Bol. Soc. Paran. Mat. (3s.) v. 32 2 (2014): 249–261.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i2.21618

Differential Transformation Method to determine Magneto Hydrodynamics flow of compressible fluid in a channel with porous walls

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abstract: In this article magneto hydrodynamics (MHD) boundary layer flow of compressible fluid in a channel with porous walls have been researched. In this study it is shown that the nonlinear Navier-Stokes equations can be reduced to an ordinary differential equation, using the similarity transformations and boundary layer approximations. Analytical solution of the developed nonlinear equation is carried out by the Differential Transformation Method (DTM). In addition to applying DTM into the obtained equation, the result of the mentioned method is compared with a type of numerical analysis as Boundary Value Problem method (BVP) and a good agreement is seen. The effects of the Reynolds number and Hartman number are investigated.

Key Words: MHD Flow, Compressible Fluid, Boundary layer, DTM, BVP

Contents

Nomenclature

BVP	*boundary value problem method*
B_0	*uniform static magnetic field*
DTM	*differential transformation method*
f	*similarity function*
H	*channel width (m)*
M	*Hartman number*
p	*pressure (pa)*
Re	*Reynolds number*
u	*x velocity (m/s)*
v	*y velocity (m/s)*

2000 *Mathematics Subject Classification:* 76NXX, 76D10

Greek symbols

μ	*dynamic viscosity* $(N.s/m^2)$
ρ	*density ((kg/m3)*
σ	*electrica conductivity (Siemens/m, where Siemens=1/*Ω)

1. Introduction

Magnetohydrodynamics is essential in plasma physics and astrophysics and studies the motion of electrically conducting media in the presence of a magnetic field. In natural systems include the Earth's core and solar flares, and in the engineering world, the electromagnetic casting of metals and the confinement of plasmas MHD effects are important [1]. Recently reactor designs commonly involve the use of electrically conducting liquid metals, in fusion engineering, are much of the interest [2].

In order to determine the velocity components, DTM is applied to solve the resulting nonlinear differential equation. Then the solution is compared with Boundary Value Problem Method. An ordinary non-linear differential equation can be derived from the governing differential equations by using similarity transformation. In semi-analytical techniques such as differential transform method (DTM), homotopy perturbation method (HPM) and etc. the differential equations will be transformed into algebraic equations so that by these methods the most problems can be solved. DTM was first applied to the engineering field by Zhou [3]. This method is based on Taylor expansion that produces a polynomial form of the main equations and requires calculating the essential derivatives of the data functions. The mentioned method includes of an iterative procedure to deal with the differential equations analytically. A.A. Joneidi and et al. [4] applied three new analytical approximate techniques for addressing nonlinear problems to Jeffery–Hamel flow. Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM) and Differential Transformation Method (DTM) were proposed and used in this research. Rahimi et al. [5] used this method for obtaining efficiency, temperature distribution, and effectiveness of conductive, convective, and radiative straight fins with temperature dependent thermal conductivity. As DTM has the ability to solve the non-linear problems, so it has been applied for the solution of the non-linear vibration problems by Chiou and Tzeng [6]. It should be explained that DTM method can also be used to solve the partial differential equations as Jang et al. [7] carried it out. Different application problems have been solved by this method [8-13].

2. Description of the problem

The two-dimensional MHD flow of a compressible fluid in a porous channel with suction and injection are investigated. The geometry of the problem is shown in figure (1-a) and (1-b). The x-axis is taken along the centerline of the channel and the y-axis transverse to these. The flow is symmetric about both axes. The porous walls of the channel are at $y = H/2$ and $y = H/2$. The fluid injection or suction takes place through the porous walls with velocity $V_0/2$. Here $V_0 > 0$ corresponds to suction and $V_0 < 0$ for injection. Let u and v be the velocity components along the x- and y-axes respectively, and B_0 is a uniform static magnetic field in Y-direction.

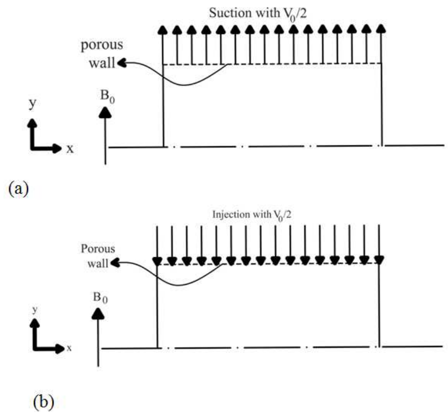

Figure 1: Axial section of the channel in case of (a) suction (b) injection

The compressible electrically conducting fluid that flows though the axial direction in the channel will induce a magnetic field in the medium in an applied magnetic field. The magnetic Reynolds number ($Re_m = \sigma_m UL$) represents the relative strength of the induced field. In the above relation the characteristics such as U and L are the scale length and velocity and μ_m is magnetic permeability. If the magnetic Reynolds number is small, the induced magnetic field will be neglected [14].

It can be assumed that the electric field is zero as no external electric field is applied and the effect of polarization of the ionized fluid is negligible. The equations for the MHD boundary layer flow of a compressible fluid with are:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\sigma B_0^2 u - \frac{\partial p}{\partial x} + \frac{4}{3}\mu\frac{\partial^2 u}{\partial x^2} + \frac{1}{3}\mu\frac{\partial^2 v}{\partial x \partial y} + \mu\frac{\partial^2 u}{\partial y^2} \tag{2.2}$$

Assuming the symmetry about the x-axis and no-slip conditions at$y = H/2$, we have:

$$\begin{aligned} &\frac{\partial u}{\partial y} = 0\,,\ v = 0 \qquad at\ \ y = 0 \\ &u = 0,\ v = \frac{V_0}{2} \qquad at\ \ y = \frac{H}{2} \end{aligned} \tag{2.3}$$

The Equation (2.4) represents the non-dimensional parameters to rewrite the Equation (2.2) in the non-dimensional form, in which $f(y^*)$ is assumed as a similarity function.

$$x^{*}=\frac{x}{H},\ \ y^{*}=\frac{y}{H},\ \ u=-V_{0}x^{*}f^{'}\left(y^{*}\right),\ \ v=V_{0}f\left(y^{*}\right) \tag{2.4}$$

Applying the above equation, Equations (2.2) and (2.3) may be written as:

$$f^{'''}+Re(f^{'2}-ff^{''})-M^{2}f^{'}=0 \tag{2.5}$$

$$f=0,\qquad f^{''}=0\quad at\ \ y=0$$
$$f=\frac{1}{2},\qquad f^{'}=0\quad at\ \ y=\frac{1}{2} \tag{2.6}$$

Where $M^2=\sigma B_0^2H^2/\mu$ and $Re=\rho HV_0/\mu$ are known as Hartman number and Reynolds number respectively. To solve Equations (2.5) and (2.6), the DTM method is employed.

3. Solution with Differential Transformation Method (DTM)

First briefly DTM method will be introduced. Let $x(t)$ be analytic function in a field that Taylor series expansion of $x(t)$ is of the form of the following [15].

$$x(t)=\sum_{k=0}^{n}\left(\frac{t}{H}\right)^{k}X(k) \tag{3.1}$$

In which the transformed function is calculated as the below equation:

$$X(k)=\sum_{k=0}^{\infty}\frac{H^{k}}{k!}\left[\frac{d^{k}x(t)}{dt^{k}}\right]_{t=0}, \tag{3.2}$$

Obviously, the concept of DTM method is based on the Taylor series expansion. Mathematical operations performed by Differential Transformation Method are listed in the Table 1.

Table 1: The fundamental operations of differential transformation method

Original function	Transformed function
$x\left(t\right)=\alpha f\left(t\right)\pm\beta g\left(t\right)$	$X\left(k\right)=\alpha F\left(k\right)\pm\beta G\left(k\right)$
$x\left(t\right)=\frac{df(t)}{dt}$	$X\left(k\right)=\left(k+1\right)F\left(k+1\right)$
$x\left(t\right)=\frac{d^2f(t)}{dt^2}$	$X\left(k\right)=\left(k+1\right)\left(k+2\right)F\left(k+2\right)$
$x\left(t\right)=f\left(t\right)g\left(t\right)$	$X\left(k\right)=\sum_{l=0}^{k}F\left(l\right)G\left(k-l\right)$
$x\left(t\right)=t^{m}$	$X\left(k\right)=\delta\left(k-m\right)=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}$

Now the explained method will be applied into Equation (2.5) considering $H=$

1.

$$
\begin{aligned}
&(j+1)(j+2)(j+3)f_{j+3} - M^2(j+1)f_{j+1} \\
&+Re\left(\sum_{i=0}^{j}(i+1)f_{i+1}\,(j-i+1)\,f_{j-i+1}\right) \\
&-Re\left(\sum_{i=0}^{j}f_i(j-i+1)(j-i+2)f_{j-i+2}\right) = 0
\end{aligned}
\tag{3.3}
$$

From boundary conditions in Equation (2.6), and performing the transformation:

$$f(0) = 0 \tag{3.4}$$

The other boundary conditions are considered as following:

$$
\begin{aligned}
f(1) &= a \\
f(2) &= b \\
f(3) &= c
\end{aligned}
\tag{3.5}
$$

Where a, b and c are constants. These parameters will be calculated with considering another boundary condition in Equation (2.6).

$$f_4 = -\frac{1}{12}Re\ ab + \frac{1}{12}M^2 b \tag{3.6}$$

$$f_5 = -\frac{1}{30}Re\ b^2 + \frac{1}{20}M^2 c \tag{3.7}$$

$$f_6 = -\frac{1}{360}Re^2 a^2 b - \frac{1}{30}Re\ bc + \frac{1}{360}M^4 b \tag{3.8}$$

$$f_7 = -\frac{1}{1260}a\,Re^2 b^2 + \frac{1}{420}Re\,aM^2 c - \frac{1}{630}ReM^2 b^2 - \frac{1}{70}Re\,c^2 + \frac{1}{840}M^4 c \tag{3.9}$$

This procedure can be continued. Inserting the Equation (3.4) to (3.9) into the main equation on the basis of DTM, the closest form of the solution will be obtained.

$$
\begin{aligned}
f(y^*) = {} & ay^* + by^{*2} + cy^{*3} + \left(-\frac{1}{12}Re\ \ ab + \frac{1}{12}M^2 b\right) y^{*4} \\
& + \left(-\frac{1}{30}Re\ \ b^2 + \frac{1}{20}M^2 c\right) y^{*5} \\
& + \left(-\frac{1}{360}Re^2 a^2 b - \frac{1}{30}Re\ \ bc + \frac{1}{360}M^4 b\right) y^{*6} \\
& + \left(-\frac{1}{1260}a\ \ Re^2 b^2 + \frac{1}{420}Re\ aM^2 c - \frac{1}{630}Re\ M^2 b^2 \right. \\
& \left. -\frac{1}{70}Re\ c^2 + \frac{1}{840}M^4 c\right) y^{*7} + \ldots
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
f'^*) = {} & a + 2by^* + 3cy^{*2} + \left(-\frac{1}{3}Reab + \frac{1}{3}M^2 b\right) y^{*3} + \left(-\frac{1}{6}Reb^2 + \frac{1}{4}M^2 c\right) y^{*4} \\
& + \left(-\frac{1}{60}Re^2 a^2 b - \frac{1}{5}Rebc + \frac{1}{60}M^4 b\right) y^{*5} \\
& + \left(-\frac{1}{180}aRe^2 b^2 + \frac{1}{60}ReaM^2 c - \frac{1}{90}ReM^2 b^2 \right. \\
& \left. -\frac{1}{10}Rec^2 + \frac{1}{120}M^4 c\right) y^{*6} + \ldots
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
f^{''}(y^*) = {} & 2b + 6cy^* + \left(-Reab + M^2 b\right) y^{*2} + \left(-\frac{2}{3}Reb^2 + M^2 c\right) y^{*3} \\
& + \left(-\frac{1}{12}Re^2 a^2 b - Rebc + \frac{1}{12}M^4 b\right) y^{*4} \\
& + \left(-\frac{1}{30}aRe^2 b^2 + \frac{1}{10}ReaM^2 c - \frac{1}{15}ReM^2 b^2 - \frac{3}{5}Rec^2 + \frac{1}{20}M^4 c\right) y^{*5} + \ldots
\end{aligned}
\tag{3.12}
$$

Substituting the boundary conditions from Equation (2.6) into Equations (3.10), (3.11) and (3.12) in$y^* = 0.5$, we have:

$$
\begin{aligned}
f(0.5) = {} & \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{1}{16}\left(-\frac{1}{12}Re\ ab + \frac{1}{12}M^2 b\right) + \frac{1}{32}\left(-\frac{1}{30}Re\ b^2 + \frac{1}{20}M^2 c\right) \\
& + \frac{1}{64}\left(-\frac{1}{360}Re^2\ a^2 b - \frac{1}{30}Re\ \ bc + \frac{1}{360}M^4 b\right) \\
& + \frac{1}{128}\left(-\frac{1}{1260}aRe^2\ b^2 + \frac{1}{420}Re\ \ aM^2 c - \frac{1}{630}Re\ \ M^2 b^2\right. \\
& \left. - \frac{1}{70}Re\ \ c^2 + \frac{1}{840}M^4 c\right) + \ldots
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
f'(0.5) = {} & a + b + \frac{3}{4}c + \frac{1}{8}\left(-\frac{1}{3}Re\ ab + \frac{1}{3}M^2 b\right) + \frac{1}{16}\left(-\frac{1}{6}Re\ b^2 + \frac{1}{4}M^2 c\right) \\
& + \frac{1}{32}\left(-\frac{1}{60}Re^2\ a^2 b - \frac{1}{5}Re\ bc + \frac{1}{60}M^4 b\right) \\
& + \frac{1}{64}\left(-\frac{1}{180}aRe^2\ b^2 + \frac{1}{60}Re\ aM^2 c - \frac{1}{90}Re\ M^2 b^2\right. \\
& \left. - \frac{1}{10}Re\ c^2 + \frac{1}{120}M^4 c\right) + \ldots
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
f''(0.5) = {} & 2b + 3c + \frac{1}{4}\left(-Re\ ab + M^2 b\right) + \frac{1}{8}\left(-\frac{2}{3}Re\ b^2 + M^2 c\right) \\
& + \frac{1}{16}\left(-\frac{1}{12}Re^2\ a^2 b - Re\ bc + \frac{1}{12}M^4 b\right) \\
& + \frac{1}{32}\left(-\frac{1}{30}aRe^2\ b^2 + \frac{1}{10}Re\ aM^2 c - \frac{1}{15}Re\ M^2 b^2 - \frac{3}{5}Re\ c^2 + \frac{1}{20}M^4 c\right) + \ldots
\end{aligned} \tag{3.15}
$$

Then Solving the above equations together and assuming$M = 1$, $Re = 4$, the values of a, b, c will be obtained.

$$a = 1.485594956, b = 0, c = -1.902424049 \tag{3.16}$$

So $f(y^*)$ will be yielded as the following:

$$
\begin{aligned}
f(y^*) = {} & 1.485594956y^* - 0.08693160073\mathrm{y}^{*9} - 0.2359936967\mathrm{y}^{*7} \\
& + 0.003149937499\mathrm{y}^{*13} + 0.002905168393\mathrm{y}^{*15} - 0.01826631831\mathrm{y}^{*11} \\
& - 1.902424049\mathrm{y}^{*3} - 0.09512120245\mathrm{y}^{*5} + 0.00007152172800\mathrm{y}^{*17}
\end{aligned} \tag{3.17}
$$

4. Result and discussion

Figure 2 represents the comparison of Differential Transformation Method (DTM) and Boundary Value Problem (BVP) for $f(y^*)$. From figure 2 and Table 2, it is considerable that DTM with fifteen orders converge to the results with a good accuracy.

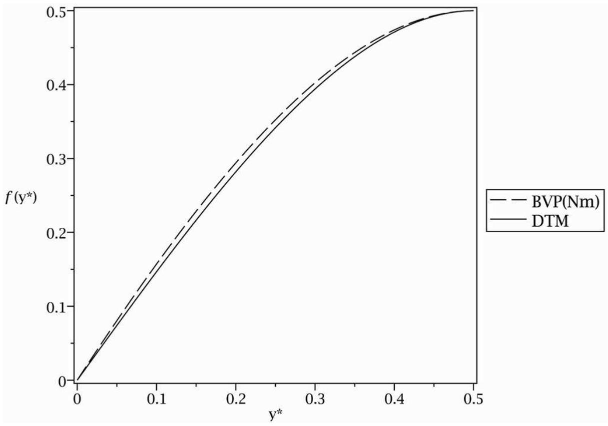

Figure 2: comparison of DTM and BVP for $f(y^*)$ on the 15th -order approximation ($Re = 4,\ M = 1$)

Table 2: Comparing the results of DTM with BVP Method for different iteration

	iteration	1		7		15	
y^*	fNM(BVP)	f DTM	Error	f DTM	Error	f DTM	Error
0	0	0	0	0	0	0	0
0.1	5.94E-02	0.07475	0.01539	0.074045	0.01468	0.07404	0.01468
0.1	0.11817	0.148	0.02983	0.146662	0.02849	0.14666	0.02849
0.2	0.175857	0.21825	0.04239	0.216419	0.04056	0.21641	0.04055
0.2	0.231861	0.284	0.05214	0.281876	0.05002	0.28187	0.05
0.3	0.285615	0.34375	0.05813	0.341577	0.05596	0.34157	0.05595
0.3	0.336543	0.396	0.05946	0.394039	0.0575	0.39403	0.05749
0.4	0.384058	0.43925	0.05519	0.437742	0.05368	0.43773	0.05368
0.4	0.427558	0.472	0.04444	0.471104	0.04355	0.4711	0.04354
0.5	0.466422	0.49275	0.02633	0.492456	0.02603	0.49245	0.02603
0.5	0.5	0.5	0	0.5	2E-10	0.5	-2E-10
$\sum$ Error			0.3833		0.37047		0.37041

In figures 3 and 4, the effect of the injection velocity on f and f' are shown. It can be seen that as the velocity injection enlarges, both f and f' increase. Although the suction case, f increases and f decreases. So it means that suction force assists the structural formation of y direction flow, in the contrary of x direction.

In figures 5 to 8 the effects of Hartman number and Reynolds number on the velocity components f and f' are investigated. From figures (5) and (6), it is observed that as the Reynolds number and Hartman number increase, the similarity function (f) decreases. In the figures 7 and 8, toward the center point from$y^* = 0$ to the suction side as the Hartman number and Reynolds number grow, f' decreases, but then this parameter increases. Hence the profile of the velocity component in x direction will have a common point that approximately takes place in $y^* = 0.25$.

So the stated point can be interpreted as a critical point in the formation of x direction flow.

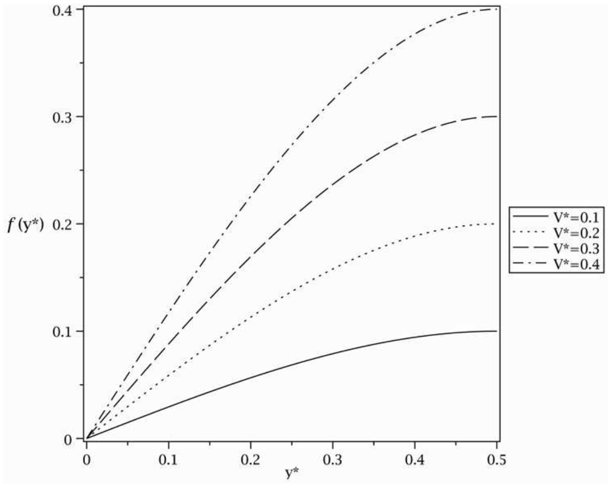

Figure 3: Effects of the injection velocity (V_0) for $f(y^*)$ on the 15th-order approximation ($M = 1$, $Re = 4$)

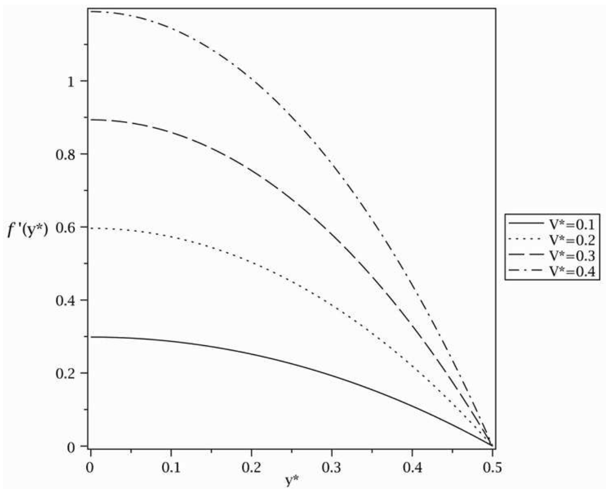

Figure 4: Effects of the injection velocity (V) for f'^{*}) on the 15th-order approximation ($M = 1$, $Re = 4$)

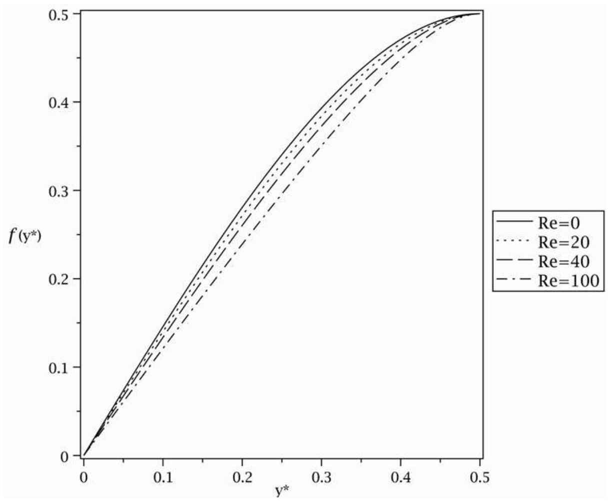

Figure 5: Effects of the Reynolds number or $f(y^*)$ on the 15th-order approximation ($M = 2$)

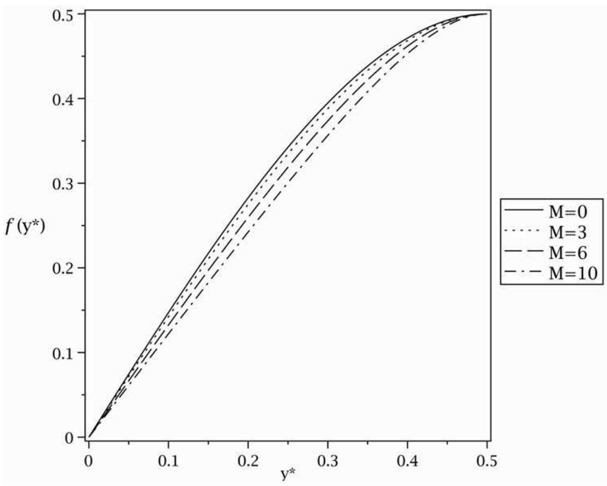

Figure 6: Effects of the Hartman number for $f(y^*)$ on the 15th-order approximation ($Re = 4$)

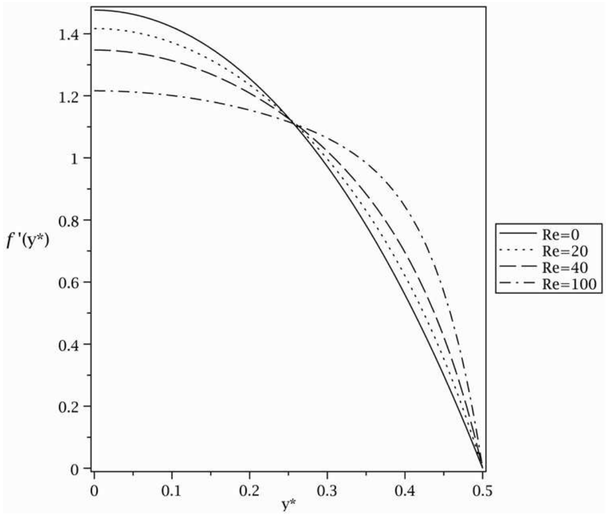

Figure 7: Effects of the Reynolds number for $f'^{*})$ on the 15th-order approximation ($M = 2$)

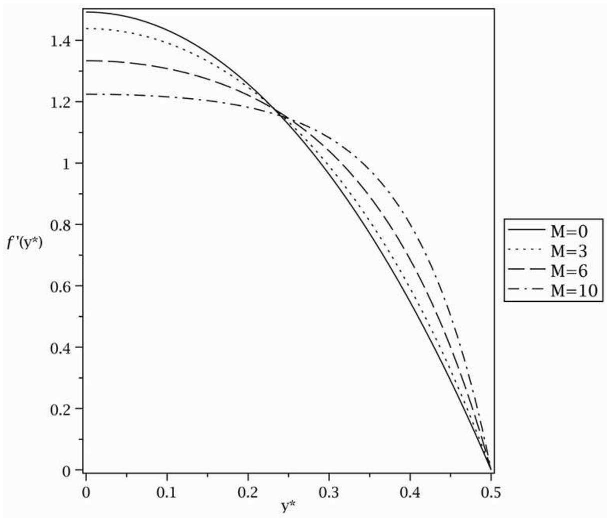

Figure 8: Effects of the Hartman number for $f'^{*})$ on the 15th-order approximation ($Re = 4$)

5. Conclusion

In this research, an analytic method for the solution of the two-dimensional magnetohydrodynamics (MHD) boundary layer flow of compressible fluid have been presented. Differential equations were transformed to algebraic equations, using Differential Transformation Method (DTM). Then DTM was compared with Boundary Value Problem (BVP) method as a numerical solution. The effects of different Reynolds number and Hartman number were investigated for the similarity functions$f$, f' used to determine the velocity components. It was found from the results, as the Hartman number and Reynolds number changed a common point appeared in the profile of the velocity component in x direction. When the velocity injection increased, it was clear that the suction force assisted the structural formation of y direction flow. This research has been also proved that DTM includes of high accuracy to solve different problems in the engineering field.

Acknowledgments

The authors are thankful to the referee for carefully reading the paper and for his suggestions and remarks.

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