Differentially positive systems

F. Forni, R. Sepulchre

***Abstract*—The paper introduces and studies differentially positive systems, that is, systems whose linearization along an arbitrary trajectory is positive. A generalization of Perron Frobenius theory is developed in this differential framework to show that the property induces a (conal) order that strongly constrains the asymptotic behavior of solutions. The results illustrate that behaviors constrained by local order properties extend beyond the well-studied class of linear positive systems and monotone systems, which both require a constant cone field and a linear state space.**

I. Introduction

Positive systems are linear behaviors that leave a cone invariant [11]. They have a rich history both because of the relevance of the property in applications (e.g., when modeling a behavior with positive variables [30], [18], [25]) and because the property significantly restricts the behavior, as established by Perron Frobenius theory: if the cone invariance is strict, that is, if the boundary of the cone is eventually mapped to the interior of the cone, then the asymptotic behavior of the system lies on a one dimensional object. Positive systems find many applications in systems and control, ranging from specific stabilization properties [48], [32], [18], [14], [26], [39] to observer design [22], [9], and to distributed control [31], [37], [42].

Motivated by the importance of positivity in linear systems theory, the present paper investigates the behavior of differentially positive systems, that is, systems whose linearization along trajectories is positive. We discuss both the relevance of the property for applications and how much the property restricts the behavior, by generalizing Perron Frobenius theory to the differential framework. The conceptual picture is that a cone is attached to every point of the state space, defining a cone field, and that contraction of that cone field along the flow eventually constrains the behavior to be one-dimensional.

Differential positivity reduces to the well-studied property of monotonicity when the state-space is a linear vector space and when the cone field is constant. First studied for closed systems [43], [24], [23], [13] and later extended to open systems [3], [5], [4], the concept of monotone systems encompasses cooperative and competitive systems [25], [36] and is extensively adopted in biology and chemistry for modeling and control purposes [15], [17], [16], [45], [6], [7]. Differential positivity is an infinitesimal characterization of monotonicity. The differential viewpoint allows for a generalization of monotonicity because the state-space needs not be linear and the cone needs not be constant in space. The generalization is relevant in a number of applications. In particular, non-constant cone fields in linear spaces and invariant cone fields on nonlinear spaces are two situations frequently encountered in applications. Like monotonicity, differential positivity induces an order between solutions. But in contrast to monotone systems, the *conal* order needs not to induce a partial order *globally*, allowing for instance to (locally) order solutions on closed curves, such as along limit cycles or in nonlinear spaces such as the circle.

A main contribution of the paper is to generalize Perron-Frobenius theory in the differential framework. The Perron-Frobenius vector of linear positive systems here becomes a vector field and the integral curves of the Perron-Frobenius vector field shape the attractors of the system. A main result of the paper is to provide a characterization of limit sets of differentially positive systems akin to Poincaré-Bendixson theorem for planar systems. Differentially positive systems can model multistable behaviors, excitable behaviors, oscillatory behaviors, but preclude for instance the existence of attractive homoclinic orbits, and a fortiori of strange attractors. In that sense, differentially positive systems single out a significant class of nonlinear systems that have a simple asymptotic behavior.

The paper is organized as follows. Section II introduces the main ideas of differential positivity on familiar phase portraits and at an intuitive level. It aims at showing that the differential concept of positivity is a natural one. Section III covers some mathematical preliminaries and notations while Section IV summarizes the main mathematical notions of order on manifolds. The next three sections contain the main results of the paper: the formal notion of differentially positive system, differential Perron-Frobenius theory, and a characterization of limit sets of differentially positive systems. Section VIII illustrates several important points of the paper on the popular nonlinear pendulum example. Proofs are in appendix. Our treatment of differential positivity is for continuous-time and discrete-time open systems. The important topic of interconnections of differentially positive systems is a rich one and will be discussed in a separate paper.

II. Differential positivity in a nutshell

Figure 1 illustrates four different phase portraits of (closed) differentially positive systems. Two of the phase portraits are represented in two different set of coordinates. The figure illustrates that for each of the phase portraits, a cone can

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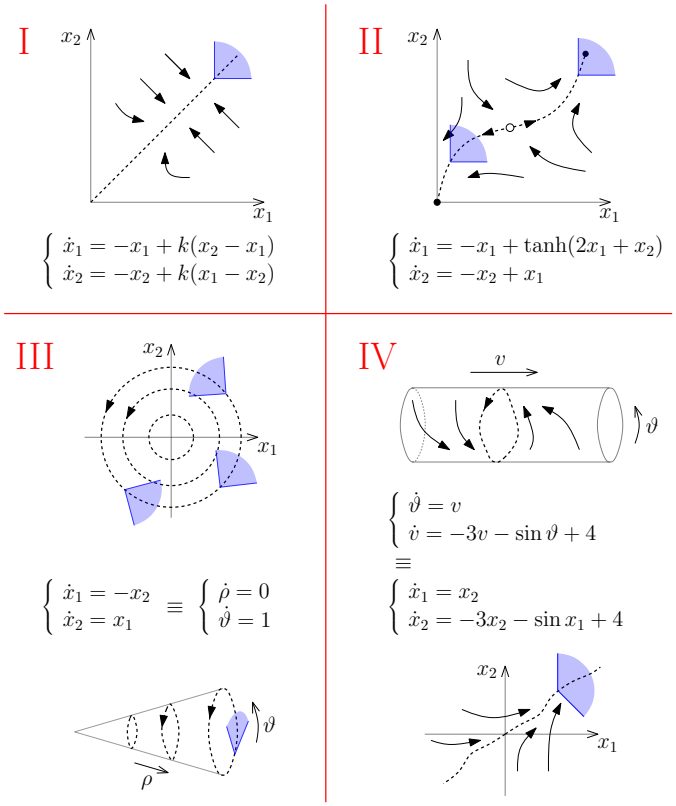

Figure 1. The phase portraits of four different planar differentially positive systems. (I) a linear consensus model. (II) a monotone bistable model. (III) the harmonic oscillator. (IV) the nonlinear pendulum with strong damping and torque.

be attached at any point in such a way that the cone is infinitesimally contracted by the flow (i.e. the cone angle shrinks under the action of the flow). Furthermore, the cones can be patched to each other to define a smooth cone field.

In the first two examples, the cone is actually the same everywhere, defining a *constant* cone field in a *linear* space. In the third example, both the state-space and the dynamics are linear but the cone rotates with the flow. It defines a *non constant* cone field in a *linear* space. In the fourth example, the cone field must be defined infinitesimally because the state space is nonlinear. At each point, a cone is defined in the tangent space. The *nonlinear* cylindrical space $\mathbb{S} \times \mathbb{R}$ is a Lie group and the cone is moved from point to point by (left) translation. The analogy between the third and fourth examples is apparent when studying the phase portrait of the harmonic oscillator in polar coordinates. The nonlinear change of coordinates makes the cone invariant on the conic nonlinear space $\mathbb{R}_+ \times \mathbb{S}$. The analogy between the first, second, and fourth examples is apparent when unwrapping the phase portrait of the nonlinear pendulum in the plane. In cartesian coordinates, that is, unwrapping the angular coordinate ϑ on the real line, the cone field becomes constant in a linear space.

The first phase portrait is the phase portrait of a linear system that leaves the positive orthant invariant. It is a strictly positive system. Its behavior is representative of consensus behaviors extensively studied in the recent years [31], [35], [41]. The second phase portrait leaves the same cone invariant but the dynamics are nonlinear. Here the cone invariance can be characterized differentially: the linearization along any trajectory is a positive linear system with respect to the positive orthant. It is an example of monotone system, representative of bistable behaviors extensively studied in decision-making processes, see e.g. [47]. The third example is the phase portrait of the harmonic oscillator. Solutions cannot be globally ordered in the state space because the trajectories are closed curves. But the positivity of the linearization is nevertheless apparent in polar coordinates. The corresponding order property will be characterized by the notion of conal order on manifolds developed in Section IV. The fourth example is the phase portrait of the nonlinear pendulum with strong damping. Positivity of the linearization and differential positivity of the nonlinear pendulum is studied in details in the last Section of the paper.

The main message of the paper is that differential positivity constrains the asymptotic behavior of the four different phase portraits in a similar way. For linear positive systems, this is Perron- Frobenius theory. The Perron-Frobenius vector attracts all solutions to a one-dimensional ray. For differentially positive systems, the generalized object is a Perron-Frobenius curve, an integral curve of the Perron-Frobenius vector field characterized in Section V. In the second phase portrait, this is the heteroclinic orbit connecting the two stable equilibria and the unstable saddle equilibrium. In the third phase portrait, every trajectory is a Perron-Frobenius curve. The differential positivity is not strict in that case. In the fourth phase portrait, all solutions except the unstable equilibrium are attracted to a single Perron-Frobenius curve, the limit cycle.

The convergence properties of differentially positive systems are a consequence of the infinitesimal contraction of cones along trajectories. The significance of the property is that it can be checked locally but that it discriminates among different types of global behaviors. The smoothness of the cone field is what connects the local property to the global property. A most important feature of differential positivity is that it allows saddle points such as in Figure 1.II, because the local order is compatible with a global smooth cone field, but that it does not allow saddle points such as in Figure 2. The homoclinic orbit makes the local order dictated by the saddle point incompatible with a global smooth cone field. This incompatibility has been recognized since the early work of Poincaré as the essence of complex behaviors. In contrast, the limit sets of differentially positive systems are simple, in a sense that is made precise in Section VII.

III. Mathematical preliminaries and basic assumptions

A (d-dimensional) manifold $\mathcal{X}$ is a couple $(\mathcal{X}, \mathcal{A}^+)$ where $\mathcal{X}$ is a set and $\mathcal{A}^+$ is a maximal atlas of $\mathcal{X}$ into $\mathbb{R}^d$, such that the topology induced by $\mathcal{A}^+$ is Hausdorff and second-countable (we refer to this topology as the manifold topology). Throughout the paper every manifold is connected. $T_x\mathcal{X}$ denotes the tangent space at x and $T\mathcal{X} := \bigcup_{x\in\mathcal{X}} \{x\} \times T_x\mathcal{X}$ denotes the tangent bundle. $\mathcal{X}$ is endowed with a Riemannian metric tensor, represented by a (smoothly varying) inner

product $\langle\cdot,\cdot\rangle_x$. $|\delta x|_x := \sqrt{\langle\delta x, \delta x\rangle_x}$, for any $\delta x \in T_x\mathcal{X}$. The Riemannian metric endows the manifold with the Riemannian distance D. We assume that $(\mathcal{X}, D)$ is a complete metric space (see e.g. [1, Section 3.6]). The metric space topology and the manifold topology agree [10, Theorem 3.1].

Given two smooth manifolds $\mathcal{X}_1$, $\mathcal{X}_2$, the differential of $f : \mathcal{X}_1 \to \mathcal{X}_2$ at x is denoted by $\partial f(x) : T_x\mathcal{X}_1 \to T_{f(x)}\mathcal{X}_2$. Given $\mathcal{X}_1 = \mathcal{Z} \times \mathcal{Y}$, the operator $\partial_z f(z, y)$ satisfies $\partial_z f(z, y)\delta z := \partial f(z, y)[\delta z, 0]$ for each $(z, y) \in \mathcal{X}_1$ and each $[\delta z, 0] \in T_{(z,y)}\mathcal{X}_1$, where $\delta z \in T_z\mathcal{Z}$. Finally, we write $f(\cdot, y)$ to denote the function in $\mathcal{Z} \to \mathcal{X}_2$ mapping each $z \in \mathcal{Z}$ into $f(z, y) \in \mathcal{X}_2$.

A *curve* or γ on $\mathcal{X}$, is a mapping $\gamma : I \to \mathcal{X}$ where either $I \subseteq \mathbb{R}$ or $I \subseteq \mathbb{Z}$. $\mathrm{dom}\gamma$ and $\mathrm{im}\gamma$ denote domain and image of γ. We say that a curve $\gamma : I \to \mathcal{X}$ is bounded if $\mathrm{im}\gamma$ is a bounded set. We sometime use $\dot\gamma(s)$ or $\frac{d\gamma(s)}{ds}$ to denote $\partial\gamma(s)1$, for $s \in \mathrm{dom}\gamma$.

Given a set $\mathcal{S} \subseteq \mathcal{X}$, $\mathrm{int}\mathcal{S}$ and $\mathrm{bd}\mathcal{S}$ denote interior and boundary of $\mathcal{S}$, respectively. Given a vector space $\mathcal{V}$, a set $\mathcal{S} \subseteq \mathcal{V}$, and a constant $\lambda \in \mathbb{R}$, $\lambda\mathcal{S}$ denotes the set $\{\lambda x \in \mathcal{V} \,|\, x \in \mathcal{S}\}$. $\mathcal{S} + \mathcal{S}$ denotes the set $\{x + y \in \mathcal{V} \,|\, x, y \in \mathcal{S}\}$. Given a point $y \in \mathcal{S}$, $\mathcal{S}\backslash\{y\} := \{x \in \mathcal{S} \,|\, x \neq p\}$. Given a sequence of sets $\mathcal{S}_n$, $\lim_{n\to\infty}\mathcal{S}_n$ is the usual set-theoretic limit [38, Chapter 4].

Let Σ be an open continuous dynamical system with (smooth) state manifold $\mathcal{X}$ and input manifold $\mathcal{U}$, represented by $\dot x = f(x, u)$, $(x, u) \in \mathcal{X} \times \mathcal{U}$, where f is a (input-dependent) vector field that assigns to each $(x, u) \in \mathcal{X} \times \mathcal{U}$ a tangent vector $f(x, u) \in T_x\mathcal{X}$. We make the *standing assumptions* that the vector field f and $u(\cdot)$ are C^2 functions. Following [10, Chapter 4, Section 4], two differentiable curves $x(\cdot) : I \subseteq \mathbb{R} \to \mathcal{X}$ (trajectory) and $u(\cdot) : I \subseteq \mathbb{R} \to \mathcal{U}$ (input) are a solution pair $(x(\cdot), u(\cdot)) \in \Sigma$ if $\dot x(t) = f(x(t), u(t))$ for all $t \in I$. An open discrete dynamical system Σ is represented by the recursive equation $x^+ = f(x, u)$, $(x, u) \in \mathcal{X} \times \mathcal{U}$. We make the *standing assumption* that $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is a C^1 function. $(x(\cdot), u(\cdot)) : [t_0, \infty) \subseteq \mathbb{Z} \to \mathcal{X} \times \mathcal{U}$ is a solution pair of Σ if $x(\cdot)$ and $u(\cdot)$ satisfy $x(t + 1) = f(x(t), u(t))$ for each $t \in [t_0, \infty)$.

In what follows, we make the simplifying assumption of *forward completeness* of the solution space, namely that every solution pair has domain $I = [t_0, \infty) \subseteq \mathbb{R}$ ($\subseteq \mathbb{Z}$). Given the solution pair $(x(\cdot), u(\cdot)) : [t_0, \infty) \to \mathcal{X} \times \mathcal{U} \in \Sigma$ we say that $\psi(\cdot, t, x(t), u(\cdot)) := x(\cdot)$ is the *trajectory* or the *integral curve* passing through $x(t)$ at time $t \geq t_0$ under the action of the input $u(\cdot)$. For constant inputs we simply write $\psi(\cdot, t, x(t), u)$ and for closed systems we use $\psi(\cdot, t, x(t))$. The *flow* of Σ is given by the quantity $\psi(t, t_0, \cdot, u)$ for any $t \geq t_0$. For any curve $\gamma(\cdot)$ and set $\mathcal{S}$, $\psi(t, t_0, \gamma(\cdot), u)$ denotes the time evolution of $\gamma(\cdot)$ along the flow of the system at time t, and $\psi(t, t_0, \mathcal{S}, u)$ denotes the set $\{\psi(t, t_0, x, u) \,|\, x \in \mathcal{S}\}$. For closed systems we say that $x_\omega \in \mathcal{X}$ is an ω-limit point of a trajectory $x(\cdot)$ if there exists a sequence of times $t_k \to \infty$ as $k \to \infty$ such that $x_\omega = \lim_{k\to\infty} x(t_k)$. In a similar way, an α-limit point of a trajectory $x(\cdot)$ is given by $\lim_{k\to\infty} x(t_k)$ for some sequence $t_k \to -\infty$ as $k \to \infty$. The ω-limit set $\omega(x_0)$ (α-limit set) is the union of the ω-limit points (α-limit points) of the trajectory $x(\cdot)$ from the initial condition $x(t_0) = x_0$.

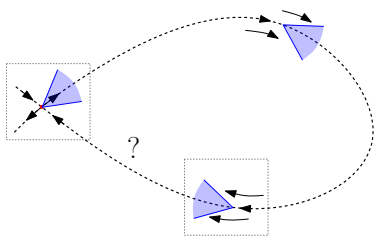

Figure 2. Strict differential positivity excludes homoclinic orbits on saddle points whose unstable manifold has dimension one. The local order imposed by the saddle point cannot be extended globally to a smooth cone field.

IV. Cone fields, conal curves, conal orders

A *conal manifold* $\mathcal{X}$ is a smooth manifold endowed with a *cone field* [28],

$$\mathcal{K}_\mathcal{X}(x) \subseteq T_x\mathcal{X} \qquad \forall x \in \mathcal{X} \ . \tag{1}$$

Like for vector fields, a cone field attaches to each point x of the manifold a cone $\mathcal{K}_\mathcal{X}(x)$ defined in the tangent space $T_x\mathcal{X}$. *Throughout the paper*, each cone $\mathcal{K}_\mathcal{X}(x) \subseteq T_x\mathcal{X}$ is closed, pointed and convex (for each $x \in \mathcal{X}$, $\mathcal{K}_\mathcal{X}(x) + \mathcal{K}_\mathcal{X}(x) \subseteq \mathcal{K}_\mathcal{X}(x)$, $\lambda\mathcal{K}_\mathcal{X}(x) \subseteq \mathcal{K}_\mathcal{X}(x)$ for any $\lambda \in \mathbb{R}_{\geq 0}$, and $\mathcal{K}_\mathcal{X}(x) \cap -\mathcal{K}_\mathcal{X}(x) = \{0\}$). To avoid pathological cases, we assume that each cone is solid (i.e. it contains n independent tangent vectors, where n is the dimension of the tangent space) and there exists a linear invertible mapping $\Gamma(x_1, x_2) : T_{x_1}\mathcal{X} \to T_{x_2}\mathcal{X}$ for each $x_1, x_2 \in \mathcal{X}$, such that $\Gamma(x_1, x_2)\mathcal{K}_\mathcal{X}(x_1) = \mathcal{K}_\mathcal{X}(x_2)$. Note that the application of a linear invertible mapping to a cone is intended as an operation on the rays of the cone, that is, $\Gamma(x_1, x_2)\mathcal{K}_\mathcal{X}(x_1) := \{\lambda\Gamma(x_1, x_2)\delta x \in T_{x_2}\mathcal{X} \,|\, \delta x \in \mathcal{K}_\mathcal{X}(x_1), \lambda \geq 0\}$.

We make the *standing assumption* that each cone field is smooth. In particular, in local coordinates,

$$\mathcal{K}_\mathcal{X}(x) = \{\delta x \in T_x\mathcal{X} \mid \forall i \in I,\ k_i(x, \delta x) \geq 0\} \tag{2}$$

where $I \subseteq \mathbb{Z}$ is an index set and $k_i : T\mathcal{X} \to \mathbb{R}$ are functions; and we say that a cone field is smooth if the functions k_i are smooth.

A curve $\gamma : I \subseteq \mathbb{R} \to \mathcal{X}$ is a *conal curve* on $\mathcal{X}$ if

$$\dot\gamma(s) \in \mathcal{K}_\mathcal{X}(\gamma(s)) \qquad \text{for all } s \in I \ . \tag{3}$$

Conal curves are integral curves of the cone field, as shown in Figure 3. They endow the manifold with a local partial order: for each $x_1, x_2 \in \mathcal{X}$, $x_1 \sqsubseteq_{\mathcal{K}_\mathcal{X}} x_2$ if and only if there exists a conal curve $\gamma : I \subseteq \mathbb{R} \to \mathcal{X}$ such that $\gamma(s_1) = x_1$ and $\gamma(s_2) = x_2$ for some $s_1 \leq s_2$.

The *conal order* $\sqsubseteq_{\mathcal{K}_\mathcal{X}}$ is the natural generalization on manifolds of the notion of partial order on vector spaces. In fact $\sqsubseteq_{\mathcal{K}_\mathcal{X}}$ is a partial order when $\mathcal{X}$ is a vector space

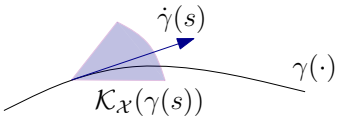

Figure 3. A conal curve satisfies (3). If $s_1 \leq s_2$ then $\gamma(s_1) \sqsubseteq_{\mathcal{K}_\mathcal{X}} \gamma(s_2)$.

and the cone field $\mathcal{K}_{\mathcal{X}}(x) = \mathcal{K}_{\mathcal{X}}$ is constant: two points $x, y \in \mathcal{X}$ satisfy $x \sqsubseteq_{\mathcal{K}_{\mathcal{X}}} y$ iff $y - x \in \mathcal{K}_{\mathcal{X}}$, as shown in [28, Proposition 1.10], which is the usual definition of a partial order on vector spaces [40, Chapter 5]. In general, $\sqsubseteq_{\mathcal{K}_{\mathcal{X}}}$ is not a (global) partial order on $\mathcal{X}$ since antisymmetry may fail. The reader is referred to [28] and [33] for a detailed exposition of the relations among cone fields, ordered manifolds, and homogeneous spaces.

Example 1: For the manifold $\mathbb{S}\times\mathbb{R}$ in Figure 1.IV, the conal order given by the cone field $\delta\theta \geq 0$, $\delta\theta + \delta v \geq 0$ is not a partial order since, for any pair of points $x, y \in \mathbb{S} \times \mathbb{R}$, there exists a conal curve connecting x to y and viceversa. However, in a sufficiently small neighborhood of any point x, the conal order is a partial order. ⌟

V. Differentially positive systems

A. Definitions

A dynamical system is differentially positive when its linearization is positive. Positivity is intended here in the sense of cone invariance [11]. More precisely, a dynamical system Σ on the conal state-input manifold $\mathcal{X} \times \mathcal{U}$ is differentially positive when the cone field

$$\mathcal{K}(x,u) = \underbrace{\mathcal{K}_{\mathcal{X}}(x,u)}_{\subseteq T_x\mathcal{X}} \times \underbrace{\mathcal{K}_{\mathcal{U}}(x,u)}_{\subseteq T_u\mathcal{U}} \subseteq T_{(x,u)}\mathcal{X} \times \mathcal{U} \tag{4}$$

is invariant along the trajectories of the linearized system. For discrete-time system $x^+ = f(x,u)$, the invariance property has a simple formulation. The mapping $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, is differentially positive if, for all $x \in \mathcal{X}$ and all $u, u^+ \in \mathcal{U}$,

$$\partial f(x,u)\mathcal{K}(x,u) \subseteq \mathcal{K}_{\mathcal{X}}(f(x,u),u^+) \ . \tag{5}$$

Indeed, $\partial f(x,u)$ is a positive linear operator, mapping each tangent vector $(\delta x, \delta u) \in \mathcal{K}(x,u) \subseteq T_{(x,u)}\mathcal{X} \times \mathcal{U}$ into $\delta x^+ := \partial f(x,u)[\delta x, \delta u] \in \mathcal{K}_{\mathcal{X}}(x^+,u^+) \subseteq T_{x^+}\mathcal{X}$. A graphical representation for closed discrete systems is provided in Figure 4. The relation between the positivity of the operator $\partial f(x,u)$ in (5) and the positivity of the linearization of Σ is justified by the fact that $\delta x^+ = \partial_x f(x,u)\delta x + \partial_u f(x,u)\delta u$, which establishes the positivity of the linearized dynamics in the sense of [11], [18], [14], [2].

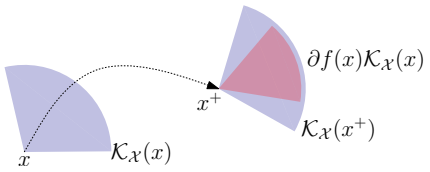

Figure 4. A graphical representation of condition (5) for the (closed) discrete time system $x^+ = f(x)$. The cone field $\mathcal{K}(x,u)$ reduces to $\mathcal{K}_{\mathcal{X}}(x)$ in this case.

For general continuous-time $\dot{x} = f(x,u) \in T_x\mathcal{X}$ and discrete-time $x^+ = f(x,u) \in \mathcal{X}$ dynamical systems Σ ($(x,u) \in \mathcal{X} \times \mathcal{U}$), the definition of differential positivity involves the *prolonged system* $\delta\Sigma$ introduced in [12]

$$\delta\Sigma : \begin{cases} (x^+) & \dot{x} &=& f(x,u) \\ (\delta x^+) & \dot{\delta x} &=& \partial_x f(x,u)\delta x + \partial_u f(x,u)\delta u \ . \end{cases} \tag{6}$$

We call *variational component* the second equation of (6).

Definition 1: Σ is a *differentially positive* dynamical system (with respect to $\mathcal{K}$ in (4)) if, for all $t_0 \in \mathbb{R}$, any solution pair $((x,\delta x)(\cdot),(u,\delta u)(\cdot)) : [t_0,\infty) \to T\mathcal{X} \times T\mathcal{U} \in \delta\Sigma$ leaves the cone field $\mathcal{K}$ invariant. Namely,

$$\begin{cases} \delta x(t_0) & \in & \mathcal{K}_{\mathcal{X}}(x(t_0),u(t_0)) \\ \delta u(t) & \in & \mathcal{K}_{\mathcal{U}}(x(t),u(t)), \ \forall t \geq t_0 \end{cases} \tag{7}$$
$$\Rightarrow$$
$$\delta x(t) \ \in \ \mathcal{K}_{\mathcal{X}}(x(t),u(t)), \ \forall t \geq t_0 \ . \quad ⌟$$

In continuous-time, differential positivity of Σ is thus positivity of the linearized system $\dot{\delta x} = A(t)\delta x + B(t)\delta u$ along any solution pairs $(x(\cdot),u(\cdot)) \in \Sigma$, where $A(t) := \partial_x f(x(t),u(t))$ and $B(t) := \partial_u f(x(t),u(t))$. For closed systems $\dot{x} = f(x)$, with cone field $\mathcal{K}_{\mathcal{X}}(x) \subseteq T_x\mathcal{X}$, we have $\dot{\delta x} = A(t)\delta x$, where $A(t) := \partial f(x(t))$, along any given solution $x(\cdot) : [t_0,\infty) \to \mathcal{X} \in \Sigma$. Therefore, the fundamental solution $\Psi_{x(\cdot)}(t,t_0)$ of the linearized dynamics [44, Appendix C.4] satisfies $\Psi_{x(\cdot)}(t,t_0)\mathcal{K}_{\mathcal{X}}(x) \subseteq \mathcal{K}_{\mathcal{X}}(x)$ for each $t \in [t_0,\infty)$, that is, $\Psi_{x(\cdot)}(t,t_0)$ is a positive linear operator.

Strict differential positivity is to differential positivity what strict positivity is to positivity. We anticipate that this (mild) property will have a strong impact on the asymptotic behavior of differentially positive systems, as shown in Section VI.

Definition 2: Σ is (uniformly) *strictly differentially positive* (with respect to $\mathcal{K}$) if differential positivity holds and there exists $T > 0$ and a cone field $\mathcal{R}_{\mathcal{X}}(x,u) \subseteq \mathrm{int}\mathcal{K}_{\mathcal{X}}(x,u) \cup \{0\}$ such that, for all $t_0 \in \mathbb{R}$, any $((x,\delta x)(\cdot),(u,\delta u)(\cdot)) : [t_0,\infty) \to T\mathcal{X} \times T\mathcal{U} \in \delta\Sigma$ satisfies

$$\begin{cases} \delta x(t_0) & \in & \mathcal{K}_{\mathcal{X}}(x(t_0),u(t_0)) \\ \delta u(t) & \in & \mathcal{K}_{\mathcal{U}}(x(t),u(t)), \ \forall t \geq t_0 \end{cases} \tag{8}$$
$$\Rightarrow$$
$$\delta x(t) \in \mathcal{R}_{\mathcal{X}}(x(t),u(t)), \ \forall t \geq t_0 + T \ .$$

We assume that the cone field $\mathcal{R}_{\mathcal{X}}$ also satisfies the following additional technical condition:

$$\Gamma(x_1,u_1,x_2,u_2)\mathcal{R}_{\mathcal{X}}(x_1,u_1) = \mathcal{R}_{\mathcal{X}}(x_2,u_2) \tag{9}$$

for each $(x_1,u_1),(x_2,u_2) \in \mathcal{X} \times \mathcal{U}$, where $\Gamma(x_1,u_1,x_2,u_2) : T_{x_1}\mathcal{X} \to T_{x_2}\mathcal{X}$ is a linear invertible mapping such that $\Gamma(x_1,u_1,x_2,u_2)\mathcal{K}_{\mathcal{X}}(x_1,u_1) = \mathcal{K}_{\mathcal{X}}(x_2,u_2)$ (see Section IV). ⌟

For open systems with output $h : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ – $\mathcal{Y}$ output manifold, endowed with the cone field $\mathcal{K}_{\mathcal{Y}}(y) \in T_y\mathcal{Y}$ – the notion of (strict) differential positivity requires the further condition that h is a *differentially positive mapping*, that is, $\partial h(x,u)\mathcal{K}(x,u) \subseteq \mathcal{K}_{\mathcal{Y}}(h(x,u))$, for each $(x,u) \in \mathcal{X} \times \mathcal{U}$.

Remark 1: Differential positivity has a geometric characterization. restricting to closed systems for simplicity, consider the cone field $\mathcal{K}_{\mathcal{X}}(x)$ represented by (2) where I is an index set and k_i are smooth functions. Then, (7) is equivalent to require that $k_i(x(t),\delta x(t)) \geq 0$ along any solution $(x(\cdot),\delta x(\cdot)) \in \delta\Sigma$, for all $i \in I$. Therefore, differential positivity for a discrete system can be established by testing that $\forall i \in I$, $k_i(x,\delta x) \geq 0$ implies $\forall i \in I$, $k_i^+ := k_i(f(x),\partial f(x)\delta x) \geq 0$, for each $(x,\delta x) \in T\mathcal{X}$. In a similar way, for continuous systems, consider any pair $(x,\delta x) \in T\mathcal{X}$ such that $k_i(x,\delta x) \geq 0$ for

all $i \in I$ and test that, for any $j \in I$, if $k_j(x, \delta x) = 0$ then $\dot{k}_j := \partial k_j(x, \delta x)[f(x), \partial f(x)\delta x] \geq 0$. ⌟

B. Examples

1) Positive linear systems are differentially positive: Consider the dynamics Σ given by $x^+ = Ax$ on the vector space $\mathcal{V}$. Positivity with respect to the cone $\mathcal{K}_\mathcal{V} \subseteq \mathcal{V}$ reads $A\mathcal{K}_\mathcal{V} \subseteq \mathcal{K}_\mathcal{V}$, [11]. A typical example is provided by the case of a matrix A with non-negative entries which guarantees the invariance of the positive orthant $\mathcal{K}_\mathcal{V} := \mathbb{R}^n_+$.

Since each tangent space of a vector space can be identified to the vector space itself, i.e. $T_x\mathcal{V} = \mathcal{V}$ for each $x \in \mathcal{V}$, consider the manifold $\mathcal{X} := \mathcal{V}$ and define the lifting of the cone $\mathcal{K}_\mathcal{V}$ to the cone field $\mathcal{K}_\mathcal{X}(x) := \mathcal{K}_\mathcal{V} \subseteq T_x\mathcal{X}$, for each $x \in \mathcal{X}$ (*constant* cone field). Then the linearized dynamics reads $\delta x^+ = A\delta x$ and the prolonged system trivially satisfies $A\mathcal{K}_\mathcal{X}(x) \subseteq \mathcal{K}_\mathcal{X}(Ax)$.

2) Monotone systems are differentially positive: A monotone dynamical system [43], [3] is a dynamical system whose trajectories preserve some partial order relation on the state space. Moving from closed [43], [24], [23], [13], [25], [36] to open systems [3], [5], [4], this wide class of systems is extensively adopted in biology and chemistry both for modeling and control [15], [17], [16], [45], [6], [7].

The partial order $\preceq$ of a monotone system is typically induced by a conic subset $\mathcal{K}_\mathcal{V} \subseteq \mathcal{V}$ of the state (vector) space $\mathcal{V}$. Precisely, two points $x, \hat{x} \in \mathcal{V}$ satisfy $x \preceq_{\mathcal{K}_\mathcal{V}} \hat{x}$ if and only if $\hat{x} - x \in \mathcal{K}_\mathcal{V}$. The preservation of the order along the system dynamics reads as follows: if $x(\cdot), \hat{x}(\cdot) \in \Sigma$ satisfy $x(t_0) \preceq_{\mathcal{K}_\mathcal{V}} \hat{x}(t_0)$ for some initial time t_0, then $x(t) \preceq_{\mathcal{K}_\mathcal{V}} \hat{x}(t)$ for all $t \geq t_0$, [43].

To show that a monotone system is differentially positive, consider $\mathcal{V}$ as a manifold endowed with the constant cone field $\mathcal{K}_\mathcal{V}(x) := \mathcal{K}_\mathcal{V}$, $x \in \mathcal{V}$. By monotonicity, the infinitesimal difference between two ordered neighboring solutions $\delta x(\cdot) := \hat{x}(\cdot) - x(\cdot)$ satisfies $\delta x(t) \in \mathcal{K}_\mathcal{V}(x(t))$, for each $t \geq t_0$. Differential positivity follows from the fact that $(x(\cdot), \delta x(\cdot))$ is a trajectory of the prolonged system $\delta\Sigma$.

Theorem 1: Given any cone $\mathcal{K}_\mathcal{V}$ on the vector space $\mathcal{V}$, the partial order $\preceq_{\mathcal{K}_\mathcal{V}}$, and the cone field $\mathcal{K}_\mathcal{X}(x) := \mathcal{K}_\mathcal{V}$, a (closed) dynamical system is monotone if and only if is differentially positive. ⌟

Proof: For constant cone fields on vector spaces recall that $\preceq_{\mathcal{K}_\mathcal{V}}$ and $\sqsubseteq_{\mathcal{K}_\mathcal{V}}$ are equivalent relations (see Section IV). Consider a conal curve $\gamma(t_0, \cdot)$ connecting two ordered initial points $\gamma(t_0, 0) := x(t_0) \preceq_{\mathcal{K}_\mathcal{V}} \hat{x}(t_0) =: \gamma(t_0, 1)$. Note that $\gamma(t_0, s_1) \sqsubseteq_{\mathcal{K}_\mathcal{V}} \gamma(t_0, s_2)$ for each $s_1 \leq s_2$. For each $s \in [0, 1]$, let $\gamma(\cdot, s)$ be a trajectory of Σ. Indeed, $\gamma(t, \cdot)$ represents the time evolution of the curve $\gamma(t_0, \cdot)$ along the flow of the system. [$\Leftarrow$] Differential positivity guarantees that $\gamma(t, \cdot)$ is a conal curve for each $t \geq t_0$. This follows from the fact that the pair $(x_s(t), \delta x_s(t)) := (\gamma(t, s), \frac{d}{ds}\gamma(t, s))$ is a trajectory of the prolonged system $\delta\Sigma$ for each $s \in [0, 1]$. Thus, $x(t) \sqsubseteq_{\mathcal{K}_\mathcal{V}} \hat{x}(t)$ for all $t \geq t_0$. [$\Rightarrow$] Monotonicity guarantees that $\gamma(t, s_1) \preceq_{\mathcal{K}_\mathcal{V}} \gamma(t, s_2)$ for all $s_1 \leq s_2$. By a limit argument, $\frac{d}{ds}\gamma(t, s) \in \mathcal{K}_\mathcal{V}(\gamma(t, s))$ for all $t \geq t_0$ and al $s \in [0, 1]$. Thus $\gamma(t, \cdot)$ is a conal curve. Note that $(x_s(t), \delta x_s(t)) := (\gamma(t, s), \frac{d}{ds}\gamma(t, s))$ is a trajectory of the prolonged system. Since $\gamma(t_0, \cdot)$ is a generic conal curve, (7) follows. ■

A similar result holds for open monotone systems, which are typically characterized by introducing two orders $\preceq_{\mathcal{K}_\mathcal{X}}$ and $\preceq_{\mathcal{K}_\mathcal{U}}$, respectively induced by the cone $\mathcal{K}_\mathcal{X}$ on the state space $\mathcal{X}$ and $\mathcal{K}_\mathcal{U}$ on the input space $\mathcal{U}$, [3, Definition II.1]. Extending the argument above it is possible to show that a dynamical system Σ is monotone with respect to $(\preceq_{\mathcal{K}_\mathcal{X}}, \preceq_{\mathcal{K}_\mathcal{U}})$ *if and only if* Σ is differentially positive on the *vector space* $\mathcal{X} \times \mathcal{U}$ endowed with the *constant* cone field $\mathcal{K}(x, u) := \mathcal{K}_\mathcal{X} \times \mathcal{K}_\mathcal{U}$, for each $(x, u) \in \mathcal{X} \times \mathcal{U}$. In this sense, differential positivity on vector spaces and constant cone fields is the differential formulation of monotonicity.

3) Differential positivity of cooperative systems and the Kamke condition: A cooperative system $\dot{x} = f(x)$ with state space $\mathcal{X} := \mathbb{R}^n$ is monotone with respect to the partial order induced by the positive orthant $\mathbb{R}^n_+$, thus differentially positive with respect to the cone field $\mathcal{K}_\mathcal{X}(x) := \mathbb{R}^n_+$, $x \in \mathcal{X}$. Exploiting the geometric conditions of Remark 1, differentially positivity with respect to $\mathcal{K}_\mathcal{X}$ holds when

$$[\partial f(x)]_{ij} \geq 0 \qquad \text{for all } 1 \leq i \neq j \leq n,\, x \in \mathcal{X}, \tag{10}$$

where $[\cdot]_{ij}$ denotes the ij component. To see this, define E_i as the vector whose i-th element is equal to one and the remaining to zero and note that the positive orthant is defined by the set of δx that satisfy $\langle E_i, \delta x\rangle \geq 0$. Then, from Remark 1, the invariance reads $\langle E_i, \delta x\rangle = 0 \Rightarrow \langle E_i, \partial f(x)\delta x\rangle \geq 0$ for any $x \in \mathcal{X}$, $\delta x \in \mathcal{K}_\mathcal{X}(x) = \mathbb{R}^n_+$, and $i \in \{1, \dots, n\}$. (10) follows by selecting $\delta x = E_j \neq E_i$. Indeed, $\partial_x f(x)$ is a Metzler matrix for each $x \in \mathcal{X}$ [3, Section VIII].

Cooperative systems typically satisfy (10), as shown in [43, Remark 1.1] on closed systems. A similar result is provided in [3, Proposition III.2] for open systems. In this sense, the pointwise geometric conditions in Remark 1 revisit and extend the comparison between cooperative systems, incrementally positive systems of [3, Section VIII], and the Kamke condition provided in [43, Chapter 3].

4) One dimensional continuous-time systems are differentially positive: This property is well-known for systems in $\mathbb{R}$: solutions are partially ordered because they cannot "pass each other". It remains true on closed manifolds such as $\mathbb{S}$, even though the conal order does not induce a (globally defined) partial order in that case.

5) Non-constant cones for oscillating dynamics: Moving from constant to non-constant cone fields opens the way to the analysis of more general limit sets such a oscillations or limit cycles. The harmonic oscillator studied in Section II provides a first simple example of differential positivity with respect to a non-constant cone field. In particular, consider $\mathcal{K}_\mathcal{X}(x) := \{\delta x \in \mathbb{R}^2 \setminus \{0\} \,|\, k_1(x, \delta x) \geq 0,\, k_2(x, \delta x) \geq 0\}$, where $k_1(x, \delta x) := -(x_1 + x_2)\delta x_1 + (x_1 - x_2)\delta x_2$ and $k_2(x, \delta x) := -(x_2 - x_1)\delta x_1 + (x_1 + x_2)\delta x_2$. The cone field is well defined on the (invariant) manifold $\mathcal{X} := \mathbb{R}^2 \setminus \{0\}$. Differential positivity with respect to $\mathcal{K}_\mathcal{X}(x)$ follows from the geometric conditions in Remark 1, since $\dot{k}_1 = 0$ and $\dot{k}_2 = 0$ everywhere.

The differential positivity of the harmonic oscillator with respect to $\mathcal{K}_{\mathcal{X}}(x)$ is not surprising if one looks at the representation of the oscillator in polar coordinates $\dot{\vartheta} = 1$, $\dot{\rho} = 0$. The state manifold becomes the cylinder $\mathbb{S} \times \mathbb{R}_+$ and the system decompose into two one-dimensional systems, which suggests the invariance of any cone field rotating with ϑ, as shown in Figure 5 (left). Indeed, polar coordinates suggest differential positivity for arbitrary decoupled dynamics $\dot{\vartheta} = f(\vartheta)$, $\dot{\rho} = g(\rho)$ with respect to the cone field $\mathcal{K}(\vartheta, \rho) := \{(\delta\vartheta, \delta\rho) \in \mathbb{R}^2) \,|\, \delta\vartheta \geq 0\,,\ \delta\rho \geq 0\}$. In fact, the linearization reads $\dot{\delta\vartheta} = \partial f(\vartheta)\delta\vartheta$, $\dot{\delta\rho} = \partial g(\rho)\delta\rho$, which guarantees that $\dot{\delta\vartheta} = 0$ for $\delta\vartheta = 0$ and $\dot{\delta\rho} = 0$ for $\delta\rho = 0$, as required by Remark 1.

Possibly, the invariance of the cone field can be strengthened to contraction by combining the two uncoupled dynamics. For example, when $f(\vartheta) = 1$ and $g(\rho) = \rho - \frac{\rho^3}{3}$, the trajectories of the variational dynamics move towards the interior of the cone field $\mathcal{K}(\vartheta, \rho) := \{(\delta\vartheta, \delta\rho) \in \mathbb{R}^2) \,|\, \delta\vartheta \geq 0\,,\ \delta\vartheta^2 - \frac{\delta\rho^2}{\rho^2} \geq 0\}$. In fact, $\frac{d}{dt}(\delta\vartheta^2 - \frac{\delta\rho^2}{\rho^2}) = \frac{4}{3}\delta\rho^2 > 0$ for each $(\delta\vartheta, \delta\rho) \in \mathrm{bd}\mathcal{K}(\vartheta, \rho) \setminus \{0\}$. Figure 5 (right) provides a representation of the (projective) contraction of the cone. We anticipate that this contraction property is tightly connected to the existence of a globally attractive limit cycle.

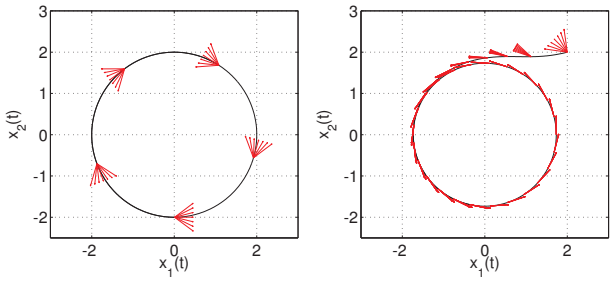

Figure 5. Simulations of the variational dynamics $\delta\Sigma$ of the two cases (i) $\dot{\rho} = 0$ (left) and (ii) $\dot{\rho} = (\rho - \frac{\rho^3}{3})$ (right), from the same initial condition. The red arrows represent the motion of the variational component along the trajectory.

Remark 2: Differential positivity requires classical positivity of the linearized dynamics at fixed points. In fact, Definition 1 shows that the cone field at any fixed point x^* is given by the invariant cone of the (positive) linearized dynamics at x^*. The harmonic oscillator is not a positive linear system, because of the presence of the two complex eigenvalues. Thus, it is not differentialy positive in $\mathbb{R}^2$. However, polar coordinates reveal that it is differentially positive in the manifold $\mathcal{X} = \mathbb{R}^2 \setminus \{0\}$. ⌟

VI. Differential Perron-Frobenius theory

A. Contraction of the Hilbert metric

Bushell [11] (after Birkhoff [8]) used the Hilbert metric on cones to show that the strict positivity of a mapping guarantees contraction among the rays of the cone, opening the way to many contraction-based results in the literature of positive operators [11], [34], [42], [9], [29], among which the reduction of the Perron-Frobenius theorem to a special case of the contraction mapping theorem [27], [11], [8]. Taking inspiration from these important results, we rely on the infinitesimal contraction properties of the Hilbert metric to study the contraction properties of differentially positive systems.

Consider the product manifold $\mathcal{X} \times \mathcal{U}$ where $\mathcal{X}$ is endowed with the cone field $\mathcal{K}_{\mathcal{X}}(x, u) \subseteq T_x\mathcal{X}$, for each $(x, u) \in \mathcal{X} \times \mathcal{U}$. Following [11], for any given (x, u), take any $\delta x, \delta y \in \mathcal{K}_{\mathcal{X}}(x, u) \setminus \{0\}$ and define the quantities

$$M_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y) := \inf\{\lambda \in \mathbb{R}_{\geq 0} \,|\, \lambda\delta y - \delta x \in \mathcal{K}_{\mathcal{X}}(x, u)\}$$
$$m_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y) := \sup\{\lambda \in \mathbb{R}_{\geq 0} \,|\, \delta x - \lambda\delta y \in \mathcal{K}_{\mathcal{X}}(x, u)\}. \tag{11}$$

$M_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y) := \infty$ when $\{\lambda \in \mathbb{R}_{\geq 0} \,|\, \lambda\delta y - \delta x \in \mathcal{K}_{\mathcal{X}}(x, u)\} = \emptyset$. The Hilbert's metric $d_{\mathcal{K}_{\mathcal{X}}(x,u)}$ induced by $\mathcal{K}_{\mathcal{X}}(x, u)$ is given by

$$d_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y) = \log\left(\frac{M_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y)}{m_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y)}\right)\,. \tag{12}$$

In each cone $\mathcal{K}_{\mathcal{X}}(x, u)$ $d_{\mathcal{K}_{\mathcal{X}}(x,u)}$ is a projective distance: for each $\delta x, \delta y, \delta z \in \mathcal{K}_{\mathcal{X}}(x, u)$, $d_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y) \geq 0$, $d_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y) = d_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta y, \delta x)$, $d_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y) \leq d_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta z) + d_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta z, \delta y)$, and $d_{\mathcal{K}_{\mathcal{X}}(x,u)}(\delta x, \delta y) = 0$ if and only if $\delta x = \lambda\delta y$ with $\lambda \geq 0$.

The following theorem is a generalization of Birkhoff result: it shows that strict differential positivity guarantees the *exponential contraction* of the metric when the input $u(\cdot)$ acts uniformly on the system (a feedforward signal). The uniform action of the input is modeled by taking the variational input $\delta u(\cdot) = 0$, since δu represents the infinitesimal mismatch between two inputs.

For readability, in what follows we denote the Hilbert metric along a solution pair $d_{\mathcal{K}_{\mathcal{X}}(x(t),u(t))}(\cdot, \cdot)$ with $d_{*(t)}(\cdot, \cdot)$.

Theorem 2: Let Σ be a dynamical system on the state/input manifold $\mathcal{X} \times \mathcal{U}$, differentially positive with respect to the cone field $\mathcal{K}(x, u) := \mathcal{K}_{\mathcal{X}}(x, u) \times \{0\}$, where $\mathcal{K}_{\mathcal{X}}(x, u) \subseteq T_x\mathcal{X}$ for each $(x, u) \in \mathcal{X} \times \mathcal{U}$. Then, for all $t \geq t_0$,

$$d_{*(t)}(\delta x_1(t), \delta x_2(t)) \leq d_{*(t_0)}(\delta x_1(t_0), \delta x_2(t_0)) \tag{13}$$

for any $(x(\cdot), \delta x_1(\cdot), u(\cdot), 0), (x(\cdot), \delta x_2(\cdot), u(\cdot), 0) \in \delta\Sigma$ with domain $[t_0, \infty)$ and $\delta x_1(t_0), \delta x_2(t_0) \in \mathcal{K}_{\mathcal{X}}(x(t_0), u(t_0))$.

If Σ is strictly differentially positive then there exist $\rho \geq 1$ and $\lambda > 0$ such that, for all $t \geq t_0$,

$$d_{*(t)}(\delta x_1(t), \delta x_2(t)) \leq \rho e^{-\lambda(t-t_0)} d_{*(t_0)}(\delta x_1(t_0), \delta x_2(t_0)) \tag{14}$$

for any $(x(\cdot), \delta x_1(\cdot), u(\cdot), 0), (x(\cdot), \delta x_2(\cdot), u(\cdot), 0) \in \delta\Sigma$ with domain $[t_0, \infty)$ and $\delta x_1(t_0), \delta x_2(t_0) \in \mathcal{K}_{\mathcal{X}}(x(t_0), u(t_0))$. Moreover, $d_{*(t)}(\delta x_1(t), \delta x_2(t)) < \infty$ for $t \geq t_0 + T$. ⌟

B. The Perron-Frobenius vector field

The Perron-Frobenius vector of a strictly positive linear map is a fixed point of the projective space. Its existence is a consequence of the contraction of the Hilbert metric, [11]. To exploit the generalized contraction of Theorem 2, we assume that the input acts uniformly on the system, that is, $\delta u(\cdot) = 0$. We endow the state manifold $\mathcal{X}$ with a (smooth) Riemannian structure and we define $\mathcal{B}(x) := \{\delta x \in T_x\mathcal{X} \,|\, |\delta x|_x = 1\} \subseteq T_x\mathcal{X}$, to make the following assumption.

Assumption 1: $(\mathcal{K}_{\mathcal{X}}(x,u) \cap \mathcal{B}(x), d_{\mathcal{K}_{\mathcal{X}}(x,u)})$ is a *complete metric space* for any given $(x,u) \in \mathcal{X} \times \mathcal{U}$ [1].

To introduce the *Perron-Frobenius vector field* we study the asymptotic behavior of $\delta\Sigma$, looking at solutions pairs $(z(\cdot),u(\cdot)) \in \Sigma$ with domain $I := (-\infty, t)$ (backward completeness of Σ). Recall that for any $(z(\cdot),u(\cdot)) \in \Sigma$, if $\delta u(\cdot) = 0$ then $\delta z(t) = \partial_{z(t_0)}\psi(t,t_0,z(t_0),u(\cdot))\delta z(t_0)$ is a trajectory of the variational components of $\delta\Sigma$ along $(z(\cdot),u(\cdot))$.

Theorem 3: Let Σ be a dynamical system on the state/input manifold $\mathcal{X} \times \mathcal{U}$. Suppose that Σ is strictly differentially positive with respect to the cone field $\mathcal{K}(x,u) := \mathcal{K}_{\mathcal{X}}(x,u) \times \{0\}$ such that $\mathcal{K}_{\mathcal{X}}(x,u) \subseteq T_x\mathcal{X}$ for each $(x,u) \in \mathcal{X} \times \mathcal{U}$ and suppose that Assumption 1 holds.

For any input $u(\cdot) : \mathbb{R} \to \mathcal{U}$ which makes Σ backward complete, there exists a time-varying vector field $\mathbf{w}_{u(\cdot)}(x,t) \in \mathrm{int}\mathcal{K}_{\mathcal{X}}(x,u(t)) \cap \mathcal{B}(x)$, $x \in \mathcal{X}$ and $t \in \mathbb{R}$, such that any solution pair $(z(\cdot),u(\cdot)) \in \Sigma$ satisfies

$$\lim_{t_0 \to -\infty} \partial_{z(t_0)}\psi(t,t_0,z(t_0),u(\cdot))\mathcal{K}_{\mathcal{X}}(z(t_0),u(t_0)) = \\ = \{\lambda \mathbf{w}_{u(\cdot)}(z(t),t) \,|\, \lambda \geq 0\} \,. \tag{15}$$

We call this vector field the *Perron-Frobenious vector field.* ⌟

Corollary 1: Under the assumptions of Theorem 3, if $u(\cdot) = u \in \mathcal{U}$ (constant), then the Perron-Frobenius vector field reduces to a continuous time-invariant vector field $\mathbf{w}_u(x)$. For linear systems the Perron-Frobenius vector field reduces to the (constant) Perron-Frobenius vector. ⌟

The evolution of an initial cone $\mathcal{K}_{\mathcal{X}}(z(t_0),u(t_0))$ along the (variational) flow of the system asymptotically converges to the span of the Perron-Frobenius vector field $\mathbf{w}_{u(\cdot)}(x,t)$ attached to each $x \in \mathcal{X}$, as illustrated in Figure 6. Decomposing the variational trajectory $\delta z(\cdot)$ along $z(\cdot)$ into a *directional* component $\vartheta(t) := \frac{\delta z(t)}{|\delta z(t)|_{z(t)}} \in \mathcal{K}_{\mathcal{X}}(z(t),u(t)) \cap \mathcal{B}_{z(t)}$, and a *magnitude* component $\rho(t) := |\delta z(t)|_{z(t)}$, Theorem 3 establishes that $\vartheta(t)$ is guaranteed to converge to $\mathbf{w}_{u(\cdot)}(z(t),t)$, for any initial condition $\vartheta(t_0)$.

For constant inputs the Perron-Frobenius vector field has a simple geometric characterization. Take any trajectory $(x(\cdot),\delta x(\cdot))$ of the prolonged system $\delta\Sigma$ under the action of the constant input u, and suppose that $d_{(x(t),u)}(\mathbf{w}_u(x(t)),\delta x(t)) = 0$ for some $t \in \mathbb{R}$. Then, from (14), $d_{(x(t+\tau),u)}(\mathbf{w}_u(x(t+\tau)),\delta x(t+\tau)) = 0$ for each $\tau \geq 0$, which shows that $\mathbf{w}_u(x(t))$ must be a time-reparametrized trajectory of Σ. Therefore, $\mathbf{w}_u(x)$ belongs to $\mathrm{int}\mathcal{K}_{\mathcal{X}}(x,u)$ for all x and satisfies the partial differential equation $[\partial_x \mathbf{w}_u(x)]f(x,u) = [\partial_x f(x,u)]\mathbf{w}_u(x) - \lambda(x,u)\mathbf{w}_u(x)$ for continuous time systems, for some $\lambda(x,u) \in \mathbb{R}$ which guarantees $|\mathbf{w}_u(x)|_x = 1$. In a similar way, for discrete dynamics we have $\mathbf{w}_u(f(x,u)) = \lambda(x,u)[\partial_x f(x,u)]w(x,u)$. As before, $\lambda(x,u) \in \mathbb{R}$ is selected to guarantee $|\mathbf{w}_u(x)|_x = 1$. Existence and uniqueness of the solution $\mathbf{w}_u(x)$ follow from the contraction of the Hilbert metric, under the assumption of backward and forward invariance of $\mathcal{X}$.

[1]The reader is referred to [11, Section 4], [29, Section 2.5], or [49] for examples of complete metric spaces on cones.

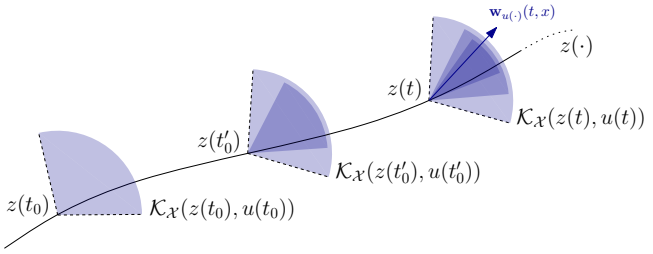

Figure 6. The contraction at time t from different initial cones, for $t_0 < t_0' < t$. Note that $\partial_{z(t_0)}\psi(t,t_0,z(t_0),u(\cdot))\mathcal{K}_{\mathcal{X}}(z(t_0),u(t_0)) \subseteq \partial_{z(t_0')}\psi(t,t_0',z(t_0'),u(\cdot))\mathcal{K}_{\mathcal{X}}(z(t_0'),u(t_0')) \subseteq \mathcal{K}_{\mathcal{X}}(z(t),u(t))$. At time t, for $t - t_0 \to \infty$, the cone reduces to a line.

VII. Limit sets of (closed) differentially positive systems

A. Behavior dichotomy

For closed continuous-time dynamical systems (or open continuous-time systems with constant inputs) the combination of the local order on the system state manifold and the projective contraction of the variational dynamics toward the Perron-Frobenius vector field $\mathbf{w}(x)$ restrict the asymptotic behavior of differentially positive systems. The next theorem characterizes the ω-limit sets of those systems.

Theorem 4: Let Σ be a closed continuous (complete) system $\dot{x} = f(x)$ with state manifold $\mathcal{X}$, strictly differentially positive with respect to the cone field $\mathcal{K}_{\mathcal{X}}(x) \subseteq T_x\mathcal{X}$. Under Assumption 1, suppose that the trajectories of Σ are bounded. Then, for every $\xi \in \mathcal{X}$, the ω-limit set $\omega(\xi)$ satisfies one of the following two properties:

(i) The vector field $f(x)$ is aligned with the Perron-Frobenius vector field $\mathbf{w}(x)$ for each $x \in \omega(\xi)$ (i.e. $f(x) = \lambda(x)\mathbf{w}(x)$, $\lambda(x) \in \mathbb{R}$), and $\omega(\xi)$ is either a fixed point or a periodic orbit or a set of fixed points and connecting arcs;

(ii) The vector field $f(x)$ is not aligned with the Perron-Frobenius vector field $\mathbf{w}(x)$ for each $x \in \omega(\xi)$ such that $f(x) \neq 0$, and either $\liminf_{t\to\infty} |\partial_x\psi(t,0,x)\mathbf{w}(x)|_{\psi(t,0,x)} = \infty$ or $\lim_{t\to\infty} f(\psi(t,0,x)) = 0$. ⌟

The interpretation of Theorem 4 is that the asymptotic behavior of Σ is either described by a *Perron-Frobenius curve* $\gamma^{\mathbf{w}}(\cdot)$, that is, a curve $\dot{\gamma}^{\mathbf{w}}(s) = \mathbf{w}(\gamma^{\mathbf{w}}(s))$ for all $s \in \mathrm{dom}\,\gamma^{\mathbf{w}}(\cdot)$; or is the union of the limit points of some trajectory $\psi(\cdot,0,\xi)$, $\xi \in \mathcal{X}$, nowhere tangent to the Perron-Frobenius vector field, as clarified in Section VII-C, and characterized by high sensitivity with respect to initial conditions, because of the unbounded linearization. The proof of Theorem 4 in Appendix, Section B, is of interest on its own since it illustrates how differential Perron-Frobenius theory impacts the behavior of Σ. In the next two subsections we further discuss the implications of Theorem 4 in case (i) and in case (ii), respectively.

B. Simple attractors of differentially positive systems

A first consequence of Theorem 4 is a result akin to Poincare-Bendixson characterization of limit sets of planar systems.

Corollary 2: Under the assumptions of Theorem 4, consider an open, forward invariant region $\mathcal{C} \subseteq \mathcal{X}$ that does not contain any fixed point. If the vector field $f(x) \in \text{int}\mathcal{K}_{\mathcal{X}}(x)$ for any $x \in \mathcal{C}$, then there exists a unique attractive periodic orbit contained in $\mathcal{C}$. ⌟

The result shows the potential of differential positivity for the analysis of limit cycles in possibly high dimensional spaces. Since stable limit cycles must correspond to Perron-Frobenius curves, stable limit cycles are excluded when Perron-Frobenius curves are open, a property always satisfied in vector spaces with constant cone field. For a differentially positive system defined in a vector space, the cone field must necessarily "rotate" with the periodic orbit in order to allow for limit cycle attractors (see, for example, Section V-B5).

Beyond isolated fixed point and limit cycles, the limit sets of differentially positive systems are severely restricted by (local) order properties, see Figure 7 for an illustration. In particular, the intuitive argument ruling out homoclinic orbits like in Figure 2 is made rigorous with Theorem 4. A limit set given by a connecting arc between two hyperbolic fixed points can exists only if it is everywhere tangent to the Perron-Frobenius vector field (Theorem 4.i), or nowhere tangent to the Perron-Frobenius vector field (Theorem 4.ii). Because any orbit between two hyperbolic fixed points must belong to the unstable manifold of its α-limit set and to the stable manifold of its ω-limit set, it can be a Perron-Frobenius curve only if, whenever it is tangent to the Perron-Frobenius eigenvector of its α-limit, it is also tangent to the Perron-Frobenius eigenvector of its ω-limit.

Corollary 3: Under the assumptions of Theorem 4, consider an orbit that connects two hyperbolic fixed points y_e, z_e, respectively as $t \to -\infty$ and $t \to \infty$. If the orbit is tangent to $\mathbf{w}(y_e)$ at y_e, then it is tangent to $\mathbf{w}(z_e)$ at z_e. ⌟

The corollary rules out the possibility of a homoclinic orbit with a one-dimensional unstable manifold, a typical ingredient of strange attractors. For system depending on parameters, the corollary rules out the possibility of homoclinic bifurcations [46, Chapter 8] where the homoclinic orbit is tangent to the dominant eigenvector of the saddle point. In accordance with Theorem 4, a limit set given by a homoclinic orbit can only exist if it is nowhere tangent to the Perron-Frobenius vector field, which rules out the possibility of being part of a simple attractor. The two situations are illustrated in Fig 8.

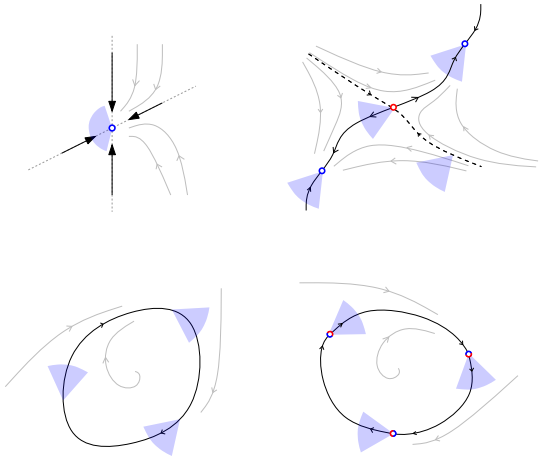

Figure 7. Simple attractors consistent with the ordering property of a Perron-Frobenius curve. Cones are shaded in blue. Top-left: fixed point with dominant eigenvector. Top-right: bistable systems, the heteroclinic orbits connecting the saddle (red) to the stable fixed points (blue) coincides with the image of some Perron-Frobenius curves. Bottom: limit cycles (left) or fixed points and connecting arcs (right) coincides with the image of some closed Perron-Frobenius curve, thus require require a rotating cone field (on vector spaces).

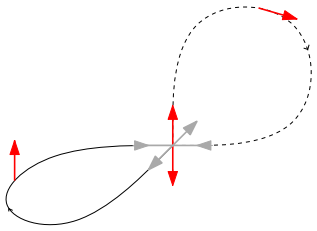

Figure 8. The stable and unstable manifolds of the saddle have dimension 1 and 2, respectively. The Perron-Frobinius vector field is represented in red. The limit set given by the homoclinic orbit on the right part of the figure (dashed line) is ruled out by Corollary 3. The limit set given by the homoclinic orbit on the left part of the figure (solid line) is compatible with Theorem 4 but the Perron-Frobenius vector field is nowhere tangent to the curve.

C. Complex limit sets of differentially positive systems are not attractors.

Part (ii) of Theorem 4 allows for more complex limit sets than those described in Part (i), but those limit sets cannot be attractors, because they are nowhere tangent to the dominant direction of the linearization. This property has been well studied for monotone systems. For instance, Smale proposed a construction to imbed chaotic behaviors in a cooperative irreducible system [43, Chapter 4]. The transversality of those limit sets to the Perron-Frobenius vector field extends to the trajectories that converge to them. For instance, consider any ω-limit set $\omega(\xi)$, $\xi \in \mathcal{X}$, satisfying Part (ii) of Theorem 4. Any trajectory whose ω-limit points belong to $\omega(\xi)$ is nowhere tangent to the Perron-Frobenius vector field. Moreover, if the trajectory does not converge to a fixed point then it shows high sensitivity with respect to initial conditions.

Corollary 4: Under the assumptions of Theorem 4, suppose that for some $\xi \in \mathcal{X}$, $\omega(\xi)$ satisfies Part (ii) of Theorem 4. Then, for any $z \in \mathcal{X}$ such that $\omega(z) \subseteq \omega(\xi)$, the trajectory $\psi(\cdot, 0, z)$ satisfies $f(\psi(t, 0, z)) \notin \mathcal{K}_{\mathcal{X}}(\psi(t, 0, z)) \setminus \{0\}$ for each $t \geq 0$. If $\omega(z)$ is not a singleton, then $\liminf_{t\to\infty} |\partial_z \psi(t, 0, z)\mathbf{w}(z)|_{\psi(t,0,z)} = \infty$. ⌟

The reason why the possibly complex limit sets of differentially positive systems are of little importance for the overall behavior is that their basin of attraction $\mathcal{W}$ seems strongly repelling. In accordance to Corollary 4, it is very "likely" for a trajectory in a small neighborhood of $\mathcal{W}$ to move away from $\mathcal{W}$ along the Perron-Frobenius vector field and "unlikely" to return to $\mathcal{W}$ at later time. The argument can be made rigorous for strongly order preserving monotone systems, allowing to recover the following celebrated result for monotone systems [43], [23].

Corollary 5: Let Σ be a continuous dynamical system of the form $\dot{x} = f(x)$ on a vector space $\mathcal{X}$, strict differentially positive with respect to the constant cone field $\mathcal{K}_{\mathcal{X}} \subseteq T_x\mathcal{X} =$

$\mathcal{X}$. Under boundedness of trajectories, the ω-limit set $\omega(\xi)$ is a fixed point for almost all $\xi \in \mathcal{X}$. ⌟

For general differentially positive systems, the above discussion leads to the following conjecture.

Conjecture 1: Under the assumptions of Theorem 4, for almost every $\xi \in \mathcal{X}$, the ω-limit set $\omega(\xi)$ is given by either a fixed point, or a limit cycle, or fixed points and connecting arcs. ⌟

The implication of Conjecture 1 would be that any limit set not covered by case (i) in Theorem 4 could at best attract a set of initial conditions of zero measure.

VIII. Extended example: differential positivity of the damped pendulum

The results of the paper are briefly illustrated on the analysis of the classical (adimensional) nonlinear pendulum model:

$$\Sigma : \left\{ \begin{array}{rcl} \dot{\vartheta} & = & v \\ \dot{v} & = & -\sin(\vartheta) - kv + u \end{array} \right. \qquad (\vartheta, v) \in \mathcal{X} := \mathbb{S}\times\mathbb{R}\,, \tag{16}$$

where $k \geq 0$ is the damping coefficient and u is the (constant) torque input.

The analysis of the state matrix $A(\vartheta, k)$ for $\vartheta \in \mathbb{S}$ of the variational system

$$\begin{bmatrix} \dot{\delta\vartheta} \\ \dot{\delta v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\cos(\vartheta) & -k \end{bmatrix}}_{=:A(\vartheta,k)} \begin{bmatrix} \delta\vartheta \\ \delta v \end{bmatrix} \qquad (\delta\vartheta, \delta v) \in T_{(\vartheta,v)}\mathcal{X} \tag{17}$$

reveals that the pendulum is strictly differentially positive for $k > 2$ and differentially positive for $k = 2$ with respect to the cone field

$$\mathcal{K}_{\mathcal{X}}(\vartheta, v) := \{(\delta\vartheta, \delta v) \in T_{(\vartheta,v)}\mathcal{X} \,|\, \delta\vartheta \geq 0, \delta\vartheta + \delta v \geq 0\}\,. \tag{18}$$

The differential positivity of (16) for $k \geq 2$ has the following simple geometric interpretation. For any $k \geq 2$ and any value of ϑ, the matrix $A(\vartheta, k)$ has only real eigenvalues. The blue and the red lines in Figure 9 show the direction of the eigenvectors of $A(\vartheta, 4)$ (left) - $A(\vartheta, 3)$ (center) - $A(\vartheta, 2)$ (right), for sampled values of $\vartheta \in \mathbb{S}$. The blue eigenvectors ($\delta\vartheta \leq 0$) are related to the smallest eigenvalues, which is negative for each ϑ. The red eigenvectors ($\delta\vartheta \geq 0$) are related to the largest eigenvalues. $\mathcal{K}_{\mathcal{X}}(\vartheta, v)$ is represented by the shaded area in Figure 9. The black arrows represent the vector field of the variational dynamics along the boundary of the cone. By continuity and homogeneity of the vector field on the boundary of the cone, Σ is strictly differentially positive for each $k > 2$. It reduces to a differentially positive system in the limit of $k = 2$. The loss of contraction in such a case has a simple geometric explanation: one of the two eigenvectors of $A(0, 2)$ belongs to the boundary of the cone and the eigenvalues of $A(0, 2)$ are both in -1. The issues is clear for $u = 0$ at the equilibrium $x_e := (0, 0)$. In such a case $A(0, 2)$ gives the linearization of Σ at x_e and the eigenvalues in -1 makes the positivity of the linearized system non strict for any selection of $\mathcal{K}_{\mathcal{X}}(x_e)$.

It is of interest to interpret differential positivity against the textbook analysis [46]. Following [46, Chapter 8], Figure 10 summarizes the qualitative behavior of the pendulum for different values of the damping coefficient $k \geq 0$ and of the constant torque input $u \geq 0$ (the behavior of the pendulum for $u \leq 0$ is symmetric). For $k > 2$, the attractors of the nonlinear pendulum for different values of the torque are compatible with Theorem 4. For example, Corollary 2 can be used to establish the existence of limit cycles for torque values larger than one. The nonlinear pendulum cannot be differentially positive for arbitrary values of the torque when $k \leq k_c$. This is because the region of bistable behaviors (coexistence of small and large oscillations) is delineated by a homoclinic orbit, which is ruled out by differental positivity (Corollary 3). For instance, looking at Figure 10, for any $k < k_c$ there exists a value $u = u_c(k)$ for which the pendulum encounters a *homoclinic bifurcation* (see [46, Section 8.5] and Figures 8.5.7 - 8.5.8 therein). In contrast, the *infinite-period bifurcation* at $k > 2$, $u = 1$ [46, Chapter 8] is compatible with differential positivity.

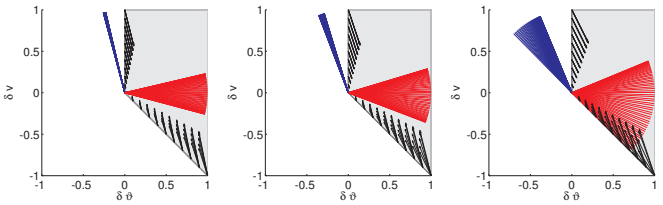

Figure 9. $\mathcal{K}_{\mathcal{X}}(\vartheta, v)$ is shaded in gray. The two eigenvector of $A(\vartheta, k)$ for a selection $\vartheta \in \mathbb{S}$ are represented in red and blue. Red eigenvectors - largest eigenvalues. Blue eigenvectors - smallest eigenvalues. Left figure: $k = 4$. Center figure: $k = 3$. Right figure: $k = 2$.

It is plausible that the "grey" area between k_c and $k = 2$ is a region where the nonlinear pendulum is differentially positive over a uniform time-horizon rather than pointwise. A detailed analysis of the region $k_c \leq k < 2$ is postponed to a further publication.

IX. Conclusions

The paper introduces the concept of differential positivity, a local characterization of monotonicity through the infinitesimal contraction properties of a cone field. The theory of differential positivity reduces to the theory of monotone systems when the state-space is linear and when the cone field is constant. The differential framework allows for a generalization of the Perron-Frobenius theory on nonlinear spaces and/or non constant cone fields. The paper focuses

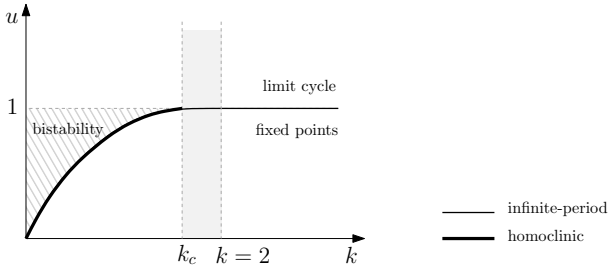

Figure 10. The qualitative behavior of the pendulum for large/small values of torque and damping, as reported in [46, p. 272].

on the characterization of limit sets of differentially positive systems, showing that those systems enjoy properties akin to the Poincare-Bendixson theory of planar systems. In particular, differential positivity is seen as a novel analysis tool for the analysis of limit cycles and as a property that precludes complex behaviors in a significant class of nonlinear systems. Many issues of interest remain to be addressed beyond the material of the present paper. The most pressing of those is probably the topic of feedback interconnections: negative feedback interconnections of monotone systems are known to provide a key mechanism of oscillation [21], [20] and it is appealing to analyze their differential positivity by inferring a (non-constant) cone field from the order properties of the subsystems and from the interconnection structure only. More generally, the construction of particular cone fields for interconnections of relevance in system theory (e.g. Lure systems) as well as the relationship between differential positivity and horizontal contraction recently studied in [19] will be the topic of further research.

APPENDIX

A. Proofs of Section VI

Proof of Theorem 2: Using $\Gamma_{(x_1,u_1,x_2,u_2)}$ to denote the linear and invertible mapping $\Gamma(x_1,u_1,x_2,u_2)$ that satisfies $\Gamma(x_1,u_1,x_2,u_2)\mathcal{K}_{\mathcal{X}}(x_1,u_1) = \mathcal{K}_{\mathcal{X}}(x_2,u_2)$ for each $(x_1,u_1),(x_2,u_2) \in \mathcal{X}\times\mathcal{U}$ (see Section IV), and $d_{*(x,u)}$ to denote $d_{\mathcal{K}_{\mathcal{X}}(x,u)}$ (for readability), note that the Hilbert metric satisfies

$$d_{*(x_2,u_2)}(v,w) = d_{*(x_1,u_1)}(\Gamma^{-1}_{(x_1,u_1,x_2,u_2)}v, \Gamma^{-1}_{(x_1,u_1,x_2,u_2)}w) \tag{19}$$

for any $(x_1,u_1),(x_2,u_2) \in \mathcal{X}\times\mathcal{U}$, and any $v,w \in \mathcal{K}_{\mathcal{X}}(x_2,u_2)$. (19) follows by the combination of (11) with the identity $\Gamma^{-1}_{(x_1,u_1,x_2,u_2)}\mathrm{bd}\mathcal{K}_{\mathcal{X}}(x_2,u_2) = \mathrm{bd}\mathcal{K}_{\mathcal{X}}(x_1,u_1)$ (by linearity).

Along any given solution pair $(x(\cdot),u(\cdot)) : [t_0,\infty) \to \mathcal{X} \in \Sigma$ define the linear operator

$$\begin{aligned} &A_{(x(\cdot),u(\cdot))}[\tau_2,\tau_1] := \\ &:= \Gamma^{-1}_{(x(\tau_1),u(\tau_1),x(\tau_2),u(\tau_2))}\partial_{x(\tau_1)}\psi(\tau_2,\tau_1,x(\tau_1),u(\cdot)) \end{aligned} \tag{20}$$

where $t_0 \le \tau_1 \le \tau_2$. Thus, using $d_{*(t)}$ to denote $d_{\mathcal{K}_{\mathcal{X}}(x(t),u(t))}$, and recalling that, in Theorem 2, $x(t) = \psi(t,t_0,x(t_0),u(\cdot))$, $\delta x_1(t) = \partial_{x(t_0)}\psi(t,t_0,x(t_0),u(\cdot))\delta x_1(t_0)$, and $\delta x_2(t) = \partial_{x(t_0)}\psi(t,t_0,x(t_0),u(\cdot))\delta x_2(t_0)$, for each $t \ge t_0$ we get

$$\begin{aligned} &d_{*(t)}(\delta x_1(t),\delta x_2(t)) = \\ &= d_{*(t_0)}(A_{(x(\cdot),u(\cdot))}[t,t_0]\delta x_1(t_0), A_{(x(\cdot),u(\cdot))}[t,t_0]\delta x_2(t_0)) \\ &\le d_{*(t_0)}(\delta x_1(t_0),\delta x_2(t_0)) \ . \end{aligned} \tag{21}$$

The identity follows by the combination of (19), (20), and differential positivity. The inequality follows from the fact that $A_{(x(\cdot),u(\cdot))}[t,t_0]$ is a linear operator in $\mathcal{K}_{\mathcal{X}}(x(t_0),u(t_0)) \to \mathcal{K}_{\mathcal{X}}(x(t_0),u(t_0))$, as in [27], [11].

Strict differential positivity guarantees that there exists $T > 0$ such that, for any given $\tau \in [t_0,\infty)$,

$$A_{(x(\cdot),u(\cdot))}[\tau+T,\tau]\mathcal{K}_{\mathcal{X}}(x(\tau),u(\tau)) \subseteq \mathcal{R}_{\mathcal{X}}(x(\tau),u(\tau)) \ . \tag{22}$$

Thus, following [11], [27], define *projective diameter* $\Delta_T := \sup\{d_{\mathcal{K}_{\mathcal{X}}(x,u)}(v_1,v_2) \,|\, v_1,v_2 \in \mathcal{R}_{\mathcal{X}}(x,u)\} < \infty$ and *contraction ratio* $\mu_T := \tanh\left(\frac{\Delta_T}{4}\right) < 1$. Then, using again $d_{*(t)}$ to denote $d_{\mathcal{K}_{\mathcal{X}}(x(t),u(t))}$, [11, Theorem 3.2] and [27, Proposition 3.14] guarantee the inequality

$$\begin{aligned} &d_{*(\tau+T)}(\delta x_1(\tau+T),\delta x_2(\tau+T)) = \\ &= d_{*(\tau)}(A_{(x(\cdot),u(\cdot))}[\tau+T,\tau]\delta x_1(\tau), A_{(x(\cdot),u(\cdot))}[\tau+T,\tau]\delta x_2(\tau)) \\ &\le \mu_T d_{*(\tau)}(\delta x_1(\tau),\delta x_2(\tau)) \ , \end{aligned} \tag{23}$$

for all $\tau \ge t_0$. By the semigroup property, for any integer k, and $t \ge t_0 + kT$ we get

$$d_{*(t)}(\delta x(t),\delta y(t)) \le \mu_T^k d_{*(t_0)}(\delta x(t_0),\delta y(t_0)) \tag{24}$$

which establishes the exponential convergence.

Finally, combining (24) and (22), for all $t \ge t_0 + (k+1)T$ we get $d_{*(t)}(\delta x(t),\delta y(t)) \le \mu_T^k \Delta_T$. ■

Proof of Theorem 3: Consider the solution pair $(z(\cdot),u(\cdot)) \in \Sigma$ such that $z(t) = x$ and define $\mathcal{C}(t,t_0,x,u(\cdot)) := \partial_{z(t_0)}\psi(t,t_0,z(t_0),u(\cdot))\mathcal{K}_{\mathcal{X}}(z(t_0),u(t_0))$. By construction, strict differential positivity guarantees that

$$\mathcal{C}(t,t-T_2,x,u(\cdot)) \subseteq \mathcal{C}(t,t-T_1,x,u(\cdot)) \tag{25}$$

for each $T_2 \ge T_1 \ge 0$. This proves the existence of the limit.

From Theorem 2, there exists $T > 0$ such that

$$\limsup_{k\to\infty}\{d_{\mathcal{K}_{\mathcal{X}}(x,u(t))}(v_1,v_2) \,|\, v_1,v_2 \in \mathcal{C}(t,t-kT,x,u(\cdot))\} = 0, \tag{26}$$

since the right-hand side is bounded from above by $\lim_{k\to\infty}\mu_T^{k-1}\Delta_T$, where $\Delta_T < \infty$ and $\mu_T < 1$ are respectively the projective diameter and the contraction ratio defined in the proof of Theorem 2. (25) and (26) guarantee that the set $\mathcal{C}(t,t-kT,x,u(\cdot)) \cap \mathcal{B}(x)$ converges to the singleton $\{\mathbf{w}_{u(\cdot)}(x,t)\} \subseteq \mathrm{int}\mathcal{K}_{\mathcal{X}}(x,u(t)) \cap \mathcal{B}(x)$ as $k \to \infty$. We write $\mathbf{w}_{u(\cdot)}(x,t)$ since the limit of the set $\mathcal{C}(t,t-kT,x,u(\cdot))\cap\mathcal{B}(x)$ for $k \to \infty$ depends only on the input signal $u(\cdot)$, the state x, and the time t. ■

Proof of Corollary 1: To show *time-invariance*, consider $u(\cdot) = u$. Under the action of the constant input u, consider the trajectories $z(\cdot)$ and $y(\cdot)$ such that $z(t) = x$ and $y(t+T) = x$, for some $t,T \in \mathbb{R}$. Using $\mathcal{B}(x) := \{\delta x \in T_x\mathcal{X} \,|\, |\delta x|_x = 1\} \subseteq T_x\mathcal{X}$, from Theorem 3, $\lim_{t_0\to-\infty}(\partial_{z(t_0)}\psi(t,t_0,z(t_0),u)\mathcal{K}_{\mathcal{X}}(z(t_0),u))\cap\mathcal{B}(x) = \mathbf{w}_u(x,t)$ and $\lim_{t_0\to-\infty}(\partial_{y(t_0)}\psi(t+T,t_0,y(t_0),u)\mathcal{K}_{\mathcal{X}}(y(t_0),u))\cap\mathcal{B}(x) = \mathbf{w}_u(x,t+T)$. However, for constant u, Σ is time invariant, therefore uniqueness of trajectories from initial conditions guarantees that $z(\tau) = y(\tau+T)$ for all $\tau \in (-\infty,t)$, which guarantees $\mathcal{K}_{\mathcal{X}}(z(t_0),u)) = \mathcal{K}_{\mathcal{X}}(y(t_0+T),u))$. $\mathbf{w}_u(x,t) = \mathbf{w}_u(x,t+T)$ follows.

To show *continuity*, consider the family of vector fields $\{g_k\}$ given by a continuous vector field $g_0(x) \in \mathcal{K}_{\mathcal{X}}(x,u) \cap \mathcal{B}(x)$ and by $g_{k+1}(\psi(T,0,x,u)) := \frac{\partial_x\psi(T,0,x,u)g_k(x)}{|\partial_x\psi(T,0,x,u)g_k(x)|_x}$. Note that each g_k is a continuous vector field because $\partial_x\psi(T,0,x,u)$ is continuous in x and the Riemannian structure is smooth in x. Moreover, $d_{\mathcal{K}_{\mathcal{X}}(x,u)}(g_{k+1}(x),\mathbf{w}_u(x)) \le \mu_T^k\Delta_T$ for all $x \in \mathcal{X}$ and all $k \ge 1$, where μ_T and Δ_T are respectively the contraction ratio and the projective diameter defined in

the proof of Theorem 2. Indeed, g_k converge uniformly to $\mathbf{w}_u$ (with respect to the Hilbert metric at x).

By contradiction, suppose that the Perron-Frobenius vector field is not continuous at x. Then, following [10, Definition 2.1], in local coordinates, the ith component $[\mathbf{w}_u(x)]_i$ of $\mathbf{w}_u(x)$ is not a continuous function: there exist $\varepsilon > 0$, a sequence of points $y_j \to x$ as $j \to \infty$, and a bound J such that the absolute value $|[\mathbf{w}_u(x)]_i - [\mathbf{w}_u(y_j)]_i| \geq \varepsilon$ for any $j \geq J$. By the uniform convergence of g_k to $\mathbf{w}_u$, there exists a bound K such that $|[g_k(x)]_i - [\mathbf{w}_u(x))]_i| \leq \frac{\varepsilon}{3}$ and $|[g_k(y_j)]_i - [\mathbf{w}_u(y_j))]_i| \leq \frac{\varepsilon}{3}$ for all $k \geq K$ and all j. Therefore, $|[g_k(x)]_i - [g_k(y_j)]_i| \geq \frac{\varepsilon}{3}$ for all $k \geq K$ and $j \geq J$, which contradicts the continuity of g_k.

Finally, the coincidence between the Perron-Frobenius vector field and the Perron-Frobenius vector for *linear systems* is a straightforward consequence of (15). ■

B. Proofs of Section VII

For readability, in what follows we use $\psi_t(x) := \psi(t, 0, x)$, $\partial\psi_t(x) := \partial_x\psi(t, 0, x)$, and $d_x(\cdot, \cdot) := d_{\mathcal{K}_{\mathcal{X}}(x)}(\cdot, \cdot)$. Recall that the pair $(\psi_t(x), \partial\psi_t(x)\delta x)$ is the trajectory of the prolonged system $\delta\Sigma$ given by $\dot{x} = f(x)$, $\dot{\delta x} = \partial f(x)\delta x$ from the initial condition $(x, \delta x) \in T\mathcal{X}$.

We develop first some technical results. The claims of the next two lemmas are about the boundedness of the trajectories of the variational systems. The claims hold for both continuous and discrete systems (closed or with constant inputs).

Lemma 1: Let $u(\cdot) = u$ be constant. Under the assumptions of Theorem 3, for any $x \in \mathcal{X}$ and any $\delta x \in T_x\mathcal{X}$, if $|\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)} < \infty$ then $\limsup_{t\to\infty} |\partial\psi_t(x)\delta x|_{\psi_t(x)} < \infty$.

Lemma 2: Let $u(\cdot) = u$ be constant. Under the assumptions of Theorem 3, let x be any point of $\mathcal{X}$ and suppose that there exists $\delta x \in \text{int}\mathcal{K}_{\mathcal{X}}(x)$ such that $\limsup_{t\to\infty} |\partial\psi_t(x)\delta x|_{\psi_t(x)} < \infty$. Then, $\limsup_{t\to\infty} |\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)} < \infty$.

Proof of Lemma 1: For the first item suppose that the implications does not hold and $|\partial\psi_t(x)\delta x|_{\psi_t(x)}$ grows unbounded. Take the vector $\delta y = \delta x + \alpha\mathbf{w}(x)$. Note that for α sufficiently large $\delta y \in \mathcal{K}_{\mathcal{X}}(x)$. These facts and the linearity of $\partial\psi_t(x)$ guarantee that $|\partial\psi_t(x)\delta y|_{\psi_t(x)}$ grows unbounded and there exists $\bar{t}$ sufficiently large such that either (i) $\partial\psi_{\bar{t}}(x)\delta y \notin \mathcal{K}_{\mathcal{X}}(\psi_{\bar{t}}(x))$, which contradicts differential positivity, or (ii) $\partial\psi_{\bar{t}}(x)\delta y \simeq \rho\psi_{\bar{t}}(x)\mathbf{w}(x)$ where $\beta \in \mathbb{R}$ is a scaling factor. Thus, for all $t \geq \bar{t}$, by linearity, $|\partial\psi_t(x)\delta y|_{\psi_t(x)} \simeq \rho|\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)}$ which grows unbounded contradicting the assumption on $|\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)}$. ■

Proof of Lemma 2: For the second item, consider any decomposition $\delta x = \alpha\mathbf{w}(x) + \beta\delta z$ where $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $\delta z \in \mathcal{K}_{\mathcal{X}}(x)$, which can always be achieved for α sufficiently small since $\delta x \in \text{int}\mathcal{K}_{\mathcal{X}}(x)$. Then, $\partial\psi_t(x)\delta x = \partial\psi_t(x)[\alpha\mathbf{w}(x) + \beta\delta z] = \alpha\partial\psi_t(x)\mathbf{w}(x) + \beta\partial\psi_t(x)\delta z$ and, by projective contraction, $\partial\psi_t(x)\delta z$ converges asymptotically to $\rho_t\mathbf{w}(\psi_t(x))$ for some $\rho_t \in \mathbb{R}_{\geq 0}$. Thus,

$$\begin{aligned}
&\limsup_{t\to\infty} |\partial\psi_t(x)\delta x|_{\psi_t(x)} = \\
&= \limsup_{t\to\infty} |\alpha\partial\psi_t(x)\mathbf{w}(x) + \beta\rho_t\mathbf{w}(\psi_t(x))|_{\psi_t(x)} \\
&= \limsup_{t\to\infty} \left|\alpha\partial\psi_t(x)\mathbf{w}(x) + \beta\rho_t \tfrac{\partial\psi_t(x)\mathbf{w}(x)}{|\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)}}\right|_{\psi_t(x)} \\
&= \limsup_{t\to\infty} \left(\alpha + \tfrac{\beta\rho_t}{|\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)}}\right) |\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)} \\
&\geq \alpha \limsup_{t\to\infty} |\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)} \ . \qquad (27)
\end{aligned}$$

■

The next lemma shows that any trajectory of a continuous and closed differentially positive system whose motion follows the Perron-Frobenius vector field either converges to a fixed point or defines a periodic orbit. In what follows we will use $\psi_t(x) := \psi(t, 0, x)$ and $\partial\psi_t(x) := \partial_x\psi(t, 0, x)$.

Lemma 3: Under the assumptions of Theorem 4, consider any x such that, for all t, $f(\psi_t(x)) = \lambda(\psi_t(x))\mathbf{w}(\psi_t(x))$ and $|\lambda(\psi_t(x))| \geq \rho > 0$. Then, the trajectory $\psi_t(x)$ is periodic.

Proof of Lemma 3: In what follows we use $\mathcal{A} := \{\psi_t(x) \,|\, t \in \mathbb{R}\}$ and $\mathcal{B}_\varepsilon(x)$ to denote a ball of radius ε centered at x: for any two points z in $\mathcal{B}_\varepsilon(x)$ there exists a curve $\gamma(\cdot)$ such that $\gamma(0) = x$, $\gamma(1) = z$ and whose length $L(\gamma(\cdot)) = \int_0^1 |\dot{\gamma}(s)|_{\gamma(s)} ds \leq \varepsilon$. We make also use of the notion of *local section* at x, which is any open set $\mathcal{S} \subseteq \mathcal{X}$ of dimension $n-1$ ($\mathcal{X}$ has dimension n) contained within a (sufficiently) small neighborhood $\mathcal{B}_\varepsilon(x)$ of x such that $x \in \mathcal{S}$ and $\mathbf{w}(z) \notin T_z\mathcal{S}$ for each $z \in \mathcal{S}$. Finally, for any given Rimannian tensor such that $\langle \delta x, \mathbf{w}(x)\rangle_x \geq 0$ for any $x \in \mathcal{X}$ and $\delta x \in \mathcal{K}_{\mathcal{X}}(x)$, define the vertical projection $W_x(\delta x) := \langle \delta x, \mathbf{w}(x)\rangle_x \mathbf{w}(x)$, and the horizontal projection $H_x(\delta x) := \delta x - W_x(\delta x)$.

1) Bounded variational dynamics: $0 \neq f(\psi_t(x)) = \lambda(\psi_t(x))\mathbf{w}(\psi_t(x))$ for all $t \geq 0$ therefore, by continuity of the vector field and boundedness of trajectories, for $\varepsilon > 0$ sufficiently small, $f(z) \in \text{int}\mathcal{K}_{\mathcal{X}}(z)$ or $-f(z) \in \text{int}\mathcal{K}_{\mathcal{X}}(z)$ for all $z \in \mathcal{E} := \bigcup_{t\geq 0} \mathcal{B}_\varepsilon(\psi_t(x))$. Note that $\mathcal{E}$ is a compact set by boundedness of trajectories. Without loss of generality consider $f(z) \in \text{int}\mathcal{K}_{\mathcal{X}}(z)$. Then, $f(\psi_t(z)) \in \text{int}\mathcal{K}(\psi_t(z))$ for all $z \in \mathcal{E}$ and $t \geq 0$, by differential positivity combined with the identity $f(\psi_t(z)) = \partial\psi_t(z)f(z)$, which makes $(\psi_t(z), f(\psi_t(z)))$ a trajectory of the prolonged system.

By boundedness of trajectories, $|\partial\psi_t(z)f(z)|_{\psi_t(z)} = |f(\psi_t(z))|_{\psi_t z}$ is necessarily bounded. Lemma 2 guarantees that $\limsup_{t\to\infty} |\partial\psi_t(z)\mathbf{w}(z)|_{\psi_t(z)} < \infty$ for all $z \in \mathcal{E}$. By Lemma 1, $\limsup_{t\to\infty} |\partial\psi_t(z)\delta z|_{\psi_t(z)} < \infty$ for all $z \in \mathcal{E}$ and $\delta z \in T_z\mathcal{X}$. Similar results can be obtained for the case $-f(z) \in \text{int}\mathcal{K}_{\mathcal{X}}(z)$ exploiting the linearity of $\partial\psi_t(z)$. Finally, consider any $z \in \mathcal{E}$ and define $\alpha_z := \sup_{\delta z \in T_z\mathcal{X}, |\delta z|_z = 1} \limsup_{t\to\infty} |\partial\psi_t(z)\delta z|_{\psi_t(z)} < \infty$. Then, for any $\delta z \in T_z\mathcal{X}$, $\limsup_{t\to\infty} |\partial\psi_t(z)\delta z|_{\psi_t(z)} < \alpha_z |\delta z|_z$. Since $\mathcal{E}$ is a compact set, there exists $\overline{\alpha} := \sup_{z\in\mathcal{E}} \alpha_z$.

2) Contraction of the horizontal component: Take $z \in \mathcal{E}$. For $\delta z \in \mathcal{K}_{\mathcal{X}}(z)$, combining the contraction property $\lim_{t\to\infty} d_{\psi_t(z)}(\partial\psi_t(z)\delta z, \mathbf{w}(\psi_t(z))) = 0$ of Theorem 2 and the bound $\limsup_{t\to\infty} |\partial\psi_t(z)\delta z|_{\psi_t(z)} \leq \overline{\alpha}|\delta z|_z$ in **1)**, we get $\lim_{t\to\infty} \partial\psi_t(z)\delta z - W_{\psi_t(z)}(\partial\psi_t(z)\delta z) = 0$, that is, $\lim_{t\to\infty} H_{\psi_t(z)}(\partial\psi_t(z)\delta z) = 0$. A similar result holds for

$\delta z \in -\mathcal{K}_{\mathcal{X}}(z)$. Consider now $\delta z \notin \mathcal{K}_{\mathcal{X}}(z)$. Define the new vector $\delta z^* = \delta z + \alpha \mathbf{w}(z)$. For α sufficiently large $\delta z^* \in \mathcal{K}_{\mathcal{X}}(z)$. Therefore $\lim_{t\to\infty} H_{\psi_t(z)}(\partial\psi_t(z)\delta z^*) = 0$ which implies $\lim_{t\to\infty} H_{\psi_t(z)}(\partial\psi_t(z)\delta z) = 0$.

3) Attractiveness of $\psi_t(x)$: Consider the case $t = 0$ since $\psi_0(x) = x$ (the argument is the same for $t > 0$) and take any curve $\gamma(\cdot) : [0, 1] \to \mathcal{E}$ such that $\gamma(0) = x$ and $L(\gamma(\cdot)) = \varepsilon$, and consider the evolution of $\gamma(\cdot)$ along the flow of the system, that is, $\psi_t(\gamma(s))$ for $s \in [0, 1]$. We observe that $\frac{d}{ds}\psi_t(\gamma(s)) = [\partial\psi_t(\gamma(s))]\dot{\gamma}(s)$. Thus, by **1)**, $\limsup_{t\to\infty} |\frac{d}{ds}\psi_t(\gamma(s))|_{\psi_t(\gamma(s))} \leq \overline{\alpha}|\dot{\gamma}(s)|_{\gamma(s)}$, which guarantees $\limsup_{t\to\infty} L(\psi_t(\gamma(\cdot))) \leq \varepsilon\overline{\alpha}$.

By **2)**, $\lim_{t\to\infty} H_{\psi_t(\gamma(s))}(\frac{d}{ds}\psi_t(\gamma(s)))|_{\psi_t(\gamma(s))} = 0$ for all $s \in [0, 1]$. Thus, $\frac{d}{ds}\psi_t(\gamma(s))$ either converges to zero or aligns to the Perron-Frobenius vector field. Precisely, three cases may occur:

- $\lim_{t\to\infty} \frac{d}{ds}\psi_t(\gamma(s))) = 0$,
- $\lim_{t\to\infty} d_{\psi_t(\gamma(s))}(\mathbf{w}(\psi_t(\gamma(s))), \frac{d}{ds}\psi_t(\gamma(s))) = 0$,
- $\lim_{t\to\infty} d_{\psi_t(\gamma(s))}(\mathbf{w}(\psi_t(\gamma(s))), -\frac{d}{ds}\psi_t(\gamma(s))) = 0$.

Furthrmore, $\lim_{t\to\infty} d_{\psi_t(\gamma(s))}(\mathbf{w}(\psi_t(\gamma(s))), f(\psi_t(\gamma(s)))) = 0$ since $f(\psi_t(\gamma(s))) \in \mathcal{K}_{\mathcal{X}}(\psi_t(\gamma(s)))$. Thus, in the limit, the image of $\psi_t(\gamma(\cdot))$ is given by the image of a (time-dependent Perron-Frobenius) curve $\gamma_t^{\mathbf{w}}(\cdot)$ that satisfies either $\frac{d}{ds}\gamma_t^{\mathbf{w}}(s) = \mathbf{w}(\gamma_t^{\mathbf{w}}(s))$ or $\frac{d}{ds}\gamma_t^{\mathbf{w}}(s) = -\mathbf{w}(\gamma_t^{\mathbf{w}}(s))$ at any fixed t. By construction, $\frac{d}{ds}\gamma_t^{\mathbf{w}}(s) = \frac{1}{\lambda(\gamma_t^{\mathbf{w}}(s))} f(\gamma_t^{\mathbf{w}}(s))$.

At each fixed t, $\gamma_t^{\mathbf{w}}(\cdot)$ is a (reparameterized) integral curve of the vector field f from the initial condition $\gamma_t^{\mathbf{w}}(0) = \psi_t(x)$. Therefore, trajectory with initial condition in $\gamma(s)$ converges asymptotically to $\mathcal{A}$, for all $s \in [0, 1]$. In particular, using the bound $L(\psi_t(\gamma(\cdot))) \leq \varepsilon\overline{\alpha}$ characterized above, each trajectory converges asymptotically to

$$\mathcal{C}_t(x) := \left\{ \psi_{t+\tau}(x) \in \mathcal{A} \,|\, -\frac{\overline{\alpha}\varepsilon}{\rho} \leq \tau \leq \frac{\overline{\alpha}\varepsilon}{\rho} \right\} . \tag{28}$$

4) Periodicity of the orbit: $\psi_t(x)$ does not converge to a fixed point and belongs to a compact set for each t, therefore there exists a point $\psi_{t_0}(x)$ whose neighborhood $\mathcal{B}_\varepsilon(\psi_{t_0}(x))$ is visited by the trajectory infinitely many times for any given $\varepsilon > 0$. For simplicity, without any loss of generality, we consider this point given at $t_0 = 0$, that is, $\psi_0(x) = x$.

Consider a local section $\mathcal{S}$ at x and consider the sequence $t_k \to \infty$ such that $\psi_{t_k}(x) \in \mathcal{S}$. Since $f(x)$ is aligned with $\mathbf{w}(x)$, for ε sufficiently small, the continuity of the system vector field f guarantees that $\mathcal{S}$ is transverse to $f(z)$ for all $z \in \mathcal{S} \cap \mathcal{B}_\varepsilon(x)$, that is, $f(z) \notin T_z\mathcal{S} \cap \mathcal{B}_\varepsilon(x)$.

By **3)**, for every $z \in \mathcal{S} \cap \mathcal{B}_\varepsilon(x)$, $\psi_{t_k}(z)$ converges asymptotically to the set $\mathcal{C}_{t_k}(x)$ as $k \to \infty$. For any positive integer N, define the ($\frac{\varepsilon}{N}$-inflated) set

$$\mathcal{C}_t^{(\varepsilon/N)}(x) := \{y \,|\, \exists\gamma : [0, 1] \to \mathcal{E}, \exists s \in [0, 1] \text{ such that } \gamma(0) \in \mathcal{C}_t(x),\, \gamma(s) = y, L(\gamma(\cdot)) \leq \varepsilon/N\} . \tag{29}$$

Then, by continuity, for every $N > 0$, there exists a $k \geq k_N$ sufficiently large such that $\psi_{t_k}(z) \in \mathcal{C}_{t_k}^{(\varepsilon/N)}(x)$ for all $z \in \mathcal{S} \cap \mathcal{B}_\varepsilon(x)$.

By the transversality of the section $\mathcal{S}$ with respect to the system vector field, for N sufficiently large, we have that the flow from $z \in \mathcal{C}_{t_k}^{(\varepsilon/N)}(x)$ satisfies $\psi_\tau(z) \in \mathcal{S}$ for some $-(\frac{\overline{\alpha}}{\rho} + \delta_N)\varepsilon \leq \tau \leq (\frac{\overline{\alpha}}{\rho} + \delta_N)\varepsilon$, where δ_N is some (small) positive constant such that $\delta_N \to 0$ as $N \to \infty$. Moreover, by continuity with respect to initial conditions, for N sufficiently large, we get

$$\psi_\tau(z) \in \mathcal{S} \cap \mathcal{B}_{\frac{\varepsilon}{3}}(\psi_{t_k}(x)) . \tag{30}$$

It follows that, for $t_k - (\frac{\overline{\alpha}}{\rho} + \delta_N)\varepsilon \leq t \leq t_k + (\frac{\overline{\alpha}}{\rho} + \delta_N)\varepsilon$, the flow $\psi_t(\cdot)$ maps every point of $\mathcal{S} \cap \mathcal{B}_\varepsilon(x)$ into $\mathcal{S} \cap \mathcal{B}_{\frac{\varepsilon}{3}}(\psi_{t_k}(x))$.

For $k \geq k_N$, denote by P_k the mapping from $\mathcal{S} \cap \mathcal{B}_\varepsilon(x)$ into $\mathcal{S} \cap \mathcal{B}_{\frac{\varepsilon}{3}}(\psi_{t_k}(x))$. Since $\psi_{t_k}(x)$ recursively visit any local section of x, eventually, for some $K \geq k_N$, the flow satisfies $\psi_{t_K}(x) \in \mathcal{S} \cap \mathcal{B}_{\frac{\varepsilon}{3}}(x)$. Using the results above, we conclude that the flow of the system maps $\mathcal{S} \cap \mathcal{B}_\varepsilon(x)$ into $\mathcal{S} \cap \mathcal{B}_{\frac{\varepsilon}{3}}(\psi_{t_K}(x)) \subseteq \mathcal{S} \cap \mathcal{B}_{\frac{2\varepsilon}{3}}(x)$, that is, P_K is a contraction. By Banach fixed-point theorem $P_K(x) = x$, that is, $\psi_{t_K}(x) = x$. ■

We are now ready for the proof of the main theorem.

Proof of Theorem 4: For any $\xi \in \mathcal{X}$ consider $\omega(\xi)$. Three cases may occur:

1) $f(x) = 0$ for some $x \in \omega(\xi)$. x is a fixed point.

2) $f(x) \in \text{int}\mathcal{K}_{\mathcal{X}}(x) \setminus \{0\}$ or $-f(x) \in \text{int}\mathcal{K}_{\mathcal{X}}(x) \setminus \{0\}$ for some $x \in \omega(\xi)$. In such a case,

$$f(z) = \lambda(z)\mathbf{w}(z) \quad \text{for all } z \in \omega(\xi) , \tag{31}$$

where $\lambda(z) \in \mathbb{R}$ is a scaling factor. To see this, consider the case $f(x) \in \text{int}\mathcal{K}_{\mathcal{X}}(x)$ (wlog). By definition of ω-limit set, there exists a sequence $t_k \to \infty$ such that $\lim_{k\to\infty} \psi_{t_k}(\xi) = x$. For $k \geq k^*$ sufficiently large, $\psi_{t_k}(\xi)$ belongs to an infinitesimal neighborhood of x therefore $f(\psi_{t_k}(\xi)) \in \mathcal{K}(\psi_{t_k}(\xi))$ by continuity of the cone field since $f(x) \in \text{int}\mathcal{K}_{\mathcal{X}}(x)$. Then, by projective contraction, $\lim_{k\to\infty} d_{\psi_{t_k}(\xi)}(f(\psi_{t_k}(\xi)), w(\psi_{t_k}(x))) = 0$, that is, $f(x) = \lambda(x)\mathbf{w}(x)$ for some scaling factor $\lambda(x) \in \mathbb{R}$. By definition of ω-limit set, starting from t_{k^*}, it is possible to find a sequence $\tau_k \to \infty$ as $k \to \infty$ such that $\lim_{k\to\infty} \psi_{t_{k^*}+\tau_k}(\xi) = z$ for any $z \in \omega(\xi)$. Thus, by the argument above, $f(z) = \lambda(z)\mathbf{w}(z)$ for all $z \in \omega(\xi)$. (31) guarantees that, for any $x \in \omega(\xi)$, the image of the trajectory $\psi_t(x)$ is a subset of the image of some Perron-Frobenius curve $\gamma^{\mathbf{w}}(\cdot)$. Note that $\lambda(\psi_t(x))$ may converge to zero. In such a case $\psi_t(x)$ converges to a fixed point. Otherwise, $|\lambda(\psi_t(x))| \geq \varepsilon > 0$ therefore, by Lemma 3, $\psi_t(x)$ is periodic.

3) It remains to consider the case $f(x) \notin \mathcal{K}_{\mathcal{X}}(x)$ (or $-f(x) \notin \mathcal{K}_{\mathcal{X}}(x)$) for some $x \in \omega(\xi)$. In such a case, from the previous item, $f(z) \notin \mathcal{K}_{\mathcal{X}}(z) \setminus \{0\}$ for all $z \in \omega(\xi)$. Then, either $\lim_{t\to\infty} f(\psi_t(x)) = 0$ ($\psi_t(x)$ converges to a fixed point) or the contraction of the Hilbert metric enforces $\liminf_{t\to\infty} |\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)} = \infty$. For the latter, consider any sequence $t_k \to \infty$ as $k \to \infty$ such that $f(\psi_{t_k}(x)) \geq \varepsilon > 0$. Take $\delta x = f(x) + \lambda\mathbf{w}(x)$. For λ sufficiently large $\delta x \in \mathcal{K}_{\mathcal{X}}(x)$. Then, $\lim_{k\to\infty} d_{\psi_{t_k}(x)}(\partial\psi_{t_k}(x)\delta x, w(\psi_{t_k}(x))) = 0$ holds only if the evolution of δx along the flow $\partial\psi_{t_k}(x)\delta x = \partial\psi_{t_k}(x)f(x) + \lambda\partial\psi_{t_k}(x)\mathbf{w}(x) = f(\psi_{t_k}(x)) + \lambda\partial\psi_{t_k}(x)\mathbf{w}(x)$

shows an unbounded growth of the component $\partial\psi_{t_k}(x)\mathbf{w}(x)$. ■

Proof of Corollary 2: Recall that $f(\psi_t(x)) = \partial\psi_t(x)f(x)$. Since $f(x) \in \text{int}\mathcal{K}_\mathcal{X}(x)$, Lemmas 1 and 2 guarantee that $\limsup_{t\to\infty} |\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)} < \infty$. Since $f(x) \neq 0$ in $\mathcal{C}$, we conclude that Case (ii) of Theorem 4 does not occur. Exploiting again the assumption $f(x) \neq 0$ in $\mathcal{C}$, Case (i) of Theorem 4 guarantees that the trajectories of Σ converge to periodic orbits. We need to prove the uniqueness.

By contradiction, suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two periodic orbits such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. Take any curve $\gamma(\cdot) : [0,1] \to \mathcal{C}$ such that $\gamma(0) \in \mathcal{A}_1$ and $\gamma(1) \in \mathcal{A}_2$, and recall that $\frac{d}{ds}\psi_t(\gamma(s)) = [\partial\psi_t(x)_{x=\gamma(s)}]\dot{\gamma}(s)$. Since $f(\psi_t(x)) = \partial\psi_t(x)f(x)$ and $f(x) \in \text{int}\mathcal{K}_\mathcal{X}(x)$, Lemmas 1 and 2 guarantee that $\limsup_{t\to\infty} |\frac{d}{ds}\psi_t(\gamma(s))|_{\psi_t(\gamma(s))} < \infty$. Since $f(x) \in \text{int}\mathcal{K}_\mathcal{X}(x)$ for any $x \in \mathcal{C}$ and the trajectories of Σ are bounded, we can use the argument in **2)** and **3)** of Lemma 3 to show that $\frac{d}{ds}\psi_t(\gamma(s))$ converges asymptotically to $\lambda_a(\psi_t(\gamma(s)))\mathbf{w}(\psi_t(\gamma(s)))$, thus to $\lambda_b(\psi_t(\gamma(s)))f(\psi_t(\gamma(s)))$, for some (bounded) scaling factors $\lambda_a(\cdot), \lambda_b(\cdot) \in \mathbb{R}$,

As a consequence, every trajectories whose initial conditions belongs to the image of $\gamma(\cdot)$ converges asymptotically to an integral curve of the system vector field $f(x)$, for $x \in \mathcal{C}$, connecting $\mathcal{A}_1$ and $\mathcal{A}_2$, since $\psi_t(\gamma(0)) \in \mathcal{A}_1$ and $\psi_t(\gamma(1)) \in \mathcal{A}_2$ for all $t \geq 0$. It follows that $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$. A contradiction. ■

Proof of Corollary 3: For some $x \in \mathcal{X}$, suppose that $y_e = \lim_{t\to\infty} \psi_{-t}(x)$ and $z_e = \lim_{t\to\infty} \psi_t(x)$ are hyperbolic fixed point.

Suppose that the orbit connecting y_e to z_e is tangential to $\mathbf{w}(y_e)$ at y_e. Take now any point y in a small neighborhood of y_e such that $y = \psi_{-T}(x)$ for some $T > 0$. By continuity, $f(y) \in \text{int}\mathcal{K}_\mathcal{X}(y)$. Thus, $\lim_{t\to\infty} d_{\psi_t(y)}(f(\psi_t(y)), \mathbf{w}(\psi_r(y))) = \lim_{t\to\infty} d_{\psi_t(y)}(f(\psi_t(x)), \mathbf{w}(x_e)) = 0$, by Theorem 2. ■

Proof of Corollary 4: Consider the trajectory $\psi_t(z)$. Following the proof of Corollary 3, necessarily, $f(\psi_t(z)) \notin \mathcal{K}_\mathcal{X}(\psi_t(z))$ for any $t \geq 0$. For instance, by contradiction, suppose that $f(\psi_t(z)) \in \mathcal{K}_\mathcal{X}(\psi_t(z))$ for some $t \geq 0$. By definition, there exists a sequence $t_k \to \infty$ as $k \to \infty$ such that $\lim_{k\to\infty} \psi_{t+t_k}(z) = x \in \omega(\xi)$ thus $\partial\psi_{t+t_k}(z)f(z) = f(x) \notin \mathcal{K}_\mathcal{X}(x)$. By continuity, since $\mathcal{K}_\mathcal{X}(x)$ is closed, there exists k^* sufficiently large $\partial\psi_{t+t_{k^*}}(z)f(\xi) \notin \mathcal{K}_\mathcal{X}(\psi_{t+t_{k^*}}(z))$. But this contradicts differential positivity.

Suppose now that $\omega(z) \subseteq \omega(\xi)$ is not a fixed point. Then, there exists a sequence $t_k \to \infty$ as $k \to \infty$ such that $\lim_{k\to\infty} f(\psi_{t_k}(z)) = f(x) \neq 0$ for some $x \in \omega(\xi)$. Take $\delta z = f(z) + \lambda\mathbf{w}(z)$. For λ sufficiently large $\delta z \in \mathcal{K}_\mathcal{X}(z)$. Then, $\lim_{t_k\to\infty} d_{\psi_{t_k}(z)}(\partial\psi_{t_k}(z)\delta z, w(\psi_{t_k}(z))) = 0$ holds only if the evolution of δz along the flow $\partial\psi_{t_k}(z)\delta z = \partial\psi_{t_k}(z)f(z) + \lambda\partial\psi_{t_k}(z)\mathbf{w}(z) = f(x) + \lambda\partial\psi_{t_k}(z)\mathbf{w}(z)$ shows an unbounded growth of the component $\partial\psi_{t_k}(z)\mathbf{w}(z)$. ■

Proof of Corollary 5: Consider Part (i) of Theorem 4. For any $x \in \omega(\xi)$, we have $f(\psi_t(x)) = \lambda(\psi_t(x))\mathbf{w}(\psi_t(x))$. On vector spaces, for constant cone fields, closed curves cannot occur because every Perron-Frobenius curve is open. Therefore, $\lim_{t\to\infty} |\lambda(\psi_t(x))| = 0$ by boundedness of solutions. Consider Part (ii) of Theorem 4 and take any $x \in \omega(\xi)$. Either $\lim_{t\to\infty} f(\psi_t(x)) = 0$, thus $\psi_t(x)$ converges to a fixed point for $t \to \infty$, or $\liminf_{t\to\infty} |\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)} = \infty$.

This last case covers attractors which are not fixed points. We show that their basin of attraction has dimension $n-1$ at most. By contradiction, let $\mathcal{A}$ be an attractor with a basin of attraction $\mathcal{B}_\mathcal{A}$ of dimension n. By Corollary 4, from every $x \in \mathcal{B}_\mathcal{A}$, $\liminf_{t\to\infty} |\partial\psi_t(x)\mathbf{w}(x)|_{\psi_t(x)} = \infty$. Let $\gamma^\mathbf{w}(\cdot)$ be any Perron-Frobenius curve such that $\gamma^\mathbf{w}(0) = x$. Since $\mathcal{B}_\mathcal{A}$ has dimension n, there exists an interval $[\underline{s}, \overline{s}] \ni 0$ such that $\gamma^\mathbf{w}(s) \in \mathcal{B}_\mathcal{A}$ for all $s \in [\underline{s}, \overline{s}]$. Also, $\liminf_{t\to\infty} |[\partial\psi_t(x)_{x=\gamma(s)}]\dot{\gamma}^\mathbf{w}(s)|_{\psi_t(x)} = \infty$ for all $s \in [\underline{s}, \overline{s}]$, that is, $\liminf_{t\to\infty} L(\psi_t(\gamma^\mathbf{w}(\cdot))) = \infty$.

For each $t > 0$, the curve $\gamma_t(\cdot) := \psi_t(\gamma^\mathbf{w}(\cdot))$ is a reparameterization of a Perron-Frobenius curve, that is, $\frac{d}{ds}\gamma_t(s) = \lambda_t(s)w(\gamma_t(s))$ where $\lambda_t(s)$ is a scalar. Thus, $\gamma_t(\cdot)$ is an open curve for each t that grows unbounded as $t \to \infty$. It follow that, for all $s \in [\underline{s}, \overline{s}]$, the trajectory $\psi_t(\gamma^\mathbf{w}(s))$ grows unbounded, contradicting the assumption on the boundedness of the trajectories of Σ. ■

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