mathfrak{e}$-module M we have $M = R\mathfrak{z}(M) = \mathfrak{z}(M)R$.

1.1.2.3. Note that for an $R^\mathfrak{e}$-module M we have an isomorphism of k-modules

$$Hom_{R^\mathfrak{e}}(R, M) \simeq \mathfrak{z}(M).$$

Any $R^\mathfrak{e}$-module M has a canonical chain of submodules $\{\mathfrak{z}_i M\}$, which satisfies the condition of 1.1.2.1. Namely, define $\mathfrak{z}_i M$ by induction as follows:

$$\mathfrak{z}_0 M := R^\mathfrak{e}\mathfrak{z}(M),$$

$$\mathfrak{z}_i M / \mathfrak{z}_{i-1} M := R^\mathfrak{e}\mathfrak{z}(M/\mathfrak{z}_{i-1}M).$$

Clearly, this is the maximal filtration satisfying the condition in 1.1.2.1. Thus M is differential iff $\bigcup \mathfrak{z}_i M = M$.

1.1.2.4 Lemma. *Any $R^\mathfrak{e}$-module M contains the biggest differential submodule M_{diff}, called the differential part of M. The correspondence $M \mapsto M_{diff}$ is functorial: for any $R^\mathfrak{e}$-module morphism $\varphi : M \to M'$, $\varphi(M_{diff}) \subseteq M'_{diff}$.*

Proof. Indeed, put $M_{diff} := \bigcup \mathfrak{z}_i M$.

1.1.2.5 Definition. Let L, N be left R-modules. Consider $Hom_k(L, N)$ as a left $R^\mathfrak{e}$-module and call $\mathrm{Diff}(L, N) = \mathrm{Diff}_k(L, N) := Hom_k(L, N)_{diff}$ the (k-linear) *differential operators* from L to N. The $R^\mathfrak{e}$-submodule $\mathrm{Diff}^n(L, N) := \mathfrak{z}_n\mathrm{Diff}(L, N) \subset \mathrm{Diff}(L, N)$ consists of differential operators of *order n*.

1.1.2.6 Definition. Given a homomorphism of k-algebras $R \to B$, we call B a *differential R-algebra* if B is a differential $R^{\mathfrak{e}}$-module.

1.1.2.7 Proposition. *a) If M, M' are differential $R^{\mathfrak{e}}$-modules, then the $R^{\mathfrak{e}}$-module $M \otimes_R M'$ is also differential.*

b) For any homomorphism of k-algebras $R \to B$, the differential part of the $R^{\mathfrak{e}}$-module B is a subring of B, i.e. B_{diff} is a differential R-algebra. More precisely, $\mathfrak{z}_i B \cdot \mathfrak{z}_j B \subset \mathfrak{z}_{i+j} B$.

c) For any left R-module L, the $R^{\mathfrak{e}}$-module $\mathrm{Diff}_k(L, L)$ is a differential R-algebra.

Proof. a) If $z \in \mathfrak{z}(M)$, $z' \in \mathfrak{z}(M')$, then $z \otimes z' \in \mathfrak{z}(M \otimes_R M')$. This implies that the image of $\mathfrak{z}_i M \otimes \mathfrak{z}_j M'$ in $M \otimes_R M'$ is contained in $\mathfrak{z}_{i+j}(M \otimes_R M')$. This proves a).

b) The multiplication map $B \otimes B \to B$ factors through the morphism of $R^{\mathfrak{e}}$-modules $B \otimes_R B \to B$. The argument in a) (and the functoriality of the filtration $\mathfrak{z}$.) imply that $\mathfrak{z}_i B \otimes_R \mathfrak{z}_j B \to \mathfrak{z}_{i+j} B$. Thus B_{diff} is a subring.

c) Given a left R-module L, we have the canonical map of $R^{\mathfrak{e}}$-modules

$$\alpha : R \to \mathfrak{z}_0 Hom_k(L, L), \quad r \mapsto (l \mapsto rl).$$

Clearly, $\alpha : R \to Hom_k(L, L)$ is a ring homomorphism, and note that the $R^{\mathfrak{e}}$-module structure on $Hom_k(L, L)$ coincides with the one induced by this homomorphism. Hence, by b) above, $\mathrm{Diff}_k(L, L)$ is a differential R-algebra.

1.1.2.8. We write $D_k(R)$, or simply $D(R)$, instead of $\mathrm{Diff}_k(R, R)$ and call it the *algebra of (k-linear) differential operators on R*. The canonical algebra homomorphism $i : R \to D(R)$ (1.1.2.7c) makes $D(R)$ a differential R-algebra. Put $D^i(R) := \mathfrak{z}_i D(R)$, $i \geq 0$. Then $D^i(R) D^j(R) \subset D^{i+j}(R)$ (1.1.2.7b), i.e. $D(R)$ is a filtered algebra by the "order of differential operators". We have $i(R) \subset D^0(R)$.

1.1.2.9 Definition. A *derivation* d of R is a k-linear map $d : R \to R$, such that $d(ab) = d(a)b + ad(b)$. Denote by $\mathrm{Der}(R)$ the k-module of derivations of R.

1.1.2.10 Proposition. *a) The center $\mathfrak{z}(Hom_k(R, R))$ of the $R^{\mathfrak{e}}$-module $Hom_k(R, R)$ consists of right multiplications by elements of R. In particular, the subring $D^0(R)$ is generated by left and right multiplications by elements of R.*

b) The $R^{\mathfrak{e}}$-module $D^1(R)$ contains $\mathrm{Der}(R)$.

Proof. All assertions are straightforward.

1.2 Localization of differential operators.

1.2.0 Assumption. Let $R \to R'$ be a homomorphism of k-algebras such that R' is the localization of R with respect to a left and right Ore set $S \subset R$.

1.2.1 Theorem. *Assume that R is a domain. Given a left $D(R)$-module L, its localization $R' \otimes_R L$ is also canonically a $D(R)$-module. In particular, there exists a canonical ring homomorphism $D(R) \to D(R')$ (preserving the filtration D^n), i.e. differential operators on R extend to R'.*

We need a lemma.

1.2.1.1 Lemma. *The canonical map of* (R', R)-*modules*

$$R' \otimes_R D(R) \to R' \otimes_R D(R) \otimes_R R'$$

is an isomorphism.

Proof. 1) *Injectivity.* Since R' is a flat right R-module, it suffices to show that the natural map

$$D(R) \to D(R) \otimes_R R'$$

is injective. Denote by $Q(R)$ the skew field of fractions of R. Obviously, $D(R)$ is a torsion free left R-module. Hence the natural map

$$D(R) \to Q(R) \otimes_R D(R)$$

is injective. Thus it suffices to show that the natural map

$$Q(R) \otimes_R D(R) \to Q(R) \otimes_R D(R) \otimes_R R'$$

is injective, or equivalently, that the right R-module $Q(R) \otimes_R D(R)$ has no torsion.
The canonical filtration by R^{c}-modules

$$0 = D^{-1}(R) \subset D^0(R) \subset D^1(R)... = D(R)$$

induces the filtration by $(Q(R), R)$-submodules

$$P^{-1} \subset P^0 \subset P^1 \subset ... = Q(R) \otimes_R D(R),$$

where $P^n := Q(R) \otimes_R D^n(R)$.
Assume that there exists $0 \neq d \in Q(R) \otimes_R D(R)$, $0 \neq s \in R$, such that $ds = 0$. Let n be such that $d \in P^n$ and $d \notin P^{n-1}$. Recall that the left R-module $D^n(R)/D^{n-1}(R)$ is generated by central elements. Hence there exists a collection of elements $\{d_i\}_i \subset D^n(R)$, which are central modulo $D^{n-1}(R)$ and such that the image in P^n/P^{n-1} of the set $\{1 \otimes d_i\}$ is a basis of the left $Q(R)$-vector space P^n/P^{n-1}. Choose one such collection.
We can write (uniquely)
$$d = \Sigma b_i \otimes d_i + \bar{d}$$

for some $b_i \in Q(R)$, $\bar{d} \in P^{n-1}$. By our assumption $b_i \neq 0$ for some i. We have

$$0 = ds = \Sigma b_i \otimes d_i s + \bar{d}s = \Sigma b_i s \otimes d_i + \bar{d}_1$$

for some $\bar{d}_1 \in P^{n-1}$. Thus $\Sigma b_i s \otimes d_i \in P^{n-1}$. But $b_i s \neq 0$ if $b_i \neq 0$, which implies a contradiction with the linear independence of the set $\{1 \otimes d_i\}$ modulo P^{n-1}. Thus we proved the injectivity.
2) *Surjectivity.* Consider the canonical filtration

$$0 \subset D^0(R) \subset D^1(R) \subset ... = D(R).$$

Recall that for each n there exists a surjection of $R^{\mathfrak{e}}$-modules

$$\oplus R \to D^n(R)/D^{n-1}(R).$$

The functors $R' \otimes_R \cdot$ and $R' \otimes_R \cdot \otimes_R R'$ are exact and commute with direct limits. Therefore, it suffices to prove the surjectivity of the canonical map

$$R' \otimes_R M \to R' \otimes_R M \otimes_R R'$$

for an $R^{\mathfrak{e}}$-module M for which there exists a surjection of $R^{\mathfrak{e}}$-modules $\oplus R \to M$. But this follows from the surjectivity of the map

$$R' \otimes_R R \to R' \otimes_R R \otimes_R R'.$$

Proof of Theorem 1.2.1. Let L, N be left R-modules, M be an $R^{\mathfrak{e}}$-module. We have a canonical isomorphism of k-bimodules

$$Hom_{R^{\mathfrak{e}}}(M, Hom_k(L, N)) \simeq Hom_R(M \otimes_R L, N).$$

Given a left $D(R)$-module L, we have a map of $R^{\mathfrak{e}}$- modules

$$D(R) \to Hom_k(L, L)$$

and therefore the corresponding map of left R-modules

$$D(R) \otimes_R L \to L.$$

This induces a map of left R'-modules

$$(*) \qquad R' \otimes_R D(R) \otimes_R L \to R' \otimes_R L.$$

We want to have a map of $R'^{\mathfrak{e}} := R' \otimes R'^0$-modules

$$R' \otimes_R D(R) \otimes_R R' \to Hom_k(R' \otimes_R L, R' \otimes_R L),$$

or, equivalently, a map of left R'-modules

$$(**) \qquad (R' \otimes_R D(R) \otimes_R R') \otimes_{R'} (R' \otimes_R L) \to R' \otimes_R L.$$

By Lemma 1.2.1.1 the canonical map of (R', R)-modules

$$R' \otimes_R D(R) \to R' \otimes_R D(R) \otimes_R R'$$

is an isomorphism. Hence the desired map $(**)$ is equal to $(*)$. This defines the action of $D(R)$ on $R' \otimes_R L$.

Note that for a differential $R^{\mathfrak{e}}$-module M, the $R'^{\mathfrak{e}}$-module $R' \otimes_R M \otimes_R R'$ is also differential. Therefore, given a left $D(R)$-module L the ring $D(R)$ acts on the R'-module $R' \otimes_R L$ again by differential operators. In particular, we have the canonical ring homomorphism $D(R) \to D(R')$.

1.3 Relation with enveloping algebras.

1.3.0 Let k be a field. Let $\mathfrak{g}$ be a Lie algebra over k, and $U(\mathfrak{g})$ its enveloping algebra. It is known that $U(\mathfrak{g})$ is a Hopf algebra with the comultiplication

$$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

defined by $g \mapsto g \otimes 1 + 1 \otimes g$ for $g \in \mathfrak{g} \subset U(\mathfrak{g})$. Let R be a k-algebra and a left $U(\mathfrak{g})$-module. We say that $U(\mathfrak{g})$ acts on R as a Hopf algebra or that R is a $U(\mathfrak{g})$-ring if

$$u(ab) = u_{(1)}(a)u_{(2)}(b),$$

where $a, b \in R$, $u \in U(\mathfrak{g})$ and $\Delta u = u_{(1)} \otimes u_{(2)} := \Sigma_i u_{(1)}^i \otimes u_{(2)}^i$. Since the algebra $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ this is equivalent to saying that

$$g(ab) = g(a)b + ag(b)$$

for all $a, b \in R$, $g \in \mathfrak{g}$, i.e. that $\mathfrak{g}$ acts by derivations. Hence a $U(\mathfrak{g})$-ring structure on R induces a homomorphism of filtered rings

$$U(\mathfrak{g}) \to D(R)$$

(prop. 1.1.2.10b)).

1.3.1 Theorem *Let R be a $U(\mathfrak{g})$-ring. Assume that R is a domain and $R \to R'$ is a localization of R by a left and right Ore set. Let L be a left $D(R)$-module (hence a left $U(\mathfrak{g})$-module). Then $R' \otimes_R L$ is naturally a $U(\mathfrak{g})$-module. In particular, R' is canonically a $U(\mathfrak{g})$-module. Moreover the $U(\mathfrak{g})$-action on R' makes it a $U(\mathfrak{g})$-ring.*

Proof. All statements except the last one follow immediately from Theorem 1.2.1. It remains to prove that the Lie algebra $\mathfrak{g} \subset U(\mathfrak{g})$ acts on R' by derivations.

Let $g \in \mathfrak{g}$. Consider g as an element in $D(R)$. Since g acts on R as a derivation, for each $r \in R$, we have the following relation in $D(R)$:

$$gr - rg = g(r).$$

Let $s^{-1}a \in R'$, where $s, a \in R$. Then

$$g(a) = g(ss^{-1}a) = sg(s^{-1}a) + g(s)s^{-1}a,$$

hence

$$g(s^{-1}a) = s^{-1}(g(a) - g(s)s^{-1}a).$$

One checks directly (see also [Dix], Prop.3.6.18) that this last formula defines a derivation of R'.

2. Graded (or β-) differential operators

2.1.0 Let k be a commutative ring and Γ be an abelian group. We want to change our world from k-modules and k-linear differential operators to $k[\Gamma]$-modules and differential operators which take into account the Γ-action. More precisely, fix a

bicharacter $\beta : \Gamma \times \Gamma \to k^*$, where k^* is the group of units of k. Note that if $M = \oplus_{a \in \Gamma} M_a$ is a Γ-graded k-module, then we have a natural homomorphism $\sigma_M : \Gamma \to \mathrm{Aut}(M)$ defined by the formula

$$\sigma_M(\gamma)|_{M_a} := \beta(\gamma, a) id_{M_a}, \quad \gamma, a \in \Gamma.$$

We call this action of Γ the *grading action*.

Let $R = \oplus_{a \in \Gamma} R_a$ be a Γ-graded k-algebra. Then Γ acts by k-algebra automorphisms of R by the grading action $\sigma = \sigma_R$. Denote by

$$R_\Gamma := k[\Gamma] \# R = \oplus_{a \in \Gamma} Ra$$

the corresponding crossed product algebra, i.e. in R_Γ we have

$$\gamma r = \sigma(\gamma)(r)\gamma = \beta(\gamma, a) r\gamma, \quad r \in R_a, \ \gamma \in \Gamma.$$

This is a Γ-graded algebra with $deg(ra) = deg(r) + a$. In particular, we may consider R_Γ as a graded R^ϵ-module.

We are going to consider categories of left (Γ-) graded R- and R^ϵ-modules. Morphisms $Hom_R(\cdot, \cdot)$ and $Hom_{R^\epsilon}(\cdot, \cdot)$ in these categories preserve the grading of modules.

Given graded R-modules L, N, we denote by $grHom(L, N)$ the k-submodule of $Hom_k(L, N)$ spanned by homogeneous elements. Clearly, $grHom(L, N)$ is a graded R^ϵ-module.

The sections 2.1.1 - 2.2.2 below essentially reproduce the material in 1.1, 1.2, but adjusted to the graded case.

2.1.1 Definition. Let M be a graded R^ϵ-module. We call M a β-*differential* module if it has a filtration by graded R^ϵ-submodules

$$0 = M_{-1} \subset M_0 \subset M_1 \subset ..., \quad \bigcup M_i = M,$$

such that, for each i, the graded R^ϵ-module M_i/M_{i-1} is a quotient of a direct sum of copies of R_Γ.

2.1.2 Definition. Let $M = \oplus_{a \in \Gamma} M_a$ be a graded R^ϵ-module. We define its β-*center* $\mathfrak{Z}_\beta(M)$ as the k-span of homogeneous elements $m \in M_\gamma$ such that

$$mr = \beta(\gamma, a) rm, \quad \text{for all} \ r \in R_a.$$

We call M β-*central* if $M = R^\epsilon \mathfrak{Z}_\beta(M)$.

Notice that for a β-central graded R^ϵ-module M, we have $M = R\mathfrak{Z}_\beta(M) = \mathfrak{Z}_\beta(M)R$.

2.1.3. Note that for a graded R^ϵ-module M we have an isomorphism of graded k-modules

$$Hom_{R^\epsilon}(R_\Gamma, M) \simeq \mathfrak{Z}_\beta(M).$$

Any R^ϵ-module M has a canonical chain of graded submodules $\{\mathfrak{Z}_{\beta i} M\}$, which satisfies the condition of 2.1.2. Namely, define $\mathfrak{Z}_{\beta i} M$ by induction as follows:

$$\mathfrak{Z}_{\beta 0} M := R^\epsilon \mathfrak{Z}_\beta(M),$$

$$\mathfrak{z}_{\beta i} M / \mathfrak{z}_{\beta i-1} M := R^{\mathfrak{c}} \mathfrak{z}_\beta (M / \mathfrak{z}_{\beta i-1} M).$$

Clearly, this is the maximal filtration satisfying the condition in 2.1.2. Thus, M is β-differential iff $\bigcup \mathfrak{z}_{\beta i} M = M$.

2.1.4 Lemma. *Any graded $R^{\mathfrak{c}}$-module M contains the biggest β-differential submodule $M_{\beta-diff}$, called the β-differential part of M. The correspondence $M \mapsto M_{\beta-diff}$ is functorial: for any graded $R^{\mathfrak{c}}$-module morphism $\varphi : M \to M'$, $\varphi(M_{\beta-diff}) \subset M'_{\beta-diff}$.*

Proof. Indeed, put $M_{\beta-diff} := \bigcup \mathfrak{z}_{\beta i} M$.

2.1.5 Definition. Let L, N be graded left R-modules. Consider $grHom(L, N)$ as a left graded $R^{\mathfrak{c}}$-module and call $\mathrm{Diff}_\beta(L, N) := grHom(L, N)_{\beta-diff}$ the (k-linear) β-*differential operators* from L, to N. The $R^{\mathfrak{c}}$- submodule $\mathrm{Diff}_\beta^m(L, N) := \mathfrak{z}_{\beta n} \mathrm{Diff}_\beta(L, N)$ of $\mathrm{Diff}_\beta(L, N)$ consists of β-differential operators of *order n*.

2.1.6 Definition. Given a homomorphism of graded k-algebras $R \to B$, we call B a β-*differential R-algebra* if B is a β-differential $R^{\mathfrak{c}}$-module.

2.1.7 Proposition. *a) If M, M' are graded β-differential $R^{\mathfrak{c}}$-modules, then the graded $R^{\mathfrak{c}}$-module $M \otimes_R M'$ is also β-differential.*

b) For any homomorphism of graded k-algebras $R \to B$, the β-differential part of the $R^{\mathfrak{c}}$-module B is a graded subring of B, i.e. $B_{\beta-diff}$ is a differential R-algebra. More precisely, $\mathfrak{z}_{\beta i} B \cdot \mathfrak{z}_{\beta j} B \subset \mathfrak{z}_{\beta i+j} B$.

c) For any graded left R-module L, the graded $R^{\mathfrak{c}}$-module $\mathrm{Diff}_\beta(L, L)$ is a β-differential R-algebra.

Proof. a) If $z \in \mathfrak{z}_\beta(M)$, $z' \in \mathfrak{z}_\beta(M')$, then $z \otimes z' \in \mathfrak{z}_\beta(M \otimes_R M')$. This implies that the image of $\mathfrak{z}_{\beta i} M \otimes \mathfrak{z}_{\beta j}(M')$ in $M \otimes_R M'$ is contained in $\mathfrak{z}_{\beta i+j}(M \otimes M')$. This proves a).

b) The multiplication map $B \otimes B \to B$ factors through the morphism of $R^{\mathfrak{c}}$-modules $B \otimes_R B \to B$. The argument in a) (and the functoriality of the filtration $\mathfrak{z}_\beta$.) imply that $\mathfrak{z}_{\beta i} B \otimes_R \mathfrak{z}_{\beta j} B \to \mathfrak{z}_{\beta i+j} B$. Thus $B_{\beta-diff}$ is a subring.

c) Given a left graded R-module L, we have the canonical map of graded $R^{\mathfrak{c}}$-modules

$$\alpha : R \to \mathfrak{z}_{\beta 0} grHom(L, L), \quad r \mapsto (l \mapsto rl).$$

Clearly, $\alpha : R \to grHom(L, L)$ is a ring homomorphism, and note that the $R^{\mathfrak{c}}$-module structure on $grHom(L, L)$ coincides with the one induced by this homomorphism. Hence, by b) above, $\mathrm{Diff}_\beta(L, L)$ is a β-differential R-algebra.

2.1.8. We write $D_\beta(R)$ instead of $\mathrm{Diff}_\beta(R, R)$ and call it the *algebra of (k-linear) β-differential operators on R*. The canonical homomorphism of graded algebras $i : R \to D_\beta(R)$ (2.1.7c) makes $D_\beta(R)$ a β-differential R-algebra. Put $D_\beta^i(R) := \mathfrak{z}_{\beta i} D(R)$, $i \geq 0$. Then $D_\beta^i(R) D_\beta^j(R) \subset D_\beta^{i+j}(R)$ (2.1.7b), i.e. $D_\beta(R)$ is a filtered algebra by the "order of differential operators". We have $i(R) \subset D_\beta^0(R)$.

2.1.9. Let d be a k-linear map $d : R \to R$ of degree $deg(d) = \gamma \in \Gamma$. We call d a *left β-derivation* if for $r \in R_a$, $r' \in R$

$$d(rr') = d(r)r' + \beta(\gamma, a) r d(r').$$

Denote by $\mathrm{Der}_\beta^l(R)$ the k-module of left β-derivations of R. Similarly, we define the k-module $\mathrm{Der}_\beta^r(R)$ of *right β-derivations* d by the formula

$$d(rr') = d(r)r'\beta(\gamma, b) + rd(r')$$

if $deg(d) = \gamma$, $deg(r') = b$.

2.1.10 Let M be a graded R^ϵ-module and $r \in R_a$. The following operators from M to M

$$m \mapsto m\beta(r) := \beta(a, \gamma)mr, \quad \text{for } m \in M_\gamma,$$

and

$$m \mapsto \beta(r)m := \beta(\gamma, a)rm, \quad \text{for } m \in M_\gamma$$

are called the *right* and *left β-multiplication* by r.

Note that

$$\mathfrak{z}_\beta(M) = \{m \in M | mr = \beta(r)m \text{ for all } r \in R\}.$$

2.1.11 Proposition. *a) The β-center $\mathfrak{z}_\beta(\mathrm{gr}Hom(R,R))$ of the graded R^ϵ-module $\mathrm{gr}Hom(R,R)$ is the k-span of right β-multiplications by homogeneous elements of R. Hence the subring $D_\beta^0(R) \subset D_\beta(R)$ is generated by left multiplication in R and right β-multiplication in R.*

b) The R^ϵ-module $D_\beta^1(R)$ contains $\mathrm{Der}_\beta^l(R)$.

Proof. All assertions are straightforward.

2.1.12 Remark. Note that if a Γ-graded R^ϵ-module M is trivially graded, i.e. $M = M_0$, then $\mathfrak{z}_\beta(M) = \mathfrak{z}(M)$. In particular, if R is trivially graded, then $D_\beta(R) = D(R)$.

2.2. Localization of β-differential operators

2.2.0 Assumption. Let $R \to R'$ be a homomorphism of Γ-graded k-algebras such that R' is the localization of R with respect to a left and right Ore set $S \subset R$ consisting of homogeneous elements.

Denote $R'^\epsilon := R' \otimes R'^0$.

2.2.1 Remarks. 1. The algebra R' is also Γ-graded. Let

$$R_\Gamma' := k[\Gamma]\#R' = \oplus_{a \in \Gamma} R'a$$

be the corresponding crossed product algebra (cf. 2.1.0). The natural map of (R', R)-modules

$$R' \otimes_R R_\Gamma \to R_\Gamma', \quad r' \otimes ra \mapsto r'ra$$

is an isomorphism. Hence $R' \otimes_R R_\Gamma$ has a natural structure of a right R'-module (which commutes with the left R'-module structure) and therefore the natural map

$$R' \otimes_R R_\Gamma \to R' \otimes_R R_\Gamma \otimes_R R'$$

is an isomorphism. Thus also

$$R_\Gamma' \simeq R' \otimes_R R_\Gamma \otimes_R R'.$$

2. If M is a β-differential $R^{\mathfrak{e}}$-module, then the $R'^{\mathfrak{e}}$-module $R' \otimes_R M \otimes_R R'$ is also β-differential. Indeed, the functor $R' \otimes_R \cdot \otimes_R R'$ is exact and by the previous remark it takes R_Γ to R'_Γ.

2.2.2 Theorem. *Assume that R is a domain. Given a left graded $D_\beta(R)$-module L, its localization $R' \otimes_R L$ is also canonically a graded $D_\beta(R)$-module. In particular, there exists a canonical ring homomorphism $D_\beta(R) \to D_\beta(R')$ (preserving the filtration D_β^n), i.e. β-differential operators on R extend to R'.*

The proof is identical to the proof of Theorem 1.2.1 (using Remarks 2.2.1) and we omit it.

2.3 Example: skew polynomial ring R.

2.3.0. Let k be a field of characteristic 0. Let $\mathbf{q}=(q_{ij})$ be an $n \times n$-matrix with entries in k^*, such that $q_{ij} = q_{ji}^{-1}$ and $q_{ii} = 1$. Let $\Gamma = \mathbb{Z}^n$ with the free generators $\gamma_1, ..., \gamma_n$. The matrix $\mathbf{q}$ defines the bicharacter $\beta : \Gamma \times \Gamma \to k^*$ by the formula

$$\beta(\gamma_i, \gamma_j) = q_{ij}.$$

Notice that $\beta(a, b) = \beta(b, a)^{-1}$.

The same matrix defines also a skew polynomial ring R in n variables. Namely, let R be a k algebra with generators $x_1, ..., x_n$ and the only relations

$$x_i x_j = q_{ij} x_j x_i.$$

The algebra R is Γ-graded in the obvious way ($deg\, x_i = \gamma_i$) and the corresponding grading action σ of Γ on R is defined by the formula

$$\sigma(\gamma_i)(x_j) = q_{ij} x_j.$$

Remarks. 1. For $r \in R_a$, $r' \in R_b$ we have

$$rr' = \beta(a, b)r'r.$$

Thus the ring R is β-commutative, in the sense that the right β-multiplication by $r \in R$ is equal to the left multiplication by r. This implies that the β-central $R^{\mathfrak{e}}$-module R is equal to its β-center (2.1.11a). Hence any β-central $R^{\mathfrak{e}}$-module is equal to its β-center.

2. The β-center of the graded $R^{\mathfrak{e}}$-module $grHom(R, R)$ coincides with left multiplications by elements of R (see 2.1.11a)) and therefore the $R^{\mathfrak{e}}$-module D_β^0 consists of left multiplications by elements of R. This means that the ring $D_\beta^0(R)$ coincides with the subring $R \subset D_\beta(R)$.

2.3.1 Lemma. *The k-module $\mathrm{Der}_\beta^l(R)$ is a left R-module.*

Proof. Let $d \in \mathrm{Der}_\beta^l(R)$ be a left β-derivation $d : R \to R$ of degree $\gamma \in \Gamma$. By definition, for $r \in R_a$, $r' \in R$ we have

$$d(rr') = d(r)r' + \beta(\gamma, a)rd(r').$$

Let $t \in R_b$. Then

$$td(rr') = td(r)r' + \beta(\gamma + b, a)rtd(r').$$

This implies that td is a left β-derivation of degree $\gamma + b$.

2.3.2 Lemma. *For each $i = 1, ..., n$ the map*

$$\partial_i(x_j) = \delta_{ij}, \quad \partial_i(1) = 0$$

extends uniquely to a left β-derivation ∂_i of R (of degree $-\gamma_i \in \Gamma$) which satisfies the relation

$$\partial_i(x_j r) = \delta_{ij} r + q_{ji} x_j \partial_i(r).$$

Proof. Straightforward.

2.3.3 Lemma. *The left R-module $\mathrm{Der}^l_\beta(R)$ (2.3.1) is a free module with the basis $\{\partial_1, ..., \partial_n\}$ defined in the last lemma.*

Proof. It is clear that β-derivations $\partial_1, ..., \partial_n$ are linearly independent over R. On the other hand, any β-derivation maps 1 to 0 and is uniquely determined by its values on $x_1, ..., x_n$. This implies that

$$\oplus_i R\partial_i = \mathrm{Der}^l_\beta(R).$$

2.3.4 Theorem. *The k-algebra $D_\beta(R)$ is generated by R (acting by left multiplications) and by the β-derivations $\partial_1, ..., \partial_n$.*

Proof. **Step 1.** Let $f, g \in \mathrm{gr}Hom_k(R, R)$ be homogeneous elements of degrees $deg(f) = a$, $deg(g) = b$. Define their β-*commutator* as

$$[f, g]_\beta := fg - \beta(a, b)gf.$$

It follows from the remarks in 2.3.0 that in this case we can give the classical definition of β-differential operators as in 1.0 replacing the commutators by β-commutators. Namely, notice that $D^n_\beta(R)$ consists of elements $f \in \mathrm{gr}Hom_k(R, R)$, such that

$$[r_1, [r_2, ...[r_{n+1}, f]_\beta, ...]_\beta]_\beta = 0$$

for any $r_1, ... r_{n+1} \in R$.

Step 2. Let $R_s \subset R$ denote the k-span of monomials in $x_1, ..., x_n$ of degree $\leq s$.

Claim. Let $D \in D^s_\beta(R)$ be such that $D|_{R_s} = 0$. Then $D = 0$.

Proof. The claim is true for $s = 0$ (2.3.0). Assume that we proved it for $s < i$ and fix $D \in D^i_\beta(R)$ of degree a such that $D|_{R_j} = 0$ for some $j \geq i$. It suffices to prove that $D|_{R_{j+1}} = 0$. Let $x_t c \in R_{j+1}$, where $c \in R_j$. Then

$$D(x_t c) = D(x_t c) - \beta(a, \gamma_t)x_t Dc + \beta(a, \gamma_t)x_t Dc = -\beta(a, \gamma_t)[x_t, D]_\beta c + 0.$$

Clearly, $[x_t, D]_\beta$ is a β-differential operator of order $i - 1$ such that $[x_t, D]_\beta|_{R_{i-1}} = 0$. Hence by induction hypothesis $[x_t, D]_\beta = 0$. This proves the claim.

Step 3. Let $A \subset D_\beta(R)$ be the k-subalgebra generated by R and by β-derivations $\partial_1, ..., \partial_n$.

Claim. R is a simple A-module.

Proof. It suffices to notice that for each $n > 0$ the map

$$\oplus \partial_i : R_n \to \oplus R_{n-1}$$

is injective (we use the fact that $\partial_i(x_i^s) = s x_i^{s-1}$ and that k has characteristic 0). Hence for every $r \in R$ there exists $d \in A$ such that $0 \neq d(r) \in k$. This proves the claim.

Step 4. Now the Jacobson's density theorem together with steps 2 and 3 finish the proof of the theorem. Namely, fix $D \in D_\beta^s(R)$. Then by the density theorem we can find $d \in A$ such that

$$d|_{R_s} = D|_{R_s}.$$

Clearly, we may assume that d is a polynomial in ∂_i's of degree $\leq s$, i.e. $d \in D_\beta^s(R)$. But then by step 2, $d = D$.

3. Quantum (or q-) differential operators

3.1.0. We keep the assumptions and notations of 2.1.0. For a Γ-graded k-module M and $\gamma \in \Gamma$, define the shifted module $M[\gamma]$ as

$$M[\gamma]_a := M_{a+\gamma}.$$

In particular this defines the action of Γ by autoequivalences on the categories of graded R-modules and $R^\mathfrak{e}$-modules.

Consider the graded $R^\mathfrak{e}$-module

$$R_\Gamma^q := \oplus_{\gamma \in \Gamma} R_\Gamma[\gamma].$$

We are going to define the quantum differential operators exactly in the same way as we defined the β-differential operators, but using the graded $R^\mathfrak{e}$-module R_Γ^q instead of R_Γ. For the reader's convenience we give the details.

3.1.1 Definition. Let M be a graded $R^\mathfrak{e}$-module. We call M a *q-differential module* if it has a filtration by graded $R^\mathfrak{e}$-submodules

$$0 = M_{-1} \subset M_0 \subset M_1 \subset ..., \quad \bigcup M_i = M,$$

such that for each i the graded $R^\mathfrak{e}$-module M_i/M_{i-1} is a quotient of a direct sum of copies of R_Γ^q.

3.1.2 Definition. Let $M = \oplus_{a \in \Gamma} M_a$ be a graded $R^\mathfrak{e}$-module. We define its q-*center* $\mathfrak{z}_q(M)$ as the k-span of homogeneous elements $m \in M_\gamma$ for which there exists $b \in \Gamma$ such that

$$mr = \beta(\gamma + b, a) rm, \quad \text{for all} \quad r \in R_a.$$

We call M q-*central* if $M = R^{\mathfrak{c}}\mathfrak{Z}_q(M)$.

Notice that for a q-central graded $R^{\mathfrak{c}}$-module M we have $M = R\mathfrak{Z}_q(M) = \mathfrak{Z}_q(M)R$.

3.1.2.1 Remark. Obviously, $\mathfrak{Z}_\beta(M) \subset \mathfrak{Z}_q(M)$ (also, $\mathfrak{Z}(M) \subset \mathfrak{Z}_q(M)$).

3.1.3. Note that for a graded $R^{\mathfrak{c}}$-module M we have an isomorphism of graded k-modules

$$Hom_{R^{\mathfrak{c}}}(R_\Gamma^q, M) \simeq \mathfrak{Z}_q(M).$$

Any $R^{\mathfrak{c}}$-module M has a canonical chain of graded submodules $\{\mathfrak{z}_{qi}M\}$, which satisfies the condition of 3.1.2. Namely, define $\mathfrak{z}_{qi}M$ by induction as follows:

$$\mathfrak{z}_{q0}M := R^{\mathfrak{c}}\mathfrak{Z}_q(M),$$

$$\mathfrak{z}_{qi}M/\mathfrak{z}_{qi-1}M := R^{\mathfrak{c}}\mathfrak{Z}_q(M/\mathfrak{z}_{qi-1}M).$$

Clearly, this is the maximal filtration satisfying the condition in 3.1.2. Thus, M is differential iff $\bigcup_i \mathfrak{z}_{qi}M = M$.

3.1.4 Lemma. *Any graded $R^{\mathfrak{c}}$-module M contains the biggest q-differential submodule M_{q-diff}, called the q-differential part of M. The correspondence $M \mapsto M_{q-diff}$ is functorial: for any graded $R^{\mathfrak{c}}$-module morphism $\varphi : M \to M'$, $\quad \varphi(M_{q-diff}) \subset M'_{q-diff}$.*

Proof. Indeed, put $M_{q-diff} := \bigcup_i \mathfrak{z}_{qi}M$.

3.1.5 Definition. Let L, N be graded left R-modules. Consider $\mathrm{gr}Hom(L, N)$ as a left graded $R^{\mathfrak{c}}$-module and call $\mathrm{Diff}_q(L, N) := \mathrm{gr}Hom(L, N)_{q-diff}$ the (k-linear) q-*differential operators* from L to N. The $R^{\mathfrak{c}}$- submodule $\mathrm{Diff}_q^n(L, N) := \mathfrak{z}_{qn}\mathrm{Diff}_d(L, N)$ of $\mathrm{Diff}_q(L, N)$ consists of q-differential operators of *order* n.

3.1.6 Definition. Given a homomorphism of graded k-algebras $R \to B$, we call B a q-*differential R-algebra* if B is a q-differential $R^{\mathfrak{c}}$-module.

3.1.7 Proposition. *a) If M, M' are graded q-differential $R^{\mathfrak{c}}$-modules, then the graded $R^{\mathfrak{c}}$-module $M \otimes_R M'$ is also q-differential.*

b) For any homomorphism of graded k-algebras $R \to B$, the q-differential part of the $R^{\mathfrak{c}}$-module B is a graded subring of B, i.e. B_{q-diff} is a differential R-algebra. More precisely, $\mathfrak{z}_{qi}B \cdot \mathfrak{z}_{qj}B \subset \mathfrak{z}_{q(i+j)}B$.

c) For any graded left R-module L, the graded $R^{\mathfrak{c}}$-module $\mathrm{Diff}_q(L, L)$ is a q-differential R-algebra.

Proof. The proof is identical to the proof of 2.1.7 except one should use the q-center instead of the β-center.

3.1.8. We write $D_q(R)$ instead of $\mathrm{Diff}_q(R, R)$ and call it the *algebra of (k-linear) quantum (or q-) differential operators on R*. The canonical homomorphism of graded algebras $i : R \to D_q(R)$ (2.1.7c) makes $D_q(R)$ a q-differential R-algebra. Put $D_q^i(R) := \mathfrak{z}_{qi}D(R)$, $i \geq 0$. Then $D_q^i(R)D_q^j(R) \subset D_q^{i+j}(R)$ (2.1.7b), i.e. $D_q(R)$ is a filtered algebra by the "order of differential operators". We have $i(R) \subset D_q^0(R)$.

3.1.9 Remark. Let L be a graded left R-module. Consider the graded R^e-module $\mathrm{gr}Hom(L, L)$. For any $\gamma \in \Gamma$, the operator $\sigma_L(\gamma)$ (of degree 0) belongs to the q-center $3_q\mathrm{gr}Hom(L, L)$. Moreover,

$$(1) \qquad 3_q\mathrm{gr}Hom(L, L) = 3_\beta\mathrm{gr}Hom(L, L)\sigma_L(\Gamma).$$

Hence, $\mathrm{Diff}_q^0(L, L) = \mathrm{Diff}_\beta^0(L, L)\sigma_L(\Gamma)$. In general, $\mathrm{Diff}_q^n(L, L) \supset \mathrm{Diff}_\beta^n(L, L)\sigma_L(\Gamma)$, and so $\mathrm{Diff}_q(L, L) \supset \mathrm{Diff}_\beta(L, L)\sigma_L(\Gamma)$.

3.1.10 Proposition. *a) The q-center $3_q(\mathrm{gr}Hom(R, R))$ of the R^e-module $\mathrm{gr}Hom(R, R)$ is the k-span of products of right multiplications by elements of R and operators $\sigma(\gamma)$ for $\gamma \in \Gamma$. Therefore the subring $D_q^0(R)$ of $D_q(R)$ is generated by left and right multiplications in R and by operators $\sigma(\Gamma)$.*
   b) The R^e-module $D_q^1(R)$ contains $\mathrm{Der}_\beta^l(R)$ and $\mathrm{Der}_\beta^r(R)$. (2.1.9)

Proof. Part a) follows from 2.1.11a) and formula (1) in the previous remark. In part b) the fact that $D_q^1(R)$ contains $\mathrm{Der}_\beta^l(R)$ follows from 2.1.11b). Let $d : R \to R$ be a right β-derivation of degree $\gamma \in \Gamma$. Then by definition

$$d(rr_1) = d(r)r_1\beta(\gamma, b) + rd(r_1)$$

for $r \in R$, $r_1 \in R_b$. Thus

$$d(rr_1) = d(r)\sigma(\gamma)(r_1) + rd(r_1).$$

Hence in $\mathrm{End}(R)$ we have the relation

$$dr - rd = d(r)\sigma(\gamma).$$

Since the RHS belongs to $D_q^0(R)$ by part a), then $d \in D_q^1(R)$. This proves the proposition.

3.1.11 Remark. Assume that R is trivially graded, i.e. $R = R_0$. Then for any graded R^e-module M we have $3_q(M) = 3(M)$. In particular, $D_q(R) = D(R)$.

3.2. Localization of quantum differential operators

3.2.0 Assumption. Let $R \to R'$ be a homomorphism of Γ-graded k-algebras such that R' is the localization of R with respect to a left and right Ore set $S \subset R$ consisting of homogeneous elements. Denote $R'^e := R' \otimes R'^0$.

3.2.1 Remarks. 1. The algebra R' is also Γ-graded. Let

$$R_\Gamma' := k[\Gamma]\#R' = \oplus_{a\in\Gamma}R'a$$

be the corresponding crossed product algebra. In Remark 2.2.1,1 we discussed the natural isomorphisms

$$R' \otimes_R R_\Gamma \xrightarrow{\sim} R' \otimes_R R_\Gamma \otimes_R R',$$

$$R_\Gamma' \xrightarrow{\sim} R' \otimes_R R_\Gamma \otimes_R R'.$$

Hence we obtain natural isomorphisms

$$R' \otimes_R R_\Gamma^q \xrightarrow{\sim} R' \otimes_R R_\Gamma^q \otimes_R R',$$

$$R_\Gamma'^q \xrightarrow{\sim} R' \otimes_R R_\Gamma^q \otimes_R R'.$$

2. If M is a q-differential R^ϵ-module, then the R'^ϵ-module $R' \otimes_R M \otimes_R R'$ is also q-differential. Indeed, the functor $R' \otimes_R \cdot \otimes_R R'$ is exact and by the previous remark it takes R_Γ^q to $R_\Gamma'^q$.

3.2.2 Theorem. *Assume that R is a domain. Given a left graded $D_q(R)$-module L, its localization $R' \otimes_R L$ is also canonically a graded $D_q(R)$-module. In particular, there exists a canonical ring homomorphism $D_q(R) \to D_q(R')$ (preserving the filtration D_q^n), i.e. q-differential operators on R extend to R'.*

The proof is identical to the proof of Theorem 1.2.1 (using Remarks 3.2.1) and we omit it.

3.3 Relation with quantum groups.

3.3.0 The quantum group U_q. Let q be an indeterminate.

Let $(a_{ij})_{i,j=1,\ldots,n}$ be a Cartan matrix of finite type (i.e. the corresponding Lie algebra $\mathfrak{g}$ is finite dimensional) and choose $d_i \in \{1,2,3\}$ such that $(d_i a_{ij})$ is symmetric. Consider the $\mathbb{Q}(q)$-algebra U_q with generators

$$E_i, F_i, K_i, K_i^{-1}, \quad i = 1,\ldots,n$$

and relations

$$K_i K_j = K_j K_i, \ K_i K_i^{-1} = 1, \quad i,j = 1,\ldots,n$$

$$K_i E_j K_i^{-1} = q^{d_i a_{ij}} E_j, \ K_i F_j K_i^{-1} = q^{-d_i a_{ij}} F_j, \quad i,j = 1,\ldots,n$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \quad i,j = 1,\ldots,n$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^r E_j E_i^s = 0, \quad i \neq j$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} F_i^r F_j F_i^s = 0, \quad i \neq j.$$

In the last two relations we have used brackets to denote Gaussian binomial coefficients. specifically, we have for $m \in \mathbb{Z}, d, t \in \mathbb{N}$,

$$[m]_d = \frac{q^{md} - q^{-md}}{q^d - q^{-d}}, \ [t]_d! = [t]_d [t-1]_d \cdots [2]_d [1]_d$$

and

$$\begin{bmatrix} m \\ t \end{bmatrix}_d = \frac{[m]_d [m-1]_d \cdots [m-t+1]_d}{[t]_d!}.$$

In fact U_q is a Hopf algebra with the comultiplication Δ, the coidentity ε, and the antipode S defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$$

$$\Delta(K_i) = K_i \otimes K_i, \quad i = 1, \dots, n$$

$$\varepsilon(E_i) = 0 = \varepsilon(F_i), \ \varepsilon(K_i) = 1, \ i = 1, \dots, n$$

$$S(E_i) = -K_i^{-1} E_i, \ S(F_i) = -F_i K_i, \ S(K_i) = K_i^{-1}$$

The algebra U_q has a triangular decomposition. Namely, let U_q^-, U_0° and U_q^+ be the subalgebras of U_q generated by F_i (resp. K_i, K_i^{-1}, resp. E_i), $i = 1, \dots, n$. Then the multiplicaiton map defines the isomorphism

$$U_q^- \otimes U_q^\circ \otimes U_q^+ \overset{\sim}{\longrightarrow} U_q.$$

3.3.1. Choose a Cartan and a Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $\alpha_1, \dots, \alpha_n$ be the corresponding set of simple roots. Put $Q := \oplus \mathbb{Z}\alpha_i$ - the corresponding root lattice. Put $(\alpha_i | \alpha_j) := d_i a_{ij}$. This defines a symmetric bilinear form $(\cdot | \cdot)$ on Q and the bicharacter

$$\beta' : Q \times Q \to \mathbb{Q}(q)^*, \quad \beta'(a,b) := q^{(a|b)}.$$

We have an isomorphism of k-algebras

$$\tau : k[Q] \overset{\sim}{\longrightarrow} U_q^0, \quad \alpha_i \mapsto K_i.$$

Notice that the quantum group U_q is Q-graded with

$$deg(K_i) = 0, \ \ deg(E_i) = \alpha_i, \ \ deg(F_i) = -\alpha_i,$$

and the corresponding grading action $\sigma : Q \to \mathrm{Aut}(U_q)$ (2.1.0) is the adjoint action of U_q^0 on U_q:

$$\sigma(a)u = \tau(a)u\tau(a)^{-1}.$$

We assume that Q is a subgroup of an abelian group Γ and β' is the restriction to $Q \times Q$ of a bicharacter $\beta : \Gamma \times \Gamma \to k^*$, where k is a field, which contains $\mathbb{Q}(q)$. If $k \neq \mathbb{Q}(q)$, then we consider U_q as the k-algebra by extending the scalars from $\mathbb{Q}(q)$ to k. Since U_q is Q-graded, it is also Γ-graded.

3.3.2. Let R be a Γ-graded k-algebra and a left U_q-module so that the action $\rho : U_q \to End_k(R)$ is a homomorphism of graded algebras. We say that U_q acts on R as a Hopf algebra or that R is a U_q-*algebra*, if

$$u(rr') = u_{(1)}ru_{(2)}r', \quad u \in U_q, \ r, r' \in R,$$

where $\Delta(u) = u_{(1)} \otimes u_{(2)} = \Sigma_i u_{(1)}^i \otimes u_{(2)}^i.$

3.3.3. Let R be a Γ-graded k-algebra and a left U_q-module with the action $\rho : U_q \to \mathrm{End}_k(R)$. Let $\sigma_R : \Gamma \to \mathrm{Aut}(R)$ be the grading action of Γ on R. We say that the U_q-action on R is *compatible with the grading* if

$$\sigma_R|_Q = \rho \cdot \tau : Q \to \mathrm{Aut}(R).$$

3.3.4 Lemma. *Assume that R is a Γ-graded k-algebra and a U_q-module, so that the U_q-action on R is compatible with the grading. Then the following are equivalent*
a) R is a U_q-algebra,
b) elements $E_i, F_i \in U_q$ act by left and right β-derivations respectively.

Proof. The compatibility of the action with the grading implies that the relation in 3.3.2 is satisfied for $u = K_i$. Hence R is a U_q-algebra iff the same relation is satisfied for E_i and F_i. But this relation for E_i (resp. for F_i) exactly means (using the compatibility of the action with the grading) that E_i (resp. F_i) acts by a left (resp. right) β-derivation.

3.3.5 Corollary. *Under the equivalent conditions of the previous lemma the quantum group U_q acts on the ring R by quantum differential operators, i.e. we have a homomorphism of k-algebras $U_q \to D_q(R)$.*

Proof. This follows immediately from Proposition 3.1.10. .

3.4.0 Let $R \to R'$ be a homomorphism of Γ-graded k-algebras such that R' is the localization of R with respect to a left and right Ore set $S \subset R$, consisting of homogeneous elements.

3.4.1 Theorem. *Assume that R is a domain. Assume that the quantum group U_q acts on R by the action which is compatible with the grading and which makes R a U_q-algebra. Then this U_q-action extends canonically to an action on R', which makes R' a U_q-algebra. Moreover, such an extension is unique and is compatible with the grading of R'.*

Proof. By Corollary 3.3.5 U_q acts on R by quantum differential operators. That is, we have the ring homomorphism $U_q \to D_q(R)$. On the other hand, we have the canonical ring homomorphism $D_q(R) \to D_q(R')$ (Theorem 3.2.1). Composing the two we get the first assertion of the theorem.

It is clear that the U_q-action on R' is compatible with the grading. Hence, by Lemma 3.3.4 it suffices to prove that elements E_i, F_i act on R' by left and right β-derivations respectively. Consider, for example, E_i. It acts on R by a left β-derivation. Hence in $D_q(R)$ we have the following relation

$$E_i r - \beta(\alpha_i, a) r E_i = E_i(r), \quad \text{for } r \in R_a.$$

Let $s^{-1}r \in R'$, where $r \in R$, $s \in R_a$. Then

$$E_i(r) = E_i(ss^{-1}r) = E_i(s)s^{-1}r + \beta(\alpha_i, a)sE_i(s^{-1}r),$$

hence

$$E_i(s^{-1}r) = \beta(\alpha_i, a)^{-1}s^{-1}\big(E_i(r) - E_i(s)s^{-1}r\big).$$

One checks directly that this formula defines a left β-derivation of R'. Similarly for F_i. The uniqueness assertion is also clear from the last formula.

3.5 Example: quantum differential operators on the skew polynomial ring.

3.5.0. Let k be a field of characteristic 0. Let $\mathbf{q} = (q_{ij})$ be an $n \times n$ matrix with entries in k^*, such that $q_{ij} = q_{ji}^{-1}$. Let $\Gamma = \mathbb{Z}^n$ with the free generators $\gamma_1, ..., \gamma_n$. The matrix $\mathbf{q}$ defines the bicharacter $\beta : \Gamma \times \Gamma \to k^*$ by the formula

$$\beta(\gamma_i, \gamma_j) = q_{ij}.$$

The same matrix defines also a skew polynomial ring R in n variables. Namely, let R be a k-algebra with generators $x_1, ..., x_n$ and the only relations

$$x_i x_j = q_{ij} x_j x_i, \quad i \neq j.$$

The algebra R is Γ-graded with $deg(x_i) = \gamma_i$ and the corresponding grading action σ of Γ on R is defined by the formula

$$\sigma(\gamma_i)(x_j) = q_{ij} x_j.$$

In case $q_{ii} = 1$ we computed the algebra $D_\beta(R)$ of β- differential operators on R (2.3.4). Here we want to discuss a more interesting case: q_{ii} is not necessarily 1. In this general case we did not succeed in writing down the algebra $D_q(R)$ (or $D_\beta(R)$) in terms of generators and relations. However, we can prove that Bernstein's theorem holds for $D_q(R)$: for any $D_q(R)$-module $M \neq 0$ its GK-dimension is at least n. This allows us to define the category of holonomic $D_q(R)$-modules M, i.e. such that $GK(M) = n$. An example of a holonomic $D_q(R)$-module is $M = R$.

3.5.1 Lemma. *For each $i = 1, ..., n$ the map*

$$\partial_i(x_j) = \delta_{ij}, \qquad \partial_i(1) = 0$$

extends uniquely to a left β-derivation ∂_i of R (of degree $-\gamma_i \in \Gamma$) which satisfies the relation

$$\partial_i(x_j r) = \delta_{ij} r + q_{ji} x_j \partial_i(r).$$

Proof. Straightforward.

3.5.2. By Proposition 3.1.10 and theorem 2.3.4 the algebra $D_q(R)$ contains $x_1, ..., x_n$ (acting by left multiplication), the β-derivations $\partial_1, ..., \partial_n$ and the automorphisms $\sigma_1 := \sigma(\gamma_1), ..., \sigma_n := \sigma(\gamma_n)$ (and their inverses). Let A be the k-subalgebra of $D_q(R)$ generated by $x_i, \partial_i, \sigma_i, \sigma_i^{-1}$. One easily checks the following relations

(1) $$x_i x_j = q_{ij} x_j x_i, \quad \partial_i \partial_j = q_{ij} \partial_j \partial_i, \quad i \neq j;$$

(2) $$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1;$$

(3) $$\partial_i x_j = q_{ji} x_j \partial_i + \delta_{ij}, \quad q_{ij} x_i \sigma_j = \sigma_j x_i, \quad \partial_i \sigma_j = q_{ji} \sigma_j \partial_i.$$

Let W be the k-algebra generated by x_i, ∂_i, σ_i, σ_i^{-1} with the set of defining relations (1)-(3) above. Thus A is a homomorphic image of W. We studied the algebra W in [LR2]. We showed that W is an example of the so-called hyperbolic algebra, and deduced Bernstein's theorem (in case q_{ij} are generic) for W from a more precise Kashiwara theorem for hyperbolic algebras (which is proved in [LR2]). That is we proved that if $M \neq 0$ is a left W-module, then the Gelfand-Kirillov dimension $GK(M)$ is at least n. The same assertion therefore holds for the algebra A instead of W. Since A is the k-subalgebra of $D_q(R)$ we obtain the following theorem .

3.5.3 Theorem. *Let $0 \neq M$ be a left $D_q(R)$-module. Then $GK(M) \geq n$.*

3.5.4. A counterexample. Here is an example, which shows that the above theorem is a subtle property.

Let $n = 1$ (that is we consider a "quantum line") and put $q_{11} = q$, $x_1 = x$, $\partial_1 = \partial$, $\sigma_1 = \sigma$. Thus $A = k < x, \partial, \sigma, \sigma^{-1} >$. Assume that $q \neq 1$. Consider the k-subalgebra $B := k < x, \partial > \subset A$. Put $\xi := (q^{-1} - 1)x\partial + 1$. We have

$$x\xi = q\xi x, \quad \partial\xi = q^{-1}\xi\partial.$$

Hence ξ generates a two-sided ideal (ξ) in B and the algebra $B/(\xi)$ is commutative. This shows that B has infinitely many modules which are finite dimensional over k (hence of GK-dimension 0). The point is that $\xi = \sigma^{-1}$. Hence $B = k < x, \partial, \sigma^{-1} >$ and passing from B to A means destroying the two-sided ideal (ξ).

4. Extension of differential endofunctors to localizations

4.0. We are going to give a categorical interpretation of the localization theorems 1.2.1, 2.2.2 and 3.2.2. More precisely, we are going to explain the meaning of the statement 4.0.1 below which is the essential part of the above mentioned theorems.

4.0.1 Statement. *Let R be a k-algebra (resp. a k-algebra graded by an abelian group Γ). Assume that R is a domain. Let $R \to R'$ be a localization with respect to a left and right Ore set (resp. consisting of homogeneous elements). Let $D(R)$ be the ring of differential operators (resp. β- or q-differential) operators on R. Then the canonical map of (R', R)-modules*

$$R' \otimes_R D(R) \to R' \otimes_R D(R) \otimes_R R'$$

is an isomorphism.

Proof. In the case of ordinary differential operators, this is a restatement of Lemma 1.2.1.1. In the case of β- or q-differential operators the proof repeats that of Lemma 1.2.1.1 using Remarks 2.2.1,1 and 3.2.1,1.

4.1. Let $\mathcal{A}$ be an abelian category with a thick subcategory $T \subset \mathcal{A}$, so that the corresponding exact localization functor $Q^* : \mathcal{A} \to \mathcal{A}/T$ has a fully faithful right adjoint $Q_* : \mathcal{A}/T \to \mathcal{A}$ (see [GZ], 4.1). Let

$$\eta : Id_\mathcal{A} \to Q_*Q^*$$

be the adjunction morphism. Denote by Σ_T the class of morphisms s of $\mathcal{A}$ such that $Ker(s)$ and $Coker(s)$ are objects of T. Or, equivalently,

$$\Sigma_T = \{s \in Hom\mathcal{A} | Qs \;\; is \;\; invertible\}$$

Let $G : \mathcal{A} \to \mathcal{A}$ be an endofunctor. By the universal property of $Q^* : \mathcal{A} \to \mathcal{A}/T$, there exists an endofunctor $\tilde{G} : \mathcal{A}/T \to \mathcal{A}/T$ such that $\tilde{G}Q^* \simeq Q^*G$, if and only if $G(\Sigma_T) \subset \Sigma_T$. If $\tilde{G}$ exists, it is unique. Define the endofunctor $G' := Q^*GQ_*$: $\mathcal{A}/T \to \mathcal{A}/T$ and consider the canonical morphism of functors

$$\eta' := Q^*G(\eta) : Q^*G \to G'Q^*.$$

4.1.0 Lemma. *The following conditions are equivalent.*
a) η' *is an isomorphism.*
b) *There exists a functor* $\tilde{G} : \mathcal{A}/T \to \mathcal{A}/T$ *such that* $Q^*G \simeq \tilde{G}Q^*$.

Proof. $a) \Rightarrow b)$. Put $\tilde{G} := G'$. Then

$$\eta' : Q^*G \xrightarrow{\sim} \tilde{G}Q^*$$

is the desired isomorphism.

$b) \Rightarrow a)$. Since Q_* is fully faithful, the adjunction morphism

$$\epsilon : Q^*Q_* \to Id$$

is an isomorphism. Hence

$$Q^*(\eta) : Q^* \to Q^*Q_*Q^*$$

is an isomorphism. This implies that for every $M \in Ob\mathcal{A}$ the morphism $\eta(M)$: $M \to Q_*Q^*(M)$ is in Σ_T. By our assumptions then $G(\eta(M)) \in \Sigma_T$. Hence $\eta'(M) = Q^*G(\eta(M))$ is an isomorphism.

4.2. We keep the notations and assumptions of statement 4.0.1 above. Let $\mathcal{A}$, $\mathcal{A}'$ be the abelian categories of left R- and R'-modules (resp. graded modules). Then $\mathcal{A}'$ is the localization of $\mathcal{A}$ in the sense of 4.1. The localization functor is $Q^* = R' \otimes_R \cdot$, and its right adjoint Q_* is the restriction of scalars from R' to R. Consider $D(R)$ as an R-bimodule and define the endofunctor

$$G := D(R) \otimes_R \cdot : \mathcal{A} \to \mathcal{A}.$$

4.2.1 Proposition. *There exists a (unique) functor* $\tilde{G} : \mathcal{A}' \to \mathcal{A}'$ *such that* $\tilde{G}Q^* \simeq Q^*G$.

Proof. By Lemma 4.1.0 the assertion is equivalent to the functorial morphism

$$\eta' : Q^*G \to Q^*GQ_*Q^*$$

being an isomorphism. And this in turn translates into the assertion that the natural map of (R', R)-modules

$$R' \otimes_R D(R) \to R' \otimes_R D(R) \otimes_R R'$$

is an isomorphism. So the proposition is equivalent to statement 4.0.1 above.

4.3 Remarks. 1. The endofunctor $D(R) \otimes_R \cdot : \mathcal{A} \to \mathcal{A}$ is an example of a *differential* endofunctor. This is the main object of study in [LR3],[LR4].

2. Proposition 4.2.1 is used in [LR1]. As its consequence we show that the cohomology of a D-module on the quantum flag variety can be computed either in the category of D-modules or "quasicoherent sheaves".

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