KYUNGPOOK Math. J. 53(2013), 459-465
http://dx.doi.org/10.5666/KMJ.2013.53.3.459

Differential Subordination Properties of Sokół-Stankiewicz Starlike Functions

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Abstract. Let $p(z)$ be an analytic function defined on the open unit disk $\mathbf{D}$ and $p(0) = 1$. Condition β in terms of complex numbers D and real E with $-1 < E < 1$ and $|D| \leq 1$ is determined such that $1 + \beta zp'(z) \prec \frac{1+Dz}{1+Ez}$ implies $p(z) \prec \sqrt{1+z}$. Furthermore, the expression $1 + \frac{\beta zp'(z)}{p(z)}$ and $1 + \frac{\beta zp'(z)}{p^2(z)}$ are considered in obtaining similar results.

1. Introduction

Let A denote the class of all analytic functions f in the open unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ and normalised by $f(0) = 0, f'(0) = 1$. An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)(z \in \mathbf{D})$, if there exists an analytic function w in $\mathbf{D}$ such that $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbf{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbf{D}) \subset g(\mathbf{D})$.

Sokół and Stankiewicz [6] introduced the class $SL^\star$ consisting of normalised analytic functions f in $\mathbf{D}$ satisfying the condition $\left|\left[\frac{zf'(z)}{f(z)}\right]^2 - 1\right| < 1, z \in \mathbf{D}$. Geometrically, a function $f \in SL^\star$ if $\frac{zf'(z)}{f(z)}$ is in the interior of the right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$. A function in the class $SL^\star$ is

* Corresponding Author.
Received August 2, 2011; accepted August 22, 2012.
2010 Mathematics Subject Classification: 30C45, 30C50 .
Key words and phrases: Analytic functions, differential subordination, Sokół-Stankiewicz starlike functions.

called Sokół-Stankiewicz starlike function. Alternatively, we can also write

$$f \in SL^{\star} \;\Leftrightarrow \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}\ .$$

Properties of functions in $SL^{\star}$ have intensively been studied by authors in [4], [7], [8], [9] and [10].

Next, we denote $S^{\star}[A,B]$ as the class of Janowski starlike functions introduced by Janowski [1] and it consists of functions $f \in A$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1).$$

For analytic function $p(z)$ in $\mathbf{D}$ with $p(0)=1$, Nunokawa et. al. [3] investigated and established the relation $1+zp'(z) \prec 1+z$ implies $p(z) \prec 1+z$. Ali et. al. [5] extended this result and obtained conditions for which $1+zp'(z) \prec \frac{1+Dz}{1+Ez}$ implies $p(z) \prec \frac{1+Az}{1+Bz}$. Recently, in [4], condition for which $1+zp'(z) \prec \sqrt{1+z}$ implies $p(z) \prec \sqrt{1+z}$ were determined. Motivated by these studies, this paper considers ascertaining condition so that $1+zp'(z) \prec \frac{1+Dz}{1+Ez}$ implies $p(z) \prec \sqrt{1+z}$. Other results involving the expression $1+\frac{\beta zp'(z)}{p(z)}$ and $1+\frac{\beta zp'(z)}{p^2(z)}$ were also looked at.

2. Main Results

In proving our results, the following lemma proved by Miller and Mocanu is used.

Lemma 2.1([2], p. 135**.** *Let q be univalent in $\boldsymbol{D}$ and let φ be analytic in a domain containing $q(\boldsymbol{D})$. Let $zq'(z)\varphi[q(z)]$ be starlike. If p is analytic in $D, p(0)=q(0)$ and satisfies $zp'(z)\varphi[p(z)] \prec zq'(z)\varphi[q(z)]$ then $p \prec q$ and q is the best dominant.*

Our first result is as follows:

Theorem 2.1. *Let p be an analytic function on $\boldsymbol{D}$ and $p(0)=1$. Let $\beta \geq \beta_0$, $\beta_0 = \frac{2\sqrt{2}|D-E|}{(1-|E|)}$ where $-1<E<1$ and $|D| \leq 1$.*

If

$$1+\beta zp'(z) \prec \frac{1+Dz}{1+Ez}\quad ,$$

then

$$p(z) \prec \sqrt{1+z}\ .$$

Proof. Let $q(z)=\sqrt{1+z}$ with $q(0)=1, q:\mathbf{D}\to C$. Since $q(\mathbf{D})$ is a convex set thus q is a convex function which implies $zq'(z)$ is starlike with respect to 0.

Lemma 2.1 suggests

$$1+\beta zp'(z) \prec 1+\beta zq'(z) \Rightarrow p(z) \prec q(z),$$

so to prove our result, it is suffice to show

$$s(z)=\frac{1+Dz}{1+Ez} \prec 1+\beta zq'(z) = 1+\frac{\beta z}{2\sqrt{1+z}} = h(z).$$

Since $s^{-1}(w)=\frac{w-1}{D-Ew}$, then

$$s^{-1}[h(z)] = \frac{\beta z}{2\sqrt{1+z}(D-E)-\beta E z} \quad .$$

For $z=e^{i\theta}, \theta \in [-\pi,\pi]$,

$$|s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]|$$

$$= \frac{\beta}{|2\sqrt{1+e^{i\theta}}(D-E)-\beta E e^{i\theta}|}$$

$$\geq \frac{\beta}{2|\sqrt{1+e^{i\theta}}||(D-E)|+\beta|E|}$$

$$= \frac{\beta}{2\sqrt{2|cos\frac{\theta}{2}|}|(D-E)|+\beta|E|}$$

It can be shown that the above expression is minimum when $\theta=0$.

Thus

$$|s^{-1}[h(z)]| \geq \frac{\beta}{2\sqrt{2}|(D-E)|+\beta|E|} \geq 1$$

for $\beta \geq \frac{2\sqrt{2}|(D-E)|}{(1-|E|)}$. Therefore $\mathbf{D} \subset s^{-1}[h(\mathbf{D})]$ or $s(\mathbf{D}) \subset h(\mathbf{D})$ implies $s(z) \prec h(z)$. Hence, the result is proven. □

Corollary 2.1. *Let* $\beta \geq \beta_0,\ \beta_0 = \frac{2\sqrt{2}|D-E|}{(1-|E|)}$ *where* $-1 < E < 1,\ |D| \leq 1,$ *and* $f \in A$.
i) If f *satisfies the following*

$$1+\beta\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+1\right) \prec \frac{1+Dz}{1+Ez}$$

then $f \in SL^\star$.

ii) If $1+\beta z f^{''}(z) \prec \frac{1+Dz}{1+Ez}$ *then* $f'(z) \prec \sqrt{1+z}$.

Proof. Define $p(z) = \frac{zf'(z)}{f(z)}$ and using Theorem 2.1, the first part of Corollary 2.1 is proved. The second part of our results in Corollary 2.1 can be derived by letting $p(z) = f'(z)$. □

Theorem 2.2. *Let* p *be an analytic function in* $\boldsymbol{D}$ *and* $p(0) = 1$. *Let* $\beta \geq \beta_0,\ \beta_0 = \frac{4|D-E|}{(1-|E|)}$, $-1 < E < 1$ *and* $|D| \leq 1$.

$$1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \sqrt{1+z}\ .$$

Proof. Let $q(z) = \sqrt{1+z}$, $q(0) = 1$. Elementary calculation will show that $\frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1+z)}$ is starlike. Thus, Lemma 2.1 can be applied as

$$1+\beta\frac{zp'(z)}{p(z)} \prec 1+\beta\frac{zq'(z)}{q(z)} \Rightarrow p(z) \prec q(z).$$

Next, we prove the subordination

$$s(z) = \frac{1+Dz}{1+Ez} \prec 1+\beta\frac{zq'(z)}{q(z)} = 1+\frac{\beta z}{2(1+z)} = h(z).$$

$$s^{-1}[h(z)] = \frac{\beta z}{2(1+z)(D-E)-\beta E z} \quad .$$

For $z = e^{i\theta}, \theta \in [-\pi, \pi]$,

$$\begin{aligned} |s^{-1}[h(z)]| &= |s^{-1}[h(e^{i\theta})]| \\ &= \frac{\beta}{|2(1+e^{i\theta})(D-E)-\beta E e^{i\theta}|} \\ &\geq \frac{\beta}{|2(1+e^{i\theta})||(D-E)|+\beta|E|} \\ &= \frac{\beta}{4|cos\frac{\theta}{2}||(D-E)|+\beta|E|} \end{aligned}$$

A straight forward computation verifies that the above expression is minimum when $\theta = 0$.
Then

$$|s^{-1}[h(z)]| \geq \frac{\beta}{4|(D-E)|+\beta|E|} \geq 1$$

for $\beta \geq \frac{4|(D-E)|}{(1-|E|)}$. Hence $s(\mathbf{D}) \subset h(\mathbf{D})$ implies $s(z) \prec h(z)$. □

Corollary 2.2. *Let* $\beta \geq \beta_0,\ \beta_0 = \frac{4|D-E|}{(1-|E|)}$, $-1 < E < 1$ *and* $|D| \leq 1$,
i)

$$1+\beta\left[1+\frac{zf^{''}(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right] \prec \frac{1+Dz}{1+Ez} \Rightarrow f \in SL^{\star}\ .$$

ii)

$$1+\beta\left[\frac{(zf(z))^{''}}{f'(z)}-\frac{2zf'(z)}{f(z)}\right] \prec \frac{1+Dz}{1+Ez} \Rightarrow \frac{z^2 f'(z)}{f^2(z)} \prec \sqrt{1+z}\quad .$$

Proof. Letting $p(z) = \frac{zf'(z)}{f(z)}$ in (i) and $p(z) = \frac{z^2 f'(z)}{f^2(z)}$ in (ii) and applying Theorem 2.2 proves the results. □

Theorem 2.3. *Let* $\beta \geq \beta_0,\ \beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)}$, $-1 < E < 1$ *and* $|D| \leq 1$.

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \sqrt{1+z}\ .$$

Proof. Let $q(z) = \sqrt{1+z}$, which implies $\frac{zq'(z)}{q^2(z)}$ is starlike.
Using Lemma 2.1,

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec 1+\beta\frac{zq'(z)}{q^2(z)} \Rightarrow p(z) \prec q(z)\ .$$

Next, let $h(z) = 1+\beta\frac{zq'(z)}{q^2(z)} = 1+\frac{\beta z}{2(1+z)^{\frac{3}{2}}}$

$$s^{-1}[h(z)] = \frac{\beta z}{2(1+z)^{\frac{3}{2}}(D-E)-\beta E z}\quad .$$

For $z = e^{i\theta}, \theta \in [-\pi,\pi]$,

$$\begin{aligned} |s^{-1}[h(z)]| &= |s^{-1}[h(e^{i\theta})]| \\ &= \frac{\beta}{|2(1+e^{i\theta})^{\frac{3}{2}}(D-E)-\beta E e^{i\theta}|} \\ &\geq \frac{\beta}{|2(1+e^{i\theta})^{\frac{3}{2}}||(D-E)|+\beta|E|} \\ &= \frac{\beta}{2|(2cos\frac{\theta}{2})^{\frac{3}{2}}||(D-E)|+\beta|E|} \end{aligned}$$

As in previous case, the above expression is minimum when $\theta = 0$.
Then

$$|s^{-1}[h(z)]| \geq \frac{\beta}{4\sqrt{2}|(D-E)|+\beta|E|} \geq 1$$

for $\beta \geq \frac{4\sqrt{2}|(D-E)|}{(1-|E|)}$. Hence $\mathbf{D} \subset s^{-1}[h(\mathbf{D})]$ implies $s(z) \prec h(z)$. □

Corollary 2.3. *Let* $\beta \geq \beta_0,\ \beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)}$, $-1 < E < 1,\ |D| \leq 1$ *and* $f \in A$,

$$1-\beta+\beta\left[\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right] \prec \frac{1+Dz}{1+Ez} \Rightarrow f \in SL^{\star}.$$

Proof. The result is obtained by taking $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2.3. □

Theorem 2.4. *Let* p *be an analytic function in* $\boldsymbol{D}$ *and* $p(0) = 1$.
Let $\beta \geq \beta_0$, $0 < \alpha \leq 1,\ \beta_0 = \frac{|1+A||1+B||D-E|}{\alpha|A-B|(1-|E|)}$, $-1 < E < 1,\ |D| \leq 1$ *and* $-1 \leq B < A \leq 1$.

$$1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\alpha} .$$

Proof. Let $q(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}$, Then

$$\frac{\beta zq'(z)}{q(z)} = \frac{\beta\alpha z(A-B)}{(1+Az)(1+Bz)} = Q(z)$$

It can easily be verified that $Q(z)$ is starlike. Lemma 2.1, we prove the subordination

$$s(z) = \frac{1+Dz}{1+Ez} \prec 1+\beta\frac{zq'(z)}{q(z)} = 1+\frac{\beta\alpha z(A-B)}{(1+Az)(1+Bz)} = h(z)$$

Since $s^{-1}(w) = \frac{w-1}{D-Ew}$ then

$$\begin{aligned}|s^{-1}[h(z)]| &= \left|\frac{\beta\alpha z(A-B)}{[(1+Az)(1+Bz)(D-E)]-\beta\alpha zE(A-B)}\right| \\ &\geq \frac{|\beta\alpha z(A-B)|}{|[(1+Az)(1+Bz)(D-E)]|+|\beta\alpha zE(A-B)|} .\end{aligned}$$

For $z = e^{i\theta}, \theta \in [-\pi,\pi]$,

$$|s^{-1}[h(e^{i\theta})]| \geq \frac{\beta\alpha|(A-B)|}{|[(1+Ae^{i\theta})(1+Be^{i\theta})(D-E)]|+\beta\alpha|E(A-B)|}$$

with minimum value being attained at $\theta = 0$.
Hence

$$|s^{-1}[h(e^{i\theta})]| \geq \frac{\beta\alpha|(A-B)|}{|[(1+A)(1+B)(D-E)]|+\beta\alpha|E(A-B)|} \geq 1$$

for $\beta \geq \frac{|[(1+A)(1+B)(D-E)]|}{\alpha|A-B|(1-|E|)}$ implies $s(z) \prec h(z)$ and the result is obtained. □

Remark. Theorem 2.4 is reduced to Theorem 2.2 when $\alpha = \frac{1}{2}$, $A = 1$ and $B = 0$.

Finally, we state the next obvious result.

Corollary 2.4. *Let* $\beta_0 = \frac{|1+A||1+B||D-E|}{\alpha|A-B|(1-|E|)}$, $-1 < E < 1$, $|D| \leq 1$ *and* $-1 \leq B < A \leq 1$.

$$1 + \beta \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right] \prec \frac{1 + Dz}{1 + Ez} \Rightarrow \frac{zf'(z)}{f(z)} \prec \left(\frac{1 + Az}{1 + Bz} \right)^{\alpha} .$$

Acknowledgements. This research was supported by IPPP/UPGP/geran(RU/PP P)/PS207/2009A University Malaya grants 2009. The authors are highly grateful to the referees and Prof. Dr. V. Ravichandran for their valuable comments and suggestions.

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