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Differential Quadrature Analysis of Free Vibration of Symmetric Cross-Ply Laminates with Shear Deformation and Rotatory Inertia

In the present work, laminates having two opposite edges simply supported are considered. The boundary conditions at the other two opposite edges may be general, and between these two edges, the thickness of the plate may be nonuniform. The theory used for the vibration analysis of such laminates includes shear deformation and rotatory inertia. The solution approach of the problem is semianalytical. By using the trigonometric functions describing the mode shapes between the simply supported edges, the governing plate equations are reduced to ordinary differential equations. The solution of the reduced equations is then sought by the differential quadrature method. The results reported in this article serve two objectives of the present investigations. One, it is demonstrated that the proposed semianalytical quadrature method offers a numerically accurate and computationally efficient technique for the title problem. Two, the relative effects of shear deformation and rotatory inertia are analyzed in a quantitative manner.

INTRODUCTION

In the classical thin plate theory (CPT), the assumptions of the neglect of (transverse) shear deformations and rotatory inertia are consistent with the underlying assumption of the plate being thin. These assumptions can no longer be considered to hold true for thick plates. However, the inadequacy of the CPT in practical applications where the plate systems may even be thin in connotation of the theory, is now well recognized. In a vibrating plate, for example, the neglect of shear deformation and rotatory inertia may not be justified due, respectively, to the wavelengths becoming so small to be comparable to plate thickness and the high frequency of vibration. The other example is of plates made of anisotropic materials, such as fibrous composites, in which the shear moduli of rigidity may be quite small relative to the in-plane elastic moduli and

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the consideration of resulting large transverse shear deformations may by itself be necessary.

The very first vibration theory of isotropic plates with transverse shear deformation was set forth by Mindlin (1951). There have been some additions in the shear deformable plate theories (Noor and Burton, 1989); however, Mindlin plate theory has been the basis of most developments in the analysis of both isotropic and anisotropic plates with shear deformation and inertia effects. The Mindlin theory was extended to laminated anisotropic plates by Yang et al. (1966). Later, Whitney and Pagano (1970) used the theory of Yang et al. for application to plates comprised of an arbitrary number of monoclinic layers and made some modification in the theory. A comparison of the two theories was made by Wang and Chou (1972) who observed that the modified theory of Whitney and Pagano produced more accurate results than that of Yang et al. The matter of shear deformation and rotatory inertia effects has also been investigated in an exact manner via three-dimensional elasticity solutions of simply supported isotropic and laminated composite plates; see, for example, the contributions of Lee and Reismann (1969), Pagano (1970), Srinivas and Rao (1970), and Iyengar and Raman (1977). Exact solutions of the Mindlin equations for simply supported plates and for plates having two opposite edges simply supported and the other two free were provided by Mindlin et al. (1956). The Mindlin plate analyses with other boundary conditions have been considered by many investigators and various techniques have been employed for the solution of the Mindlin equations. One may see for example, the isotropic plate solutions based on the Rayleigh–Ritz method (Dawe and Roufaiel, 1980; Liew et al., 1993), the finite element method (Greimann and Lynn, 1970; Hinton and Bicanic, 1979), the finite strip method (Benson and Hinton, 1976; Dawe, 1978; Roufaiel and Dawe, 1980), the collocation method (Mikami and Yoshimura, 1984), the finite difference method (Aksu and Al-Kaabi, 1987), and the spline strip method (Mizusawa, 1993). On the analysis of anisotropic and laminated plates, one may find the exact (Bert and Chen, 1978) and the finite element (Reddy, 1979) solutions for simply supported antisymmetric angle-ply laminates, the finite strip method solution for laminates with two opposites edges simply supported (Hinton, 1976), and the Rayleigh–Ritz and finite strip analyses of symmetric laminates (Craig and Dawe, 1986).

The present study concerns the free vibration analysis of symmetric cross-ply laminates including the effects of shear deformation and rotatory inertia. The laminate configurations considered in this work are the ones having two opposite edges simply supported and with general boundary conditions at the other two edges between which the plate thickness may be varying. The study was undertaken with a few objectives. It presents the very first application of the differential quadrature method (DQM) for the problem under consideration. During recent years, research interests in plate vibration problems have reactivated. The major interests are in the evaluation of free vibration characteristics of plates with complicating effects, such as thickness nonuniformity, shear deformation and rotatory inertia, general polygonal boundaries, nonclassical boundary conditions, material anisotropy, and so on. However, with inclusion of one or more complicating effects, the task of generating vibration characteristic data for all possible combinations of geometric and/or material parameters may become boundless. From a practical viewpoint, the database generated for some limited values of design parameters often serves little purpose because the interpolated characteristics for the parameters of the actual problem may not be sufficiently accurate and the interpolation process itself may be time consuming. Further, the data base may need substantial computer storage. A better alternative to handle this situation is to have computer codes based on accurate and efficient numerical solution techniques that may be used for real time analysis and design. The problem undertaken in the present work, even with a few limitations on the laminations and the boundary conditions, can have unlimited combinations of the geometry and material parameters. As shown by the results of the present work, the DQM offers a highly accurate technique and due to the small computation times required in the evaluation of vibration characteristics data, it has the potential for real time analysis and design purposes.

It is known that of the two complicating effects, the shear deformation is a dominating effect compared to the effect of rotatory inertia. This study presents a comprehensive quantitative comparison of the effects of shear deformation only vis-à-vis shear deformation with rotatory inertia vibration characteristics. The results conform to the known facts; however, more importantly, it is shown that the neglect of rotatory inertia can be advantageous for efficient evaluation of the vibration characteristics.

The article includes results on tapered specially orthotropic laminates. To the knowledge of the present investigators, tapered orthotropic plates with shear deformation and rotatory inertia effects have not been analyzed earlier, and as such, the analysis and the results should be new to the plate vibration literature.

As mentioned earlier, in this work, DQM is used for the solution of the title problem and, therefore, a brief review of the method would be in order. The DQM was proposed in the early 1970s (Bellman and Casti, 1971; Bellman et al., 1972) as a technique for the rapid numerical solution of nonlinear differential equations. Bert et al. (1988, 1989) and Jang et al. (1989) introduced the method as a tool for structural analysis. During recent years, there have been many publications on the development of the method itself, such as the techniques for implementation of boundary conditions (Wang and Bert, 1993), the use of harmonic test functions (Bert et al., 1993), and on the new applications of the method (Wang et al., 1994; Malik and Bert, 1994; Laura and Gutierrez, 1993, 1994). These published works have placed the DQM on a strong footing and the method is fast developing to be a potential alternative to the conventional numerical techniques.

In the following, the relevant governing equations, solution details, and results of the investigations are presented.

THE GOVERNING EQUATIONS

Consider a symmetric cross-ply laminate having sides of lengths a and b along the x and y axes, respectively, and let its thickness h be varying in the y direction, that is, $h = h(y)$. The governing equations of free vibrations of the laminate may be derived following the analysis of Whitney and Pagano (1970), recalling, however, that in the case of a symmetric cross-ply laminate, all coupling stiffness coefficients vanish, that is,

$$B_{ij} = 0; \quad i, j = 1, 2, 6;$$
$$A_{16} = A_{26} = D_{16} = D_{26} = 0; \quad A_{45} = A_{54} = 0$$

where A_{ij}, B_{ij}, and D_{ij} used here and in the later equations, are the symbols for the plate stiffness coefficients as commonly employed in the laminate theories (Whitney, 1987). Further, due to nonuniform thickness in the y direction, the nonzero stiffness coefficients of the laminate are also varying in the y direction. The derived equations may be written in dimensionless form as:

$$\begin{aligned}&-12k_5^2\beta^2\bar{A}_{55}\frac{\partial^2 W}{\partial X^2} - 12k_4^2\beta^2\lambda^2\bar{A}_{44}\left(\frac{\partial^2 W}{\partial Y^2} + \frac{dH}{dY}\frac{\partial W}{\partial Y}\right)\\ &+12k_5^2\beta^3\bar{A}_{55}\frac{\partial \Phi}{\partial X} + 12k_4^2\beta^3\lambda\bar{A}_{44}\left(\frac{\partial \Psi}{\partial Y} + \frac{1}{H}\frac{dH}{dY}\Psi\right) = \Omega^2 W\end{aligned} \tag{1}$$

$$\begin{aligned}&-144k_5^2\beta^3\bar{A}_{55}\frac{1}{H^2}\frac{\partial W}{\partial X} - 12\beta^2\bar{D}_{11}\frac{\partial^2 \Phi}{\partial X^2} - 12\beta^2\lambda^2\bar{D}_{66}\left(\frac{\partial^2 \Phi}{\partial Y^2} + \frac{3}{H}\frac{dH}{dY}\frac{\partial \Phi}{\partial Y}\right)\\ &+144k_5^2\beta^4\bar{A}_{55}\frac{1}{H^2}\Phi - 12\beta^2\lambda(\bar{D}_{12} + \bar{D}_{66})\frac{\partial^2 \Psi}{\partial X \partial Y} - 36\beta^2\lambda\bar{D}_{66}\frac{1}{H}\frac{dH}{dY}\frac{\partial \Psi}{\partial X} = \Omega^2\Phi\end{aligned} \tag{2}$$

$$\begin{aligned}&-144k_4^2\beta^3\lambda\bar{A}_{44}\frac{1}{H^2}\frac{\partial W}{\partial Y} - 12\beta^2\lambda(\bar{D}_{12} + \bar{D}_{66})\frac{\partial^2 \Phi}{\partial X \partial Y} - 36\beta^2\lambda\bar{D}_{12}\frac{1}{H}\frac{dH}{dY}\frac{\partial \Phi}{\partial X}\\ &-12\beta^2\bar{D}_{66}\frac{\partial^2 \Psi}{\partial X^2} - 12\beta^2\lambda^2\bar{D}_{22}\left(\frac{\partial^2 \Psi}{\partial Y^2} + \frac{3}{H}\frac{dH}{dY}\frac{\partial \Psi}{\partial Y}\right) + 144k_4^2\beta^4\bar{A}_{44}\frac{1}{H^2}\Psi = \Omega^2\Psi\end{aligned} \tag{3}$$

where it is further assumed that the plies of the laminate are of the same orthotropic material and that the vibratory motion of the laminate is harmonic. In these equations, $X = x/a$ and $Y = y/b$ are the dimensionless coordinates, $\lambda = a/b$ is the aspect ratio, $H = H(Y) = h/h_o$ is the dimensionless thickness function, h_o is some reference plate thickness, and $\beta = a/h_o$. Further, $W = W(X, Y) = w(x, y)/h_o$ is the lateral displacement mode function, $\Phi = \Phi(X, Y)$ and $\Psi = \Psi(X, Y)$ are the mode functions of cross-sectional rotations in the x and y directions, respectively, and

k_4^2 and k_5^2 are shear correction factors that account for the nonuniformity of the transverse shear strain distributions through the laminate thickness. The dimensionless frequency Ω is defined as

$$\Omega^2 = \frac{\rho h_o a^4}{D_o} \omega^2 \quad (4)$$

where ω is the circular frequency of free vibrations (in rad/s), ρ is the density of the laminate material, and D_o is the characteristic flexural stiffness:

$$D_o = \frac{E_L h_o^3}{12(1 - \nu_{LT}\nu_{TL})} \quad (5)$$

in which the subscripts L and T refer, respectively, to the directions parallel and perpendicular to the fibers in the plane of the laminate. The symbols E and ν denote the elastic modulus and Poisson's ratio, respectively.

The dimensionless flexural stiffnesses of the plate are defined as:

$$\begin{aligned} \bar{D}_{ij} &= \frac{1}{D_o} \int_{-h_o/2}^{h_o/2} Q_{ij}^l z^2 \, dz; \quad i, j = 1, 2; \\ \bar{D}_{66} &= \frac{1}{D_o} \int_{-h_o/2}^{h_o/2} Q_{66}^l z^2 \, dz \\ \bar{A}_{44} &= \frac{1}{12D_o} \int_{-h_o/2}^{h_o/2} Q_{44}^l z \, dz, \\ \bar{A}_{55} &= \frac{1}{12D_o} \int_{-h_o/2}^{h_o/2} Q_{55}^l z \, dz \end{aligned} \quad (6)$$

where l designates the layer-wise coefficients; Q_{11}^l, Q_{22}^l, Q_{12}^l, and Q_{66}^l are plane-stress reduced-stiffness coefficients; and Q_{44}^l and Q_{55}^l are shear-stiffness coefficients. In general, these coefficients are different for each ply of the laminate and the layer-wise integrations in Eqs. (6) are implied. Expressions for these coefficients in terms of orthotropic elastic constants are given in Table 1. Note that the plies are transversely isotropic so that $G_{Lz} = G_{LT}$ where z is the through-thickness coordinate.

Equations (1), (2), and (3) are applicable to isotropic material plates with

$$\bar{D}_{11} = \bar{D}_{22} = 1, \quad \bar{D}_{12} = \nu, \\ \bar{D}_{66} = \bar{A}_{44} = \bar{A}_{55} = (1 - \nu)/2.$$

For isotropic materials, the shear correction factors in the two transverse directions are of course the same, that is, $k_4^2 = k_5^2 = k^2$. A commonly used value of k^2 for isotropic materials, following the work of Mindlin (1951), is $\pi^2/12$. For laminated plates, the shear correction factors would generally be different in the two transverse directions; the two factors depend on the properties of the individual layers and the laminate construction. The matter of the shear correction factors of laminated constructions has been considered and procedures for their evaluation under static bending have been given by some authors; see for example, Chow (1971), Whitney (1973), and Bert (1973).

The particular plate configurations under consideration are the ones having two x edges, $X = 0$ and 1, simply supported. In that case, one may represent the mode functions in the following well-known forms:

$$W = \bar{W}(Y)\sin m\pi X, \quad \Phi = \bar{\Phi}(Y)\text{cox } m\pi X, \\ \Psi = \bar{\Psi}(Y)\sin m\pi X \quad (7)$$

where m is an integer and represents the number of half-waves between the two x edges of the vibrating laminate. Also, the functions $\bar{W}$, $\bar{\Phi}$, and

Table 1. Plane Stress Reduced Stiffness and Shear Stiffness Coefficients in Terms of Elastic Constants of Orthotropic Ply

Stiffness Coefficient	Expressions in terms of Elastic Constants for	
	0° ply	90° ply
Q_{11}^l	$E_L/(1 - \nu_{LT}\nu_{TL})$	$E_T/(1 - \nu_{LT}\nu_{TL})$
Q_{22}^l	$E_T/(1 - \nu_{LT}\nu_{TL})$	$E_L/(1 - \nu_{LT}\nu_{TL})$
Q_{12}^l	$\nu_{LT}E_T/(1 - \nu_{LT}\nu_{TL}) = \nu_{TL}E_L/(1 - \nu_{LT}\nu_{TL})$	
Q_{66}^l	G_{LT}	G_{LT}
Q_{44}^l	G_{TT}	G_{LT}
Q_{55}^l	G_{LT}	G_{TT}

$\bar{\Psi}$ define the mode shapes between the two y edges.

Equations (7) satisfy the boundary conditions of the two x edges. Further, using Eqs. (7), the eigenvalue partial differential equations (1), (2), and (3) are reduced to the following ordinary differential equations:

$$-12k_4^2\beta^2\lambda^2\bar{A}_{44}\left(\frac{d^2\bar{W}}{dY^2}+\frac{dH}{dY}\frac{d\bar{W}}{dY}\right)+12k_5^2\beta^2\bar{A}_{55}(m\pi)^2\bar{W}$$
$$-12k_5^2\beta^3\bar{A}_{55}(m\pi)\bar{\Phi}+12k_4^2\beta^3\lambda\bar{A}_{44}\left(\frac{d\bar{\Psi}}{dY}+\frac{1}{H}\frac{dH}{dY}\bar{\Psi}\right)=\Omega_{mn}^2\bar{W} \tag{8}$$

$$-144k_5^2\beta^3\bar{A}_{55}(m\pi)\frac{1}{H^2}\bar{W}-12\beta^2\lambda^2\bar{D}_{66}\left(\frac{d^2\bar{\Phi}}{dY^2}+\frac{3}{H}\frac{dH}{dY}\frac{d\bar{\Phi}}{dY}\right)$$
$$+12\beta^2\left[\bar{D}_{11}(m\pi)^2+12k_5^2\beta^2\bar{A}_{55}\frac{1}{H^2}\right]\bar{\Phi}-12\beta^2\lambda(\bar{D}_{12}+\bar{D}_{66})(m\pi)\frac{d\bar{\Psi}}{dY} \tag{9}$$
$$-36\beta^2\lambda\bar{D}_{66}(m\pi)\frac{1}{H}\frac{dH}{dY}\bar{\Psi}=\Omega_{mn}^2\bar{\Phi}$$

$$-144k_4^2\beta^3\lambda\bar{A}_{44}\frac{1}{H^2}\frac{d\bar{W}}{dY}+12\beta^2\lambda(\bar{D}_{12}+\bar{D}_{66})(m\pi)\frac{d\bar{\Phi}}{dY}$$
$$+36\beta^2\lambda\bar{D}_{12}(m\pi)\frac{1}{H}\frac{dH}{dY}\bar{\Phi}-12\beta^2\lambda^2\bar{D}_{22}\left(\frac{d^2\bar{\Psi}}{dY^2}+\frac{3}{H}\frac{dH}{dY}\frac{d\bar{\Psi}}{dY}\right) \tag{10}$$
$$+12\beta^2\left[\bar{D}_{66}(m\pi)^2+12k_4^2\beta^2\bar{A}_{44}\frac{1}{H^2}\right]\bar{\Psi}=\Omega_{mn}^2\bar{\Psi}$$

where the dimensionless frequency is now denoted by Ω_{mn} as one associated with the m n mode; n being the number of half-waves in the y direction of the vibrating laminate.

The boundary conditions considered at the two y edges are the combinations of simply supported (SS), clamped (C), and free (F) edge conditions. These conditions, in terms of $\bar{W}$, $\bar{\Phi}$, and $\bar{\Psi}$ functions, are given as: *simply supported edge*

$$\bar{W}=0,\quad \bar{\Phi}=0,\quad \frac{d\bar{\Psi}}{dY}=0; \tag{11}$$

clamped edge

$$\bar{W}=0,\quad \bar{\Phi}=0,\quad \bar{\Psi}=0; \tag{12}$$

free edge

$$\lambda\frac{d\bar{W}}{dY}-\beta\bar{\Psi}=0,$$
$$\lambda\frac{d\bar{\Phi}}{dY}+(m\pi)\bar{\Psi}=0, \tag{13}$$
$$\lambda\bar{D}_{22}\frac{d\bar{\Psi}}{dY}-\bar{D}_{12}(m\pi)\bar{\Phi}=0.$$

Thus, it may be seen that, at either of the two y edges, $Y=0$ and 1, three boundary conditions exist for a given edge condition. As usual, with dual combination of the three type of edge conditions at the two y edges, six types of plate configurations exist. The total number of boundary conditions for any plate type are six. It may also be noted that the solution domain of the governing equations, (8), (9) and (10), is $0 \le Y \le 1$.

The solution of the eigenvalue differential equations, (8), (9), and (10), in conjunction with the boundary conditions is sought by the differential quadrature method; the details of the quadrature formulation are given in the following section.

DIFFERENTIAL QUADRATURE FORMULATION

Consider a set of N sampling points $Y_i (i = 1, 2, \ldots, N)$ in the domain $0 \le Y \le 1$. Let the values of a function $F(Y)$ (representing the mode functions $\bar{W}$, $\bar{\Phi}$, and $\bar{\Psi}$) at an ith point be $F_i = F(Y_i)$. An rth-order function derivative at a point Y_i may be expressed by the *quadrature rule* as

$$\left.\frac{d^rF}{dY^r}\right|_{Y=Y_i} = \sum_{j=1}^{N} B_{ij}^{(r)} F_j \tag{14}$$

where $B_{ij}^{(r)}$ are the weighting coefficients of the rth-order derivative associated with the ith sampling point. These weighting coefficients may be obtained by having appropriate approximations (the test functions) of the function $F(Y)$.

Using the quadrature rule, Eq. (14), for various derivatives in Eqs. (8), (9), and (10), one obtains the *quadrature analog* of the three differential equations at an ith point of the solution domain as the following sets of linear algebraic equations:

$$\begin{aligned}&-12k_4^2\beta^2\lambda^2\bar{A}_{44}\sum_{j=1}^{N}\left(B_{ij}^{(2)} + \frac{dH}{dY}B_{ij}^{(1)}\right)\bar{W}_j + 12k_5^2\beta^2\bar{A}_{55}(m\pi)^2\bar{W}_i\\&-12k_5^2\beta^3\bar{A}_{55}(m\pi)\bar{\Phi}_i + 12k_4^2\beta^3\lambda\bar{A}_{44}\left(\sum_{j=1}^{N}B_{ij}^{(1)}\bar{\Psi}_j + \frac{1}{H}\frac{dH}{dY}\bar{\Psi}_i\right) = \Omega_{mn}^2\bar{W}_i\end{aligned} \tag{15}$$

$$\begin{aligned}&-144k_5^2\beta^3\bar{A}_{55}(m\pi)\frac{1}{H^2}\bar{W} - 12\beta^2\lambda^2\bar{D}_{66}\sum_{j=1}^{N}\left(B_{ij}^{(2)} + \frac{3}{H}\frac{dH}{dY}B_{ij}^{(1)}\right)\bar{\Phi}_j\\&+12\beta^2\left[\bar{D}_{11}(m\pi)^2 + 12k_5^2\beta^2\bar{A}_{55}\frac{1}{H^2}\right]\bar{\Phi}_i - 12\beta^2\lambda(\bar{D}_{12} + \bar{D}_{66})(m\pi)\sum_{j=1}^{N}B_{ij}^{(1)}\bar{\Psi}_j\\&-36\beta^2\lambda\bar{D}_{66}(m\pi)\frac{1}{H}\frac{dH}{dY}\bar{\Psi}_i = \Omega_{mn}^2\bar{\Phi}_i\end{aligned} \tag{16}$$

$$\begin{aligned}&-144k_4^2\beta^3\lambda\bar{A}_{44}\frac{1}{H^2}\sum_{j=1}^{N}B_{ij}^{(1)}\bar{W}_j + 12\beta^2\lambda(\bar{D}_{12} + \bar{D}_{66})(m\pi)\sum_{j=1}^{N}B_{ij}^{(1)}\bar{\Phi}_j\\&+36\beta^2\lambda\bar{D}_{12}(m\pi)\frac{1}{H}\frac{dH}{dY}\bar{\Phi}_i - 12\beta^2\lambda^2\bar{D}_{22}\sum_{j=1}^{N}\left(B_{ij}^{(2)} + \frac{3}{H}\frac{dH}{dY}B_{ij}^{(1)}\right)\bar{\Psi}_j\\&+12\beta^2\left[\bar{D}_{66}(m\pi)^2 + 12k_4^2\beta^2\bar{A}_{44}\frac{1}{H^2}\right]\bar{\Psi}_i = \Omega_{mn}^2\bar{\Psi}_i\end{aligned} \tag{17}$$

where, in these equations, $i = 2, 3, \ldots, (N - 1)$, that is, in obtaining the quadrature analog equations (15), (16), and (17) from the eigenvalue differential equations, the end points $i = 1$ and N are omitted. Thus, Eqs. (15), (16), and (17) yield $(3N - 6)$ equations for $3N$ discrete values of the mode functions. The remaining six equations are obtained from the quadrature analog equations of the total of six boundary conditions at the two end points and, thereby, the boundary conditions are invoked. The quadrature analogs of the boundary conditions for the three types of edges, obtained from Eqs. (11), (12), and (13) are: *simply supported edge*

$$\bar{W}_i = 0,\quad \bar{\Phi}_i = 0,\quad \sum_{j=1}^{N} B_{ij}^{(1)}\bar{\Psi}_j = 0; \tag{18}$$

clamped edge

$$\bar{W}_i = 0,\quad \bar{\Phi}_i = 0,\quad \bar{\Psi}_i = 0; \tag{19}$$

free edge

$$\begin{aligned}&\lambda\sum_{j=1}^{N}B_{ij}^{(1)}\bar{W}_j - \beta\bar{\Psi}_i = 0,\\&\lambda\sum_{j=1}^{N}B_{ij}^{(1)}\bar{\Phi}_j + (m\pi)\bar{\Psi}_i = 0,\\&\lambda\bar{D}_{22}\sum_{j=1}^{N}B_{ij}^{(1)}\bar{\Psi}_j - \bar{D}_{12}(m\pi)\bar{\Phi}_i = 0\end{aligned} \tag{20}$$

where, in the above equations, $i = 1$ for the end $Y = 0$ and $i = N$ for the end $Y = 1$.

SOLUTION OF QUADRATURE ANALOG EQUATIONS

The set of quadrature analog equations from the differential equation and the boundary conditions yield a system of $3N$ algebraic equations that may be arranged in matrix form as:

$$\begin{bmatrix} [S_{bb}] & [S_{bd}] \\ [S_{db}] & [S_{dd}] \end{bmatrix} \begin{Bmatrix} \{F_b\} \\ \{F_d\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \Omega_{mn}^2\{F_d\} \end{Bmatrix}. \quad (21)$$

in which $\{F_d\}$ is a $(3N - 6) \times 1$ column vector comprising three sets of $(N - 2)$ values, one each of the three mode function values $\bar{W}_i$, $\bar{\Phi}_i$, and $\bar{\Psi}_i$ at the sampling points $i = 2, 3, \ldots, (N - 1)$. The column vector $\{F_b\}$ is of the six function values, $\bar{W}_i$, $\bar{\Phi}_i$, and $\bar{\Psi}_i$ at the boundary points $i = 1$ and N. By eliminating the $\{F_b\}$ column from Eq. (21), one obtains the following eigenvalue equation:

$$[S]\{F_d\} - \Omega_{mn}^2[I]\{F_d\} = 0 \quad (22)$$

in which the matrix $[S]$ is of the size $(3N - 6) \times (3N - 6)$. The eigenvalues, the frequency squared values, of the $[S]$ matrix are obtained by inverse iteration with shifting (Bathe, 1982). The solution yields, along with the eigenvalues, the corresponding eigenvector $\{F_d\}$ from which the number of half-waves n in the y direction become known. Obviously, one needs to specify the half-waves m in the x direction as input data for the construction of the quadrature analog equations. Thus, at the end of the solution, one gets to know the free vibration characteristics in terms of the frequency Ω_{mn} and the corresponding mode pattern m n of the vibrating laminate.

In the foregoing analysis and solution procedure, shear deformation and rotatory inertia are accounted together. As mentioned in the introduction, for quantitative comparison of the effects of shear deformation and rotatory inertia, the results being presented in the following section have been obtained in two ways: one with the inclusion of shear deformation and rotatory inertia together, and the other, by the neglect of rotatory inertia and with shear deformation alone. The rotatory inertia may be neglected by taking simply the right sides of Eqs. (16) and (17) equal to zero. In that case, $\{F_d\}$ in Eq. (21) becomes a $(N - 2) \times 1$ column comprised of the mode function values $\bar{W}_i$, $i = 2, 3, \ldots, (N - 1)$. On the other hand $\{F_b\}$ becomes a $(2N + 2) \times 1$ vector comprised of $2N$ values $\bar{\Phi}_i$, $\bar{\Psi}_i$, $i = 1, 2, \ldots, N$, and the two end values $\bar{W}_1$ and $\bar{W}_N$. After the elimination of the $\{F_b\}$ vector, the size of the eigenvalue matrix $[S]$ in Eq. (22) with shear deformation only will be $(N - 2) \times (N - 2)$ compared to the size of $(3N - 6) \times (3N - 6)$ for the case of both shear deformation and rotatory inertia. The evaluation of frequencies for shear deformation only would obviously be less time consuming than those for both shear deformation and rotatory inertia. It should be noted that the time saving cannot be directly related to the reduction in the size of the eigenvalue matrix because elimination of the $\{F_b\}$ vector in the case of shear deformation only would be much more time consuming than in case of shear deformation and rotatory inertia. However, overall time saving should be expected because evaluation of the frequencies is by an iteration process.

Two extensively decisive factors for the successful application of the differential quadrature method are: one, the accuracy of the weighting coefficients, and two, the choice of sampling points. For the usual polynomial test functions, the weighting coefficients are determined most accurately from the explicit formulae developed by Quan and Chang (1989) and Shu and Richards (1992); the same are used in the present work.

A natural, and often convenient, choice for the sampling points is that of the equally spaced points; these are given by

$$Y_i = \frac{i - 1}{N - 1}, \quad i = 1, 2, \ldots, N. \quad (23)$$

Quite frequently, the differential quadrature solutions exhibit better convergence and deliver more accurate results with unequally spaced sampling points. Although such points may be selected by trials (Sherbourne and Pandey, 1991), a rational basis for the sampling points is provided by the zeros of the orthogonal polynomials such as the Legendre and Chebyshev polynomials. The sampling points used in the present work are given by

$$Y_i = \frac{1}{2}\left(1 - \cos\frac{(i - 1)\pi}{N - 1}\right), \quad i = 1, 2, \ldots, N \quad (24)$$

RESULTS AND DISCUSSION

The results based on the analysis and the solution method proposed are now produced to meet the set objectives of the present study. It should be mentioned here that a similar quadrature solution has been used by the present investigators (Bert and Malik, 1994) to analyze isotropic and specially orthotropic plates using the CPT. The plates considered were of a similar type, that is, those having two opposite edges simply supported and with general boundary conditions at the other two edges between which the thickness may be varying. Excellent comparisons have been established with the published analytical solution results of uniform thickness isotropic (Leissa, 1973) and specially orthotropic (Huffington and Hoppmann, 1959) plates and numerical solution results of isotropic tapered plates (Bhat et al., 1990). Furthermore, the solution method has been found to be highly efficient with respect to computational time.

At the outset, the primary issue that needs to be resolved is the convergence analysis of the solution method with respect to the number of sampling points N and, consequently, determining the type and the number of sampling points that could be used uniformly in all calculations. Comprehensive convergence studies were carried out for isotropic and specially orthotropic plates and symmetric cross-ply laminates, all of which could be analyzed by the same computer program. In all the cases of different types of plates, at least nine modes, given by $m, n = 1, 2, 3$, were considered. Both equally and unequally spaced points, given by Eqs. (23) and (24), respectively, were considered and the number of sampling points varied from 7–23. Some selected results of these studies, obtained by the unequally spaced sampling points, are given in Table 2. These results pertain to the mode $m = 2$, $n = 3$ of six isotropic plates. Shear deformation and rotatory inertia are accounted for with a shear correction factor of $\pi^2/12$. The plates have linearly varying thickness of the type:

$$H = 1 + \alpha Y \tag{25}$$

where α is the taper ratio $(h_1 - h_o)/h_o$, h_o and h_1 being thicknesses of the plate at the ends $y = 0$ and $y = b$, respectively. It may be observed from Table 2 that with increasing number of sampling points, the proposed method leads to converged solutions. The convergence of the eigenvalues indeed depends on how accurately the eigenvector represents the vibration mode shape. This in turn depends on the number and distribution of sampling points. Consequently, the convergence behavior may not exhibit clear monotonic pattern for smaller numbers of sampling points. This is clearly seen from Table 2 that for the plate cases 1–4, the convergence is not really monotonic for sampling points up to approximately $N = 13$.

As a matter of illustration of the dependence of the convergence behavior of quadrature solutions on the distribution of sampling points, the frequencies of SS-C-SS-F and SS-F-SS-F plates obtained from equally spaced sampling points are given in Table 3. It may be seen that the quadrature solution with equally spaced sampling points does not really exhibit a stable convergence be-

Table 2. Semianalytical DQ Solution Convergence: Free Vibration Frequency of Thick Isotropic Rectangular Plates vs. Number of Sampling Points

Mode Sequence	Number of y Direction Sampling Points, N					
	7	9	13	17	21	23
		1. SS-SS-SS-SS Plate: $\lambda = 0.5$				
8	32.818404	33.055088	33.054339	33.054359	33.054359	33.054359
		2. SS-C-SS-C Plate: $\lambda = 1.0$				
8	89.025162	90.500521	90.508339	90.508329	90.508329	90.508329
		3. SS-C-SS-SS Plate: $\lambda = 2.0$				
11	300.62062	307.33162	307.38560	307.38650	307.38651	307.38651
		4. SS-SS-SS-F Plate: $\lambda = 0.5$				
8	30.144783	30.161505	30.166019	30.165950	30.165949	30.165949
		5. SS-C-SS-F Plate: $\lambda = 1.0$				
6	72.123917	71.864308	71.862195	71.861323	71.861308	71.861308
		6. SS-F-SS-F Plate: $\lambda = 2.0$				
8	146.50125	146.41820	146.33942	146.33630	146.33625	146.33625

Plates are designated by edge conditions in the order $x = 0$, $y = 0$, $x = a$, and $y = b$. $m = 2$, $n = 3$ ($\alpha = 1.0$, $\nu = 0.3$, $h_o/b = 0.2$).

Table 3. Semianalytical DQ Solution with Equally Spaced Sampling Points for Thick Isotropic Rectangular Plates

	SS-C-SS-F Plate: λ = 1.0	SS-F-SS-F Plate: λ = 2.0		SS-C-SS-F Plate: λ = 1.0	SS-F-SS-F Plate: λ = 2.0
N	Ω_{23}		N	Ω_{23}	
7	78.147257	146.25772	18	71.859188	146.33538
9	71.776217	147.60372	19	71.862351	146.33998
11	71.999825	146.86918	20	71.860834	146.33606
13	71.910492	146.53280	21	71.861529	146.33703
15	71.877602	146.39679	22	71.861212	146.33621
17	71.865722	146.35226	23	71.861351	146.33640

$\alpha = 1.0$, $\nu = 0.3$, $h_o/b = 0.2$.

havior. On the other hand, similar to what is observed from Table 2, in all the cases studied, full convergence could always be achieved with N = 19–21 unequally spaced sampling points in that for a larger number of sampling points, there was no further change in frequency values rounded to eight digits.

Identical trends were found for uniformly thick symmetric cross-ply and tapered specially orthotropic laminates (of the materials for which detailed results are given later). Having assured the convergence for both isotropic and composite plates, the results, as reported in succeeding tables, were obtained with an N = 21 unequally spaced sampling point. It should be remarked, however, that for the fundamental ($m = n = 1$) and two or three higher modes, the same level of accuracy could be obtained with smaller sampling points of N = 9 or 11.

Subsequent to the convergence analysis was the issue of establishing the numerical accuracy of the proposed quadrature solution, and results to this effect are given in Tables 4 and 5. In Table 4, the comparison is given with some of the results of Mizusawa (1993) who provided free vibration characteristics data for the first eight modes of six tapered isotropic plates having two opposite edges simply supported. In his work, Mizusawa (1993) used the spline strip method and compared his results with those of Mikami and Yoshimura (1984). The shear correction was taken as $\pi^2/12$. The results were provided to five significant digits. To keep the comparison meaningful, the quadrature solution results in Table 4 are given to six significant digits. It may be seen that the results of the present calculations, when rounded off to five significant digits, become identical to those of Mizusawa (1993).

In Table 5, the comparison is given with the vibration characteristics of symmetric five-layer 0°/90°/0°/90°/0° cross-ply laminates reported by Craig and Dawe (1986). In these laminates, the thickness of each 0° layer was taken to be two-thirds of each 90° layer and the elastic properties of the orthotropic material, typical of a high-modulus-fiber composite, were chosen as:

$$E_L/E_T = 30, \quad \nu_{LT} = 0.25, \quad G_{LT}/E_T = 0.6, \quad G_{TT}/E_T = 0.5.$$

Further, Craig and Dawe have defined the dimensionless frequency in the following form:

$$\bar{\Omega}^2 = \frac{\rho a^4}{h_o^2 Q_{11}} \omega^2$$

where

$$Q_{11} = \frac{1}{h_o} \int_{-h_o/2}^{h_o/2} Q_{ij}^l \, dz$$

and is related to dimensionless frequency of the present work, Eq. (4), as:

$$\bar{\Omega}^2 = \frac{E_L}{6(E_L + E_T)} \Omega^2 = \frac{1}{6.2} \Omega^2.$$

Craig and Dawe (1986) used two methods for the laminate vibration analysis: the Rayleigh–Ritz method and the finite-strip method. In Table 5, the results of both of these methods are included where the results of the finite strip method are those obtained by running the strips between the pair of simply supported edges; see Craig and Dawe (1986) for further details. Also the values of the shear correction factors were found, based on the procedure of Chow (1971) and Whitney

Table 4. Accuracy of Semianalytical DQ Solution: First Eight Free Vibration Frequencies of Thick Isotropic Rectangular Plates Compared with Published Data

1. SS-SS-SS-SS Plate: $\lambda = 0.5$, $\alpha = 0.25$							
1 1[a]	1 2	1 3	2 1	1 4	2 2	2 3	1 5
10.1865[b]	14.5489	20.4970	24.4999	27.2432	27.2522	31.3669	34.3650
10.186[b]	14.549	20.497	24.500	27.243	27.252	31.367	34.365
2. SS-C-SS-C Plate: $\lambda = 1.0$, $\alpha = 0.5$							
1 1	2 1	1 2	2 2	3 1	1 3	3 2	2 3
25.2103	44.4435	48.3282	63.4006	71.4409	74.8192	85.9908	87.1065
25.210	44.444	48.328	63.401	71.441	74.819	85.991	87.107
3. SS-C-SS-SS Plate: $\lambda = 2.0$, $\alpha = 0.75$							
1 1	2 1	3 1	1 2	4 1	2 2	3 2	5 1
69.6385	95.2485	134.573	172.110	183.003	190.163	219.231	236.880
69.638	95.248	134.57	172.11	183.00	190.16	219.23	236.88
4. SS-SS-SS-F Plate: $\lambda = 0.5$, $\alpha = 0.5$							
1 1	1 2	1 3	1 4	2 1	2 2	1 5	2 3
9.65360	12.1572	16.9038	22.7456	24.7841	26.1564	28.6409	29.3136
9.6536	12.157	16.904	22.746	24.784	26.157	28.641	29.314
5. SS-C-SS-F Plate: $\lambda = 1.0$, $\alpha = 0.75$							
1 1	1 2	2 1	2 2	1 3	2 3	3 1	3 2
14.9347	29.7546	41.1024	51.5365	53.9251	70.6787	71.0678	78.3017
14.935	29.755	41.102	51.537	53.925	70.679	71.068	78.302
6. SS-F-SS-F Plate: $\lambda = 2.0$, $\alpha = 1.0$							
1 1	1 2	2 1	2 2	3 1	1 3	3 2	2 3
14.1787	35.2551	50.2759	76.3358	97.8341	116.422	125.560	146.336
14.179	35.255	50.276	76.336	97.834	116.42	125.56	146.34

($\nu = 0.3$, $h_o/b = 0.2$.)

[a]Two digit numbers indicate the m n values corresponding to each mode.

[b]First line values are of DQ solution; second line values are of Mizusawa (1993).

Table 5. Accuracy of semianalytical DQ Solution: Frequencies of Five Free Vibration Modes of Five-Layer Cross-Ply Square Laminates Compared with Published Data

h_o/a	Solution Method	Mode Pattern m n				
		1 1	1 2	2 1	2 2	1 3
				1. SS-SS-SS-SS Plate		
0.10	DQ	3.6035	7.0928	8.8375	10.7977	11.7133
	EXACT	3.60353	7.09284	8.83751	10.79772	11.71327
	FSM	3.604	7.093	8.838	10.799	11.715
	RRM	3.604	7.093	8.838	10.799	11.714
				2. SS-C-SS-C Plate		
0.10	DQ	4.4889	7.9089	9.2127	11.3256	12.2011
	FSM	4.489	7.910	9.213	11.327	12.203
	RRM	4.489	7.911	9.213	11.328	12.202
0.01	DQ	6.1836	14.3675	14.4989	19.8256	27.1312
	FSM	6.184	14.382	14.500	19.835	27.168
	RRM	6.184	14.369	14.500	19.828	27.134
				4. SS-SS-SS-SS Plate		
0.10	DQ	2.9067	4.3948	8.5411	9.2475	8.5081
	FSM	2.907	4.396	8.541	9.249	8.510
	RRM	2.908	4.402	8.542	9.255	8.515
0.01	DQ	3.4076	5.1948	13.3883	14.2909	12.1139
	FSM	3.408	5.197	13.389	14.294	12.125
	RRM	3.408	5.197	13.390	14.294	12.117

DQ, present quadrature solution; FSM, finite strip method SS-series solution; and RRM, Rayleigh–Ritz method solution. FSM and RRM results are of Craig and Dawe (1986).

(1973), to be $k_4^2 = 0.87323$ and $k_5^2 = 0.59139$; the same values are used in the present calculations for Table 5 and for the later results of cross-ply laminates.

For the case of a laminate simply supported on all four edges, the mode functions may be taken as:

$$\begin{aligned} W &= \tilde{W} \sin m\pi X \sin n\pi Y, \\ \Psi &= \tilde{\Phi} \cos m\pi X \sin n\pi Y, \\ \Psi &= \tilde{\Psi} \sin m\pi X \cos n\pi Y, \end{aligned} \tag{26}$$

which satisfy all the simply supported boundary conditions. In addition, if the laminate is of uniform thickness ($h = h_o$, $H = 1$), then by substitution of Eq. (26) in Eqs. (1), (2), and (3), only may obtain in a usual manner a cubic equation in Ω^2. The least root of this equation, corresponding to the dominant transverse mode, gives the desired frequency. In Table 5, these values, which are the exact frequencies of a simply supported laminate calculated for the present comparisons, are also included.

The dimensionless frequencies in Table 5 are from the definition of Craig and Dawe (1986). It may be observed that all the quadrature solution results compare very well with the results of Craig and Dawe for the three types of symmetric five-layer cross-ply laminates, namely, SS-SS-SS-SS, SS-C-SS-C, and SS-SS-SS-F laminates. For the case of an SS-SS-SS-SS laminate, the quadrature solution results were found to match with exact solution results to at least six decimal places (in Table 5, values are given to a smaller number of decimal places). It may be remarked here that, as mentioned by Craig and Dawe, the Rayleigh–Ritz method results of a simply supported laminate ought to be exact. However, there is some discrepancy in the Rayleigh–Ritz method values of Craig and Dawe and the presently calculated exact values for '2 2' and '1 3' modes. In view of very close matching of the frequencies from the quadrature solution with the exact solution frequencies for the case of simply supported laminates, it may be anticipated that for other boundary conditions (at the two opposite y edges), the quadrature method yields results of higher accuracy than those of Craig and Dawe.

The results of Tables 4 and 5 establish one of the objectives of the present study, that is, the proposed quadrature solution method is a highly accurate technique; a related issue of computational efficiency will be taken up later.

The results of further investigations are now presented for uniform thickness symmetric cross-ply laminates in Table 6 and for tapered specially orthotropic laminates in Table 7. It should be noted that the dimensionless frequencies given in these tables are from Eq. (4). The basic issue that is considered in these results is the quantitative comparison of the effects of the shear deformation and rotatory inertia on the free vibration frequencies. For this purpose free vibration frequencies were calculated by the inclusion of both shear deformation and rotatory inertia and then by neglecting rotatory inertia. The quantitative comparison of these two effects is then expressed as

$$\%\text{diff} = 100 \times \frac{(\Omega_{mn})_{\text{SD}} - (\Omega_{mn})_{\text{SDRI}}}{(\Omega_{mn})_{\text{SDRI}}} \tag{27}$$

where, $(\Omega_{mn})_{\text{SD}}$ and $(\Omega_{mn})_{\text{SDRI}}$ denote, the shear deformation only and shear deformation with rotatory inertia frequencies, respectively. Note that the comparison in Eq. (27) is with respect to $(\Omega_{mn})_{\text{SDRI}}$ because within the confines of the thin and thick plate theories, the shear deformation with inertia frequencies are most exact.

The results in Table 6 are given for nine modes; m, n = 1, 2, 3. Note that these are not meant to be the values in the particular order of increasing magnitudes of the frequencies. These results are for six combinations of the boundary conditions at the y edges and for one aspect ratio $\lambda = 1.0$. Also included in Table 6 are the results of the thin plate theory, that is, the frequencies without shear deformation and rotatory inertia. These have been obtained using an earlier analysis of the present investigators (Bert and Malik, 1994).

The results of Table 6 are for the 0°/90°/0°/90°/0° symmetric laminate of Craig and Dawe (1986) that was used for accuracy analysis in Table 5. The results in these tables are for two values of the thickness parameter $h_o/a = 0.1$ and 0.2.

The dimensionless stiffnesses, needed for the calculation of frequency data of Table 7, are obtained using the following values of the orthotropic elastic constants (Pagano, 1974):

$$\begin{aligned} &E_L = 30 \times 10^6 \text{ psi}, \quad E_T = 3 \times 10^6 \text{ psi}, \\ &\nu_{LT} = 0.25, \quad G_{LT} = 1.5 \times 10^6 \text{ psi}, \\ &G_{TT} = 0.6 \times 10^6 \text{ psi}. \end{aligned}$$

Following Whitney (1987), the shear correction factors for the orthotropic laminate are taken as

Table 6. Comparison of Classical (CPT), Shear Deformation (SD) only, and Shear Deformation with Rotatory Inertia (SDRI) Solutions for Nine (m, n = 1, 2, 3) Free Vibration Frequencies of Five-Layer Cross-Ply Square Laminates

			$h_o/a = 0.1$			$h_o/a = 0.2$		
m	n	CPT	SD	SDRI	%diff	SD	SDRI	%diff
				1. SS-SS-SS-SS Laminate				
1	1	10.49076	9.01441	8.97271	0.465	6.76397	6.72153	0.631
1	2	24.47284	17.76853	17.66102	0.609	11.53297	11.47727	0.485
1	3	51.40165	29.32437	29.16581	0.544	17.05818	16.99902	0.348
2	1	34.45915	22.11374	22.00523	0.493	13.44912	13.38681	0.465
2	2	41.96304	27.05589	26.88610	0.631	16.36243	16.30256	0.367
2	3	62.81008	35.78356	35.57863	0.576	20.61488	20.56131	0.260
3	1	76.03202	36.78848	36.63974	0.406	20.41150	20.33985	0.352
3	2	80.59362	40.02376	39.80668	0.545	22.42911	22.36387	0.293
3	3	94.41684	46.36951	46.13707	0.504	25.68154	25.62984	0.202
				2. SS-C-SS-C Laminate				
1	1	15.47442	11.21195	11.17721	0.311	7.46229	7.43922	0.310
1	2	36.23460	19.78893	19.69303	0.487	11.78776	11.72251	0.557
1	3	69.08506	30.53144	30.38049	0.497	17.17941	17.11388	0.383
2	1	36.37969	23.02042	22.93939	0.353	13.76561	13.74030	0.184
2	2	49.93278	28.34633	28.20055	0.517	16.54137	16.47295	0.415
2	3	78.13026	36.73205	36.53763	0.532	20.71379	20.65403	0.289
3	1	76.98888	37.28994	37.18611	0.279	20.58731	20.56336	0.116
3	2	85.18969	40.85706	40.68013	0.435	22.56012	22.48898	0.316
3	3	105.4558	47.07750	46.86225	0.459	25.76199	25.70434	0.224
				3. SS-C-SS-SS Laminate				
1	1	12.48319	10.04074	10.00031	0.404	7.08658	7.05014	0.517
1	2	29.96464	18.84519	18.74546	0.532	11.67625	11.61814	0.500
1	3	59.88165	29.94210	29.78697	0.521	17.11417	17.04904	0.382
2	1	35.17842	22.52221	22.42422	0.437	13.59516	13.54646	0.359
2	2	45.49168	27.72944	27.57469	0.561	16.46060	16.39936	0.373
2	3	70.02581	36.26703	36.06709	0.554	20.66151	20.60216	0.288
3	1	76.40268	37.01507	36.88434	0.354	20.49395	20.44115	0.258
3	2	82.58066	40.45308	40.26010	0.479	22.49913	22.43418	0.289
3	3	99.48636	46.72933	46.50538	0.482	25.72031	25.66313	0.223
				4. SS-SS-SS-F Laminate				
1	1	8.50107	7.26170	7.23773	0.331	5.40807	5.37173	0.676
1	2	12.96556	11.04067	10.94301	0.892	8.22030	8.10472	1.426
1	3	30.33351	21.40161	21.18493	1.023	13.66263	13.50303	1.182
2	1	33.58823	21.34529	21.26719	0.377	12.81146	12.75377	0.452
2	2	35.85317	23.22663	23.02598	0.871	14.32471	14.16517	1.126
2	3	46.45286	29.74191	29.44340	1.014	17.98280	17.81025	0.969
3	1	75.40230	36.27492	36.17151	0.286	19.98676	19.92717	0.299
3	2	77.08523	37.59186	37.31559	0.740	21.06124	20.88049	0.866
3	3	83.75997	41.99945	41.62588	0.897	23.68471	23.50157	0.779
				5. SS-C-SS-F Laminate				
1	1	8.78726	7.52005	7.49383	0.345	5.62613	5.59553	0.547
1	2	15.82988	12.17559	12.07389	0.842	8.40582	8.28328	1.479
1	3	36.58569	22.37723	22.17315	0.920	13.81210	13.67234	1.022
2	1	33.69610	21.43959	21.35946	0.375	12.88798	12.84335	0.347
2	2	37.15355	23.75981	23.56481	0.827	14.43201	14.26617	1.162
2	3	50.91804	30.40007	30.12122	0.926	18.08288	17.93099	0.847
3	1	75.46748	36.33050	36.22633	0.287	20.02468	19.98074	0.220
3	2	77.81567	37.90567	37.64215	0.700	21.13417	20.94964	0.881
3	3	86.47633	42.43588	42.09089	0.820	23.75221	23.58861	0.694

Table 6. (*Continued*)

m	n	CPT	$h_o/a = 0.1$ SD	SDRI	%diff	$h_o/a = 0.2$ SD	SDRI	%diff
				6. SS-F-SS-F Laminate				
1	1	8.36225	7.14542	7.12978	0.219	5.31183	5.29754	0.270
1	2	8.90796	7.58940	7.53837	0.677	5.67410	5.56668	1.930
1	3	16.39039	13.86514	13.68169	1.341	10.26315	10.08588	1.758
2	1	33.45027	21.24755	21.19036	0.270	12.72168	12.70388	0.141
2	2	34.00426	21.63226	21.48694	0.676	13.07466	12.88900	1.440
2	3	37.93464	24.91153	24.59559	1.284	15.62732	15.40136	1.467
3	1	75.26498	36.18405	36.11162	0.201	19.89884	19.88413	0.074
3	2	75.81571	36.54763	36.34410	0.560	20.25112	20.04605	1.023
3	3	78.63580	38.79070	38.36082	1.121	22.00639	21.74674	1.194

Table 7. Comparison of Classical (CPT), Shear Deformation (SD) only, and Shear Deformation with Rotatory Inertia (SDRI) Solutions for Nine (m, n = 1, 2, 3) Free Vibration Frequencies of Specially Orthotropic Square Laminates (h_o/a = 0.1)

m	n	Uniform Thickness, $\alpha = 0.0$ CPT	SD	SDRI	%diff	Linearly Varying Thickness, $\alpha = 0.5$ CPT	SD	SDRI	%diff
					1. SS-SS-SS-SS Laminate				
1	1	11.46213	10.44713	10.38388	0.609	14.21632	12.40573	12.30506	0.818
1	2	18.71325	16.56853	16.35865	1.283	23.28259	19.46681	19.15850	1.609
1	3	33.23398	27.21053	26.71773	1.844	41.22467	31.08570	30.45832	2.060
2	1	40.80697	30.44006	30.20851	0.767	49.04088	33.69347	33.40438	0.865
2	2	45.84854	34.11515	33.67508	1.307	57.67455	38.10351	37.56124	1.444
2	3	56.74379	41.34589	40.62477	1.775	70.90599	45.84041	45.00482	1.857
3	1	90.09968	53.96555	53.59763	0.686	104.8637	57.32893	56.91440	0.728
3	2	94.43670	56.61606	55.98151	1.133	118.0174	60.71355	60.03708	1.127
3	3	103.1592	61.72825	60.79009	1.543	130.4246	66.13560	65.14101	1.527
					2. SS-C-SS-C Laminate				
1	1	13.31412	11.79968	11.73451	0.555	16.48777	13.85874	13.76385	0.689
1	2	24.29512	19.57263	19.36591	1.067	30.11453	22.31616	22.04195	1.244
1	3	42.44674	30.57575	30.12834	1.485	52.56532	33.90934	33.36550	1.630
2	1	41.52746	30.87875	30.66047	0.712	50.59943	34.23029	33.97364	0.755
2	2	48.82050	35.46590	35.07309	1.120	60.94076	39.33267	38.87756	1.171
2	3	63.14191	43.31286	42.68022	1.482	78.53128	47.46986	46.76259	1.512
3	1	90.52119	54.17505	53.83139	0.638	107.2114	57.64336	57.28829	0.620
3	2	96.21140	57.30633	56.75210	0.977	120.3243	61.31743	60.75035	0.933
3	3	107.3018	62.85289	62.04560	1.301	134.6444	67.06875	66.24190	1.248
					3. SS-C-SS-SS Laminate				
1	1	12.20512	11.02307	10.95713	0.602	15.29843	13.10593	13.00009	0.814
1	2	21.26436	18.04254	17.83122	1.185	26.30711	20.89058	20.58202	1.499
1	3	37.61563	28.92230	28.44974	1.661	46.42943	32.62189	32.02063	1.878
2	1	41.11500	30.63549	30.40775	0.749	50.52808	34.10714	33.83782	0.796
2	2	47.17973	34.76393	34.34435	1.222	59.59860	38.75201	38.22882	1.369
2	3	59.72090	42.32816	41.65024	1.628	74.64873	46.72361	45.93101	1.726
3	1	90.28892	54.06227	53.70351	0.668	107.2110	57.61258	57.24982	0.634
3	2	95.24617	56.94851	56.35054	1.061	120.1846	61.06408	60.41812	1.069
3	3	105.0844	62.28572	61.41221	1.422	133.1513	66.64896	65.71058	1.428

Table 7. (*Continued*)

		Uniform Thickness, α = 0.0				Linearly Varying Thickness, α = 0.5			
m	*n*	CPT	SD	SDRI	%diff	CPT	SD	SDRI	%diff
		4. SS-SS-SS-F Laminate							
1	1	10.14732	9.25742	9.22159	0.389	13.46366	11.59213	11.53561	0.490
1	2	13.35100	12.04930	11.91423	1.134	17.02222	14.42901	14.19950	1.616
1	3	22.50955	19.57338	19.16805	2.115	28.04295	22.82142	22.17133	2.932
2	1	39.74475	29.66236	29.48618	0.597	49.03453	33.56579	33.32081	0.735
2	2	42.44787	31.60062	31.24021	1.154	56.14699	35.79822	35.40541	1.109
2	3	49.33046	36.44576	35.74760	1.953	62.98033	40.73878	39.77650	2.419
3	1	89.07874	53.36078	53.06749	0.553	104.8637	57.28341	56.89772	0.678
3	2	91.64108	54.89126	54.35388	0.989	117.9785	59.36845	58.93081	0.743
3	3	97.57935	58.44134	57.49900	1.639	128.1650	62.79764	61.68823	1.798
		5. SS-C-SS-F Laminate							
1	1	10.29045	9.37604	9.33811	0.406	13.89485	11.83837	11.78002	0.495
1	2	14.59329	12.92350	12.77320	1.177	18.46903	15.29812	15.03819	1.728
1	3	25.52595	21.16723	20.75021	2.010	31.58284	24.33216	23.66730	2.809
2	1	39.80947	29.70544	29.52819	0.600	50.49967	33.88279	33.67236	0.625
2	2	43.08072	31.95637	31.59330	1.149	57.11930	36.12672	35.70240	1.188
2	3	51.15664	37.22608	36.54171	1.873	65.20886	41.47495	40.51721	2.364
3	1	89.11944	53.38310	53.09013	0.552	107.2110	57.53727	57.21493	0.563
3	2	92.04695	55.07857	54.55013	0.969	120.0765	59.54243	59.09216	0.762
3	3	98.76855	58.86525	57.95632	1.568	129.7116	63.20684	62.11060	1.765
		6. SS-F-SS-F Laminate							
1	1	9.85561	9.00207	8.97610	0.289	11.50187	10.26708	10.21436	0.516
1	2	10.97089	9.95198	9.88164	0.712	14.37243	12.23503	12.13312	0.840
1	3	15.76809	14.06844	13.81359	1.845	19.78447	16.71872	16.30231	2.554
2	1	39.44848	29.45209	29.29937	0.521	43.36480	31.32266	31.07080	0.811
2	2	40.58928	30.25211	29.99717	0.850	52.20670	34.53303	34.24238	0.849
2	3	44.72430	33.15738	32.59684	1.720	58.10979	37.20051	36.49345	1.937
3	1	88.78037	53.18459	52.92410	0.492	95.31040	55.07612	54.66605	0.750
3	2	89.91682	53.86106	53.45789	0.754	110.4058	58.34068	57.89406	0.771
3	3	93.81413	56.16227	55.36611	1.438	122.2703	60.41269	59.61535	1.337

$k_4^2 = k_5^2 = 5/6$. Also, the results in Table 7 are for a uniform thickness (α = 0.0) and tapered (α = 0.5) laminate; the thickness parameter in both cases is $h_o/a = 0.1$.

It may be seen that both shear deformation and rotatory inertia lower the CPT frequencies, and, thus, the shear with inertia frequencies are actually less than the shear only frequencies. However, in conformity to the known results, the effect of shear deformation clearly outweighs the effect of rotatory inertia. Consider, for example, two extreme cases of $m = n = 1$ and $m = n = 3$ in, say, SS-C-SS-C laminate with $h_o/a = 0.2$ (Table 6). The reduction in frequencies due to shear deformation only, relative to the CPT frequencies are 51.78% in Ω_{11} and 75.57% in Ω_{33}. On the other hand, with the inclusion of rotatory inertia together with shear deformation, the relative reductions are changed only slightly to 51.93% in Ω_{11} and 75.63% in Ω_{33}. The relative effects are further exemplified by the %diff values in these results (Tables 6); these values signify the overestimate in using the shear deformation only frequencies with the respect to more exact shear with inertia frequencies. It should be interesting to note that these overestimates are small and are of such magnitudes that, possibly, the shear deformation values can be used with some appropriate correction or safety factor without the fear of design conservatism. Also, in general terms, one may talk of the effect of shear deformation only, rather than the effect of both shear deformation and rotatory inertia.

The relative reduction in CPT frequency due to inclusion of shear deformation (or the overestimation by the CPT due to the exclusion of this effect) is indicative of the significance of the effect of shear deformation. The effect of shear deformation increases with an increase in thickness. However, more significant is the effect of the number of half-waves. The larger the number of half-waves (i.e., higher mode), the more effect of shear deformation. This may be seen by taking once again the example of an SS-C-SS-C laminate. The reductions in Ω_{11} values relative to the CPT value are 27.55% and 51.78% for h_o/a equal to 0.1 and 0.2, respectively. On the other hand, the relative reductions in Ω_{33} values are 55.36% for $h_o/a = 0.1$ and 75.75% for $h_o/a = 0.2$.

The CPT overestimates free vibration frequencies due to neglect of shear deformation in both cross-ply (Table 6) and specially orthotropic (Table 7) laminates. However, the overestimates in specially orthotropic plates are not as large as in the cross-ply laminates. This implies that the shear deformation effects in the orthotropic laminates are not as large as in the cross-ply laminates. This is simply verified by the values of shear stiffness coefficients that are $\bar{A}_{44} = \bar{A}_{55} = 0.018295$ for cross-ply laminate, and $\bar{A}_{44} = 0.019875$ and $\bar{A}_{55} = 0.049687$ for the orthotropic laminate. These values indicate that for the chosen materials, the specially orthotropic laminates are stiffer in transverse shear than the cross-ply laminates. In fact, if the material of the orthotropic laminates is taken to be the same as that of the cross-ply laminates, then the shear stiffness coefficients would be $\bar{A}_{44} = 0.016632$ and $\bar{A}_{55} = 0.019958$ and it has been checked that, in that case, the shear deformation effects are much greater than those in Table 7 for the chosen material.

The computer programs for the present work were developed and executed on DECstations 5000/25 (operating system: Ultrix 4.2a) at the University of Oklahoma. With the inclusion of both shear deformation and rotatory inertia, the average CPU time for obtaining 25 different mode frequencies (Ω_{mn}; $m, n = 1, 2, \ldots, 5$) of a given laminate with $N = 21$ was found to be 2.54 s. This CPU time is of course quite small and shows the high computational efficiency of the proposed method. However, with the exclusion of rotatory inertia, the CPU time required for the evaluation of 25 shear deformation only frequencies (Ω_{mn}; $m, n = 1, 2, \ldots, 5$) is reduced to only 0.52 s. In view of the earlier discussion on the shear deformation only frequencies, it is obvious that neglect of rotatory inertia can be used quite advantageously in terms of saving on computer time for evaluation of vibration characteristics that may be good enough for design purposes. In fact, if one has to determine only the fundamental and a few higher mode frequencies, calculations can be much faster because for the same level of accuracy, fewer sampling points ($N = 9$ or 11) are needed.

CONCLUSION

The analysis of this article was applicable to isotropic plates as well as the specially orthotropic and symmetric cross-poly laminates. The plate configurations considered in this work were of the type in which two opposite edges were simply supported and the other two edges could have general boundary conditions along with thickness variation between those two edges. This permitted reduction of the governing partial differential equations to ordinary differential equations. The solution of the reduced equations was sought by the differential quadrature method. This was in fact a semianalytical approach for the quadrature solution.

The present study was undertaken with some definitive objectives. It is believed that this is the very first application of the differential quadrature method to the isotropic and laminated composite plate vibration problem that includes the effects of shear deformation and rotatory inertia. The accuracy of the proposed method was checked extensively by comparison of its calculations with the available results and the method was found to yield results of very high numerical accuracy.

The free vibration frequencies of several modes of the symmetric cross-ply and specially orthotropic laminates were determined in two ways: one including both shear deformation and rotatory inertia and the other by neglecting rotatory inertia and including shear deformation only. The vibration frequencies so obtained presented a clear quantitative comparison of the relative contributions of shear and inertia as well as the effect of the two on the frequencies of classical plate theory. The effect of rotatory inertia is considerably smaller in comparison to that of shear deformation. The shear deformation frequencies are small overestimates on the frequencies from the more exact case of shear deformation with rotatory inertia; however, these overestimated values can possibly be used for design purposes. Needless to say, the frequencies of classical plate theory may be very high overestimates of the frequencies based on shear deformation only or shear deformation with rotatory inertia and as such, outside the confines of CLP, the use of CLP frequencies ought to be unacceptable.

The proposed solution method is found to be computationally efficient as shown by small CPU times. The computational efficiency can be greatly enhanced by neglect of rotatory inertia. The high numerical accuracy of the proposed solution method coupled with computational efficiency indicate that it may be used for development of computer codes for use in real time analysis and design.

The article also included the results on tapered specially orthotropic thick laminates that provided new additional data in the plates literature.

Although, in this study the matter of the numerical accuracy of the quadrature solution for solution of the title problem was considered in detail, no attempt was made to evaluate the computational efficiency of the method in relation to the other numerical methods. The evaluation of the numerical accuracy of the method has been done by comparisons mainly with the available semianalytical finite solutions and the latter part was not the goal of the present work. The superior computational efficiency of DQM over the numerical solution methods such as the finite difference and finite element methods and the approximate methods is indeed well established and has been dealt with in detail in other works; see for example, Malik and Bert (1994) and Malik and Civan (1995).

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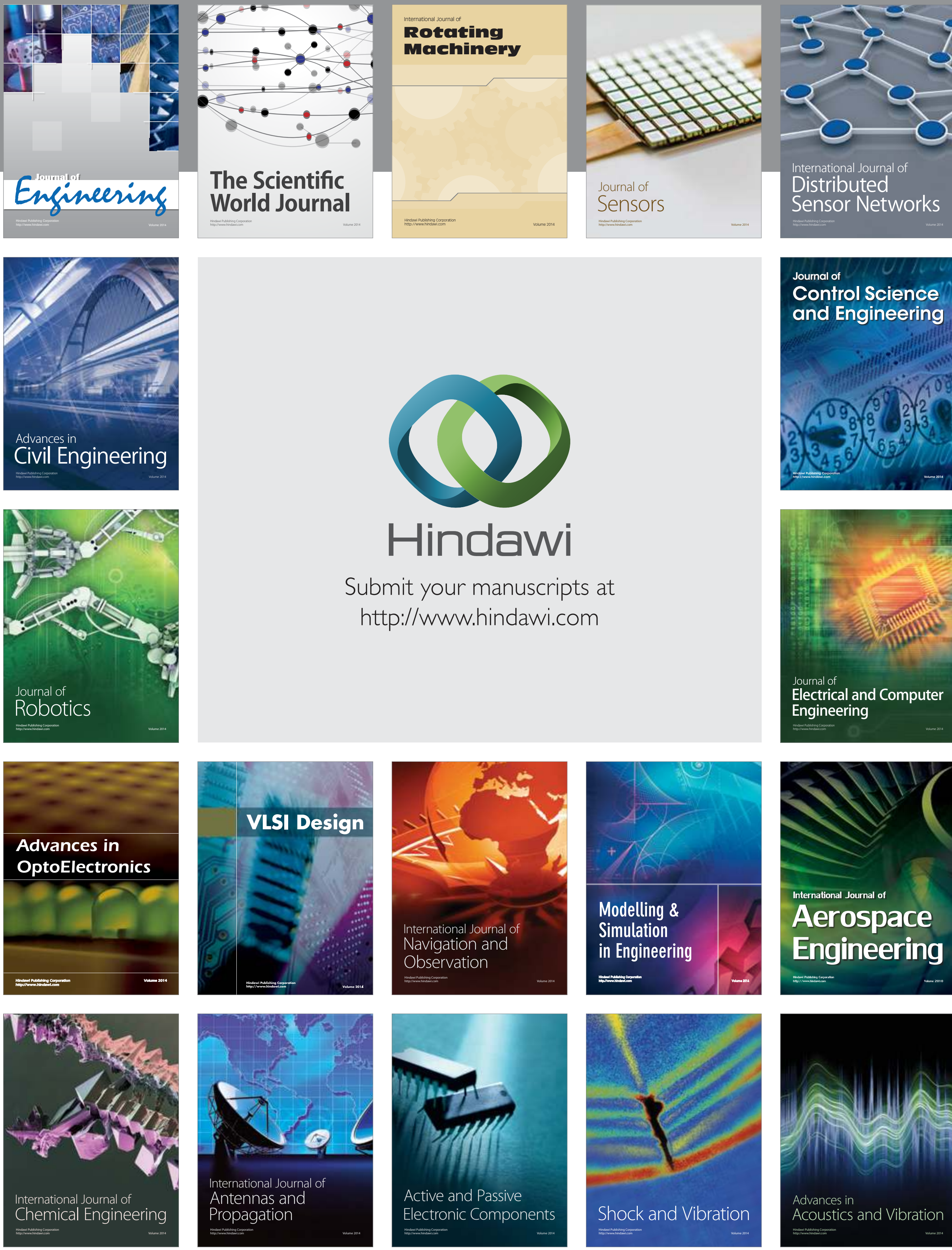
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