Geophys. J. R. astr. Soc. (1983) **74**, 301–312

Differential rotation produced by potential vorticity mixing in a rapidly rotating fluid

R. Hide *Geophysical Fluid Dynamics Laboratory, Meteorological Office (Met O 21), Bracknell, Berkshire RG12 2SZ*

I. N. James *Department of Meteorology, University of Reading, Whiteknights Park, Reading, Berkshire RG6 2AU*

Received 1982 December 6

Summary. The differential rotation of a rapidly rotating spherical shell of incompressible fluid of low viscosity subject to large-scale mixing is investigated by considering the dynamical behaviour of axial filaments of fluid. Owing to the gyroscopic constraints expressed by the Proudman-Taylor theorem and the Ertel theorem, each filament retains its coherence and undergoes little change in its potential vorticity over time-scales of typical displacements perpendicular to the rotation axis. The form of the profile of the latitudinal variation of the mean zonal flow velocity depends on several factors, including the coupling between the fluid shell and the underlying surface and the thickness of the shell, strong positive jets being found near the equator when the fluid shell is thin and at mid-latitudes when the shell is thick. It is remarkable that such a simple model can reproduce many of the observed features of the differential rotation of the Earth, Jupiter, Saturn and the Sun.

1 Introduction

Differential rotation in a partially or wholly fluid astronomical body such as a planet or star is associated with energetic processes involving the transformations between gravitational potential energy, kinetic energy and thermal energy. In the absence of the internal or external energy sources required to drive these processes, the body would rotate rigidly at a constant rate Ω_0 (say) about its fixed axis of maximum moment of inertia through its centre of mass. Relative to that frame of reference, all components of the Eulerian flow velocity $\mathbf{u}(R, \theta, \lambda, t) = (w, -v, u)$ would vanish, where (R, θ, λ) are spherical polar coordinates of a general point, R being distance from the centre of mass, θ co-latitude and λ east-longitude. Relative to any other frame which rotates steadily with constant angular speed ω with respect to this basic frame about the polar axis, including an inertial frame, for which $\omega = -\Omega_0$, we have $(w, -v, u) = (0, 0, -\omega R \sin\theta)$.

A major objective in the construction of theoretical models of hydrodynamical motions in planetary and stellar atmospheres and interiors is the determination from first principles of the magnitude and distribution of the mean differential rotation, as specified by

$$\bar{\Omega}(R, \theta) \equiv [\bar{u}(R, \theta)]/R \sin \theta = (2\pi T)^{-1} \int_0^T \int_0^{2\pi} (R \sin \theta)^{-1} u(R, \theta, \lambda, t)\, d\lambda\, dt \tag{1.1}$$

where the length of time T over which the average is taken is long in comparison with typical time-scales associated with $\mathbf{u}(R, \theta, \lambda, t)$ but is otherwise arbitrary. (We are here following a conventional notation of using an overbar to denote time average and square bracket to denote longitudinal average.)

The dependence of $[\bar{u}]$ on R and θ would of course emerge from a full solution of the governing equations of hydrodynamics, thermodynamics and (in the case of electrically conducting fluids) electrodynamics, under appropriate boundary conditions. But these equations are highly intractable and have only been solved in simplified cases. Possibly the most advanced work in this connection is that done by dynamical meterologists in their numerical studies of the general circulation of the Earth's atmosphere, in which are reproduced $[\bar{u}(R, \theta)]$ and other principal features of atmospheric flow.

The Earth's atmosphere is the only natural system for which observations are sufficient to enable direct determinations of $\bar{\Omega}(R, \theta)$ to be made (see, e.g. Lorenz 1967). On average it rotates faster than the solid Earth; $[\bar{u}]$ (if measured relative to the underlying surface) is found to be positive nearly everywhere, with an average value of about $10\,\mathrm{m\,s^{-1}}$, but with negative values in certain regions, including the Trade Winds at low levels in the tropics. The highest values of $[\bar{u}]$ in the troposphere, about $30\,\mathrm{m\,s^{-1}}$, are associated with mid-latitude jet streams.

In the cases of the atmospheres of Jupiter and Saturn, observations of the motions of markings on the visible surface of dense cloud going back many decades provide limited information about the dependence of $[u]$ at the (horizontally variable) cloud level as a function of t and θ. Both planets have strong equatorial jet-streams at their visible surfaces, which attain speeds as high as about $100\,\mathrm{m\,s^{-1}}$ relative to the deep interior for Jupiter and $400\,\mathrm{m\,s^{-1}}$ for Saturn (see, e.g. Gehrels 1976, 1983), the speeds of rotation of these interiors having been determined from radioastronomical observations. The jet streams are positive (i.e. westerly) in direction, and this implies that they must be produced by non-axisymmetric processes, involving the action of local west–east pressure gradients (Hide 1969).

Comparable information on the dependence of $[u]$ on θ and t for the solar atmosphere can be obtained from observations of sunspot motions and from spectroscopic data (see, e.g. Howard & Harvey 1970). The visible surface of the Sun rotates most quickly at the equator and $[u]$ exhibits a general decrease with distance from the equator that is more gradual than the corresponding latitudinal variation of zonal flow at the visible surfaces of Jupiter and Saturn. Some theories of the origin of magnetic fields of planets and stars invoke differential rotation in their electrically conducting fluid interiors as the main amplification process, but there are no direct observations of $[u]$ in these regions (see, e.g. Moffatt 1978; Parker 1979).

Departures from axial symmetry in the pattern of relative motion of a rapidly rotating fluid are to be expected even when the boundary conditions are axisymmetric (Hide 1982). But the correct quantitative representation of the effects of non-axisymmetric features on the magnitude and form of the differential rotation is by no means straightforward and presents serious technical difficulties. Some of these can be overcome by the introduction of a 'mixing hypothesis', which leads to considerable theoretical simplifications without sacrificing essentials. In the present paper we investigate differential rotation in a rotating spherical shell of incompressible fluid by assuming that non-axisymmetric motions act in such a way as to smooth out latitudinal gradients in potential vorticity (see equation 3.1). The latitudinal profile of $\bar{\Omega}$ depends *inter alia* on the thickness of the shell, exhibiting strong jets near the equator when the shell is thin and at mid-latitudes when the shell is thick.

Differential rotation in geophysical and astrophysical systems has been the subject of numerous theoretical studies and a comprehensive and critical review of previous work lies beyond the scope of the present paper. Many relevant references can be found in recent papers by Stewart & Thomson (1977), Glatzmeier & Gilman (1981), Busse (1982), Busse & Hood (1982), Rüdiger (1982), Schmidt (1982) and in Gehrels (1976, 1983). The very simple model discussed in this paper was developed over ten years ago as an improvement on one proposed much earlier by Rossby (1947). He considered the effects of horizontal mixing of *radial* filaments of fluid on the profile of mean zonal flow and derived expressions for such profiles on the assumption that mixing eliminates gradients of the vertical component of absolute vorticity poleward of a certain arbitrary latitude. In our model, in keeping with the constraints of the Proudman-Taylor theorem, we consider the behaviour of *axial* filaments of fluid (see Hide 1966), assuming that each filament retains its coherence and, owing to the weakness of frictional effects, undergoes little change in its potential vorticity (see equation 3.1) over time-scales of typical displacements perpendicular to the rotation axis. These displacements are associated with local pressure gradients which, in a rapidly rotating fluid, act at right-angles to the displacements. We publish this work now because of the growing interest in differential rotation, particularly in planetary atmospheres. It is remarkable that such a simple model can reproduce many of the observed features of the differential rotation of the Earth, Jupiter, Saturn and the Sun. (Whether or not internal dynamical processes such as those studied in this paper can account for the enormous value of the super-rotation of the atmosphere of Venus, at over 10 times the speed of the underlying planet, is a matter for further investigation. Gold & Soter (1971), for example, argue that such high values cannot be explained without invoking the action of external couples and have developed a model based on the action of the Sun's gravitational field on non-axisymmetric density variations associated with thermal tides.)

2 The Proudman-Taylor theorem and Ertel's potential vorticity theorem

The Eulerian continuity and momentum equations governing the flow of an incompressible fluid of variable density $\hat{\rho}(1+\vartheta)$ relative to a frame of reference which rotates with steady angular velocity $\hat{\mathbf{\Omega}}$ relative to an inertial frame are as follows:

$$\nabla \cdot \mathbf{u} = 0 \tag{2.1}$$

and

$$\partial \mathbf{u}/\partial t + (2\hat{\mathbf{\Omega}} + \boldsymbol{\xi}) \times \mathbf{u} = -\nabla(P + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}) + \mathbf{g}\vartheta + \mathbf{F} \tag{2.2}$$

where $\boldsymbol{\xi} \equiv \nabla \times \mathbf{u}$, the relative vorticity, t denotes time, $\mathbf{g}$ is the acceleration due to gravity plus centripetal effects, $\hat{\rho}\nabla P$ is equal to the pressure gradient minus $\mathbf{g}\hat{\rho}$ and $\mathbf{F}$ represents frictional effects due to viscosity (and in the case of an electrically conducting fluid, Lorentz forces due to the presence of electric currents within the fluid).

Equation (2.1) is a satisfactory representation of the full mass continuity equation, $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0$ only when the speed of sound greatly exceeds the *absolute* motion (not the relative motion), see Hide (1969), but for the sake of simplicity in the present work we shall ignore effects due to compressibility. Equation (2.2) incorporates the so-called Boussinesq approximation, which is valid when accelerations are so small in comparison with $\mathbf{g}$ that density variations can be neglected in all but the buoyancy term (see, e.g. Pedlosky 1979).

Equation (2.2) expresses the balance of forces acting on individual fluid elements. The corresponding torque balance is expressed by the vorticity equation, obtained by taking the *curl* of equation (2.2); thus:

$$\partial \boldsymbol{\xi}/\partial t + (\mathbf{u}\cdot\nabla)\,\boldsymbol{\xi} - \{(2\hat{\boldsymbol{\Omega}} + \boldsymbol{\xi})\cdot\nabla\}\,\mathbf{u} = -\mathbf{g}\times\nabla\vartheta + \nabla\times\mathbf{F}. \tag{2.3}$$

Now introduce a quantity known as the potential vorticity and defined as $(2\hat{\boldsymbol{\Omega}} + \boldsymbol{\xi})\cdot\nabla\Lambda$ where Λ is any scalar quantity satisfying $\partial\Lambda/\partial t + (\mathbf{u}\cdot\nabla)\Lambda \equiv D\Lambda/Dt = 0$. By equation (2.3) we have

$$D\{(2\hat{\boldsymbol{\Omega}} + \boldsymbol{\xi})\cdot\nabla\Lambda\}/Dt = -(\mathbf{g}\times\nabla\vartheta)\cdot\nabla\Lambda + \nabla\times\mathbf{F}\cdot\nabla\Lambda, \tag{2.4}$$

which reduces to Ertel's theorem expressing the conservation of potential vorticity by individual fluid elements

$$D\{(2\hat{\boldsymbol{\Omega}} + \boldsymbol{\xi})\cdot\nabla\Lambda\}/Dt = 0 \tag{2.5}$$

when effects due to density inhomogeneities and friction are negligible (*cf.* Gill 1982, Hide 1983 and Pedlosky 1979).

In regions where the relative acceleration and frictional terms in equation (2.2) are much smaller than the Coriolis term, quasi-geostrophic flow occurs, characterized by the approximate balance

$$2\hat{\boldsymbol{\Omega}}\times\mathbf{u} \doteqdot -\nabla P + \mathbf{g}\vartheta. \tag{2.6}$$

The corresponding vorticity equation (*cf.* equation 2.3) is

$$(2\hat{\boldsymbol{\Omega}}\cdot\nabla)\,\mathbf{u} \doteqdot \mathbf{g}\times\nabla\vartheta, \tag{2.7}$$

the first two components of which comprise the familiar 'thermal wind equation'. In the limit of strictly geostrophic flow of a homogeneous fluid, the last equation gives the Proudman-Taylor theorem

$$(2\hat{\boldsymbol{\Omega}}\cdot\nabla)\,\mathbf{u} = 0, \tag{2.8}$$

implying axial coherence of the motion (*cf.* Hide 1971).

3 Mathematical model

We envisage the configuration illustrated in Fig. 1, in which a fluid layer is of depth d and whose outer radius is a. The fluid is taken to be homogeneous, incompressible and inviscid. The whole system is supposed to be rotating rapidly, so that motions in the fluid layer will be constrained by the Proudman-Taylor theorem. As a result, the fluid will move as coherent filaments aligned parallel to the rotation axis. The potential vorticity of a filament will be defined as

$$q = \zeta/l \tag{3.1}$$

where ζ is the axial component of $2\hat{\boldsymbol{\Omega}} + \boldsymbol{\xi}$, its absolute vorticity, and l its length. By equation (2.5) with Λ equal to the axial distance of a point from one of the bounding surfaces divided by l, we have $Dq/Dt = 0$.

In this section we shall describe the flow which results if non-axisymmetric eddy motions are assumed thoroughly to mix potential vorticity, so that q becomes uniform. Less extreme assumptions are considered in Section 5.

In view of the relevance of the Proudman-Taylor theorem, the distance r of a fluid filament from the rotation axis is a natural coordinate to use. However, it will prove more

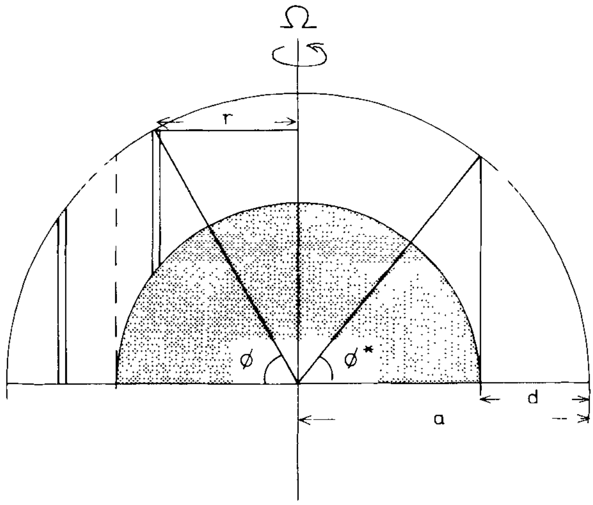

Figure 1. The configuration under consideration. A thick fluid shell of outer radius a overlies a core (shaded) of radius $(a-d)$. Symmetry about the equator is assumed. Fluid filaments a distance r from the rotation axis are distinguished by their latitude ϕ at the surface of the planet.

convenient to work in terms of the latitude of the filament at the planet's outer surface

$$\phi = \cos^{-1}(r/a) = \pi/2 - \theta. \tag{3.2}$$

When $(a-d) < r < a$ filaments are no longer in contact with the core of the planet. The 'critical latitude' ϕ^* at which $r = a-d$ separates regions of different flow regime and is the basic variable parameter of the system. It is related to a and d by

$$d/a = 1 - \cos\phi^*. \tag{3.3}$$

The length of the fluid filament is given by

$$\left.\begin{aligned} l &= a\,\{\sin\phi - (\sin^2\phi - \sin^2\phi^*)^{1/2}\}, \quad && \phi > \phi^* \\ l &= a\sin\phi, && \phi < \phi^*. \end{aligned}\right\} \tag{3.4}$$

Let us assume that non-axisymmetric eddy processes result in a large-scale mixing of potential vorticity. If we consider the zonally averaged flow, denoted by square brackets, the absolute vorticity of an element can be expressed in terms of the rotation rate $\Omega(r)$ of a cylindrical shell of fluid of radius r:

$$[\zeta] = \frac{1}{r}\frac{d}{dr}\{\Omega(r)\,r^2\}. \tag{3.5}$$

Transforming the independent variable from r to ϕ, and substituting into the zonally averaged form of (3.1) leads to

$$2\Omega - \cot\phi\,\frac{d\Omega}{d\phi} = [q]\,al. \tag{3.6}$$

The filament length l is a known function of ϕ (see equation 3.4), and the so differential equation (3.6) can, in principle, be solved to yield Ω.

Our hypothesis is that q should be well-mixed meridionally, so that $[q]$ is constant with respect to variations of ϕ. For the moment, we shall allow the possibility that $[q]$ may have different values in the polar (i.e. $\phi > \phi^*$) and equatorial ($\phi < \phi^*$) regions. Denoting these values by q_p and q_e respectively, we derive the basic equations describing the flow as a function of latitude:

$$\left.\begin{aligned} 2\Omega - \cot\phi \frac{d\Omega}{d\phi} &= aq_p\{\sin\phi - (\sin^2\phi - \sin^2\phi^*)^{1/2}\}, \quad && \phi > \phi^* \\ 2\Omega - \cot\phi \frac{d\Omega}{d\phi} &= aq_e \sin\phi, && \phi < \phi^*. \end{aligned}\right\} \tag{3.7}$$

Elementary integration techniques yield the solutions to these equations:

$$\Omega = \frac{A\{\sin^3\phi - (\sin^2\phi - \sin^2\phi^*)^{3/2}\} + B}{\cos^2\phi}, \quad \phi > \phi^* \tag{3.8a}$$

$$\Omega = \frac{C\sin^3\phi + D}{\cos^2\phi}, \quad \phi < \phi^*. \tag{3.8b}$$

The constants A and C are proportional to the potential vorticities q_p and q_e, while B and D are constants of integration. All these constants can be defined in terms of ϕ^*, given suitable boundary conditions.

A sufficient set of boundary conditions is obtained by requiring Ω to be continuous at $\phi = \phi^*$, and also that Ω should be finite as $\phi \to \pi/2$. The constants A and C are replaced by setting $\Omega_p = \Omega(\pi/2)$ and $\Omega_e = \Omega(0)$. Using these boundary conditions we obtain

$$\left.\begin{aligned} \Omega &= \frac{2\Omega_p\{1 - \cos^3\phi^* - \sin^3\phi^* + (\sin^2\phi - \sin^2\phi^*)^{3/2}\}}{3(1-\cos\phi^*)\cos^2\phi}, && \phi > \phi^* \\ \Omega &= \frac{\{2\Omega_p(1 - \cos^3\phi^* - \sin^3\phi^*) - 3\Omega_e(1-\cos\phi^*)\}\sin^3\phi + 3\Omega_e(1-\cos\phi^*)}{3(1-\cos\phi^*)\sin^3\phi^*\cos^2\phi}, && \phi < \phi^* \end{aligned}\right\} \tag{3.9}$$

All the angular velocities in these expressions can be scaled by Ω_p. Consequently two independent parameters remain to be determined, namely ϕ^* and Ω_e/Ω_p. In the next section we shall mention some ways of determining Ω_e/Ω_p.

4 Some equatorial jets

An indication of the possible zonal flow profiles given by the expressions (3.9) is gained by determining Ω_e/Ω_p and ϕ^* arbitrarily. Fig. 2 illustrates some typical results. The solid curves are for $\Omega_e = 1.5\,\Omega_p$, with ϕ^* varying between 10° and 70°. The dashed curves are for $\phi^* = 30°$ and various values of Ω_e/Ω_p. For $\phi > \phi^*$, the curves of course do not depend on Ω_e. Clearly, for small values of ϕ^* and rather large values of Ω_e, an equatorial jet structure is produced. Its shape can be reminiscent of the Jovian jet with the the characteristic slight reduction of the super-rotation at the equator, and a rapid drop of the zonal velocity around $\phi = \phi^*$. For larger values of ϕ^*, the maximum super-rotation occurs at mid-latitudes (though always where $\phi < \phi^*$).

The choice of Ω_e can be rendered less arbitrary. Suppose that before the mixing ensues, the atmosphere is in a state of solid body rotation at some initial rotation rate Ω_i. If there is no friction at the planetary core, the total angular momentum of the atmosphere will be

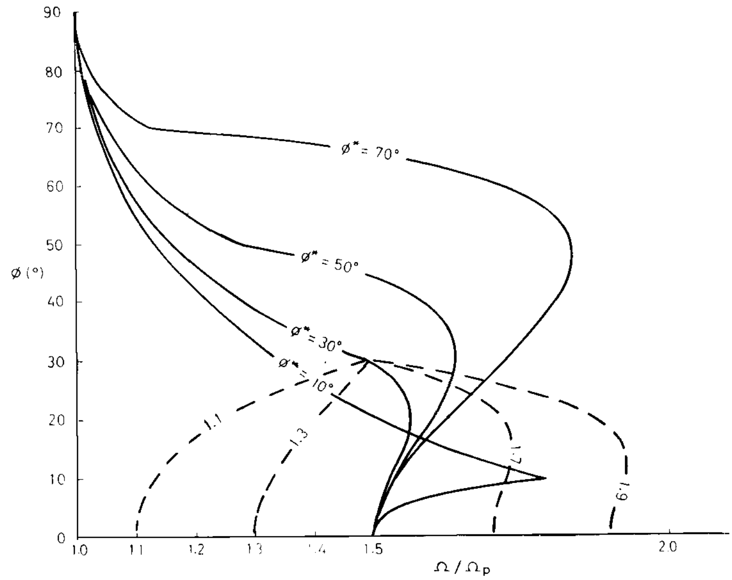

Figure 2. A series of flow profiles obtained from equation (2.9). The solid curves are for $\Omega_e = 1.5\,\Omega_p$ and various values of ϕ^*. The dashed curves are for $\phi^* = 30°$ and a range of Ω_e. Note the discontinuity of $\partial\Omega/\partial\phi$ at $\phi = \phi^*$.

conserved. This principle can be invoked to yield an expression for Ω_e in terms of Ω_i and ϕ^*:

$$\frac{\Omega_e}{\Omega_p} = \frac{(1-\cos^5\phi^*)}{5\sin^3\phi^*}\frac{\Omega_i}{\Omega_p} - \frac{2(1-\cos^3\phi^* - \sin^3\phi^*)(1-\cos^3\phi^*)}{3(1-\cos\phi^*)\sin^3\phi^*}. \tag{4.1}$$

The results of applying this formula to formulae (3.9) are shown in Fig. 3. The solid curves

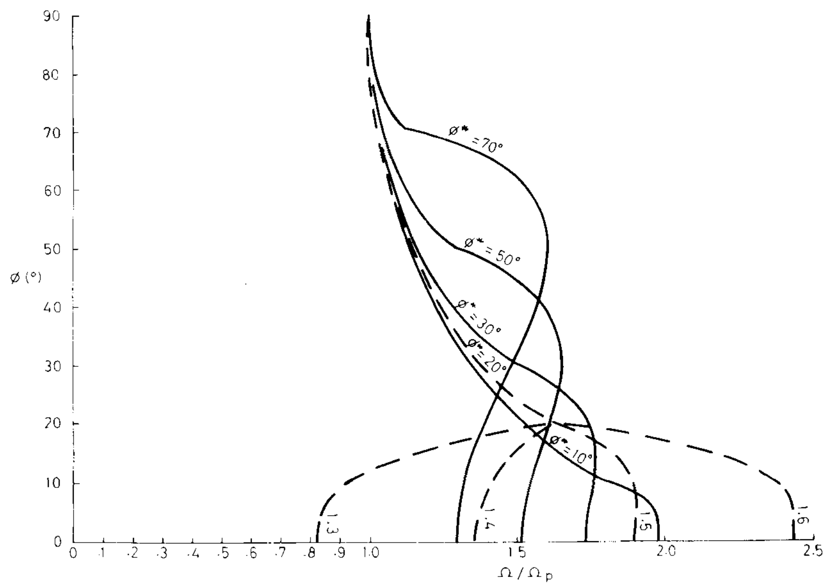

Figure 3. A series of flow profiles assuming no interaction with the core, and conservation of angular momentum for a state of initial solid body rotation, rate Ω_i. The solid curves are for $\Omega_i = 1.5\,\Omega_p$ and various ϕ^*; the dashed curves are for $\phi^* = 20°$ and various Ω_i.

show the effects of varying ϕ^* for $\Omega_i = 1.5\,\Omega_p$. The critical latitude ϕ^* was held at 20° and Ω_i varied to obtain the dashed curves. When Ω_i is held constant and ϕ^* varied, the strength of the jet increases as it narrows. For large ϕ^*, the maximum is in mid-latitudes, but is located on the equator when ϕ^* is sufficiently small. This is accentuated when Ω_i is increased.

This procedure is unsatisfying, since it still leaves two free parameters at our disposal. We have merely replaced Ω_e by Ω_i, which is equally arbitrary unless further assumptions can be justified.

If some weak frictional coupling with the core is presumed, an atmosphere which is initially co-rotating with the core at angular velocity Ω_c will evolve until the net torque exerted on the core is zero. This constrains the flow for $\phi > \phi^*$ and in fact serves to determine Ω_p in terms of Ω_c and ϕ^*. Referring to equation (3.9b), the flow in this region is independent of Ω_e. Consequently, without further assumptions, Ω_e is undetermined and the equations remain incomplete. In any case, the resulting flow for $\phi > \phi^*$ will only be a solution provided the drag is very ineffective, compared to the mixing process, in modifying the potential vorticity of the atmosphere. We shall comment on these assumptions in the next section.

Until this point, q_p and q_e, the potential vorticities of the polar and equatorial latitudes, have been, in general, different. Indeed, the discontinuity in $dl/d\phi$ at $\phi = \phi^*$ will greatly inhibit mixing across the critical latitude, assuming the planet is rapidly rotating. The effect is similar to the formation of a region of stagnant fluid when rapidly rotating barotropic flow passes over a sufficiently high isolated hill. Fluid is trapped over the hill, with little mixing with the embedding flow. Unless some external torque is acting on the fluid in the equatorial region, it is difficult to argue that no diffusion of potential vorticity across $\phi = \phi^*$ will occur. After a sufficiently long time, it must be presumed that $q_p = q_e$.

Such a constraint enables Ω_e to be determined in terms of Ω_p and ϕ^*, and so we obtain a single parameter family of solutions. Clearly, l is continuous at $\phi = \phi^*$; furthermore, our boundary conditions took Ω to be continuous at $\phi = \phi^*$. It follows that continuity of q is obtained by requiring $d\Omega/d\phi$ to be continuous at $\phi = \phi^*$. Calculating $d\Omega/d\phi$ from (3.9) and setting $\phi = \phi^*$ enables this condition to be written, after some tedious algebra, as:

$$\frac{\Omega_e}{\Omega_p} = \frac{2(1-\cos^3\phi^*)}{3(1-\cos\phi^*)}. \tag{4.2}$$

Inspection, of this formula reveals that Ω_e/Ω_p lies in the range 2/3–2 for all ϕ^*. Ω_e exceeds Ω_p for $\phi^* < 68.5°$. The family of solutions is illustrated by Fig. 4. When ϕ^* is less than 30° or so, a set of blunt profiled equatorial jets results. The equatorial acceleration increases as the jet narrows. For large ϕ^*, smooth profiles with a mid-latitude maximum are obtained.

This set of solutions is perhaps the most satisfactory obtained in this section. Nevertheless, the 'equatorial acceleration' is enormous, with Ω_e near $2\Omega_p$ for profiles of a realistic shape. The observed value of Ω_e is $1.008\,\Omega_p$ for Jupiter and $1.04\,\Omega_p$ for Saturn.

5 Limited mixing hypothesis

In this section, we present an elaboration of our model which permits jets of more moderate velocity, but with widths comparable to those observed on Jupiter and Saturn. This is achieved at the expense of a second disposable parameter. In addition to the geometrical parameter ϕ^*, our second parameter may be thought of as summarizing the dynamical properties of the mixing eddies.

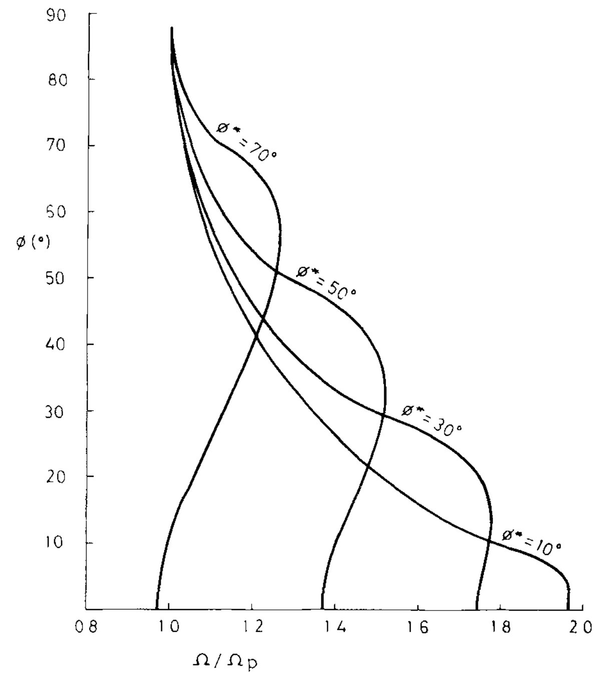

Figure 4. Flow profiles assuming the equatorial and polar potential vorticities are equal ($q_e = q_p$) for various values of ϕ^*.

We suppose that there is some frictional coupling between the core and the atmosphere for $\phi > \phi^*$; the core is regarded as being very massive and rotating at an angular velocity Ω_c (so that its angular momentum is effectively infinite). In the region $\phi < \phi^*$, only radial diffusion of potential vorticity takes place. The evolution of potential vorticity may be written as

$$\frac{\partial [q]}{\partial t} = \left\{\frac{2\Omega_c}{l} - [q]\right\} \Big/ \tau(\phi) + D([q]). \tag{5.1}$$

The first term represents the tendency of the atmosphere to spin up on a time-scale τ, until it rotates with the core. τ is a function of ϕ and will clearly be infinite for $\phi < \phi^*$. D represents some diffusion operator which parameterizes radial mixing by the eddies. For the sake of simplicity, we assume that τ is constant for $\phi > \phi^*$ and $D \equiv K\nabla^2$. If $K/a^2 \ll \tau^{-1}$, the mixing will be confined to a narrow range of latitudes of width

$$\Delta\phi = 0\ \{(k\tau)^{1/2}/a\} \tag{5.2}$$

in the steady state. The complementary case, $K/a^2 \gg \tau^{-1}$, was commented upon in the previous section when Ω_p was related to Ω_c by the requirement of no net torque on the core.

Rather than attempt a full solution of the time dependent diffusion problem defined by (5.1), we shall adapt the model described in Section 3 by assuming that mixing in the region $\phi > \phi^*$ only extends to some latitude ϕ_m, rather than to the pole. Clearly, $(\phi_m - \phi^*) = 0(\Delta\phi)$. The solutions (2.8) are reworked, this time applying the boundary condition of Ω and $d\Omega/d\phi$ continuous at $\phi = \phi^*$ and $\phi = \phi_m$ and assuming $\Omega(\phi_m) = \Omega_c$. The algebra is entirely straightforward; we quote the expressions for the constants A, B, C and D in formulae (3.8) for the sake of completeness:

$$A = \Omega_c \cos^2\phi_m/[\sin^3\phi_m - (\sin^2\phi_m - \sin^2\phi^*)^{3/2} - M], \tag{5.3a}$$

$$B = AM, \tag{5.3b}$$

$$C = \Omega_\phi^* \cos\phi^*/3 \sin^2\phi^* - 2\Omega^*/3 \sin\phi^*, \tag{5.3c}$$

$$D = \Omega^*[\cos^2\phi^* (2/3) \sin^2\phi^*] - \Omega_\phi^*[\cos\phi^* \sin\phi^*]/3, \tag{5.3d}$$

where

$$M = 3 \cos^2\phi_m [\sin\phi_m - (\sin^2\phi_m - \sin^2\phi^*)^{1/2}]/2 + \sin\phi_m [\sin^3\phi_m - (\sin^2\phi_m - \sin^2\phi^*)^{3/2}], \tag{5.4a}$$

$$\Omega^* \equiv \Omega(\phi^*) = (A \sin^3\phi^* + B)/\cos\phi^*, \tag{5.4b}$$

$$\Omega_\phi^* \equiv d\Omega(\phi^*)/d\phi = \frac{[A(3\cos^2\phi^* \sin\phi^* + 2\sin^4\phi^*) + 2B] \sin\phi^*}{\cos^3\phi}. \tag{5.4c}$$

It might be thought that the simplest case of all would be to set $\phi_m = \phi^*$, so that mixing is entirely confined to the equatorial latitudes. In this case, Ω_ϕ^* is clearly zero and $\Omega^* = \Omega_c$. Hence, from (3.8d) and (5.3d)

$$\Omega_e = D = \Omega_c[1 - (\sin^2\phi^*)/3]. \tag{5.5}$$

Ω_e is always less than Ω_c, and becomes smaller as ϕ^* increases. No equatorial jet can result.

However, as soon as $\phi_m - \phi^*$ becomes non-zero, Ω_e increases and becomes positive. For values of ϕ_m slightly in excess of ϕ^*, it is possible to produce some very realistic jets with a small super-rotation of a few per cent, and widths of around 10° of latitude. Some examples are shown in Fig. 5. The strength of the jet increases both with $\phi_m - \phi^*$ and with ϕ^*. When ϕ_m is sufficiently close to ϕ^* (or, equivalently, when ϕ^* is large enough) a characteristic retardation is seen actually on the equator, with the maximum located a little way either side of the equator. Such features are observed both on Jupiter and Saturn (see Gehrels 1976, 1983).

Although this form of the model can be tuned to produce realistic jets, the need to adjust two parameters leaves it somewhat unsatisfactory. Further progress requires an independent determination of ϕ^*, together with more detailed discussions of the nature of the eddy motions and of the core–atmosphere coupling, so that realistic bounds can be placed on $\phi_m - \phi^*$. Nevertheless, the model serves to demonstrate that eddies in a rapidly rotating shell of barotropic fluid have no difficulty in forming equatorial jets. Rather the problem is to identify the mechanisms which would oppose the tendency to mix potential vorticity and so reduce the strengths of the jets to more acceptable values.

6 Concluding remarks

A detailed discussion of the zonal flow in the fluid regions of particular planets and stars would require solutions of the full hydrodynamic equations, with explicit representation of energy sources, mixing processes and coupling with bounding surfaces, a task of considerable magnitude even with the aid of the most modern computers. In our investigation of the simple model introduced in this paper we have found a remarkable range of apparently relevant flow types by varying just two basic parameters. This indicates that the model could provide a useful basis for further work in which effects we have neglected in the first instance, such as compressibility, baroclinicity, and (in the case of electrically

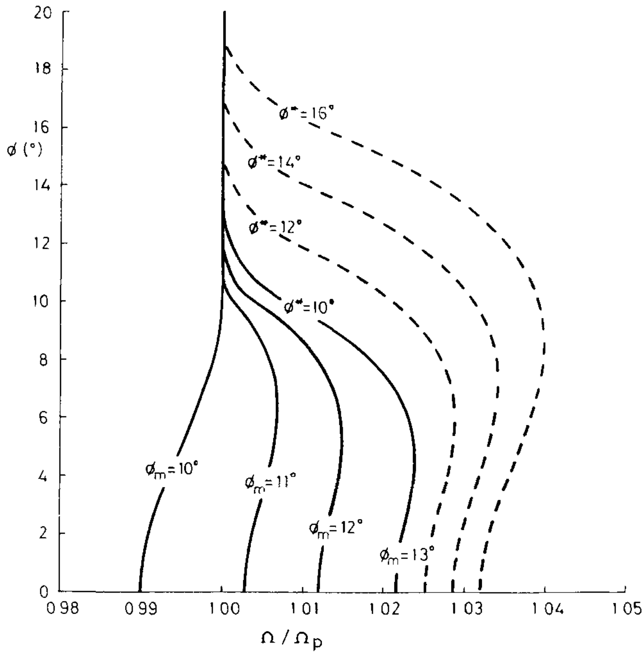

Figure 5. Flow profiles obtained using the hypothesis that potential vorticity mixing in the polar region extends only to latitude ϕ_m. The solid curves show the effect of varying ϕ_m when ϕ^* is set to 10°. For the dashed curves, ϕ^* was varied, but keeping $\phi_m - \phi^* = 3°$.

conducting fluids such as the solar atmosphere and interior and planetary cores) magnetohydrodynamics effects are systematically taken into account. The assumption of coherence of axial fluid filaments is weakened by the presence of density inhomogeneities (see Hide 1977), and in some circumstances, notably when there is a strong stable vertical gradient of potential density, it is more likely that it is the gradient of the *vertical* component of potential vorticity that tends to be smoothed out by large-scale mixing, as in the case discussed by Rossby (1947), rather than that of the axial component.

Late note

Ideas similar to some of those introduced in the present paper have been employed in an investigation by Rhines & Young (1982), a report of which has just appeared.

References

Busse, F. H., 1982. On the problem of stellar rotation, *Astrophys. J.*, **259**, 759–766.

Busse, F. H. & Hood, L. L., 1982. Differential rotation driven by convection in a rapidly-rotating annulus, *Geophys. Astrophys. Fluid Dyn.*, **21**, 59–74.

Gehrels, T. (ed.), 1976. *Jupiter*, University of Arizona Press.

Gehrels, T. (ed.), 1983. *Saturn*, University of Arizona Press.

Gill, A. E., 1982. *Atmosphere–Ocean Dynamics*, Academic Press, London.

Glatzmeier, G. A. & Gilman, P. A., 1981. Compressible convection in a rotating spherical shell. II. A linear anelastic model, *Astrophys. J. Suppl.*, **45**, 351–380.

Gold, T. & Soter, S., 1971. Atmospheric tides and the 4-day circulation on Venus, *Icarus*, **14**, 16–20.

Hide, R., 1966. Free hydromagnetic oscillations of the Earth's core and the theory of the geomagnetic secular variation, *Phil. Trans. R. Soc. A*, **259**, 615–647.

Hide, R., 1969. Dynamics of the atmospheres of the major planets, *J. atmos. Sci.*, **26**, 841–847.

Hide, R., 1971. On geostrophic motion of a non-homogeneous fluid, *J. Fluid Mech.*, **49**, 745–751.

Hide, R., 1982. Rotating fluids in geophysics and planetary physics, *Q. Jl R. astr. Soc.*, **23**, 220–235.

Hide, R., 1983. The magnetic analogue of Ertel's potential vorticity theorem, *Annls Geophys.*, **1**, 59–60.

Howard, R. & Harvey, J., 1970. Spectroscopic determinations of solar rotation, *Sol. Phys.*, **12**, 23–51.

Lorenz, E. N., 1967. *The Nature and Theory of the General Circulation of the Atmosphere*, World Meteorological Organization (Geneva), Publication No. 218.

Moffatt, H. K., 1978. *Magnetic Field Generation by Fluid Motions*, Cambridge University Press.

Pedlosky, J., 1979. *Geophysical Fluid Dynamics*, Springer-Verlag, Heidelberg.

Parker, E. N., 1979. *Cosmical Magnetic Fields*, Clarendon Press, Oxford.

Rhines, P. B. & Young, W. R., 1982. Homogenization of potential vorticity in planetary gyres, *J. Fluid Mech.*, **122**, 347–367.

Rossby, C. G., 1947. On the distribution of angular velocity in gaseous envelopes under the influence of large-scale horizontal mixing, *Bull. Am. meteorol. Soc.*, **28**, 53–68.

Rüdiger, G., 1982. Reynolds stresses and differential rotation II. Mean field methods, *Geophys. Astrophys. Fluid Dyn.*, **21**, 1–25.

Schmidt, W., 1982. Models of solar differential rotation, *Geophys. Astrophys. Fluid Dyn.*, **21**, 27–58.

Stewart, R. W. & Thomson, R. E., 1977. Re-examination of vorticity transfer theory, *Proc. R. Soc. A*, **354**, 1–8.