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# Differential Operators with Non Dense Domain

G. DA PRATO - E. SINISTRARI

## Introduction and Notation

The aim of this paper is the study of the initial value problem in the Banach space  $E$ :

$$(0.1) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

where  $A : D_A \subseteq E \rightarrow E$  is a closed linear operator,  $f : [0, T] \rightarrow E$  and  $u_0 \in E$  are given.

This problem has been extensively studied in the case in which  $A$  is the generator of a semigroup and the Hille-Yosida theorem gives the necessary and sufficient conditions in order that this occurs: among these conditions there is the density of  $D_A$  in  $E$ . In this paper we show that this is not necessary (in a certain sense) to solve problem (0.1): in other words, if we assume the Hille-Yosida conditions with the exception of the density of  $D_A$  in  $E$ , then we can obtain for problem (0.1) existence and uniqueness results which are even more general than those known in the case  $\overline{D_A} = E$  (for a more detailed comparison with the classical theory see remark 8.5).

The paper is divided into three parts: in the first one the case mentioned before, which is called the hyperbolic case (because of the applications to the partial differential equations of this type), is studied; in the second part we consider the more particular situation in which  $A$  generates an analytic semigroup not necessarily strongly continuous at the origin (because  $\overline{D_A} \subseteq E$ ): this is called the parabolic case (for reasons analogous to those of the previous situation). Finally we give several examples of differential operators with non dense domain satisfying the Hille-Yosida estimates and we make some applications to the study of partial differential equations of hyperbolic and ultraparabolic type, ending with an equation arising from the stochastic control theory.

Let us introduce some notations which will be used in this paper.

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We consider a Banach space  $E$  with norm  $\|\cdot\|$  and  $A$  a closed linear operator in  $E$  with domain  $D_A$ , which will be endowed with the graph norm  $\|u\|_{D_A} = \|u\| + \|Au\|$ .  $\mathcal{L}(E)$  is the Banach space of the continuous linear operators from  $E$  to  $E$  with the uniform norm;  $\lambda \in \mathbb{C}$  belongs to  $\rho(A)$  if  $(\lambda - A)^{-1} \in \mathcal{L}(E)$  and we set  $(\lambda - A)^{-1} = R(\lambda, A)$ . If  $E_1$  and  $E_2$  are Banach spaces,  $E_1 \hookrightarrow E_2$  means that  $E_1$  is continuously embedded in  $E_2$ ; if also  $E_2 \hookrightarrow E_1$  we write  $E_1 \simeq E_2$ . To state our results we need to introduce some notations about spaces of functions with values in  $E$ :

$$L^p(0, T; E) = \{u : [0, T] \rightarrow E; u \text{ is Bochner measurable and } \|u(\cdot)\|^p \text{ is integrable}\} (1 \leq p < \infty) \text{ with } \|u\|_{L^p(0, T; E)} = \left( \int_0^T \|u(t)\|^p dt \right)^{1/p},$$

$$B(0, T; E) = \{u : [0, T] \rightarrow E; u \text{ is bounded}\} \text{ with } \|u\|_{B(0, T; E)} = \sup_{0 \leq t \leq T} \|u(t)\|,$$

$$C(0, T; E) = \{u : [0, T] \rightarrow E, u \text{ is continuous}\} \text{ with } \|u\|_{C(0, T; E)} = \sup_{0 \leq t \leq T} \|u(t)\|,$$

$$C^\alpha(0, T; E) = \{u : [0, T] \rightarrow E; [u]_{C^\alpha(0, T; E)} = \sup_{0 \leq t \leq s \leq T} \frac{\|u(t) - u(s)\|}{|t - s|^\alpha} < \infty\} \\ \text{with } \|u\|_{C^\alpha(0, T; E)} = \|u\|_{C(0, T; E)} + [u]_{C^\alpha(0, T; E)}, \quad (0 < \alpha < 1),$$

$$h^\alpha(0, T; E) = \{u : [0, T] \rightarrow E; \lim_{\delta \rightarrow 0} \sup_{\substack{0 < |t-s| \leq \delta \\ t, s \in [0, T]}} \frac{\|u(t) - u(s)\|}{|t-s|^\alpha} = 0\}, \quad (0 < \alpha < 1),$$

$$\text{with } \|u\|_{h^\alpha(0, T; E)} = \|u\|_{C^\alpha(0, T; E)},$$

$$C^n(0, T; E) = \{u : [0, T] \rightarrow E; u^{(k)} \in C(0, T; E), k = 1, 2, \dots, n\}, \text{ where } n \in \mathbb{N} \\ \text{and } u^{(k)} \text{ denotes the Fréchet derivative}\},$$

$$C^{1+\alpha}(0, T; E) = \{u : [0, T] \rightarrow E; u' \in C^\alpha(0, T; E)\}, \quad (0 < \alpha < 1).$$

The elements of  $h^\alpha(0, T; E)$  are called little Hölder functions and it can be proved that the closure of  $C^1(0, T; E)$  in  $C^\alpha(0, T; E)$  is  $h^\alpha(0, T; E)$  (see [11] theorem 5.3).

We will also need the  $E$ -valued Sobolev spaces

$$W^{1,p}(0, T; E) = \{u : [0, T] \rightarrow E; u(t) = u_0 + \int_0^t u'(s) ds, t \in [0, T] \text{ for some} \\ u_0 \in E \text{ and } u' \in L^p(0, T; E)\}, \quad (1 \leq p < \infty), \text{ with} \\ \|u\|_{W^{1,p}(0, T; E)} = \|u\|_{L^p(0, T; E)} + \|u'\|_{L^p(0, T; E)}.$$

If  $X(0, T; E)$  denotes one of the spaces just introduced we set:

$$X_0(0, T; E) = \{u \in X(0, T; E); u(0) = 0\},$$

$$X_\#(0, T; E) = \{u \in X(0, T; E); u(0) = u(T)\},$$

$$X(0^+, T; E) = \{u : ]0, T[ \rightarrow E; u \in X(\varepsilon, T; E) \text{ for each } \varepsilon \in ]0, T[ \},$$

and we also set

$$X(0, T; \mathbb{R}) = X(0, T).$$

If  $\Omega \subset \mathbb{R}^n$  is an open and bounded set with regular boundary  $\Gamma$  we will use spaces of functions from  $\Omega$  (or  $\bar{\Omega}$ ) to  $\mathbb{R}$  noted as  $L^p(\Omega)$ ,  $C(\bar{\Omega})$ ,  $C^\alpha(\bar{\Omega})$ ,  $W^{1,p}(\Omega)$  defined in a way similar to the corresponding spaces previously introduced. We also set

$$C_0(\bar{\Omega}) = \{u \in C(\bar{\Omega}); \quad u = 0 \text{ on } \Gamma\}$$

and similarly for  $C_0^\alpha(\bar{\Omega})$ .

When the space  $E$  is made of functions of the variable  $x \in D$  (a subset of  $\mathbb{R}^n$ ) and we are given a function  $u(t, x)$  from  $[0, T] \times D$  into  $\mathbb{R}$ , then a function  $u : [0, T] \rightarrow E$ ,  $t \rightarrow u(t)$  can be defined by setting

$$u(t)(x) = u(t, x).$$

The use of the same symbol  $u$  for both functions originates no confusion because the variables on which they depend are different.

## Hyperbolic case

### 1. - The Hille-Yosida conditions

In the first part of the paper we will suppose that a linear operator  $A : D_A \subseteq E \rightarrow E$  is given in the Banach space  $E$  (with norm  $\|\cdot\|$ ) such that all the conditions of the Hille-Yosida theorem ([15], Ch. IX, 7) are satisfied with the exception of the density of  $D_A$  in  $E$ . Hence  $A$  satisfies the property:

$$(1.1) \quad \begin{cases} \lambda > 0 \Rightarrow (\lambda - A)^{-1} \in \mathcal{L}(E) \\ \sup_{k \in \mathbb{N}, \lambda > 0} \|\lambda^k (\lambda - A)^{-k}\|_{\mathcal{L}(E)} = M < +\infty. \end{cases}$$

We are interested in solving the problem

$$(1.2) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

where  $f : [0, T] \rightarrow E$  and  $u_0 \in E$  are given.

When  $\bar{D}_A = E$ , a classical procedure to solve it is to consider first (1.2) with  $f = 0$ , and  $A$  substituted by its Yosida approximation  $A_n = nAR(n, A)$ ; then, we have to prove that the solution of this approximate problem converges to a function  $u(t, u_0)$  which is a semigroup of linear operators in  $E$  and, for fixed  $u_0 \in D_A$ , is a solution of (1.2) with  $f = 0$ . After that, the inhomogeneous problem (1.2) can be solved by means of the classical method of the constant's

variation when  $u_0 \in D_A$  and  $f \in C^1(0, T; E)$  (Phillips theorem; see [8] Ch. IX, 5). This procedure cannot be used when  $\overline{D_A} \neq E$  since in this case the solution of the above mentioned approximating problem converges only for  $u_0 \in \overline{D_A}$ : so, this method cannot be applied when  $f(t) \notin \overline{D_A}$  (see also Remark 8.5). The method used in the first part of this paper to solve (1.2) when  $\overline{D_A} \neq E$  employs the Yosida approximations of the time derivative, considered as an operator in the Banach space  $L^p(0, T; E)$ : this method lets us find a strict solution for each  $u_0 \in D_A$  and  $f \in W^{1,1}(0, T; E)$  such that  $Au_0 + f(0) \in \overline{D_A}$  (note that this last condition is necessary to get a solution of (1.2) up to  $t=0$ ): when  $\overline{D_A} = E$  this result gives a generalization of the Phillips theorem. In this first part we introduce also two other types of solutions i.e. the  $F$ -solution (or the solution in the sense of Friedrichs) and the integral solution: we prove their existence for each  $f \in L^1(0, T; E)$  and  $u_0 \in \overline{D_A}$ , and in addition we show that they coincide and reduce to the mild solution (as defined e.g. in 2.3 of [10]) when  $\overline{D_A} = E$ . Let us observe that the reduction of the Hille-Yosida conditions is illusory when  $E$  is reflexive: in fact we have the following result due to Kato [7]:

PROPOSITION 1.1. *Let  $A : D_A \subseteq E \rightarrow E$  be a linear operator in the reflexive Banach space  $E$  such that there exist  $\omega, M > 0$  verifying the property:*

$$(1.3) \quad \lambda > \omega \Rightarrow (\lambda - A)^{-1} \in \mathcal{L}(E) \text{ and } \|(\lambda - A)^{-1}\|_{\mathcal{L}(E)} \leq \frac{M}{\lambda - \omega}$$

then  $\overline{D_A} = E$ .

PROOF. Let us fix  $u$  in  $E$ : as the sequence  $\|nR(n, A)u\|$ ,  $n \in \mathbb{N}$ ,  $n > \omega$  is bounded, there exists a subsequence  $\{n_k R(n_k, A)u\}$  converging weakly to some  $v \in E$ : hence  $\{AR(n_k, A)u\}$  converges weakly to  $v - u$ . As  $R(n_k, A)u \rightarrow 0$  we have  $u = v$  because  $A$  is weakly closed, so  $u$  is in the weak closure of  $D_A$  and the conclusion follows.

Finally let us observe that condition (1.1) can be replaced by the following:

$$(1.4) \quad \left\{ \begin{array}{l} \text{there exists } \omega \in \mathbb{R} \text{ such that for } \lambda > \omega \text{ we have } (\lambda - A)^{-1} \\ \in \mathcal{L}(E) \text{ and } \sup_{k \in \mathbb{N}, \lambda > \omega} \|(\lambda - \omega)^k (\lambda - A)^{-k}\| = M < +\infty \end{array} \right.$$

with no real difference in the results with the exception of some estimates: for more details see Appendix at the end of the paper.

## 2. - F-Solutions and strict solutions

Let  $A : D_A \subseteq E \rightarrow E$  be a closed linear operator in the Banach space  $E$  and choose  $f \in L^p(0, T; E)$  ( $1 \leq p < \infty$ ) and  $u_0 \in E$ : a *strict solution* in  $L^p$  of

$$(2.1) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \text{ a.e.} \\ u(0) = u_0 \end{cases}$$

is a function  $u \in W^{1,p}(0, T; E) \cap L^p(0, T; D_A)$  verifying (2.1).

A function  $u \in L^p(0, T; E)$  is called an *F-solution in  $L^p$*  of (2.1) if, for each  $k \in \mathbb{N}$ , there is  $u_k \in W^{1,p}(0, T; E) \cap L^p(0, T; D_A)$  such that by setting

$$(2.2) \quad \begin{cases} u'_k(t) - Au_k(t) = f_k(t), & t \in [0, T] \text{ a.e.} \\ u_k(0) = u_{0k} \end{cases}$$

we have

$$(2.3) \quad \lim_{k \rightarrow \infty} (\|u_k - u\|_{L^p(0, T; E)} + \|f_k - f\|_{L^p(0, T; E)} + \|u_{0k} - u_0\|) = 0.$$

Let us suppose now that  $f \in C(0, T; E)$  and  $u_0 \in E$  : a *strict solution in  $C$*  of

$$(2.4) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

is a function  $u \in C^1(0, T; E) \cap C(0, T; D_A)$  verifying (2.4).

A function  $u \in C(0, T; E)$  is called an *F-solution in  $C$*  of (2.4) if for each  $k \in \mathbb{N}$  there exists  $u_k \in C^1(0, T; E) \cap C(0, T; D_A)$  such that by setting

$$(2.5) \quad \begin{cases} u'_k(t) - Au_k(t) = f_k(t), & t \in [0, T] \\ u_k(0) = u_{0k} \end{cases}$$

we have

$$(2.6) \quad \lim_{k \rightarrow \infty} (\|u_k - u\|_{C(0, T; E)} + \|f_k - f\|_{C(0, T; E)} + \|u_{0k} - u_0\|) = 0.$$

From this it follows that  $u(0) = u_0$ .

Let us observe that a strict solution in  $C$  of (2.4) is also a strict solution in  $L^p$  and the same is true for the F-solutions. Moreover a strict solution is also an F-solution. The converse is not true in general: however we will show that an F-solution in  $L^p$  of (2.1) is continuous and that it verifies  $u(0) = u_0$ . Finally let us observe that  $u_0 \in \overline{D_A}$  is a necessary condition for the existence of any kind of solution.

### 3. - Yosida Approximations of the Time-Derivative

Set  $X_p = L^p(0, T; E)$ ,  $1 \leq p < \infty$  and let  $B : D_B \subset X_p \rightarrow X_p$  be the linear operator defined as

$$(3.1) \quad \begin{cases} Bu = -u' \\ D_B = \{u \in W^{1,p}(0, T; E) ; u(0) = 0\} \end{cases}$$

For each  $\lambda \in \mathbb{C}$  there exists  $(\lambda - B)^{-1} = R(\lambda, B) \in \mathcal{L}(X_p)$  and we have for each  $u \in X_p$

$$(3.2) \quad (R(\lambda, B)u)(t) = \int_0^t e^{-\lambda(t-s)} u(s) ds, \quad t \in [0, T].$$

From Young's inequality we get

$$(3.3) \quad \|R(\lambda, B)\|_{\mathcal{L}(X_p)} \leq \frac{1}{\operatorname{Re} \lambda} \text{ for } \operatorname{Re} \lambda > 0.$$

Analogously in the Banach space  $X_\infty = C(0, T; E)$  we define the linear operator  $B : D_B \subset X_\infty \rightarrow X_\infty$  as

$$(3.4) \quad \begin{cases} Bu = -u' \\ D_B = \{u \in C^1(0, T; E) ; u(0) = 0\}. \end{cases}$$

For each  $\lambda \in \mathbb{C}$  there exists  $(\lambda - B)^{-1} = R(\lambda, B) \in \mathcal{L}(X_\infty)$  and (3.2)-(3.3) hold.

PROPOSITION 3.1. *Let us consider the Yosida approximations of  $B$ :*

$$(3.5) \quad B_n = n^2 R(n, B) - n = nBR(n, B), \quad n \in \mathbb{N}$$

We have  $B_n \in \mathcal{L}(X_p)$  and

$$(3.6) \quad \lim_{n \rightarrow \infty} \|B_n u - Bu\|_{X_p} = 0 \Leftrightarrow u \in D_B \text{ and } Bu \in \overline{D_B}.$$

PROOF. If  $v \in D_B$  we get from (3.3)  $\|nR(n, B)v - v\|_{X_p} = \|R(n, B)Bv\|_{X_p} \leq n^{-1} \|Bv\|_{X_p}$  hence

$$(3.7) \quad \lim_{n \rightarrow \infty} \|nR(n, B)v - v\|_{X_p} = 0.$$

As  $\|nR(n, B)\|_{\mathcal{L}(E)} \leq 1$  we conclude that (3.7) is true also for  $v \in \overline{D_B}$ .

If  $u \in D_B$  and  $Bu \in \overline{D_B}$  we can set  $v = Bu$  in (3.7) and obtain

$$(3.8) \quad \lim_{n \rightarrow \infty} \|B_n u - Bu\|_{X_p} = 0.$$

Conversely, if this is true for some  $u \in D_B$ , then  $Bu \in \overline{D_B}$  because in this case  $B_n u \in D_B$ .

As  $\overline{D_B} \neq X_p$  if and only if  $p = \infty$  we deduce the following

COROLLARY 3.2. *We have*

$$(3.9) \quad \lim_{n \rightarrow \infty} \|B_n u - Bu\|_{X_p} = 0, \quad \forall u \in D_B$$

if and only if  $p < \infty$ .

REMARK 3.3. It can be checked that proposition 3.1 is true if  $B : D_B \subseteq E \rightarrow E$  is any linear operator in a Banach space  $E$  such that

$$\sup_{n \in \mathbb{N}} \|nR(n, B)\| < \infty.$$

In the following sections we want to prove first the existence of an F-solution in  $L^p$ : according to its definition this requires the proof of the existence of a strict solution of a suitable approximating problem: this will be done in the next section; in the subsequent one we will see that for appropriate  $f$  and  $u_0$  the solutions of this approximating problem satisfy condition (2.3) and in this way we obtain the F-solutions in  $L^p$ .

#### 4. - Approximating problem

We will consider now (2.1) as a functional equation of type

$$-Bu = Au + f$$

with  $f \in X_p$ . If we want to replace this problem with another one in which  $B$  is substituted by its Yosida approximation in  $L^p(0, T; E)$ , we must also take into account the initial condition  $u(0) = 0$  contained in the definition of  $D_B$ ; for this reason  $Bu$  will be replaced by  $B_n(u - u_0)$ . We shall prove that the approximating problem obtained in this way has a solution for each  $f \in X_p$  and  $u_0 \in E$  and satisfies an estimate which is very important in the proof of the convergence of all the methods employed in the sequel.

THEOREM 4.1. *Given  $f \in L^p(0, T; E)$  and  $u_0 \in E$  there exists for each  $n \in \mathbb{N}$  a unique  $v_n \in L^p(0, T; D_A)$  verifying*

$$(4.1) \quad B_n(v_n - u_0) + Av_n + f = 0$$

and the following estimates hold

$$(4.2) \quad \|v_n(t)\| \leq M(\|u_0\| + \frac{\|f(t)\|}{n}) + \int_0^t \|f(s)\| ds, \quad t \in [0, T] \text{ a.e.}$$



$$(4.3) \quad \|v_n\|_{L^p(0,T;E)} \leq M(1+T)(\|u_0\| + \|f\|_{L^p(0,T;E)})$$

If in addition  $f \in C(0, T; E)$  then  $v_n \in C(0, T; D_A)$ ; so (4.2) holds for each  $t \in [0, T]$  and we get

$$(4.4) \quad \|v_n\|_{C(0,T;E)} \leq M(1+T)(\|u_0\| + \|f\|_{C(0,T;E)})$$

Finally if  $f \in W^{1,p}(0, T; E)$  then  $v_n \in W^{1,p}(0, T; D_A)$ .

PROOF. As  $B_n = n^2 R(n, B) - n$  and (3.2) holds we can write equation (4.1) in an equivalent way as follows:

$$(4.5) \quad n^2 \int_0^t e^{-n(t-s)} v_n(s) ds + n e^{-nt} u_0 - (n - A)v_n(t) + f(t) = 0, \\ t \in [0, T] \text{ a.e.}$$

If there exists a solution  $v_n \in L^p(0, T; D_A)$  of this equation, by applying  $R(n, A)$  to both sides we deduce

$$(4.6) \quad v_n(t) = n^2 e^{-nt} \int_0^t e^{ns} R(n, A) v_n(s) ds + n R(n, A) e^{-nt} u_0 + R(n, A) f(t), \\ t \in [0, T] \text{ a.e.}$$

Setting

$$(4.7) \quad w_n(t) = \int_0^t e^{ns} R(n, A) v_n(s) ds, \quad t \in [0, T]$$

we have  $w_n \in W^{1,p}(0, T; D_A)$  and

$$(4.8) \quad w_n'(t) = e^{nt} R(n, A) v_n(t) \quad t \in [0, T] \text{ a.e.}$$

If we substitute  $v_n(t)$  with the right-hand side of (4.6) by using (4.7) we get a differential equation satisfied by  $w_n$ :

$$(4.9) \quad w_n'(t) = n^2 R(n, A) w_n(t) + n R^2(n, A) u_0 + e^{nt} R^2(n, A) f(t), \\ t \in [0, T] \text{ a.e.}$$

As  $w_n(0) = 0$  and  $n^2 R(n, A) \in \mathcal{L}(E)$  we deduce that

$$(4.10) \quad w_n(t) = \int_0^t e^{n^2 R(n, A)(t-s)} R^2(n, A) [n u_0 + e^{ns} f(s)] ds, \quad t \in [0, T]$$

By virtue of (1.1) we get the following estimate for  $0 \leq s \leq t$

$$(4.11) \quad \|e^{n^2 R(n,A)(t-s)} R^2(n, A)\| = \left\| \sum_{k=0}^{\infty} \frac{n^{2k} R^{k+2}(n, A)(t-s)^k}{k!} \right\| \leq \\ \leq M \sum_{k=0}^{\infty} \frac{n^{2k}(t-s)^k}{n^{k+2} k!} = \frac{M e^{n(t-s)}}{n^2}.$$

Now (4.6) can be written (by using (4.7)) as

$$(4.12) \quad v_n(t) = n^2 e^{-nt} w_n(t) + n R(n, A) e^{-nt} u_0 + R(n, A) f(t)$$

and therefore from the uniqueness of a solution  $w_n$  in  $W^{1,p}(0, T; D_A)$  of (4.9) we deduce the uniqueness of a solution  $v_n$  in  $L^p(0, T; D_A)$  of (4.5). By using (4.12), (4.10), (4.11) and (1.1) we get for  $t \in [0, T]$  a.e.

$$\|v_n(t)\| \leq n^2 e^{-nt} \int_0^t \frac{M e^{n(t-s)}}{n^2} (n \|u_0\| + e^{ns} \|f(s)\|) ds + M e^{-nt} \|u_0\| \\ + \frac{M}{n} \|f(t)\| = M n \|u_0\| \int_0^t e^{-ns} ds + M \int_0^t \|f(s)\| ds + M e^{-nt} \|u_0\| \\ + \frac{M}{n} \|f(t)\|$$

so (4.2) is proved; from (4.2) and the Schwarz-Hölder inequality we deduce

$$(4.13) \quad \|v_n\|_{L^p(0, T; E)} \leq M \left\{ T^{\frac{1}{p}} \|u_0\| + \left(T^{\frac{1}{p}} + \frac{1}{n}\right) \|f\|_{L^p(0, T; E)} \right\}$$

hence (4.3).

Let us prove now that there exists a solution  $v_n$  of (4.5) for each  $f \in L^p(0, T; E)$  and  $u_0 \in E$ . If we define  $w_n$  by means of (4.10) we have  $w_n \in W^{1,p}(0, T; D_A)$  and equation (4.9) is verified: hence setting

$$(4.14) \quad v_n(t) = (n - A) e^{-nt} w_n'(t)$$

we deduce (4.7) so (4.9) implies (4.6) by using (4.14) and (4.7): by applying  $n - A$  to both sides of (4.6) we get (4.5) which (as a consequence) has a unique solution in  $L^p(0, T; D_A)$ . From (4.6) we see that if  $f \in W^{1,p}(0, T; E)$  then  $v_n \in W^{1,p}(0, T; D_A)$  and if  $f \in C(0, T; E)$  then  $v_n \in C(0, T; D_A)$ , in this latter case (4.2) is true for each  $t \in [0, T]$  and (4.4) follows.

Let us end this section with a property of the solutions  $v_n$  of (4.1): they approach (in  $L^p(0, T; E)$ ) each possible F-solution in  $L^p$  of (2.1) which therefore

is unique. This uniqueness result will be proved also later (see theorem 5.1) in an independent way.

**THEOREM 4.2.** *Given  $f \in L^p(0, T; E)$  and  $u_0 \in E$  let  $v_n$  be the solution of the approximating problem (4.1). If  $u$  is an F-solution in  $L^p$  of (2.1) we have*

$$(4.15) \quad \lim_{n \rightarrow \infty} \|u - v_n\|_{L^p(0, T; E)} = 0.$$

**PROOF.** Let  $u$  be an F-solution in  $L^p$  of (2.1) and let  $u_k$  verify (2.2)-(2.3). By using (2.2) and (4.1) we have for  $k, n \in \mathbb{N}$

$$B_n(u_k - v_n - u_{0k} + u_0) + A(u_k - v_n) + B_n(u_{0k} - u_k) - f + f_k - u'_k = 0$$

and therefore from (4.3)

$$\begin{aligned} \|u_k - v_n\|_{L^p(0, T; E)} &\leq M(1+T)(\|B_n(u_{0k} - u_k) - f + f_k - u'_k\|_{L^p(0, T; E)} \\ &\quad + \|u_0 - u_{0k}\|) \leq M(1+T)(\|B_n(u_{0k} - u_k) - u'_k\|_{L^p(0, T; E)} \\ &\quad + \|f - f_k\|_{L^p(0, T; E)} + \|u_0 - u_{0k}\|). \end{aligned}$$

Given  $\varepsilon > 0$ , let  $\bar{k} \in \mathbb{N}$  verify (see (2.3))

$$\|u - u_{\bar{k}}\|_{L^p(0, T; E)}, \|f - f_{\bar{k}}\|_{L^p(0, T; E)}, \|u_0 - u_{0\bar{k}}\| < \varepsilon$$

so that for each  $n \in \mathbb{N}$  we have

$$\begin{aligned} \|u - v_n\|_{L^p(0, T; E)} &\leq \|u - u_{\bar{k}}\|_{L^p(0, T; E)} + \|u_{\bar{k}} - v_n\|_{L^p(0, T; E)} \\ &\leq \varepsilon + M(1+T)[\|B_n(u_{0\bar{k}} - u_{\bar{k}}) - u'_{\bar{k}}\|_{L^p(0, T; E)} + 2\varepsilon]. \end{aligned}$$

As  $u_{0\bar{k}} - u_{\bar{k}} \in D_B$  we deduce from (3.9) that  $\lim_{n \rightarrow \infty} B_n(u_{0\bar{k}} - u_{\bar{k}}) = u'_{\bar{k}}$  in  $L^p(0, T; E)$ : from this and the last estimate we get (4.14).

## 5. - An a priori estimate

In this section we will show that an F-solution in  $L^p$  is a continuous function with values in  $\overline{D}_A$  and verifies the initial condition in the usual sense; then we will prove an a priori estimate which will be useful to get the existence results of section 7.

**THEOREM 5.1.** *If  $u$  is an F-solution in  $L^p$  of (2.1) then  $u \in C(0, T; E)$ ,  $u(t) \in \overline{D}_A$  for each  $t \in [0, T]$ ,  $u(0) = u_0$  and*

$$(5.1) \quad \|u(t)\| \leq M(\|u(0)\|) + \int_0^t \|f(s)\| ds, \quad t \in [0, T];$$

so the  $F$ -solution in  $L^p$  is unique. In addition, if  $u_k$  verify (2.2)-(2.3) then

$$(5.2) \quad \lim_{k \rightarrow \infty} \|u_k - u\|_{C(0,T;E)} = 0.$$

PROOF. If  $u$  is also a strict solution in  $L^p$  of (2.1), then  $u$  is a function in  $W^{1,p}(0,T;E) \cap L^p(0,T;D_A)$ , and for each  $n \in \mathbb{N}$

$$(5.3) \quad B_n(u - u_0) + Au = u' + B_n(u - u_0) - f,$$

so, from (4.2):

$$(5.4) \quad \|u(t)\| \leq M(\|u_0\| + \frac{1}{n}\|u'(t) + B_n(u - u_0)(t) - f(t)\| + \int_0^t \|u'(s) + B_n(u - u_0)(s) - f(s)\| ds), \quad t \in [0, T] \text{ a.e.}$$

As  $u - u_0 \in D_B$ , from (3.9) we have in  $L^p(0, T; E)$

$$\lim_{n \rightarrow \infty} B_n(u - u_0) = B(u - u_0) = -u'$$

in  $L^p(0, T; E)$ , hence there is a subsequence  $\{n_k\}$  such that

$$\lim_{k \rightarrow \infty} B_{n_k}(u - u_0)(t) = -u'(t), \quad t \in [0, T] \text{ a.e.}$$

and therefore, from (5.4) we get (5.1) for  $t \in [0, T]$  a.e.; but as both sides of (5.1) are continuous functions, we deduce that (5.1) is true for all  $t \in [0, T]$ . Suppose now that  $u_k$  verifies (2.2)-(2.3); in particular  $u_k$  is in  $W^{1,p}(0, T; E) \cap L^p(0, T; D_A)$  and it is an  $F$ -solution in  $L^p$  of (2.2); from the estimate just proved we get for  $h, k \in \mathbb{N}$

$$\|u_h(t) - u_k(t)\| \leq M(\|u_h(0) - u_k(0)\| + \int_0^t \|f_h(s) - f_k(s)\| ds), \quad t \in [0, T];$$

thus from (2.3) we deduce that  $\{u_k\}$  converges also in  $C(0, T; E)$  to  $u$ : in particular  $u_k(0) \rightarrow u(0)$  hence  $u(0) = u_0$ . Moreover as  $u_k(t) \in \overline{D_A}$  for  $t \in [0, T]$  we have  $u(t) \in \overline{D_A}$  for  $t \in [0, T]$ .

From the first part of the proof we get for  $k \in \mathbb{N}$  and  $t \in [0, T]$ :

$$\|u_k(t)\| \leq M(\|u_k(0)\| + \int_0^t \|f_k(s)\| ds),$$

so for  $k \rightarrow \infty$  we have (5.1).

## 6. - Integral solutions

In this section we will give another definition of the solution of (2.1), which is suggested by the formal integration of both sides of (2.1) and is very useful to prove some regularity results. Given  $f \in L^1(0, T; E)$  and  $u_0 \in E$  we say that  $u : [0, T] \rightarrow E$  is an *integral solution* of (2.1) if

$$(6.1) \quad u \in C(0, T; E)$$

$$(6.2) \quad \int_0^t u(s) ds \in D_A \text{ for } t \in [0, T]$$

$$(6.3) \quad u(t) = u_0 + A \int_0^t u(s) ds + \int_0^t f(s) ds, \quad t \in [0, T].$$

In other words  $u \in C(0, T; E)$  is an integral solution of (2.1) if and only if  $v(t) = \int_0^t u(s) ds$ ,  $t \in [0, T]$  is a strict solution in  $C$  of

$$(6.4) \quad \begin{cases} v'(t) = Av(t) + u_0 + \int_0^t f(s) ds & t \in [0, T] \\ v(0) = 0. \end{cases}$$

Let us remark that an integral solution has values in  $\overline{D}_A$  because from (6.1) and (6.2) we get  $u(t) = \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} u(s) ds \in \overline{D}_A$ . From this we deduce that if an integral solution of (2.1) exists then necessarily  $u_0 \in \overline{D}_A$  because from (6.3) we obtain  $u(0) = u_0$ .

**THEOREM 6.1.** *The integral solution is unique.*

**PROOF.** Let  $u$  verify (6.1) (6.3) with  $u_0 = 0$  and  $f \equiv 0$ ; then  $v(t) = \int_0^t u(s) ds$  is a strict solution in  $C$  of (6.4) with  $u_0 = f = 0$ ; from (5.1) we deduce that  $v \equiv 0$  and also  $u \equiv 0$ .

**THEOREM 6.2.** *If  $u$  is an  $F$ -solution in  $L^p$  of (2.1) then it is also an integral solution (the converse will be proved in Corollary 7.3).*

PROOF. Let  $u_k$  verify (2.2)–(2.3). For  $t \in [0, T]$  we get

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^t u_k(s) ds &= \int_0^t u(s) ds \\ \int_0^t u_k(s) ds &\in D_A, \quad k \in \mathbb{N} \\ A \int_0^t u_k(s) ds &= \int_0^t A u_k(s) ds = \int_0^t u'_k(s) ds - \int_0^t f_k(s) ds = \\ &= u_k(t) - u_k(0) - \int_0^t f_k(s) ds, \end{aligned}$$

hence from (5.2) we deduce the existence of

$$\lim_{k \rightarrow \infty} A \int_0^t u_k(s) ds = u(t) - u(0) - \int_0^t f(s) ds, \quad t \in [0, T].$$

As  $A$  is closed we obtain (6.2)–(6.3); (6.1) is a consequence of Theorem 5.1.

Now we prove that an integral solution is a strict solution if it is sufficiently regular. This result will be used later (see lemma 7.1).

**THEOREM 6.3.** *Let  $f \in L^p(0, T; E)$  and  $u_0 \in E$ . If  $u$  is an integral solution of (2.1) belonging to  $W^{1,p}(0, T; E)$  or to  $L^p(0, T; D_A)$  then  $u$  is a strict solution in  $L^p$  of (2.1).*

*Let  $f \in C(0, T; E)$  and  $u_0 \in E$ . If  $u$  is an integral solution of (2.4) belonging to  $C^1(0, T; E)$  or to  $C(0, T; D_A)$  then  $u$  is a strict solution in  $C$  of (2.4).*

PROOF. Let  $u$  be an integral solution of (2.1) and  $u \in W^{1,p}(0, T; E)$ : for  $t, t+h \in [0, T]$  with  $h \neq 0$  we have

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} u(s) ds &= u(t) \\ \frac{1}{h} \int_t^{t+h} u(s) ds &\in D_A \end{aligned}$$

and from (6.3) and the fact that  $u \in W^{1,p}(0, T; E)$  we deduce also the existence

of

$$\lim_{h \rightarrow 0} A \frac{1}{h} \int_t^{t+h} u(s) ds = u'(t) - f(t), \quad t \in [0, T] \text{ a.e.}$$

As  $A$  is closed we obtain  $u(t) \in D_A$  and  $Au(t) = u'(t) - f(t), t \in [0, T]$  a.e., so  $u$  is a strict solution in  $L^p$  of (2.1).

If  $u \in L^p(0, T; D_A)$  we get from (6.3)

$$(6.4) \quad u(t) = u_0 + \int_0^t Au(s) ds + \int_0^t f(s) ds, \quad t \in [0, T];$$

this implies (2.1) and again  $u$  is a strict solution in  $L^p$ . The second part of the theorem is a consequence of the first part.

## 7. - Existence of F-solutions in $L^p$

In this section we shall prove that the solutions of the approximating problem of section 4 can be used to obtain an F-solution (which is even strict) of (2.4) when  $u_0 = 0$  and  $f$  is very regular and vanishes at  $t = 0$  with its derivatives: this result will be sufficient to obtain an F-solution in  $L^p$  in the general case. Also now the main tool of the proofs is the estimate of theorem 4.1.

LEMMA 7.1. *If  $f \in C^3(0, T; E)$ ,  $f(0) = f'(0) = f''(0) = 0$  and  $u_0 = 0$  then problem (2.4) has a strict solution in  $C$ .*

PROOF. From theorem 4.1 we deduce the existence, for each  $n \in \mathbb{N}$ , of a unique  $v_n \in C(0, T; D_A)$  verifying

$$(7.1) \quad B_n v_n + A v_n + f = 0$$

hence (see (4.6))

$$(7.2) \quad v_n(t) = n^2 R(n, A) \int_0^t e^{-ns} v_n(t-s) ds + R(n, A) f(t), \quad t \in [0, T]$$

As  $f \in C^2(0, T; E)$ ,  $f(0) = f'(0) = 0$  we have  $v_n \in C^2(0, T; D_A)$ ,  $v_n(0) = v_n'(0) = 0$  and

$$(7.3) \quad v_n''(t) = n^2 R(n, A) \int_0^t e^{-ns} v_n''(t-s) ds + R(n, A) f''(t), \quad t \in [0, T].$$

By applying  $n - A$  to both sides we obtain (using (3.5)),

$$(7.4) \quad B_n v_n'' + A v_n'' + f'' = 0.$$

Hence we can use (4.4) and, as  $v_n \in D_{B^2}$ , we get

$$(7.5) \quad \|B^2 v_n\|_{C(0,T;E)} = \|v_n''\|_{C(0,T;E)} \leq M(1+T)\|f''\|_{C(0,T;E)}.$$

Now from (7.1) we have for  $n, m \in \mathbb{N}$

$$B_n(v_n - v_m) + (B_n - B_m)v_m + A(v_n - v_m) = 0$$

and from this, by virtue of (4.4),

$$\|v_n - v_m\|_{C(0,T;E)} \leq M(1+T)\|(B_n - B_m)v_m\|_{C(0,T;E)},$$

but from (1.1)

$$\begin{aligned} \|(B_n - B_m)v_m\|_{C(0,T;E)} &= \|(m-n)R(n, B)R(m, B)B^2 v_m\|_{C(0,T;E)} \leq \\ &\leq M^2 \left| \frac{1}{n} - \frac{1}{m} \right| \|f''\|_{C(0,T;E)}; \end{aligned}$$

hence there exists  $u \in C(0, T; E)$  such that

$$(7.6) \quad \lim_{n \rightarrow \infty} \|v_n - u\|_{C(0,T;E)} = 0.$$

Let us first prove that  $u$  is an F-solution in  $C$  of (2.4) with  $u_0 = 0$  by showing that (2.5)-(2.6) are true when  $u_k = v_k$  and  $u_0 = 0$ . In fact by using (7.1) and (7.5) we have

$$\begin{aligned} \|v_n' - A v_n - f\|_{C(0,T;E)} &= \|-B v_n + B_n v_n\|_{C(0,T;E)} = \|R(n, B)B^2 v_n\|_{C(0,T;E)} \\ &\leq \frac{M(1+T)}{n} \|f''\|_{C(0,T;E)} \end{aligned}$$

and therefore

$$(7.7) \quad \lim_{n \rightarrow \infty} \|v_n' - A v_n - f\|_{C(0,T;E)} = 0.$$

As  $v_n \in C^1(0, T; E) \cap C(0, T; D_A)$  and  $v_n(0) = 0$  from (7.6) and (7.7) we deduce that  $u$  is an F-solution in  $C$  of (2.4) when  $u_0 = 0$ .

As  $f' \in C^2(0, T; E)$  and  $f'(0) = f''(0) = 0$  we can proceed as above to prove for each  $n \in \mathbb{N}$  the existence of  $w_n \in C(0, T; D_A)$  verifying

$$(7.8) \quad B_n w_n + A w_n + f' = 0$$



and also the existence of  $w \in C(0, T; E)$  such that

$$(7.9) \quad \lim_{n \rightarrow \infty} \|w_n - w\|_{C(0, T; E)} = 0.$$

Now from (7.2) we get (as  $v_n(0) = 0$ )

$$v'_n(t) = n^2 R(n, A) \int_0^t e^{-ns} v'_n(t-s) ds + R(n, A) f'(t)$$

hence

$$B_n v'_n + A v'_n + f' = 0.$$

From the uniqueness of the solution of (7.8) (see theorem 4.1) we have

$$(7.10) \quad w_n = v'_n.$$

Now (7.6), (7.9) and (7.10) imply that  $u \in C^1(0, T; E)$  and the conclusion follows from theorem 6.2 and 6.3.

We are now in the position to prove the existence of an F-solution in  $L^p$  of (2.1) as we can construct  $\{f_k\}$  approximating  $f$  in  $L^p$  and such that (2.2) can be solved with the aid of the preceding lemma: the convergence of  $u_k$  to  $u$  will be a consequence of the a priori estimate proved in Section 5

**THEOREM 7.2.** *Problem (2.1) has a unique F-solution in  $L^p$  for each  $f \in L^p(0, T; E)$  and  $u_0 \in \overline{D_A}$ .*

**PROOF.** Let  $u_{0k} \in D_A$  be such that  $\lim_{k \rightarrow \infty} \|u_{0k} - u_0\| = 0$ . There exists  $f_k \in C^3(0, T; E)$  verifying  $f_k(0) = -A u_{0k}$ ,  $f'_k(0) = 0$ ,  $f''_k(0) = 0$  and

$$(7.11) \quad \lim_{k \rightarrow \infty} \|f_k - f\|_{L^p(0, T; E)} = 0.$$

From the previous lemma we deduce the existence of  $v_k \in C^1(0, T; E) \cap C(0, T; D_A)$  such that

$$\begin{cases} v'_k(t) = A v_k(t) + f_k + A u_{0k}, & t \in [0, T] \\ v_k(0) = 0. \end{cases}$$

Setting  $u_k(t) = v_k(t) + u_{0k}$  we have  $u_k \in C^1(0, T; E) \cap C(0, T; D_A)$  and

$$(7.12) \quad \begin{cases} u'_k(t) = A u_k(t) + f_k(t), & t \in [0, T] \\ u_k(0) = u_{0k} \end{cases}$$

hence, from estimate (5.1), we have for  $h, k \in \mathbb{N}$  and  $t \in [0, T]$

$$\|u_k(t) - u_h(t)\| \leq M(\|u_{0k} - u_{0h}\| + \int_0^t \|f_k(s) - f_h(s)\| ds)$$

and therefore

$$\|u_k - u_h\|_{C(0,T;E)} \leq M(\|u_{0k} - u_{0h}\| + T^{\frac{p-1}{p}} \|f_k - f_h\|_{L^p(0,T;E)}).$$

This and (7.11) imply the existence of  $u \in C(0, T; E)$  such that

$$(7.13) \quad \lim_{n \rightarrow \infty} \|u_k - u\|_{C(0,T;E)} = 0$$

and  $u$  is an F-solution in  $L^p$  of (2.1) by virtue of (7.11), (7.12), (7.13) and  $\lim_{k \rightarrow \infty} \|u_{0k} - u_0\| = 0$ .

**COROLLARY 7.3.** *Given  $f \in L^p(0, T; E)$  and  $u_0 \in \overline{D}_A$ , there exists a unique integral solution of (2.1) which coincides with the F-solution in  $L^p$  of (2.1).*

**PROOF.** Given  $f \in L^p(0, T; E)$  and  $u_0 \in \overline{D}_A$ , problem (2.1) has an F-solution in  $L^p$  given by the preceding theorem; this is also an integral solution of (2.1) by virtue of Theorem 6.2 and it is unique (see Theorem 6.1.).

### 8. - Existence of strict solutions

We prove now our main result: the existence of a strict solution in  $C$  of problem (2.1) for each  $u_0 \in D_A$  and  $f$  sufficiently regular, provided a necessary compatibility condition is verified (see remarks 8.2 and 8.4). This result is obtained as a consequence of the existence of the F-solutions in  $L^p$  and the properties of the integral solutions proved in theorem 6.3. Our first result is of temporal regularity i.e.  $f$  is assumed to belong to  $W^{1,p}(0, T; E)$  for some  $p \geq 1$ .

**THEOREM 8.1.** *Let  $f \in W^{1,p}(0, T; E)$ ,  $u_0 \in D_A$  and*

$$(8.1) \quad Au_0 + f(0) \in \overline{D}_A.$$

*Then there exists a unique  $u \in C^1(0, T; E) \cap C(0, T; D_A)$  verifying*

$$(8.2) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0. \end{cases}$$

*Moreover  $v = u'$  is an F-solution in  $L^p$  of the problem*

$$(8.3) \quad \begin{cases} v'(t) = Av(t) + f'(t), & t \in [0, T] \text{ a.e.} \\ v(0) = Au_0 + f(0). \end{cases}$$

Hence we have, for each  $t \in [0, T]$ ,  $u'(t) \in \overline{D}_A$  and

$$(8.4) \quad \|u(t)\| \leq M(\|u_0\| + \int_0^t \|f(s)\| ds)$$

$$(8.5) \quad \|u'(t)\| \leq M(\|Au_0 + f(0)\| + \int_0^t \|f'(s)\| ds).$$

PROOF. By virtue of Theorem 7.2 there exists an F-solution in  $L^p$  of problem

$$(8.6) \quad \begin{cases} v'(t) = Av(t) + f'(t), & t \in [0, T] \text{ a.e.} \\ v(0) = Au_0 + f(0). \end{cases}$$

As  $v$  is also an integral solution (see Theorem 6.2.) we have for  $t \in [0, T]$

$$(8.7) \quad \begin{aligned} v(t) &= Au_0 + f(0) + A \int_0^t v(s) ds + \int_0^t f'(s) ds = \\ &= A(u_0 + \int_0^t v(s) ds) + f(t). \end{aligned}$$

Setting  $u(t) = u_0 + \int_0^t v(s) ds$ ,  $t \in [0, T]$  we deduce  $u \in C^1(0, T; E) \cap C(0, T; D_A)$  and  $u' = v$ : hence (8.7) shows that  $u$  is a strict solution of (8.2) and  $u'$  is an F-solution in  $L^p$  of (8.6) so that (8.4)-(8.5) are a consequence of (5.1).

REMARK 8.2. Condition (8.1) is a necessary compatibility condition between  $f$  and  $u_0$ : in other words if there exists a strict solution in  $C$  of (8.2) then  $Au_0 + f(0) \in \overline{D}_A$  because  $Au_0 + f(0) = u'(0) = \lim_{t \rightarrow 0^+} t^{-1}(u(t) - u(0)) \in \overline{D}_A$ .

Our next result is of spatial regularity i.e.  $f(t)$  is supposed to belong to  $D_A$  for  $t \in [0, T]$  a.e. and  $\|Af(\cdot)\|$  to be integrable in  $[0, T]$ .

THEOREM 8.3. Let  $f \in L^p(0, T; D_A)$ ,  $u_0 \in D_A$  and  $Au_0 \in \overline{D}_A$ . Then there exists a unique  $u \in W^{1,p}(0, T; E) \cap C(0, T; D_A)$  such that

$$(8.8) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \text{ a.e.} \\ u(0) = u_0. \end{cases}$$

Moreover  $v = Au$  is an F-solution in  $L^p$  of the problem

$$(8.9) \quad \begin{cases} v'(t) = Av(t) + Af(t), & t \in [0, T] \text{ a.e.} \\ v(0) = Au_0. \end{cases}$$

Hence we have for each  $t \in [0, T]$ ,  $Au(t) \in \overline{D}_A$  and

$$(8.10) \quad \|u(t)\| \leq M(\|u_0\| + \int_0^t \|f(s)\| ds)$$

$$(8.11) \quad \|Au(t)\| \leq M(\|Au_0\| + \int_0^t \|Af(s)\| ds).$$

If in addition  $f \in L^p(0, T; D_A) \cap C(0, T; E)$  then we have also  $u \in C^1(0, T; E) \cap C(0, T; D_A)$  and (8.8)<sub>1</sub> holds for every  $t \in [0, T]$ .

PROOF. From Theorem 7.2 we deduce the existence of an F-solution in  $L^p$  of

$$(8.12) \quad \begin{cases} w'(t) = Aw(t) + (A - 1)f(t), & t \in [0, T] \text{ a.e.} \\ w(0) = (A - 1)u_0. \end{cases}$$

As  $w$  is also an integral solution (see Theorem 6.2) we have

$$w(t) = (A - 1)u_0 + A \int_0^t w(s) ds + \int_0^t (A - 1)f(s) ds, \quad t \in [0, T]$$

hence

$$(8.13) \quad (A - 1)^{-1}w(t) = u_0 + \int_0^t w(s) ds + (A - 1)^{-1} \int_0^t w(s) ds + \int_0^t f(s) ds, \\ t \in [0, T].$$

Setting  $u(t) = (A - 1)^{-1}w(t)$  we deduce that  $u \in W^{1,p}(0, T; E) \cap C(0, T; D_A)$ ,  $u(0) = u_0$  and, for  $t \in [0, T]$ ,

$$(8.14) \quad u(t) = u_0 + \int_0^t w(s) ds + \int_0^t u(s) ds + \int_0^t f(s) ds$$

hence for  $t \in [0, T]$  a.e.

$$(8.15) \quad u'(t) = w(t) + u(t) + f(t) = Au(t) + f(t)$$

i.e.  $u$  verifies (8.9). If, in addition,  $f \in C(0, T; E)$ , (8.14) implies also  $u \in C^1(0, T; E)$  and (8.15) holds for each  $t \in [0, T]$ .

Finally we have that  $v = w + u$  is an F-solution in  $L^p$  of (8.9) and (8.10), (8.11) are consequence of (5.1).

REMARK 8.4. Concerning the condition  $Au_0 \in \overline{D}_A$ , we must observe that this is necessary in order to have such a solution under the assumptions of the theorem. In fact we have  $Au(t) \in \overline{D}_A$  for each  $t \in [0, T]$ .

REMARK 8.5. When  $A$  verifies condition (1.1) but  $\overline{D}_A \neq E$ , it is interesting to examine what can be deduced from the application of the classical theorems of Hille-Yosida and Phillips to the part of  $A$  in  $\overline{D}_A$ . To this purpose let us define the Banach space  $E_0$  and the linear operator  $A_0 : D_{A_0} \subseteq E_0 \rightarrow E_0$  as follows:

$$(8.16) \quad \begin{cases} E_0 = \overline{D}_A \text{ with the norm of } E \\ D_{A_0} = \{u \in D_A; Au \in \overline{D}_A\} \\ A_0 u = Au \end{cases}$$

Now  $A_0 : D_{A_0} \subseteq E_0 \rightarrow E_0$  verifies all the Hille-Yosida conditions (the density of the domain included), so we can apply the usual theory to the study of:

$$(8.17) \quad \begin{cases} u'(t) = A_0 u(t) + f(t), & t \in [0, T] \\ u(0) = u_0. \end{cases}$$

But this problem can replace problem (1.2) only if  $u_0, f(t) \in \overline{D}_A$ ,  $t \in [0, T]$ . This was the case considered in theorem 8.3, but even in this case, if we want to apply only the classical results for (8.17) (see e.g. [10] theorem 2.9 pag. 109), we need to impose on  $f$  the condition  $f(t) \in D_{A_0}$ ,  $t \in [0, T]$  a.e. and not only  $f(t) \in D_A$ , as in theorem 8.3. Also in the particular case  $\overline{D}_A = E$  some of our results (e.g. Theorem 8.1) give a generalization of the classical theory (see [10] pag. 107).

## 9. - Existence of F-solutions in $C$

We can prove now the existence of F-solutions in  $C$  of problem (2.4) with the aid of the strict solutions in  $L^p$  of problem (2.1) obtained in the previous section.

**THEOREM 9.1.** *There exists a unique F-solution in  $C$  of (2.4) for each  $u_0 \in \overline{D}_A$  and  $f \in C(0, T; E)$ .*

**PROOF.** As we proved in Section 3, we have  $\lim_{n \rightarrow \infty} nR(n, A)x = x$  for each  $x \in \overline{D}_A$ , so if we set

$$u_{0n} = nR(n, A)[u_0 - R(1, A)f(0)] + R(1, A)f(0)$$

we have

$$(9.1) \quad \lim_{n \rightarrow \infty} \|u_{0n} - u_0\| = 0.$$

Moreover  $u_{0n} \in D_A$  and  $Au_{0n} = -n[u_0 - R(1, A)f(0)] + n^2R(n, A)[u_0 - R(1, A)f(0)] - f(0) + (1 - A)^{-1}f(0)$  and therefore we deduce

$$(9.2) \quad Au_{0n} + f(0) \in \overline{D_A}.$$

Let  $f_n \in W^{1,p}(0, T; E)$  be such that  $f_n(0) = f(0)$  and

$$(9.3) \quad \lim_{n \rightarrow \infty} \|f_n - f\|_{C(0, T; E)} = 0.$$

By virtue of theorem 8.1 there exists  $u_n \in C^1(0, T; E) \cap C(0, T; D_A)$  solution of

$$\begin{cases} u'_n(t) = Au_n(t) + f_n(t), & t \in [0, T] \\ u_n(0) = u_{0n} \end{cases}$$

and from (8.4) we get for  $t \in [0, T]$  and  $n, m \in \mathbb{N}$

$$\|u_n(t) - u_m(t)\| \leq M(\|u_{0n} - u_{0m}\| + \int_0^t \|f_n(s) - f_m(s)\| ds).$$

Therefore  $\{u_n\}$  converges in  $C(0, T; E)$  to a function  $u$ : in particular  $u(0) = \lim_{n \rightarrow \infty} u_n(0) = \lim_{n \rightarrow \infty} u_{0n} = u_0$  by virtue of (9.1). In conclusion  $u$  is an  $F$ -solution in  $C$  of (2.4).

We will consider now the following problem

$$(9.4) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u(T) \end{cases}$$

and we will say that  $u : [0, T] \rightarrow E$  is an  $F$ -solution or a *strict solution* of (9.4) in  $L^p$  or in  $C$  if there exists  $u_0 \in E$  such that  $u$  is a solution of the same type of

$$(9.5) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0. \end{cases}$$

The results relative to problem (9.5) together with Remark 8.5 let us give an existence and uniqueness theorem for each kind of solutions of (9.4)

**THEOREM 9.2.** *Let  $A$  verify condition (1.1) and let  $\exp(A_0t)$  be the semigroup generated by  $A_0$  in  $E_0$  (see definition 8.16). Under the assumption*

$$(9.6) \quad (1 - \exp(A_0T))^{-1} \in \mathcal{L}(E_0)$$

- (i) if  $f \in L^p(0, T; E)$  there exists a unique F-solution in  $L^p$  of (9.4),
- (ii) if  $f \in C(0, T; E)$  there exists a unique F-solution in  $C$  of (9.4),
- (iii) if  $f \in L^p(0, T; D_A)$  there exists a unique  $u \in W^{1,p}(0, T; E) \cap C(0, T; D_A)$  solution of (9.4) a.e. in  $[0, T]$ ,
- (iv) if  $f \in L^p(0, T; D_A) \cap C(0, T; E)$  there exists a unique strict solution in  $C$  of (9.4),
- (v) if  $f \in W^{1,p}(0, T; E)$  there exists a unique strict solution in  $C$  of (9.4).

PROOF. To prove the uniqueness let us suppose that there exists  $u_0 \in \overline{D}_A$  such that  $u$  is an F-solution in  $L^p$  of

$$(9.7) \quad \begin{cases} u'(t) = Au(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

verifying

$$(9.8) \quad u(0) = u(T).$$

Then we must have

$$(9.9) \quad u(t) = \exp(A_0 t)u_0$$

because  $t \rightarrow \exp(A_0 t)u_0$  is an F-solution in  $C$  of (9.7): in fact as  $u_0 \in \overline{D}_A$  we can choose  $u_{0k} \in D_{A_0}$  such that  $\lim_{k \rightarrow \infty} \|u_{0k} - u_0\| = 0$ : setting  $u_k(t) = \exp(A_0 t)u_{0k}$  we have  $u_k \in C^1(0, T; E) \cap C(0, T; D_A)$  and  $\lim_{k \rightarrow \infty} \|u_k - u\|_{C(0, T; E)} = 0$ . (9.8) implies  $(1 - \exp(A_0 T))u_0 = 0$  so  $u_0 = 0$  by virtue of (9.6), therefore  $u = 0$  and the uniqueness for problem (9.4) is proved.

To prove existence let us begin with case (i): from theorem 7.2 we know that there exists an F-solution in  $L^p$  of

$$(9.10) \quad \begin{cases} v'(t) = Av(t) + f(t), & t \in [0, T] \\ v(0) = 0 \end{cases}$$

As  $v(T) \in \overline{D}_A$  (see Theorem 5.1) we can define

$$(9.11) \quad u_0 = (1 - \exp(A_0 T))^{-1}v(T)$$

and deduce as above that  $t \rightarrow \exp(A_0 t)u_0$  is an F-solution in  $L^p$  of (9.7). Hence

$$(9.12) \quad u(t) = \exp(A_0 t)u_0 + v(t), \quad t \in [0, T]$$

is an F-solution in  $L^p$  of (9.5) and verifies (9.8), i.e., by definition,  $u$  is an F-solution in  $L^p$  of (9.4). Case (ii) can be treated in the same way by using theorem 9.1 instead of theorem 7.2. In case (iii) problem (9.10) has a solution

$$v \in W^{1,p}(0, T; E) \cap C(0, T; D_A)$$

verifying (9.10)<sub>1</sub> a.e. in  $[0, T]$  and such that  $Av(T) \in \overline{D_A}$  (see theorem 8.3), hence  $v(T) \in D_{A_0}$ . This implies that  $u_0$ , defined by (9.11), is also in  $D_{A_0}$ . In fact as for  $x \in D_{A_0}$

$$(1 - \exp(A_0T))(1 - A_0)x = (1 - A_0)(1 - \exp(A_0T))x$$

we deduce

$$(1 - \exp(A_0T))^{-1}(1 - A_0)^{-1} = (1 - A_0)^{-1}(1 - \exp(A_0T))^{-1}$$

hence

$$\begin{aligned} u_0 &= (1 - \exp(A_0T))^{-1}v(T) = (1 - \exp(A_0T))^{-1}(1 - A_0)^{-1}(1 - A_0)v(T) \\ &= (1 - A_0)^{-1}(1 - \exp(A_0T))^{-1}(1 - A_0)v(T) \end{aligned}$$

is in  $D_{A_0}$ . This implies that  $t \rightarrow \exp(A_0t)u_0$  is a strict solution in  $C$  of (9.7): therefore  $u$  defined by (9.12) verifies the conditions of (iii). Case (iv) is proved in the same way and similarly is case (v) for which theorem 8.1 can be used.

We will consider in the next part of this paper the case in which  $A$  verifies a more restrictive spectral property (but  $D_A$  is again not necessarily dense in  $E$ ): with this property we can construct a semigroup of operators in  $E$  (in contrast with the previous situation) and this fact lets us write an explicit formula for the possible solutions.

## Parabolic case

### 10. - Analytic semigroups

In what follows we shall consider again problem

$$(10.1) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

where  $A : D_A \subseteq E \rightarrow E$  is a linear operator in a Banach space  $E$  with not (necessarily) dense domain  $D_A$  and verifies the condition:

$$(10.2) \quad \begin{cases} \text{there exist } \phi \in ]\frac{\pi}{2}, \pi[ \quad \text{and } \overline{M} > 0 \text{ such that if} \\ \lambda \in S_\phi = \{z \in \mathbb{C}; z \neq 0, |\arg z| \leq \phi\} \text{ then } \|R(\lambda, A)\|_{\mathcal{L}(E)} \leq \frac{\overline{M}}{|\lambda|}. \end{cases}$$

When  $A$  has this property, (10.1) is called a parabolic abstract evolution equation. We will show that condition (10.2) is stronger than (1.1) (see Theorem



10.4) and lets us define a semigroup  $e^{At} \in \mathcal{L}(E)$ ,  $t > 0$ , in the usual way (see [8] p. 487):

$$(10.3) \quad e^{At} = \frac{1}{2\pi i} \int_{+C} e^{\lambda t} R(\lambda, A) d\lambda, \quad t > 0$$

where  $+C$  is a suitable oriented path in the complex plane. We will say that  $e^{At}$  is the semigroup generated by  $A$ : later it will be shown that  $e^{At}$  cannot be generated (in this way) by another operator (see theorem 10.3). Many of the properties of the classical analytic semigroups still hold:

$$(10.4) \quad e^{At}x \in D_{A^k} \text{ for } t > 0, x \in E, k \in \mathbb{N},$$

$$(10.5) \quad \text{given } k \in \mathbb{N} \text{ there is } M_k \text{ (depending also on } \overline{M} \text{ and } \phi) \text{ such that } \|A^k e^{At}\|_{\mathcal{L}(E)} \leq \frac{M_k}{t^k} \text{ for } t > 0,$$

$$(10.6) \quad \text{for each } k \in \mathbb{N} \text{ and } t > 0 \text{ there exists } \frac{d^k e^{At}}{dt^k} = A^k e^{At} \text{ and } t \rightarrow e^{At} \text{ can be extended analytically in a sector containing the positive real semiaxis,}$$

$$(10.7) \quad Ae^{At}x = e^{At}Ax \text{ for } t > 0 \text{ and } x \in D_A,$$

$$(10.8) \quad R(\lambda, A)e^{At} = e^{At}R(\lambda, A) \text{ for } t > 0 \text{ and } \lambda \in S_\phi.$$

The main difference with the usual analytic semigroups is in the behavior of  $e^{At}x$  when  $t$  approaches 0:

**THEOREM 10.1.** *When  $A$  verifies (10.2), the following properties hold:*

$$(10.9) \quad \text{if } x \in \overline{D}_A \text{ the } \lim_{t \rightarrow 0} e^{At}x = x. \text{ Conversely if there exists } \lim_{t \rightarrow 0} e^{At}x = y \text{ then } y \in \overline{D}_A \text{ and } y = x$$

$$(10.10) \quad \text{for each } x \in E \text{ and } t > 0 \text{ we have } \int_0^t e^{As}x \, ds \in D_A \text{ and}$$

$$A \int_0^t e^{As}x \, ds = e^{At}x - x$$

$$(10.11) \quad \text{if } x \in D_A \text{ and } Ax \in \overline{D}_A \text{ then } \lim_{t \rightarrow 0} \frac{e^{At}x - x}{t} = Ax.$$

*Conversely if there exists*  $\lim_{t \rightarrow 0} \frac{e^{At}x - x}{t} = y$  *then*  $x \in D_A$ ,  $Ax \in \overline{D}_A$  *and*  $Ax = y$ .

**PROOF.** See Proposition 1.2 of [12].

We will extend to our situation a classical result which refers to the Laplace transform of  $e^{At}$ .

**THEOREM 10.2.** *For each  $x \in E$  and  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}\lambda > 0$  we have*

$$(10.12) \quad R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda t} e^{At} x \, dt.$$

**PROOF.** From (10.5) we get  $\|e^{-\lambda t} e^{At} x\| \leq e^{-\operatorname{Re}\lambda t} M_0 \|x\|$ , hence for  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}\lambda > 0$

$$R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} e^{At} x \, dt$$

is well defined.

By using (10.6) and (10.9) we get for each  $y \in D_A$ :

$$\begin{aligned} R(\lambda)Ay &= \int_0^{+\infty} e^{-\lambda t} e^{At} Ay \, dt = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{+\infty} e^{-\lambda t} \frac{d}{dt}(e^{At}y) dt = \lim_{\varepsilon \rightarrow 0} [-e^{-\lambda\varepsilon} e^{A\varepsilon}y] + \\ &+ \lambda \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{+\infty} e^{-\lambda t} e^{At}y \, dt = -y + \lambda R(\lambda)y. \end{aligned}$$

Setting  $y = R(\lambda, A)x$ , with  $x$  arbitrarily chosen in  $E$ , we have

$$R(\lambda)AR(\lambda, A)x = -R(\lambda, A)x + \lambda R(\lambda)R(\lambda, A)x$$

from which we deduce  $R(\lambda)x = R(\lambda, A)x$ .

As first consequence we can prove that there is a one-to-one correspondence between the semigroup  $e^{At}$  and its generator also in the case of non dense domain.

**THEOREM 10.3.** *Let  $A : D_A \subseteq E \rightarrow E$  and  $B : D_B \subseteq E \rightarrow E$  verify (10.2) and let  $e^{At}$  and  $e^{Bt}$  the semigroups generated by them through (10.3). If  $e^{At} = e^{Bt}$ ,  $t > 0$  then  $D_A = D_B$  and  $Ax = Bx$  for  $x \in D_A = D_B$ .*

**PROOF.** If  $e^{At} = e^{Bt}$  we get from (10.12) that  $R(\lambda, A)x = R(\lambda, B)x$ ,  $x \in E$  and  $\lambda > 0$  hence  $D_A = R(\lambda, A)E = R(\lambda, B)E$ . In addition, if  $x \in D_A$ , setting  $y = Ax$  we have  $x = R(\lambda, A)(\lambda x - y) = R(\lambda, B)(\lambda x - y)$  and therefore  $(\lambda - B)x = \lambda x - y$  i.e.  $Ax = Bx$ .

Another consequence of theorem 10.2 is the proof that condition (10.2) is more stringent than (10.1)

**THEOREM 10.4.** *If  $A$  verifies (10.2) then also (1.1) is true with  $M = M_0$ .*

**PROOF.** From (10.12) we deduce the existence for  $\operatorname{Re}\lambda > 0$  of  $(\lambda - A)^{-1} \in \mathcal{L}(E)$  and also for each  $k \in \mathbb{N}$

$$R^k(\lambda, A)x = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} R(\lambda, A)x = \frac{(-1)^{k-1}}{(k-1)!} \int_0^{+\infty} (-t)^{k-1} e^{-\lambda t} e^{At} x \, dt$$

hence from (10.5)

$$\begin{aligned} \|R^k(\lambda, A)x\| &\leq \frac{M_0\|x\|}{(k-1)!} \int_0^{+\infty} t^{k-1} e^{-\operatorname{Re}\lambda t} dt = \frac{M_0\|x\|}{(\operatorname{Re}\lambda)^k (k-1)!} \int_0^{\infty} s^{k-1} e^{-s} ds = \\ &= \frac{M_0\|x\|}{(\operatorname{Re}\lambda)^k} \end{aligned}$$

and (1.1) is true with  $M = M_0$ .

### 11. - Intermediate spaces

To state necessary and sufficient conditions for the regularity of the solutions of (10.1) we must introduce two families of intermediate spaces between  $D_A$  and  $E$ . The proofs of the results are given in Section 1.3 of [12].

DEFINITION 11.1. For each  $\theta \in ]0, 1[$  we define the Banach space

$$D_A(\theta, \infty) = \{x \in E; \|x\|_{\theta} = \sup_{t>0} \|t^{1-\theta} A e^{At} x\| < \infty\}$$

with norm

$$\|x\|_{D_A(\theta, \infty)} = \|x\| + \|x\|_{\theta}.$$

DEFINITION 11.2. For each  $\theta \in ]0, 1[$  we define the Banach space

$$D_A(\theta) = \{x \in E; \lim_{t \rightarrow 0} t^{1-\theta} A e^{At} x = 0\}$$

with norm

$$\|x\|_{D_A(\theta)} = \|x\| + \|x\|_{\theta}.$$

We have  $D_A(\theta) \subset D_A(\theta, \infty)$  because  $t \rightarrow e^{At}$  is bounded. More generally we have for  $0 < \theta_1 < \theta_2 < 1$

$$(11.1) \quad D_A \hookrightarrow D_A(\theta_2, \infty) \hookrightarrow D_A(\theta_1) \hookrightarrow D_A(\theta_1, \infty) \hookrightarrow \bar{D}_A$$

and  $D_A(\theta)$  can be characterized as the closure of  $D_A$  in  $D_A(\theta, \infty)$ . Other characterizations which are important in the study of evolution equations are given by:

$$(11.2) \quad x \in D_A(\theta, \infty) \Leftrightarrow t \rightarrow e^{At} x \text{ is in } C^{\theta}(0, T, E) \text{ for each } T > 0$$

$$(11.3) \quad x \in D_A(\theta) \Leftrightarrow t \rightarrow e^{At} x \text{ is in } h^{\theta}(0, T, E) \text{ for each } T > 0.$$

A very important property of these spaces is given by the fact that they depend only on  $A$  and  $E$  (in contrast with the fractional powers of  $-A$ )

**THEOREM 11.3.** *Let  $A : D_A \subseteq E \rightarrow E$  and  $B : D_B \subseteq E \rightarrow E$  verify (10.2) and  $D_A = D_B$ . Then*

$$(11.4) \quad D_A(\alpha, \infty) \simeq D_B(\alpha, \infty) \text{ and } D_A(\alpha) = D_B(\alpha), \quad \forall \alpha \in ]0, 1[.$$

Moreover if  $\overline{M}, \gamma > 0$  and  $\phi \in ]\frac{\pi}{2}, \pi[$  are such that if  $z \in \mathbb{C}$  and  $|\arg z| \leq \phi$  then  $(z - A)^{-1}, (z - B)^{-1} \in \mathcal{L}(E)$ ;  $\|z(z - A)^{-1}\|_{\mathcal{L}(E)}, \|z(z - B)^{-1}\|_{\mathcal{L}(E)} \leq M$  and for each  $x \in D_A = D_B$

$$(11.5) \quad \gamma^{-1}(\|x\| + \|Bx\|) \leq \|x\| + \|Ax\| \leq \gamma(\|x\| + \|Bx\|)$$

then there exists  $\delta = \delta(M, \phi, \alpha, \gamma)$  such that for each  $x \in D_A(\alpha, \infty) = D_B(\alpha, \infty)$

$$(11.6) \quad \delta^{-1}\|x\|_{D_B(\alpha, \infty)} \leq \|x\|_{D_A(\alpha, \infty)} \leq \delta\|x\|_{D_B(\alpha, \infty)}$$

**PROOF.** The result could be deduced from proposition 1.15 of [12] but we give here a simpler proof (due to G. Di Blasio).

We can suppose that (10.2) is verified by  $A$  and  $B$  with the same constants  $\overline{M}$  and  $\phi$ . Moreover as  $A$  and  $B$  are closed,  $D_A = D_B$  implies  $D_A \simeq D_B$ , so there exists  $\gamma > 0$  such that (11.5) is true. In addition we know that there exists  $M_k (k = 0, 1, 2)$  depending on  $\overline{M}$  and  $\phi$  such that

$$(11.7) \quad \|t^k A^k e^{At}\|_{\mathcal{L}(E)} \leq M_k, \quad \|t^k B^k e^{Bt}\|_{\mathcal{L}(E)} \leq M_k, \quad t > 0.$$

Let us suppose that  $x \in D_B(\alpha, \infty)$  and set

$$[x] = \sup_{t>0} \|t^{1-\alpha} B e^{Bt} x\|.$$

Fix  $t \in ]0, 1[$  and consider the function  $\phi : [t, +\infty[ \rightarrow E$  defined as

$$\phi(s) = A e^{As} e^{B(s-t)} x$$

$\phi$  is continuous in  $[t, +\infty[$ ,  $\phi(t) = A e^{At} x$  and  $\phi(+\infty) = 0$ . For  $s > t$

$$\phi'(s) = A^2 e^{As} e^{B(s-t)} x + A e^{As} B e^{B(s-t)} x$$

and therefore, if  $t < a < 1 < b$  we get

$$(11.8) \quad \begin{aligned} \|\phi(b) - \phi(a)\| &= \left\| \int_a^b \phi'(s) ds \right\| = \left\| \int_a^1 A^2 e^{As} e^{B(s-t)} x ds \right. \\ &\quad \left. - \int_1^b A^2 e^{As} e^{B(s-t)} x ds + \int_a^b A e^{As} B e^{B(s-t)} x ds \right\| \leq I_1 + I_2 + I_3 \end{aligned}$$

with

$$\begin{aligned}
I_1 &\leq M_1 \int_a^1 \|Ae^{B(s-t)}x\| \frac{ds}{s} \leq \gamma M_1 \int_a^1 (\|e^{B(s-t)}x\| + \|Be^{(s-t)B}x\|) \frac{ds}{s} \\
&\leq \gamma M_0 M_1 \|x\| \int_t^1 \frac{ds}{s} + \gamma M_1 [x] \int_t^1 (s-t)^{\alpha-1} \frac{ds}{s} \leq \gamma M_0 M_1 \|x\| t^{\alpha-1} \int_0^1 \frac{ds}{s^\alpha} + \\
&\gamma M_1 [x] t^{\alpha-1} \int_1^{1/t} (s-1)^{\alpha-1} \frac{ds}{s} \leq \gamma M_0 M_1 \|x\| (1-\alpha)^{-1} t^{\alpha-1} + M_1 [x] c(\alpha) t^{\alpha-1},
\end{aligned}$$

where

$$(11.9) \quad c(\alpha) = \int_1^{+\infty} (s-1)^{\alpha-1} s^{-1} ds,$$

$$\begin{aligned}
I_2 &\leq M_0 M_2 \|x\| \int_1^b \frac{ds}{s^2} \leq M_0 M_2 \|x\| t^{\alpha-1} \int_1^{+\infty} s^{-1-\alpha} ds = M_0 M_2 \|x\| \alpha^{-1} t^{\alpha-1}, \\
I_3 &\leq M_1 [x] \int_t^{+\infty} (s-t)^{\alpha-1} \frac{ds}{s} \leq M_1 [x] c(\alpha) t^{\alpha-1}.
\end{aligned}$$

Letting  $a \rightarrow t^+$  and  $b \rightarrow +\infty$  in (11.8) we get

$$\sup_{0 < t < 1} \|t^{1-\alpha} A e^{At} x\| \leq M_0 [\gamma M_1 (1-\alpha)^{-1} + M_2 \alpha^{-1}] \|x\| + (1+\gamma) M_1 c(\alpha) [x].$$

As

$$\sup_{t \geq 1} \|t^{1-\alpha} A e^{At} x\| \leq M_1 \|x\|$$

we conclude that  $x \in D_A(\alpha, \infty)$  and  $\|x\|_{D_A(\alpha, \infty)} \leq \delta \|x\|_{D_B(\alpha, \infty)}$  with

$$(11.10) \quad \delta = 1 + M_1 + M_0 (\gamma M_1 (1-\alpha)^{-1} + M_2 \alpha^{-1}) + (1+\gamma) M_1 c(\alpha).$$

If we change  $A$  with  $B$  in the preceding proof we get  $D_A(\alpha, \infty) \simeq D_B(\alpha, \infty)$  and (11.6). As  $D_A(\alpha)$  is the closure of  $D_A$  in  $D_A(\alpha, \infty)$ , it coincides with the closure of  $D_B$  in  $D_B(\alpha, \infty)$  i.e. with  $D_B(\alpha)$ .

## 12. - Mild and classical solutions

When condition (1.1) is verified and  $\overline{D_A} = E$  then  $A$  generates a semigroup by virtue of the Hille-Yosida theorem. The same is true when

(10.2) holds as we showed in Section 10. In these cases the classical variation of constant formula suggests another definition of solution of problem

$$(12.1) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0. \end{cases}$$

DEFINITION 12.1. Let  $A$  generate the semigroup  $e^{At}$ ,  $f \in L^1(0, T; E)$  and  $u_0 \in \overline{D}_A$ . The continuous function defined by

$$(12.2) \quad u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}f(s)ds, \quad t \in [0, T]$$

is called the *mild solution* of problem (12.1) (see e.g. [10] pag. 106).

When  $A$  generates an analytic semigroup one can define another type of solution which is the abstract version of the solution of a parabolic partial differential equation when the initial datum is not regular: in this case the equation is not required to be satisfied for  $t = 0$  (see [8] pag. 491).

DEFINITION 12.2. Let  $A$  verify (10.2),  $f \in L^p(0, T; E)$  and  $u_0 \in \overline{D}_A$ . A function  $u \in C(0, T; E) \cap W^{1,p}(0^+, T; E) \cap L^p(0^+, T; D_A)$  when verifies

$$(12.3) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \text{ a.e.} \\ u(0) = u_0 \end{cases}$$

is called a *classical solution in  $L^p$*  of (12.3).

When, in addition,  $f \in C(0, T; E)$ , a function  $u \in C(0, T; E) \cap C^1(0^+, T; E) \cap C(0^+, T; D_A)$  verifying

$$(12.4) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in ]0, T] \\ u(0) = u_0 \end{cases}$$

is called a *classical solution in  $C$*  of (12.4).

Note that if  $u$  is a classical solution in  $L^p$  then  $u(t) \in \overline{D}_A$ ,  $\forall t \in [0, T]$ . A strict solution in  $C$  (in  $L^p$ ) is also a classical solution in  $C$  (in  $L^p$ ) and a classical solution in  $C$  is also a classical solution in  $L^p$  for each  $p \geq 1$ .

PROPOSITION 12.3. Let  $A$  generate the semigroup  $e^{At}$  and let  $f \in L^p(0, T; E)$ ,  $u_0 \in \overline{D}_A$ . Then the mild solution of (12.1) coincides with the  $F$ -solution.

PROOF. We will first prove that a classical solution in  $L^p$  of (12.3) is necessarily a mild solution: in particular it is unique. The proof can be easily adapted from the analogous proof of the case  $\overline{D}_A = E$ . Let  $u$  be a classical solution in  $L^p$  of (12.3): given  $t \in ]0, T]$  and  $\varepsilon \in ]0, \frac{t}{2}]$  [ let us consider the function

$$v(s) = e^{A(t-s)}u(s), \quad s \in [\varepsilon, t - \varepsilon]$$

By virtue of (10.6) and (12.3) we have for  $s \in [\varepsilon, t - \varepsilon]$  a.e.

$$v'(s) = e^{A(t-s)}u'(s) - Ae^{A(t-s)}u(s) = e^{A(t-s)}f(s).$$

Moreover, from

$$v(t - \varepsilon) - v(\varepsilon) = \int_{\varepsilon}^{t-\varepsilon} v'(s)ds = \int_{\varepsilon}^{t-\varepsilon} e^{A(t-s)}f(s)ds$$

we deduce

$$e^{A\varepsilon}u(t - \varepsilon) - e^{A(t-\varepsilon)}u(\varepsilon) = \int_{\varepsilon}^{t-\varepsilon} e^{A(t-s)}f(s)ds.$$

As  $u(t) \in \overline{D}_A$  for each  $t \in [0, T]$ , letting  $\varepsilon \rightarrow 0^+$  we obtain (12.2) for  $t \in ]0, T[$ ; for  $t = 0$  (12.2) is true by definition of classical solution and so  $u$  is the mild solution. Suppose now that  $u$  is an F-solution in  $L^p$  of (12.3); by definition there exists  $u_k$ , strict solution in  $L^p$  of (2.2), such that (2.3) holds; from what proved above  $u_k$  are mild solutions of (2.2) and so

$$u_k(t) = u_{0k} + \int_0^t e^{A(t-s)}f_k(s)ds, \quad t \in [0, T].$$

From (2.3) we deduce that  $u_k$  converges in  $C(0, T; E)$  necessarily to  $u$ : therefore (12.2) is true and  $u$  is the mild solution of (12.3).

By using the existence result of the F-solutions in  $L^p$  (theorem 7.2) and the uniqueness of the mild solution one deduces that a mild solution is also an F-solution.

The previous result together with Corollary 7.3 proves that the mild solution is equivalent to the integral solution. But it can be interesting to prove this without the existence theorem used in the proof of the preceding proposition.

**PROPOSITION 12.4.** *Let  $A$  generate the semigroup  $e^{At}$  and let  $f \in L^p(0, T; E)$ ,  $u_0 \in \overline{D}_A$ . Then the mild solution of (12.1) coincides with the integral solution.*

**PROOF.** Denoting by  $u$  the mild solution of (12.1) and integrating both

sides of (12.2) from 0 to  $\tau \in [0, T]$  we get

$$\begin{aligned} \int_0^\tau u(t)dt &= \int_0^\tau e^{At}u_0dt + \int_0^\tau dt \int_0^t e^{A(t-s)}f(s)ds = \\ &= \int_0^\tau e^{At}u_0dt + \int_0^\tau ds \int_s^\tau e^{A(t-s)}f(s)dt = \\ &= \int_0^\tau e^{At}u_0dt + \int_0^\tau ds \int_0^{\tau-s} e^{At}f(s)dt \end{aligned}$$

Now, by using (10.10) we deduce  $\int_0^\tau u(t)dt \in D_A$  and

$$A \int_0^\tau u(t)dt = e^{A\tau}u_0 - u_0 + \int_0^\tau [e^{A(\tau-s)}f(s) - f(s)]ds = -u_0 + u(\tau) - \int_0^\tau f(s)ds$$

This shows that  $u$  is an integral solution.

Conversely let us assume that  $u$  is an integral solution of (12.1). This means that

$$v(t) = \int_0^t u(s)ds, \quad t \in [0, T]$$

is a strict solution in  $C$  of

$$\begin{cases} v'(t) = Av(t) + u_0 + g(t), & t \in [0, T] \\ v(0) = 0 \end{cases}$$

where

$$g(t) = \int_0^t f(s)ds, \quad t \in [0, T].$$

From Proposition 12.3 we get

$$v(t) = \int_0^t e^{As}u_0ds + \int_0^t e^{As}g(t-s)ds, \quad t \in [0, T].$$

As  $u_0 \in \overline{D}_A$ ,  $g \in W^{1,p}(0, T; E)$  and  $g(0) = 0$  we deduce that the right-hand side is differentiable in  $[0, T]$  and

$$v'(t) = e^{At}u_0 + \int_0^t e^{As}g'(t-s)ds$$

i.e. (12.2) and therefore  $u$  is the mild solution of (12.1).



### 13. - Existence, uniqueness, and maximal regularity results

In this section we want to collect the results about the existence of solutions of problem (12.1) in the parabolic case. For the sake of conciseness we will consider only the solutions in  $C$  and refer the reader to [5] for the result in  $L^p$ . Therefore we will assume that  $f$  is at least continuous from  $[0, T]$  to  $E$ .

Let us remark that the mild solution is for  $t > 0$  more than continuous with values in  $E$ .

**THEOREM 13.1.** *Let (10.2) hold. Given  $f \in C(0, T; E)$  and  $u_0 \in \overline{D}_A$ , the mild solution  $u$  of (12.1) given by (12.2) belongs to  $C^\alpha(0^+, T; D_A(1 - \alpha))$  for each  $\alpha \in ]0, 1[$ .*

*The proof is given in [12], theorem 3.4.*

*We state now two theorems which give conditions for the existence of a classical or a strict solution of (12.1).*

**THEOREM 13.2.** *Let (10.2) hold. Given  $f \in C^\alpha(0, T; E)$  and  $u_0 \in \overline{D}_A$  there exists a unique classical solution  $u$  in  $C$  of (12.1) and  $u' \in C^\alpha(0^+, T; E) \cap B(0^+, T; D_A(\alpha, \infty))$ . This solution is also strict if and only if  $u_0 \in D_A$  and  $Au_0 + f(0) \in \overline{D}_A$ .*

*For a proof see [12] theorems 4.4 and 4.5.*

**THEOREM 13.3.** *Let (10.2) hold. Given  $f \in C(0, T; E) \cap B(0, T; D_A(\alpha, \infty))$  and  $u_0 \in \overline{D}_A$  there exists a unique classical solution  $u$  in  $C$  of (12.1) and  $Au \in C^\alpha(0^+, T; E) \cap B(0^+, T; D_A(\alpha, \infty))$ . This solution is also strict if and only if  $u_0 \in D_A$  and  $Au_0 \in \overline{D}_A$ .*

*The proof is given in [12], theorems 5.4 and 5.5.*

For the parabolic equation (12.1) the maximal regularity property can be defined as follows:

**DEFINITION 13.4.** Let  $X$  be a subset of  $C(0, T; E)$ . There is the *maximal regularity property for problem (12.1) in  $X$*  if for each  $f \in X$  there exists a unique strict solution in  $C$  of (12.1) such that  $u'$  and  $Au$  belong to  $X$  (provided  $u_0$  satisfies a necessary compatibility condition with  $f$ ).

**THEOREM 13.5.** *Let (10.2) hold. The maximal regularity property for (1.2) holds in*

- (i)  $X = C^\alpha(0, T; E)$  if and only if  $u_0 \in D_A$  and  $Au_0 + f(0) \in D_A(\alpha, \infty)$ ,
- (ii)  $X = h^\alpha(0, T; E)$  if and only if  $u_0 \in D_A$  and  $Au_0 + f(0) \in D_A(\alpha)$ ,
- (iii)  $X = C(0, T; E) \cap B(0, T; D_A(\alpha, \infty))$  if and only if  $u_0 \in D_A$  and  $Au_0 \in D_A(\alpha, \infty)$ ,
- (iv)  $X = C(0, T; D_A(\alpha))$  if and only if  $u_0 \in D_A$  and  $Au_0 \in D_A(\alpha)$ .

The proofs of these results are given in [12], theorems 4.5 and 5.5.

REMARK 13.6. In general  $X = C^1(0, T; E)$  cannot be considered a space of maximal regularity. In fact for each  $f \in C(0, T; E)$  set  $g(t) = \int_0^t f(s)ds$  and consider problem

$$\begin{cases} v'(t) = Av(t) + g(t), & t \in [0, T] \\ v(0) = 0. \end{cases}$$

As  $g \in C^1(0, T; E)$  and  $g(0) = 0$  there exists a strict solution in  $C$  (see Theorem 13.2) given by  $v(t) = \int_0^t e^{As}g(t-s)ds$  and we have  $v'(t) = \int_0^t e^{As}f(t-s)ds$ . If also  $v' \in C^1(0, T; E)$  then the mild solution of

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = 0 \end{cases}$$

belongs to  $C^1(0, T; E)$  and therefore it is a strict solution in  $C$  (see theorem 6.3 and proposition 12.3); but this is not possible for every  $f \in C(0, T; E)$  when  $E$  is a general Banach space (see [1]).

## Applications

### 14. - Differential operators with non dense domain

We will consider some examples of operators  $A$  with non dense domain verifying the Hille-Yosida estimates (1.1); it is well known that it is easier to check the more restrictive condition

$$(14.1) \quad \|(\lambda - A)^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{\lambda}, \quad \lambda > 0$$

and this will be done in this section.

Let us begin with the differentiation operator in a one-dimensional compact interval of the  $x$ -axis.

EXAMPLE 14.1. Setting

$$(14.2) \quad E \rightsquigarrow C(0, 1)$$

$$(14.3) \quad Au = -u'$$

$$(14.4) \quad D_A = C_0^1(0, 1)$$

we have  $\overline{D}_A = C_0(0, 1) \neq E$ . Moreover for each  $\lambda > 0$  and  $v \in E$ :

$$(14.5) \quad u(x) = (R(\lambda, A)v)(x) = \int_0^x e^{-\lambda y} v(x-y) dy, \quad x \in [0, 1]$$

hence

$$(14.6) \quad |u(x)| \leq \|v\|_E \int_0^{+\infty} e^{-\lambda y} dy = \frac{1}{\lambda} \|v\|_E.$$

which implies (14.1).

EXAMPLE 14.2. Let us set for some  $\alpha \in ]0, 1[$ :

$$(14.7) \quad E = C_0^\alpha(0, 1)$$

$$(14.8) \quad Au = -u'$$

$$(14.9) \quad D_A = \{u \in C^{1+\alpha}(0, 1); u(0) = u'(0) = 0\}$$

We have  $\overline{D}_A = h_0^\alpha(0, 1) \neq E$  and for each  $\lambda > 0$  and  $v \in E$ , (14.5) holds: hence from (14.6) we get

$$(14.10) \quad \|u\|_{C(0,1)} \leq \frac{1}{\lambda} \|v\|_{C(0,1)}$$

and for  $0 \leq x_1 < x_2 \leq 1$  (using the fact that  $v(0) = 0$ ):

$$\begin{aligned} |u(x_2) - u(x_1)| &\leq \left| \int_0^{x_1} e^{-\lambda y} [v(x_2 - y) - v(x_1 - y)] dy \right| + \\ &\quad + \left| \int_{x_1}^{x_2} e^{-\lambda y} v(x_2 - y) dy \right| \leq [v]_{C^\alpha(0,1)} [(x_2 - x_1)^\alpha \int_0^{x_1} e^{-\lambda y} dy + \\ &\quad + \int_{x_1}^{x_2} e^{-\lambda y} (x_2 - y)^\alpha dy] \leq [v]_{C^\alpha(0,1)} (x_2 - x_1)^\alpha \int_0^{x_2} e^{-\lambda y} dy \leq \\ &\leq \frac{1}{\lambda} [v]_{C^\alpha(0,1)} (x_2 - x_1)^\alpha \end{aligned}$$

which gives

$$(14.11) \quad [u]_{C^\alpha(0,1)} \leq \frac{1}{\lambda} [v]_{C^\alpha(0,1)}.$$

Now (14.10) and (14.11) imply (14.1).

In the previous example  $C_0^\alpha(0, 1)$  cannot be replaced by  $C^\alpha(0, 1)$  in the definition of  $E$  because the Hille-Yosida estimates are not true in this case as the next proposition shows.

This result is analogous to that found by Von Wahl for the second derivative with Dirichlet boundary conditions (see [14])

**PROPOSITION 14.3.** *Set  $E = C^\alpha(0, 1)$ ,  $Au = -u'$  and  $D_A = C_0^{1+\alpha}(0, 1)$ . Then there exist no constants  $M, \omega > 0$  such that*

$$(14.12) \quad \|(\lambda - A)^{-1}\|_{\mathcal{L}(E)} \leq \frac{M}{\lambda}, \quad \lambda > \omega.$$

**PROOF.** Also in this case (14.5) is true for each  $\lambda > \omega$  and  $v \in E$ : choosing  $v(x) \equiv 1$  we get

$$u(x) = \frac{1}{\lambda}(1 - e^{-\lambda x})$$

and if  $\lambda > 1$  we obtain also

$$[u]_{C^\alpha(0,1)} \geq \sup_{0 < x \leq 1} \frac{|u(x) - u(0)|}{x^\alpha} = \sup_{0 < x \leq 1} \frac{1 - e^{-\lambda x}}{\lambda x^\alpha} \geq \frac{1 - e^{-1}}{\lambda^{1-\alpha}}.$$

If (14.12) holds we deduce for  $\lambda > \omega$  and  $\lambda > 1$ :

$$\frac{1 - e^{-1}}{\lambda^{1-\alpha}} \leq [u]_{C^\alpha(0,1)} \leq \frac{M}{\lambda}$$

and so  $\lambda^\alpha \leq \frac{M}{1 - e^{-1}}$  which, for  $\lambda \rightarrow +\infty$ , yields a contradiction.

Let us consider now the simplest cases of generators of analytic semigroups with non dense domain obtained from the Laplace operator with homogeneous Dirichlet boundary conditions in spaces of continuous functions. We shall begin with the one-dimensional case

**EXAMPLE 14.4.** Set

$$(14.13) \quad E = C(0, 1)$$

$$(14.14) \quad Au = u''$$

$$(14.15) \quad D_A = \{u \in C^2(0, 1); u(0) = u(1) = 0\}$$

then (10.2) is true with each  $\phi \in ]\frac{\pi}{2}, \pi[$  and  $\overline{M} = (\cos \phi/2)^{-1}$ . For a proof see Section 8.1 of [2]. In this case we have

$$\overline{D}_A = \{u \in C(0, 1); u(0) = u(1) = 0\} \neq E$$

**EXAMPLE 14.5.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with regular boundary  $\Gamma$  and define

$$(14.16) \quad E = C(\overline{\Omega})$$

$$(14.17) \quad Au = \Delta u$$

$$(14.18) \quad D_A = D_\Delta = \{u \in C(\bar{\Omega}); u = 0 \text{ on } \Gamma; \Delta u \in C(\bar{\Omega})\};$$

here  $\Delta$  is the Laplacian in the sense of distributions on  $\Omega$ . In this case we have  $\overline{D_A} = C_0(\bar{\Omega}) \neq E$  and it has been proved in [13] that (10.2) is true for each fixed  $\phi$  in  $] \frac{\pi}{2}, \pi[$ . As mentioned before, estimate (10.2) does not hold if  $C(\bar{\Omega})$  is replaced by  $C^\alpha(\bar{\Omega})$  (see [14]).

We shall prove here a result needed in the sequel:

PROPOSITION 14.6. *Let  $\Delta : D_A \subseteq C(\bar{\Omega}) \rightarrow C(\bar{\Omega})$  be defined as in (14.16)-(14.18), then  $\Delta$  generates a contraction analytic semigroup  $e^{\Delta t}$  in  $C(\bar{\Omega})$ , i.e.*

$$(14.19) \quad \|e^{\Delta t}\|_{\mathcal{L}(C(\bar{\Omega}))} \leq 1, \quad t > 0$$

and

$$(14.20) \quad \|R(\lambda, \Delta)\|_{\mathcal{L}(C(\bar{\Omega}))} \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

PROOF. For each  $p \in ]1, \infty[$  let us define  $\Delta_p : D(\Delta_p) \subseteq L^p(\Omega) \rightarrow L^p(\Omega)$  as

$$\Delta_p u = \Delta u$$

in the distribution sense ( $\Delta_p$  must not be confused with the iterated Laplacian) and

$$D(\Delta_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

It is known that  $\lambda \in \rho(\Delta) \cap \rho(\Delta_p)$  if  $\lambda$  is not negative real; moreover for each  $u \in C(\bar{\Omega})$  we have  $(\lambda - \Delta)^{-1}u = (\lambda - \Delta_p)^{-1}u$ ; hence for  $t > 0$  we get from (10.3):

$$e^{\Delta t}u = \frac{1}{2\pi i} \int_{+C} e^{\lambda t} (\lambda - \Delta)^{-1} u d\lambda = \frac{1}{2\pi i} \int_{+C} e^{\lambda t} (\lambda - \Delta_p)^{-1} u d\lambda = e^{\Delta_p t}u.$$

It is also known that  $\|e^{\Delta_p t}\|_{\mathcal{L}(L^p(\Omega))} \leq 1$  (see [10] pag. 215), so

$$\|e^{\Delta t}u\|_{L^p(\Omega)} \leq \|e^{\Delta_p t}u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)},$$

but  $e^{\Delta t}u \in C(\bar{\Omega})$  hence we obtain (14.19) for  $p < +\infty$ .

Estimate (14.20) is a consequence of (14.19) and the representation formula (10.12).

We study now the heat-operator considered as acting on functions of time and space: estimate (14.1) will be proved with the aid of the existence theorem 8.1 for the strict solution in  $L^1$ .

EXAMPLE 14.7. Let us consider the Banach space of functions  $u : [0, T] \rightarrow C(\bar{\Omega})$  defined as

$$(14.21) \quad E = W_0^{1,p}(0, T; C(\bar{\Omega}))$$

and the heat-operator

$$(14.22) \quad Au = \Delta u - u'$$

(hence  $(Au(t, x) = \Delta u(t, x) - u_t(t, x))$  with domain

$$(14.23) \quad D_A = \{u \in C_0(0, T; D_\Delta) \cap C^1(0, T; C(\bar{\Omega})); \Delta u - u' \in W_0^{1,p}(0, T; C(\bar{\Omega}))\}$$

where  $D_\Delta$  is defined by (14.18). It can be shown that

$$\bar{D}_A = W_0^{1,p}(0, T; C_0(\bar{\Omega})) \neq E.$$

Let us prove that (14.1) holds. Given  $v \in W_0^{1,p}(0, T; C(\bar{\Omega}))$  and  $\lambda > 0$  we must find  $u \in D_A$  verifying

$$(14.24) \quad \begin{cases} u'(t) = (\Delta - \lambda)u(t) + v(t), & t \in [0, T] \\ u(0) = 0. \end{cases}$$

Setting  $\bar{u}(t) = e^{\lambda t}u(t)$  and  $\bar{v}(t) = e^{\lambda t}v(t)$ , we have  $\bar{v} \in W_0^{1,p}(0, T; C(\bar{\Omega}))$  and (14.24) becomes

$$(14.25) \quad \begin{cases} \bar{u}'(t) = \Delta \bar{u}(t) + \bar{v}(t), & t \in [0, T] \\ \bar{u}(0) = 0. \end{cases}$$

As  $\bar{v}(0) = 0$  by virtue of (14.20) and Theorem 8.1, this problem has a unique solution  $\bar{u} \in C^1(0, T; C(\bar{\Omega})) \cap C_0(0, T; D_\Delta)$ ; hence there is a unique solution  $u \in D_A$  of (14.24) and from (8.4), (8.5) we get

$$(14.26) \quad \begin{aligned} \|\bar{u}(t)\| &\leq \int_0^t \|\bar{v}(s)\| ds, & t \in [0, T] \\ \|\bar{u}'(t)\| &\leq \int_0^t \|\bar{v}'(s)\| ds, & t \in [0, T] \end{aligned}$$

i.e.

$$\begin{aligned} \|u(t)\| &\leq \int_0^t e^{-\lambda(t-s)} \|v(s)\| ds \\ \|u'(t)\| &\leq \int_0^t e^{-\lambda(t-s)} \|v'(s)\| ds. \end{aligned}$$

From Young's inequality we deduce

$$(14.27) \quad \begin{aligned} \|u\|_{L^p(0,T;E)} &\leq \|v\|_{L^p(0,T;E)} \int_0^T e^{-\lambda s} ds \leq \\ &\leq \|v\|_{L^p(0,T;E)} \int_0^{+\infty} e^{-\lambda s} ds = \frac{1}{\lambda} \|v\|_{L^p(0,T;E)}. \end{aligned}$$

Analogously we have

$$(14.28) \quad \|u'\|_{L^p(0,T;E)} \leq \frac{1}{\lambda} \|v'\|_{L^p(0,T;E)}$$

hence

$$(14.29) \quad \|u\|_{W^{1,p}(0,T;E)} \leq \frac{1}{\lambda} \|v\|_{W^{1,p}(0,T;E)}, \quad \lambda > 0$$

which proves (14.1).

In the next example we study the same heat-operator but in a different space: in this case we need the maximal regularity results of theorem 13.5 to get the resolvent estimate (14.1).

EXAMPLE 14.8. Let us define the space of function  $u : [0, T] \rightarrow C(\bar{\Omega})$  as

$$(14.30) \quad E = C_0^\alpha(0, T; C(\bar{\Omega}))$$

and the heat-operator

$$Au = \Delta u - u'$$

with domain

$$(14.31) \quad D_A = \{u \in C^\alpha(0, T; D_\Delta) \cap C^{1+\alpha}(0, T; C(\bar{\Omega})); u(0) = u'(0) = 0\}.$$

We have  $\bar{D}_A = h_0^\alpha(0, T; C_0(\bar{\Omega})) \neq E$ .

If we consider the resolvent equation (14.24) with  $v \in C_0^\alpha(0, T; C(\bar{\Omega}))$  and  $\lambda > 0$  and perform the transformation which leads to (14.25) we can use (i) of theorem 13.5 and get a solution  $\bar{u} \in C_0^\alpha(0, T; D_\Delta) \cap C^{1+\alpha}(0, T; C(\bar{\Omega}))$ : hence there exists a unique solution  $u \in D_A$  of (14.24) given by

$$(14.32) \quad u(t) = e^{-\lambda t} \bar{u}(t) = e^{-\lambda t} \int_0^t e^{\Delta s} \bar{v}(t-s) ds = \int_0^t e^{\Delta s} e^{-\lambda s} v(t-s) ds.$$

By virtue of (14.19) we deduce

$$(14.33) \quad \|u(t)\| \leq \|v\|_{C(0,T;C(\bar{\Omega}))} \int_0^{+\infty} e^{-\lambda s} ds = \frac{1}{\lambda} \|v\|_{C(0,T;C(\bar{\Omega}))}$$

and for  $0 \leq t_1 < t_2 \leq T$  we get (as  $v(0) = 0$ ):

$$\begin{aligned} \|u(t_2) - u(t_1)\| &= \left\| \int_0^{t_1} e^{\Delta s} e^{-\lambda s} [v(t_2 - s) - v(t_1 - s)] \, ds + \right. \\ &\quad \left. + \int_{t_1}^{t_2} e^{\Delta s} e^{-\lambda s} v(t_2 - s) \, ds \right\| \leq [v]_{C^\alpha(0, T; C(\bar{\Omega}))} (t_2 - t_1)^\alpha \\ &\quad \left( \int_0^{t_1} e^{-\lambda s} \, ds + \int_{t_1}^{t_2} e^{-\lambda s} \, ds \right) \leq [v]_{C^\alpha(0, T; C(\bar{\Omega}))} (t_2 - t_1)^\alpha \int_0^{+\infty} e^{-\lambda s} \, ds \end{aligned}$$

hence

$$(14.34) \quad [u]_{C^\alpha(0, T; C(\bar{\Omega}))} \leq \frac{1}{\lambda} [v]_{C^\alpha(0, T; C(\bar{\Omega}))}.$$

From (14.33) and (14.34) we get

$$(14.35) \quad \|u\|_{C^\alpha(0, T; C(\bar{\Omega}))} \leq \frac{1}{\lambda} \|v\|_{C^\alpha(0, T; C(\bar{\Omega}))}$$

which is (14.1).

As in example 14.2 we cannot choose  $C^\alpha(0, T; C(\bar{\Omega}))$  as space  $E$  because the Hille-Yosida estimates are not satisfied. In fact the following theorem holds:

**PROPOSITION 14.9.** *Let  $E = C^\alpha(0, T; C(\bar{\Omega}))$ ,  $Au = \Delta u - u'$  and  $D_A = C^\alpha(0, T; D_\Delta) \cap C^{1+\alpha}(0, T; C(\bar{\Omega}))$ . Then there exist no constants  $M, \omega > 0$  such that*

$$(14.36) \quad \|(\lambda - A)^{-1}\|_{\mathcal{L}(E)} \leq \frac{M}{\lambda}, \quad \lambda > \omega$$

**PROOF.** We can repeat the arguments of example 14.8 and prove that for  $\lambda > 0$ ,  $(\lambda - A)^{-1} \in \mathcal{L}(E)$  and  $u = (\lambda - A)^{-1}v$  is given by formula (14.32). Let us choose an element  $y \in D_\Delta(\alpha, \infty)$ , then  $v(t) = e^{\Delta t}y$ ,  $t \in [0, T]$  belongs to  $C^\alpha(0, T; C(\bar{\Omega}))$  (see (11.3)) and we deduce from (14.32)

$$(R(\lambda, A)v)(t) = \frac{1 - e^{-\lambda t}}{\lambda} e^{\Delta t}y.$$

Now suppose that (14.36) holds: then for  $\lambda > \omega$  and  $\lambda \geq \frac{1}{T}$  we have

$$\|v\|_E \frac{M}{\lambda} \geq \sup_{0 < t \leq T} \frac{1 - e^{-\lambda t}}{\lambda} \frac{\|e^{\Delta t}y\|_{C(\bar{\Omega})}}{t^\alpha} \geq \frac{1 - e^{-1}}{\lambda^{1-\alpha}} \|e^{\Delta/\lambda}y\|_{C(\bar{\Omega})}$$

hence, setting  $\varepsilon = \frac{1}{\lambda}$ , we get

$$\|e^{\Delta\varepsilon}y\|_{C(\bar{\Omega})} \leq \frac{M\|v\|_E}{1 - e^{-1}} \varepsilon^\alpha$$



and therefore  $\lim_{\varepsilon \rightarrow 0^+} \|e^{\Delta\varepsilon} y\|_{C(\bar{\Omega})} = 0$ : this implies  $y = 0$  (see (10.9)) and we have a contradiction because  $D_\Delta(\alpha, \infty) = C_0^{2\alpha}(\bar{\Omega})$  (see [9])

Let us consider now the periodic versions of examples 14.7 and 14.8:

EXAMPLE 14.10. Consider again a space of functions  $u : [0, T] \rightarrow C(\bar{\Omega})$

$$(14.37) \quad E = W_{\#}^{1,p}(0, T; C(\bar{\Omega}))$$

and the heat operator

$$(14.38) \quad Au = \Delta u - u'$$

with domain

$$(14.39) \quad D_A = \{u \in C^1(0, T; C(\bar{\Omega})) \cap C_{\#}(0, T; D_\Delta); \Delta u - u' \in W_{\#}^{1,p}(0, T; C(\bar{\Omega}))\}$$

which is not dense in  $E$  because we have

$$(14.40) \quad \bar{D}_A = W_{\#}^{1,p}(0, T; C_0(\bar{\Omega})).$$

To show that 14.1 holds in this case too we will use the results of theorem 9.2.

Given  $v \in W_{\#}^{1,p}(0, T; C(\bar{\Omega}))$  and  $\lambda > 0$ , let us consider problem

$$(14.41) \quad \begin{cases} \lambda u(t) - Au(t) = v(t), & t \in [0, T] \\ u(0) = u(T) \end{cases}$$

or, setting

$$(14.42) \quad \begin{cases} B : D_B \subset E \rightarrow E \\ D_B = D_\Delta \\ Bu = \Delta u - \lambda u, \end{cases}$$

problem

$$(14.43) \quad \begin{cases} u'(t) = Bu(t) + v(t), & t \in [0, T] \\ u(0) = u(T). \end{cases}$$

From example 14.5 we deduce (see Appendix) that  $B$  verifies (1.1) and

$$(14.44) \quad \|e^{B_0 t}\|_{\mathcal{L}(E_0)} \leq e^{-\lambda t}, \quad t \geq 0$$

where  $E_0 = C_0(\bar{\Omega})$  with the sup-norm and  $B_0$  is the part of  $B$  in  $E_0$ , according to definition (8.16); from (14.44) we get (9.6) with  $A_0 = B_0$ . Therefore we can

use theorem 9.2, case (v) to get a solution  $u \in C^1(0, T; C(\bar{\Omega}) \cap C_{\#}(0, T; D_{\Delta}))$  of (14.43). As  $v(0) = v(T)$  we have

$$(14.45) \quad \Delta u - u' = \lambda u - v \in W_{\#}^{1,p}(0, T; C(\bar{\Omega})),$$

so  $u \in D_A$ .

To prove estimate (14.1) let us observe that from (14.43) we deduce by virtue of Theorem A.2 of the Appendix, for each  $t \in [0, T]$

$$(14.46) \quad \begin{aligned} \|u(t)\| &\leq e^{-\lambda t}(\|u(0)\| + \int_0^t e^{\lambda s} \|v(s)\| ds) = \\ &= e^{-\lambda t} \|u(T)\| + \int_0^t e^{-\lambda(t-s)} \|v(s)\| ds \end{aligned}$$

and

$$(14.47) \quad \begin{aligned} \|u'(t)\| &\leq e^{-\lambda t}(\|Bu(0) + v(0)\| + \int_0^t e^{\lambda s} \|v'(s)\| ds) \\ &= e^{-\lambda t} \|u'(T)\| + \int_0^t e^{-\lambda(t-s)} \|v'(s)\| ds. \end{aligned}$$

From (14.46) with  $t = T$  we get

$$\|u(T)\| \leq (e^{\lambda T} - 1)^{-1} \int_0^T e^{\lambda s} \|v(s)\| ds$$

and therefore, from (14.46),

$$(14.48) \quad \begin{aligned} \|u(t)\| &\leq (e^{\lambda T} - 1)^{-1} \int_0^T e^{-\lambda(t-s)} \|v(s)\| ds + \int_0^t e^{-\lambda(t-s)} \|v(s)\| ds \\ &= \int_0^T K(t-s) \|v(s)\| ds \end{aligned}$$

with

$$K(t-s) = \begin{cases} e^{-\lambda(t-s)} [(e^{\lambda T} - 1)^{-1} + 1] & \text{for } 0 \leq s \leq t \leq T \\ e^{-\lambda(t-s)} (e^{\lambda T} - 1)^{-1} & \text{for } 0 \leq t \leq s \leq T. \end{cases}$$

Hence

$$\|u\|_{L^p(0,T;E)} \leq \|v\|_{L^p(0,T;E)} \sup_{0 \leq t \leq T} \int_0^t K(t-s) ds = \frac{1}{\lambda} \|v\|_{L^p(0,T;E)}.$$

Analogously (14.47) implies

$$\|u'\|_{L^p(0,T;E)} \leq \frac{1}{\lambda} \|v'\|_{L^p(0,T;E)}.$$

The two last estimates give (14.1).

Now we change the space  $E$  and use the parabolic theory to get the resolvent estimate (14.1).

EXAMPLE 14.11. Setting

$$(14.49) \quad \begin{cases} E = C_{\#}^{\alpha}(0, T; C(\overline{\Omega})) \\ Au = \Delta u - u' \\ D_A = \{u \in C_{\#}^{\alpha}(0, T; D_{\Delta}) \cap C^{1+\alpha}(0, T; C(\overline{\Omega})); \\ \Delta u - u' \in C_{\#}^{\alpha}(0, T; C(\overline{\Omega}))\} \end{cases}$$

we have  $\overline{D}_A = h_{\#}^{\alpha}(0, T; C_0(\overline{\Omega})) \neq E$  and

$$(14.50) \quad \|(\lambda - A)^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{\lambda} \text{ for } \lambda > 0$$

To prove this we could use the method of the previous example, but we prefer to use a different one. Given  $v \in C_{\#}^{\alpha}(0, T; E)$  let us denote by  $\bar{v}$  the periodic extension of  $v$  to  $] - \infty, T]$ . We have for  $t_1, t_2 \in ] - \infty, T]$ :

$$\|\bar{v}(t_2) - \bar{v}(t_1)\| \leq [v]_{\alpha} |t_2 - t_1|^{\alpha}$$

(because there exists  $t'_1, t'_2 \in [0, T]$  such that  $\bar{v}(t_1) = v(t'_1)$ ,  $\bar{v}(t_2) = v(t'_2)$  and  $|t'_2 - t'_1|^{\alpha} \leq |t_2 - t_1|^{\alpha}$ ). Let  $e^{\Delta t}$  be the analytic semigroup generated by  $\Delta$  (see example 14.5) and set, for  $t \in [0, T]$ ,

$$(14.51) \quad u(t) = \int_{-\infty}^t e^{(\Delta - \lambda)(t-s)} \bar{v}(s) ds = \int_0^{+\infty} e^{(\Delta - \lambda)s} \bar{v}(t-s) ds$$

(for each  $\lambda > 0$  and  $u \in E$  we set  $e^{(\Delta - \lambda)s} u = e^{\Delta s} e^{-\lambda s} u$ ,  $s \geq 0$ ).

For  $t \in [0, T]$  we have

$$\begin{aligned} u(t) &= \int_{-\infty}^0 e^{(\Delta-\lambda)(t-s)} \bar{v}(s) ds + \int_0^t e^{(\Delta-\lambda)(t-s)} \bar{v}(s) ds = \\ &= e^{(\Delta-\lambda)t} \int_{-\infty}^0 e^{-(\Delta-\lambda)s} \bar{v}(s) ds + \int_0^t e^{(\Delta-\lambda)(t-s)} v(s) ds = \\ &= e^{(\Delta-\lambda)t} u(0) + \int_0^t e^{(\Delta-\lambda)(t-s)} v(s) ds \end{aligned}$$

As  $v \in C^\alpha(0, T; E)$  we get (by using theorem 13.2)  $u \in C(0^+, T; E)$  and

$$(14.52) \quad u'(t) = (\Delta - \lambda)u(t) + v(t), \quad 0 < t \leq T$$

$$(14.53) \quad u'(t) \in \bar{D}_\Delta, \quad 0 < t \leq T.$$

Moreover we have  $u(0) = u(T)$  because

$$\begin{aligned} u(0) &= \int_{-\infty}^0 e^{-(\Delta-\lambda)s} \bar{v}(s) ds = \int_{-\infty}^0 e^{-(\Delta-\lambda)s} \bar{v}(s+T) ds = \\ &= \int_{-\infty}^T e^{(\Delta-\lambda)(T-s)} \bar{v}(s) ds = u(T) \end{aligned}$$

and therefore  $u(0) = u(T) \in D_\Delta$ . From this and (14.53) we get

$$(\Delta - \lambda)u_0 + f(0) = (\Delta - \lambda)u(T) + f(T) = u'(T) \in \bar{D}_\Delta.$$

From theorem 13.12 we deduce that  $u \in C^1(0, T; C(\bar{\Omega})) \cap C(0, T; D_\Delta)$  and (14.52) holds also for  $t=0$ : from this it follows that  $u \in D_A$ .

From the estimate

$$(14.54) \quad \|u(t)\| \leq \|\bar{v}\|_{C(-\infty, T; C(\bar{\Omega}))} \int_{-\infty}^t e^{-\lambda(t-s)} ds = \|v\|_{C(0, T; C(\bar{\Omega}))} \frac{1}{\lambda}$$

which holds for each  $t \in [0, T]$ , we get the uniqueness of the solution of (14.54). We also have for  $0 \leq t_1 \leq t_2 \leq T$

$$\begin{aligned} \|u(t_2) - u(t_1)\| &= \left\| \int_0^{+\infty} e^{(\Delta-\lambda)s} [\bar{v}(t_2-s) - \bar{v}(t_1-s)] ds \right\| \leq \\ &[v]_\alpha (t_2 - t_1)^\alpha \int_0^{+\infty} e^{-\lambda s} ds = [v]_\alpha (t_2 - t_1)^\alpha \frac{1}{\lambda}; \end{aligned}$$

thus, by using also (14.54), we deduce

$$\|u\|_{C^\alpha(0,T;C(\bar{\Omega}))} \leq \frac{1}{\lambda} \|v\|_{C^\alpha(0,T;C(\bar{\Omega}))}$$

and (14.50) follows.

Let us end this section with a very simple example in a space of functions defined in the whole real axis.

EXAMPLE 14.12. Setting

$$(14.55) \quad \begin{cases} E = L^\infty(\mathbb{R}) \\ Au = -u' \\ D_A = \{u \in L^\infty(\mathbb{R}), u \text{ is absolutely continuous and } u' \in L^\infty(\mathbb{R})\} \end{cases}$$

we have  $\bar{D}_A \neq E$  and

$$(14.56) \quad \|(\lambda - A)^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

In fact, as  $D_A$  is contained in the set of bounded and continuous functions on  $\mathbb{R}$ , we have  $\bar{D}_A \neq E$ . For each  $\lambda > 0$  and  $v \in L^\infty(\mathbb{R})$  it is easy to see that

$$(14.57) \quad u(x) = \int_0^{+\infty} e^{-\lambda y} v(x-y) dy, \quad x \in \mathbb{R}$$

satisfies the resolvent equation

$$(14.58) \quad u'(x) = -\lambda u(x) + v(x), \quad x \in \mathbb{R} \text{ a.e.}$$

Moreover  $u \in D_A$  and for each  $x \in \mathbb{R}$

$$|u(x)| \leq \sup_{x \in \mathbb{R}} |v(x)| \int_0^{+\infty} e^{-\lambda y} dy$$

hence

$$\|u\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\lambda} \|v\|_{L^\infty(\mathbb{R})}.$$

Conversely if  $u \in D_A$  satisfies (14.58) for each  $\xi \in \mathbb{R}$  we have

$$\frac{d}{dx} [u(x)e^{-\lambda(\xi-x)}] = e^{-\lambda(\xi-x)} v(x), \quad x \in \mathbb{R} \text{ a.e.}$$

As  $u$  is absolutely continuous, integrating from  $a \in \mathbb{R}$  to  $\xi$  we get

$$u(\xi) - u(a)e^{-\lambda(\xi-a)} = \int_a^\xi e^{-\lambda(\xi-x)}v(x)dx$$

for  $a \rightarrow -\infty$  we obtain

$$u(\xi) = \int_{-\infty}^\xi e^{-\lambda(\xi-x)}v(x)dx = \int_0^{+\infty} e^{-\lambda x}v(\xi-x)dx$$

and therefore (14.57) is the unique solution in  $D_A$  of (14.58).

In the following sections we will see some of the possible applications of the abstract existence theorems and the examples just considered.

For the sake of conciseness we will consider only the strict solutions.

### 15. - Linear partial differential equations of the first order

We want to apply our abstract methods to a very simple partial differential equation: we will show how to obtain the classical solution as an easy application of Theorem 8.1.

**THEOREM 15.1.** *Let  $f : [0, T]^2 \rightarrow \mathbb{R}$  and  $u_1, u_2 : [0, T] \rightarrow \mathbb{R}$  be functions verifying the following properties:*

- (15.1)  $f(t, x) = f_1(t, x) + f_2(t, x)$  such that setting  $f_1(t)(x) = f_1(t, x)$  and  $f_2(x)(t) = f_2(t, x)$  we have  $f_1, f_2 \in W^{1,1}(0, T; C(0, T))$
- (15.2)  $u_1, u_2 \in C^1(0, T)$
- (15.3)  $u_1(0) = u_2(0)$
- (15.4)  $u'_1(0) + u'_2(0) = f(0, 0)$ .

Then there exists a unique  $u \in C^1([0, T]^2)$  such that

$$(15.5) \quad \begin{cases} u_t(t, x) + u_x(t, x) = f(t, x), & (t, x) \in [0, T]^2 \\ u(0, x) = u_1(x) & , \quad x \in [0, T] \\ u(t, 0) = u_2(t) & , \quad t \in [0, T] \end{cases}$$

Let us remark that (15.2)-(15.4) are necessary conditions for the existence of a solution belonging to  $C^1([0, T]^2)$ .

**PROOF.** We can suppose that

$$(15.6) \quad u'_1(0) = f_1(0, 0), \quad u'_2(0) = f_2(0, 0)$$

otherwise we substitute  $f_1$  and  $f_2$  with

$$\begin{aligned}\tilde{f}_1(t, x) &= f_1(t, x) - f_1(0, 0) + u_1'(0) \\ \tilde{f}_2(t, x) &= f_2(t, x) - f_2(0, 0) + u_2'(0)\end{aligned}$$

which satisfy again (15.1) and (15.4).

Let us first consider the case in which

$$(15.7) \quad u_1(0) = u_2(0) = 0.$$

If we define

$$(15.8) \quad \begin{cases} E = C(0, T) \\ D_A = C_0^1(0, T) \\ Au = -u' \end{cases}$$

then, setting  $v(t)(x) = v(t, x)$  and  $f_1(t)(x) = f_1(t, x)$ , problem

$$(15.9) \quad \begin{cases} v_t(t, x) = -v_x(t, x) + f_1(t, x), & (t, x) \in [0, T]^2 \\ v(0, x) = u_1(x), & x \in [0, T] \\ v(t, 0) = 0, & t \in [0, T] \end{cases}$$

can be written as

$$\begin{cases} v'(t) = Av(t) + f_1(t), & t \in [0, T] \\ v(0) = u_1. \end{cases}$$

From (15.1) we have  $f_1 \in W^{1,1}(0, T; E)$ , from (15.2) and (15.7) we get  $u_1 \in D_A$  and from (15.6) we deduce  $Au_1 + f_1(0) \in \overline{D}_A = C_0(0, 1)$ : hence theorem 8.1 gives a solution  $v \in C^1(0, T; E) \cap C(0, T; D_A)$ , so  $v(t, x)$  belongs to  $C^1([0, T]^2)$ . Analogously problem

$$(15.10) \quad \begin{cases} w_t(t, x) = -w_x(t, x) + f_2(t, x), & (t, x) \in [0, T]^2 \\ w(0, x) = 0, & x \in [0, T] \\ w(t, 0) = u_2(t), & t \in [0, T] \end{cases}$$

can be written as

$$\begin{cases} w'(x) = Aw(x) + f_2(x), & x \in [0, T] \\ w(0) = u_2 \end{cases}$$

after setting  $w(x)(t) = w(t, x)$  and  $f_2(x)(t) = f_2(t, x)$  and proceeding as above we find a solution  $w(t, x)$  in  $C^1([0, T]^2)$ . By adding (15.9) and (15.10) we obtain a solution  $u = v + w$  of (15.5) in  $C^1([0, T]^2)$ .

Let us consider the case in which (15.7) is not true. Setting

$$\begin{cases} \tilde{u}_1(x) = u_1(x) - u_1(0), & x \in [0, T] \\ \tilde{u}_2(x) = u_2(x) - u_2(0), & x \in [0, T] \end{cases}$$

we can use the result just obtained to deduce the existence of  $u \in C^1([0, T]^2)$  solution of

$$\begin{cases} \tilde{u}_t(t, x) + \tilde{u}_x(t, x) = f(t, x), & (t, x) \in [0, T]^2 \\ \tilde{u}(0, x) = \tilde{u}_1(x) & , \quad x \in [0, T] \\ \tilde{u}(t, 0) = \tilde{u}_2(t) & , \quad t \in [0, T] \end{cases}$$

We can check that  $u(t, x) = \tilde{u}(t, x) + u_1(0)$  is the solution of problem (15.5). Its uniqueness is a consequence of the fact that the only function  $u \in C^1(0, T; E) \cap C(0, T; D_A)$  which verifies

$$\begin{cases} u'(t) = Au(t), & t \in [0, T] \\ u(0) = 0 \end{cases}$$

is zero (see Theorem 8.1).

REMARK 15.2. Let us remark that to solve (15.5) with the aid of the usual semigroup theory we must suppose  $f(t, x) = 0$  for all  $t \in [0, T]$  or for all  $x \in [0, T]$  because we need to substitute in definition (15.8)  $E = C(0, T)$  with  $E = C_0(0, T)$  and  $D_A = C_0^1(0, T)$  with  $D_A = \{u \in C^1(0, T); u(0) = u'(0) = 0\}$  in order to verify property  $\overline{D}_A = E$ .

Obviously, problem (15.5) can be solved by the characteristics methods, although its justification under the above conditions on  $f$  is not straightforward: moreover with the abstract methods we can find weaker solutions under very mild conditions on  $f, u_1$  and  $u_2$ .

### 16. - Ultraparabolic partial differential equations

We will consider in this section an initial value problem for an ultraparabolic equation. In the last years a sufficiently large amount of papers have been devoted to this subject but under assumptions different from ours (see e.g. [6] and references therein).

The problem could be treated by using either the abstract hyperbolic theory or the parabolic one. Let us begin with the first method:

THEOREM 16.1. *Let  $\Omega$  be a bounded open set of  $\mathbb{R}^n$  with regular boundary  $\Gamma, T > 0$  and let  $f(t, \tau, x), u_1(\tau, x)$  and  $u_2(t, x)$  be such that setting*

$$\begin{aligned} f_1(t)(\tau, x) &= f(t, \tau, x), & f_2(\tau)(t, x) &= f(t, \tau, x) \\ u_1(\tau)(x) &= u_1(\tau, x) & , & \quad u_2(t)(x) = u_2(t, x) \end{aligned}$$



we have

$$(16.1) \quad f_1, f_2 \in W^{1,1}(0, T; W^{1,1}(0, T; C(\bar{\Omega})))$$

$$(16.2) \quad u_1, u_2 \in C(0, T; D_A) \cap C^1(0, T; C(\bar{\Omega}))$$

and also

$$(16.3) \quad u_1(0, x) = u_2(0, x), \quad x \in \bar{\Omega}$$

$$(16.4) \quad u_1(\tau, x) = u_2(t, x) = 0, \quad \tau, t \in [0, T], \quad x \in \Gamma$$

$$(16.5) \quad \Delta u_1(\tau, x) + f(0, \tau, x) = \Delta u_2(t, x) + f(t, 0, x) = 0; \quad \tau, t \in [0, T], x \in \Gamma$$

$$(16.6) \quad \frac{\partial u_1(\tau, x)}{\partial \tau} \Big|_{\tau=0} + \frac{\partial u_2(t, x)}{\partial t} \Big|_{t=0} = \Delta u_1(0, x) + f(0, 0, x) = \Delta u_2(0, x) + f(0, 0, x), \quad x \in \bar{\Omega}$$

Then there exists a unique  $u(t, \tau, x)$  such that:

$$(16.7) \quad u_t, u_\tau, \Delta u \text{ are continuous in } [0, T]^2 \times \bar{\Omega}$$

and verify

$$(16.8) \quad \begin{cases} u_t(t, \tau, x) + u_\tau(t, \tau, x) = \Delta u(t, \tau, x) + f(t, \tau, x); & t, \tau \in [0, t], \\ & x \in \bar{\Omega} \\ u(0, \tau, x) = u_1(\tau, x) & ; \quad \tau \in [0, T], \quad x \in \bar{\Omega} \\ u(t, 0, x) = u_2(t, x) & ; \quad t \in [0, T], \quad x \in \bar{\Omega} \\ u(t, \tau, x) = 0 & ; \quad t, \tau \in [0, T], \quad x \in \Gamma \end{cases}$$

PROOF. Let us first consider problem

$$(16.9) \quad \begin{cases} u_t(t, \tau, x) + u_\tau(t, \tau, x) = \Delta u(t, \tau, x) + f(t, \tau, x) - f(t, 0, x); \\ & t, \tau \in [0, T], \quad x \in \bar{\Omega} \\ u(0, \tau, x) = v_1(\tau, x) & ; \quad \tau \in [0, T], \quad x \in \bar{\Omega} \\ u(t, 0, x) = 0 & ; \quad t \in [0, T], \quad x \in \bar{\Omega} \\ u(t, \tau, x) = 0 & ; \quad t, \tau \in [0, T], \quad x \in \Gamma \end{cases}$$

with  $v_1$  such that

$$(16.10) \quad \begin{cases} \frac{\partial v_1(\tau, x)}{\partial \tau} = \Delta v_1(\tau, x) + f(0, \tau, x) - f(0, 0, x); & \tau \in [0, T], \quad x \in \bar{\Omega} \\ v_1(0, x) = 0 & ; \quad x \in \bar{\Omega} \\ v_1(\tau, x) = 0 & ; \quad \tau \in [0, T], \quad x \in \Gamma \end{cases}$$

This problem can be solved by using the results of example 14.5. In fact, setting

$$(16.11) \quad \begin{cases} v_1(\tau)(x) = v_1(\tau, x) \\ g(\tau)(x) = f(0, \tau, x) - f(0, 0, x) \end{cases}$$

(16.10) can be written as a problem in the Banach space  $C(\overline{\Omega})$ :

$$(16.12) \quad \begin{cases} v_1'(\tau) = Av_1(\tau) + g(\tau), & \tau \in [0, T] \\ v_1(0) = 0. \end{cases}$$

We can use Theorem 8.1 because (16.1) implies

$$(16.13) \quad g \in W_0^{1,1}(0, T; C(\overline{\Omega}))$$

thus there exists

$$(16.14) \quad v_1 \in C^1(0, T; C(\overline{\Omega})) \cap C_0(0, T; D_\Delta)$$

(see (14.18) for the definition of  $D_\Delta$ ) solution of (16.12).

To solve problem (16.9) we can use the results of example 14.7 by setting

$$(16.15) \quad \begin{cases} E = W_0^{1,1}(0, T; C(\overline{\Omega})) \\ Au = \Delta u - u' \text{ i.e. } (Au)(\tau, x) = u(\tau, x) - u_\tau(\tau, x) \\ D_A = \{u \in C_0(0, T; D_\Delta) \cap C^1(0, T; C(\overline{\Omega}))\}; \Delta u - u' \in \\ \qquad \qquad \qquad W_0^{1,1}(0, T; C(\overline{\Omega}))\} \\ u(t)(\tau, x) = u(t, \tau, x) \\ h(t)(\tau, x) = f(t, \tau, x) - f(t, 0, x) \end{cases}$$

and write (16.9) as an initial value problem in  $E$

$$(16.16) \quad \begin{cases} u'(t) = Au(t) + h(t), & t \in [0, T] \\ u(0) = v_1. \end{cases}$$

The assumptions of Theorem 8.1 are verified. In fact  $v_1 \in D_A$  by virtue of (16.14) and (16.1); moreover  $Av_1 + h(0) = 0$  and  $h \in W^{1,1}(0, T; E)$ : hence there exists a solution

$$(16.17) \quad u \in C^1(0, T; E) \cap C(0, T; D_A)$$

of (16.16) and therefore  $u(t, \tau, x) = u(t)(\tau, x)$  is a solution of (16.9).

Analogously we can consider problem

$$(16.18) \quad \begin{cases} u_t(t, \tau, x) + u_\tau(t, \tau, x) = \Delta u(t, \tau, x) + f(t, 0, x) - f(0, 0, x); & t, \tau \in [0, T], \quad x \in \overline{\Omega} \\ u(0, \tau, x) = 0 & \tau \in [0, T], \quad x \in \overline{\Omega} \\ u(t, 0, x) = v_2(t, x) & t \in [0, T], \quad x \in \overline{\Omega} \\ u(t, \tau, x) = 0 & t, \tau \in [0, T], \quad x \in \Gamma \end{cases}$$

where  $v_2(t, x)$  is solution of

$$(16.19) \quad \begin{cases} \frac{\partial v_2(t, x)}{\partial t} = \Delta v_2(t, x) + f(t, 0, x) - f(0, 0, x); & t \in [0, T], \quad x \in \overline{\Omega} \\ v_2(0, x) = 0 & ; \quad x \in \overline{\Omega} \\ v_2(t, x) = 0 & ; \quad t \in [0, T], \quad x \in \Gamma \end{cases}$$

This problem can be solved by setting

$$(16.20) \quad \begin{cases} v_2(t)(x) = v_2(t, x) \\ k(t)(x) = f(t, 0, x) - f(0, 0, x) \end{cases}$$

and writing (16.9) as an abstract equation in the Banach space  $C(\overline{\Omega})$ :

$$(16.21) \quad \begin{cases} v_2'(t) = \Delta v_2(t) + h(t) \\ v_2(0) = 0. \end{cases}$$

From (16.1) we deduce

$$(16.22) \quad k \in W_0^{1,1}(0, T; C(\overline{\Omega}))$$

so Theorem 8.1 implies the existence of

$$(16.23) \quad v_2 \in C^1(0, T; C(\overline{\Omega})) \cap C(0, T; D_\Delta)$$

solution of (16.21).

Let us examine problem (16.18): set as in the example 14.7

$$(16.24) \quad \begin{cases} E = W_0^{1,1}(0, T; C(\overline{\Omega})) \\ Au = \Delta u - u' \text{ i.e. } (Au)(t, x) = \Delta u(t, x) - u_t(t, x) \\ D_A = \{u \in C_0(0, T; D_\Delta) \cap C^1(0, T; C(\overline{\Omega}))\}; \Delta u - u' \in W_0^{1,1}(0, T; C(\overline{\Omega}))\} \\ u(\tau)(t, x) = u(t, \tau, x) \\ \omega(t, x) = f(t, 0, x) - f(0, 0, x) \end{cases}$$

and write (16.18) as an initial value problem in  $E$

$$(16.25) \quad \begin{cases} u'(\tau) = Au(\tau) + \omega(\tau), \quad \tau \in [0, T] \\ u(0) = v_2. \end{cases}$$

Also in this case we can check that the assumptions of Theorem 8.1 are verified because  $v_2 \in D_A$  by virtue of (16.19) and because (16.1) implies  $\omega \in W_0^{1,1}(0, T; C(\overline{\Omega}))$ ; moreover  $Av_2 + \omega = 0$  and therefore there exists a solution

$$(16.26) \quad u \in C^1(0, T; E) \cap C(0, T; D_A)$$

of (16.25): hence  $u(t, \tau, x) = u(\tau)(t, x)$  is a solution of (16.18).

Let us set now for  $t, \tau \in [0, T]$  and  $x \in \bar{\Omega}$

$$(16.27) \quad \begin{cases} w_1(\tau, x) = u_1(\tau, x) - v_1(\tau, x) \\ w_2(t, x) = u_2(t, x) - v_2(t, x) \end{cases}$$

and

$$(16.28) \quad w_1(\tau)(x) = w_1(\tau, x), \quad w_2(t)(x) = w_2(t, x).$$

From (16.3), (16.10)<sub>2</sub> and (16.19)<sub>2</sub> we deduce

$$(16.29) \quad w_1(0) = w_2(0)$$

whereas (16.5) and (16.10) imply

$$\Delta w_1(\tau, x) + f(0, 0, x) = -\frac{\partial v_1(\tau, x)}{\partial \tau}, \quad \tau \in [0, T], \quad x \in \Gamma.$$

As (16.14) gives

$$(16.30) \quad v_1'(\tau) \in \bar{D}_A$$

setting

$$(16.31) \quad \gamma(x) = f(0, 0, x), \quad x \in \bar{\Omega}$$

we obtain:

$$(16.32) \quad \Delta w_1(\tau) + \gamma \in \bar{D}_A.$$

In the same way we get

$$(16.33) \quad v_2'(t) \in \bar{D}_A$$

and

$$(16.34) \quad \Delta w_2(t) + \gamma \in \bar{D}_A.$$

If we set for  $t, \tau \in [0, T]$  and  $x \in \bar{\Omega}$

$$(16.35) \quad \begin{cases} u_1(\tau)(x) = u_1(\tau, x) \\ u_2(t)(x) = u_2(t, x) \end{cases}$$

condition (16.6) can be written as

$$(16.36) \quad u_1'(0) + u_2'(0) = \Delta u_1(0) + \gamma = \Delta u_2(0) + \gamma.$$

From (16.12)<sub>2</sub> and (16.21)<sub>2</sub> we deduce

$$(16.37) \quad \Delta v_1(0) = \Delta v_2(0) = 0$$

hence

$$(16.38) \quad w_1'(0) + w_2'(0) = \Delta w_1(0) + \gamma = \Delta w_2(0) + \gamma.$$

Let us observe that (16.2), (16.14) and (16.23) imply

$$(16.39) \quad w_1, w_2 \in C(0, T; D_A) \cap C^1(0, T; E)$$

and so for  $t, \tau \in [0, T]$

$$(16.40) \quad w_1'(\tau), w_2'(t) \in \overline{D}_\Delta.$$

Finally let us consider problem

$$(16.41) \quad \begin{cases} u_t(t, \tau, x) + u_\tau(t, \tau, x) = \Delta u(t, \tau, x) + f(0, 0, x); & t, \tau \in [0, T], x \in \overline{\Omega} \\ u(0, \tau, x) = w_1(\tau, x) & ; \tau \in [0, T], x \in \overline{\Omega} \\ u(t, 0, x) = w_2(t, x) & ; t \in [0, T], x \in \overline{\Omega} \\ u(t, \tau, x) = 0 & ; t, \tau \in [0, T], x \in \Gamma. \end{cases}$$

Setting

$$(16.42) \quad u(t, \tau)(x) = u(t, \tau, x)$$

(16.41) can be written as a first-order partial differential equation with values in the Banach space  $C(\overline{\Omega})$

$$(16.43) \quad \begin{cases} u_t(t, \tau) + u_\tau(t, \tau) = \Delta u(t, \tau) + \gamma & ; t, \tau \in [0, T] \\ u(0, \tau) = w_1(\tau) & ; \tau \in [0, T] \\ u(t, 0) = w_2(t) & ; t \in [0, T]. \end{cases}$$

As  $\Delta$  generates an analytic semigroup in  $C(\overline{\Omega})$  (see example 14.5) we easily obtain a formula for the possible solution

$$(16.44) \quad u(t, \tau) = \begin{cases} e^{\Delta t} w_1(\tau - t) + \int_0^t e^{\Delta s} \gamma ds, & 0 \leq t \leq \tau \leq T \\ e^{\Delta \tau} w_2(t - \tau) + \int_0^\tau e^{\Delta s} \gamma ds, & 0 \leq \tau \leq t \leq T. \end{cases}$$

This definition is consistent for  $t = \tau$  by virtue of (16.29). Let us prove that  $u(t, \tau)$  verify (16.43). By using (16.32), (16.39) and (16.40) we deduce the

existence of

$$(16.45) \quad u_t(t, \tau) = \begin{cases} -e^{\Delta t} w_1'(\tau - t) + e^{\Delta t} (\Delta w_1(\tau - t) + \gamma), & 0 \leq t \leq \tau \leq T \\ e^{\Delta \tau} w_2'(t - \tau) & , \quad 0 \leq \tau \leq t \leq T \end{cases}$$

and of

$$(16.46) \quad u_r(t, \tau) = \begin{cases} e^{\Delta t} w_1'(\tau - t) & , \quad 0 \leq t \leq \tau \leq T \\ -e^{\Delta t} w_2'(t - \tau) + e^{\Delta \tau} (\Delta w_2(t - \tau) + \gamma), & 0 \leq \tau \leq t \leq T \end{cases}$$

which are consistent for  $t = \tau$  by virtue of (16.38). In addition by using (10.10) we get

$$(16.47) \quad \Delta u(t, \tau) = \begin{cases} e^{\Delta t} (\Delta w_1(\tau - t) + \gamma) - \gamma, & 0 \leq t \leq \tau \leq T \\ e^{\Delta \tau} (\Delta w_2(t - \tau) + \gamma) - \gamma, & 0 \leq \tau \leq t \leq T. \end{cases}$$

From (16.45)-(16.47) we derive (16.43). In addition from (16.34), (16.39) and (16.40) we deduce that  $(t, \tau) \rightarrow u_t(t, \tau)$ ,  $u_r(t, \tau)$ ,  $\Delta u(t, \tau)$  are continuous from  $[0, T]^2$  to  $C(\bar{\Omega})$ ; the same property is satisfied by the solutions of (16.9) and (16.18). In conclusion the sum of the solutions of (16.9), (16.18) and (16.41) is a solution of (16.8) and satisfies (16.7). The uniqueness of this kind of solution is easily proved because if  $u$  verifies (16.7) and (16.8) with  $f = u_1 = u_2 = 0$  then necessarily  $u$  is given by (16.44) with  $w_1 = w_2 = \gamma = 0$  and therefore  $u = 0$ .

REMARK 16.2. It can be checked that if (16.1) holds and there exists a solution  $u$  of (16.8) with property (16.7) then (16.2)-(16.6) must be necessarily satisfied.

We can study problem (16.8) by means of the parabolic theory without supposing the differentiability of  $f$ : in this case  $u$  is differentiable only along the characteristics thus the left-hand side of (16.8) must be interpreted as the derivative of  $u$  along their direction: this has a physical meaning in some biological problems arising in the study of age-structured populations (see [4]).

THEOREM 16.3. Let  $\Omega$  be a bounded open set of  $\mathbb{R}^n$  with regular boundary  $\Gamma$ ,  $T > 0$  and let  $f(t, \tau, x)$ ,  $u_1(\tau, x)$  and  $u_2(t, x)$  be such that, setting

$$(16.48) \quad \begin{cases} f_1(t, \tau)(x) = f(t, \tau, x) \\ u_1(\tau)(x) = u_1(\tau, x) \\ u_2(t)(x) = u_2(t, x), \end{cases}$$

we have  $f : [0, T]^2 \rightarrow C(\bar{\Omega})$ ;  $u_1, u_2 : [0, T] \rightarrow C(\bar{\Omega})$ . Moreover assume that

$$(16.49) \quad \sup\{|\delta|^{-\alpha} \|f(t + \delta, \tau + \delta) - f(t, \tau)\|_{C(\bar{\Omega})}; (t, \tau), (t + \delta, t + \delta) \in [0, T]^2\} < +\infty$$

for some  $\alpha \in ]0, 1[$ . Then, if

$$(16.50) \quad u_1(\tau), u_2(t) \in D_\Delta; \quad t, \tau \in [0, T]$$

and

$$(16.51) \quad \Delta u_1(\tau, x) + f(0, \tau, x) = \Delta u_2(t, x) + f(t, 0, x) = 0, \quad t, \tau \in [0, T], \quad x \in \Gamma$$

(where  $D_\Delta$  is defined in (14.18)), there exists a unique  $u(t, \tau, x)$  continuous in  $[0, T]^2 \times \bar{\Omega}$  such that  $u(t, \tau, \cdot) \in D_\Delta$  for  $(t, \tau) \in [0, T]^2$  and verifies

$$(16.52) \quad \begin{cases} Du(t, \tau, x) = \Delta u(t, \tau, x) + f(t, \tau, x) & ; t, \tau \in [0, T], \quad x \in \bar{\Omega} \\ u(0, \tau, x) = u_1(\tau, x) & , \tau \in [0, T] \quad , x \in \bar{\Omega} \\ u(t, 0, x) = u_2(t, x) & , t \in [0, T] \quad , x \in \bar{\Omega} \\ u(t, \tau, x) = 0 & , t, \tau \in [0, T] \quad , x \in \Gamma \end{cases}$$

where

$$(16.53) \quad Du(t, \tau, x) = \lim_{h \rightarrow 0} \frac{u(t+h, \tau+h, x) - u(t, \tau, x)}{h}.$$

PROOF. Setting

$$(16.54) \quad u(t, \tau)(x) = u(t, \tau, x)$$

and using (16.48) we can write (16.52) as a problem in the Banach space  $C(\bar{\Omega})$

$$(16.55) \quad \begin{cases} Du(t, \tau) = \Delta u(t, \tau) + f(t, \tau), & t, \tau \in [0, T] \\ u(0, \tau) = u_1(\tau) & , \tau \in [0, T] \\ u(t, 0) = u_2(t) & , t \in [0, T]. \end{cases}$$

If there exists a solution (in the sense specified above) choosing  $(t_0, \tau_0) \in [0, T]^2$  with  $t_0 = 0$  or  $\tau_0 = 0$ , there exists  $T_0 > 0$  such that if  $h \in [0, T_0]$ , then  $(t_0 + h, \tau_0 + h) \in [0, T]^2$ ; hence, setting

$$(16.56) \quad \begin{cases} u(h) = u(t_0 + h, \tau_0 + h), & h \in [0, T_0] \\ f(h) = f(t_0 + h, \tau_0 + h), & h \in [0, T_0], \end{cases}$$

we deduce from (16.55) (with  $t = t_0 + h$  and  $\tau = t_0 + h$ ) that  $u(\cdot)$  satisfies

$$(16.57) \quad \begin{cases} u'(h) = \Delta u(h) + f(h), & h \in [0, T_0] \\ u(0) = u_0 \end{cases}$$

where  $u_0 = u_1(\tau_0)$  if  $t_0 = 0$  and  $u_0 = u_2(t_0)$  if  $\tau_0 = 0$ . From the results of section 12 we deduce the uniqueness of the solution: to prove its existence we can use Theorem 13.2 with  $E = C(\bar{\Omega})$  and  $A = \Delta$ . In fact, for each  $(t_0, \tau_0)$

we have  $f(\cdot) \in C^\alpha(0, T_0; C(\bar{\Omega}))$  by virtue of (16.49) whereas (16.50) implies  $u_0 \in D_\Delta$  and from (16.51) we deduce  $\Delta u_0 + f(0) = \Delta u_1(\tau_0) + f(0, \tau_0) \in \bar{D}_\Delta$  when  $t_0 = 0$  and  $\Delta u_0 + f(0) = \Delta u_2(t_0) + f(t_0, 0) \in \bar{D}_\Delta$  when  $\tau_0 = 0$ . Therefore problem (16.57) has a solution  $u \in C^1(0, T_0; C(\bar{\Omega})) \cap C(0, T_0; D_\Delta)$ . Now defining  $u(t, \tau, x)$  by means of (16.54) we obtain a solution of (16.52): in fact for each given  $(t, \tau) \in [0, T]^2$  we can set  $t_0 = t - \tau$ ,  $\tau_0 = 0$ ,  $h = \tau$  if  $t \geq \tau$  and  $t_0 = 0$ ,  $\tau_0 = \tau - t$ ,  $h = t$  if  $t \leq \tau$ . In this way (16.57) reduces to (16.55) which in turn implies (16.52) because  $u(t, \tau) \in \bar{D}_\Delta$ .

In the next example we want to exhibit a situation in which the hyperbolic theory (with non dense domain) seems to be the unique way of studying the evolution problem.

**17. - Generalized Laplacian in infinite dimensional spaces**

The problem which will be introduced here arises in the theory of stochastic control: we only sketch the proofs and refer the reader to [3] for more details and motivations.

Let  $H$  be a separable Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and a complete orthonormal set  $\{e_k\}_{k \in \mathbb{N}}$ .

Let  $E$  be the Banach space of functions  $\phi$ , uniformly continuous and bounded from  $H$  to  $\mathbb{R}$ , endowed with the sup-norm.

For each  $k \in \mathbb{N}$  let us define the linear operators  $D_k$  and  $A_k$  as follows:

$$(17.1) \quad \left\{ \begin{array}{l} D_k : D(D_k) \subseteq E \rightarrow E \\ D(D_k) = \{ \phi \in E; \text{there exists } \lim_{h \rightarrow 0^+} \frac{\phi(x + he_k) - \phi(x)}{h} \text{ uniformly for} \\ \hspace{15em} x \in H \} \\ (D_k \phi)(x) = \lim_{h \rightarrow 0^+} \frac{\phi(x + he_k) - \phi(x)}{h} \end{array} \right.$$

$$(17.2) \quad \left\{ \begin{array}{l} A_k : D(A_k) \subseteq E \rightarrow E \\ D(A_k) = \{ \phi \in D(D_k); A_k \phi \in D(D_k) \} \\ A_k \phi = \frac{1}{2} D_k(D_k \phi). \end{array} \right.$$

$A_k$  generates an analytic semigroup in  $E$  given by

$$(17.3) \quad (e^{A_k t} \phi)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2t}\right) \phi(x - y e_k) dy, \quad t > 0.$$

Let  $S$  be a positive nuclear operator in  $H$  such that

$$(17.4) \quad S e_k = \lambda_k e_k, \quad k \in \mathbb{N}$$



with  $\lambda_k > 0$  verifying  $\sum_{k=1}^{\infty} \lambda_k < +\infty$ . The generalized Laplacian in  $E$  (corresponding to  $S$ ) is the operator

$$(17.5) \quad \begin{cases} \Delta : D(\Delta) \subseteq E \rightarrow E \\ D(\Delta) = \{\phi \in E; \phi', \phi'' \in E\} \\ \Delta\phi = \frac{1}{2} \text{Tr}(S\phi'') = \sum_{k=1}^{\infty} \lambda_k A_k \phi \end{cases}$$

(here  $\phi'$  and  $\phi''$  are the Fréchet-derivatives).

It can be proved that the closure of  $\Delta$  coincides with the generator

$$(17.6) \quad \mathcal{A} : D(\mathcal{A}) \subset E \rightarrow E$$

of the strongly continuous semigroup in  $E$  defined by

$$(17.7) \quad T(t)\phi = \lim_{n \rightarrow \infty} \prod_{i=1}^n e^{\lambda_i A_i t} \phi, \quad t > 0.$$

Finally, let us suppose that

$$(17.8) \quad B : D(B) \subset H \rightarrow H$$

is the generator of a strongly continuous semigroup  $e^{Bt}$  in  $H$  and consider the problem

$$(17.9) \quad \begin{cases} u_t(t, x) = \frac{1}{2} \text{Tr}(S u_{xx}(t, x)) - \langle Bx, u_x(t, x) \rangle + f(t, x); \\ u(0, x) = u_0(x), \quad x \in H \end{cases} \quad \begin{matrix} t \in [0, T], \\ x \in H \end{matrix}$$

where  $f : [0, T] \times H \rightarrow \mathbb{R}$  and  $u_0 : H \rightarrow \mathbb{R}$  are given. To write it as an evolution problem in  $E$  let us introduce the operator

$$(17.10) \quad \begin{cases} \mathcal{B} : D(\mathcal{B}) \subset E \rightarrow E \\ D(\mathcal{B}) = \left\{ \phi \in E; \text{there exists } \lim_{h \rightarrow 0^+} \frac{\phi(e^{Bh}x) - \phi(x)}{h} \text{ uniformly for } \right. \\ \left. x \in H \right\} \\ (\mathcal{B}\phi)(x) = \lim_{h \rightarrow 0^+} \frac{\phi(e^{Bh}x) - \phi(x)}{h} \end{cases}$$

Now  $\phi \in D(\mathcal{B})$  if and only if there exists

$$(17.11) \quad \lim_{h \rightarrow 0^+} \frac{\phi(x + hBx) - \phi(x)}{h} \stackrel{\text{def}}{=} \langle Bx, \phi'(x) \rangle$$

uniformly for  $x \in D(\mathcal{B})$ : hence for each  $\phi \in D(\mathcal{B})$ , the mapping  $x \rightarrow$

$\langle Bx, \phi'(x) \rangle$  can be extended to  $E$ . In conclusion we can write the abstract version in  $E$  of (17.9) as follows:

$$(17.12) \quad \begin{cases} u'(t) = Au(t) + Bu(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases}$$

where we have set  $u(t)(x) = u(t, x)$  and  $f(t)(x) = f(t, x)$ . If  $B$  is unbounded then  $B$  verifies estimate (14.1) but  $\overline{D(B)} \neq E$ ; moreover it can be proved that  $A + B$  with domain  $D(A) \cap D(B)$  is closable in  $E$  and if  $\lambda > 0$  then  $\lambda \in \phi(\overline{A + B})$  and

$$(17.13) \quad \|(\lambda - \overline{A + B})^{-1}\|_{\mathcal{L}(E)} \leq \frac{1}{\lambda}$$

This lets us apply the hyperbolic theory to problem (17.12) and to get solutions of (17.9) by using the existence theorems of sections 7-9 (see [3]).

### Appendix

In many applications it is important to consider operators  $B$  such that  $A = B - \omega I$  satisfies (1.1) for some  $\omega \in \mathbb{R}$ . The extension of the previous theory to this situation is very simple but it will be useful to write explicitly some less obvious results (as the estimates for the solutions).

Let us suppose that

$$B : D_B \subseteq E \rightarrow E$$

is a linear operator such that there exists  $\omega \in \mathbb{R}$  verifying

$$(A.1) \quad \begin{cases} \lambda > \omega \Rightarrow (\lambda - B)^{-1} \in \mathcal{L}(E) \\ \sup_{k \in \mathbb{N}, \lambda > \omega} \|(\lambda - \omega)^k (\lambda - B)^{-k}\|_{\mathcal{L}(E)} = M < \infty \end{cases}$$

If we consider problem

$$(A.2) \quad \begin{cases} v'(t) = Bv(t) + g(t), & t \in [0, T] \\ v(0) = v_0 \end{cases}$$

where  $g : [0, T] \rightarrow E$  and  $v_0 \in E$  are given, we can define in an obvious way all the types of solutions introduced for problem (1.2).

If we set

$$(A.3) \quad \begin{cases} A : D_A \subseteq E \rightarrow E \\ D_A = D_B \\ Au = Bu - \omega u \end{cases}$$

then  $A$  verifies (1.1) and it can be checked that to each solution  $u$  of problem

$$(A.4) \quad \begin{cases} u'(t) = Au(t) + e^{-\omega t}g(t), & t \in [0, T] \\ u(0) = v_0 \end{cases}$$

corresponds a solution  $v$  (of the same type) of (A.2) given by

$$(A.5) \quad v(t) = e^{\omega t}u(t), \quad t \in [0, T]$$

and conversely.

From theorems 5.1, 7.2 and 9.1 we deduce directly:

**THEOREM A.1.** *Let (A.1) hold. Problem (A.2) has a unique F-solution  $v$  in  $L^p$  for each  $g \in L^p(0, T; E)$  and  $u_0 \in \overline{D}_B$ ; this solution verifies the estimate*

$$(A.6) \quad \|v(t)\| \leq M e^{\omega t}(\|v(0)\| + \int_0^t \|e^{-\omega s}g(s)\| ds), \quad t \in [0, T].$$

*If in addition  $f \in C(0, T; E)$  then  $v$  is an F-solution also in  $C$ .*

About the strict solutions we have the following result

**THEOREM A.2.** *Let (A.1) hold and let  $g \in W^{1,p}(0, T; E)$ ,  $v_0 \in D_B$  and  $Bv_0 + g(0) \in \overline{D}_B$ : then there exists a unique solution  $v \in C^1(0, T; E) \cap C(0, T; D_B)$  of (A.2). Moreover we have  $v'(t) \in \overline{D}(A)$  and*

$$(A.7) \quad \|v'(t)\| \leq M e^{\omega t}(\|Bv_0 + g(0)\| + \int_0^t \|e^{-\omega s}g'(s)\| ds); \quad t \in [0, T]$$

as  $w = v'$  is an F-solution in  $L^p$  of

$$\begin{cases} w'(t) = Bw(t) + g'(t), & t \in [0, T] \text{ a.e.} \\ w(0) = Bv_0 + g(0). \end{cases}$$

**PROOF.** We can proceed as in the proof of Theorem 8.1 by showing that if  $w$  is the integral solution of

$$(A.8) \quad \begin{cases} w'(t) = Bw(t) + g'(t), & t \in [0, T] \\ w(0) = Bv_0 + g(0) \end{cases}$$

then  $v(t) = v_0 + \int_0^t w(s) ds$  is a strict solution in  $C$  of (A.2).

Estimate (A.7) is a consequence of the fact that  $w$  is an F-solution of (A.8) and therefore (A.6) can be used.

We end with an existence result in the case in which  $g$  has values in  $D_B$ : its proof is similar to that of Theorem 8.3:

THEOREM A.3. Let (A.1) hold,  $g \in L^p(0, T; D_B)$ ,  $v_0 \in D_B$  and  $Bv_0 \in \bar{D}_B$ . Then there exists a unique  $v \in W^{1,p}(0, T; E) \cap C(0, T; D_B)$  such that (A.2) holds a.e. in  $[0, T]$  and the following estimate is true

$$(A.9) \quad \|Bv(t)\| \leq M e^{\omega t} (\|Bv_0\| + \int_0^t \|e^{-\omega s} Bg(s)\| ds), \quad t \in [0, T].$$

Moreover  $w = Bv$  is F-solution in  $L^p$  of

$$(A.10) \quad \begin{cases} w'(t) = Bw(t) + Bg(t), & t \in [0, T] \text{ a.e.} \\ w(0) = Bv_0 \end{cases}$$

and therefore  $Bw(t) \in \overline{D(A)}$  for  $t \in [0, T]$ .

If in addition  $g \in L^p(0, T; D_B) \cap C(0, T; E)$  then we have also  $v \in C^1(0, T; E) \cap C(0, T; D_B)$  and (A.2) holds for each  $t \in [0, T]$ .

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