Differentially Private Histograms in the Shuffle Model from Fake Users

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Abstract

There has been much recent work in the shuffle model of differential privacy, particularly for approximate d-bin histograms. While these protocols achieve low error, the number of messages sent by each user—the message complexity—has so far scaled with d or the privacy parameters. The message complexity is an informative predictor of a shuffle protocol's resource consumption. We present a protocol whose message complexity is two when there are sufficiently many users. The protocol essentially pairs each row in the dataset with a fake row and performs a simple randomization on all rows. We show that the error introduced by the protocol is small, using rigorous analysis as well as experiments on real-world data. We also prove that corrupt users have a relatively low impact on our protocol's estimates.

Introduction 1

Given that statistical computations often involve data sourced from human users, an analyst could execute differentially private algorithms in the central model (also called *centrally private* algorithms). Originally defined by Dwork, McSherry, Nissim, and Smith [12], these algorithms provide quantifiable protection to data contributors at a small price in terms of accuracy. As an example, there exists an (ε, δ) -centrally private algorithm that computes d-bin histograms from n users up to maximum (ℓ_{∞}) error $O(\frac{1}{\varepsilon n} \log \frac{1}{\delta})$ [6].

We focus on computing accurate histograms since they allow approximate top-t selection, the set of t data values that occur most frequently in a population. One application is smart-phone autocomplete. Because devices are resource constrained, a keyboard offers word corrections from a smaller pool than the entire vocabulary. To obtain a list of the most common words, user devices could participate in a differentially private computation that estimates word frequencies.

Users contributing to a centrally private algorithm need to trust that the analyst correctly executes the algorithm and does not leak their data. To collect data from less trusting users, analysts can instead implement *locally private protocols*: each user applies a differentially private algorithm on their data and sends a message containing the algorithm's output to the analyst. This weaker trust assumption comes at a price: there are lower bounds that show locally private protocols have significantly more error than the centrally private counterparts. Returning to the histogram example, Bassily & Smith show $(\varepsilon, o(1/n))$ -local privacy incurs a maximum error of

 $\Omega(\frac{1}{\varepsilon}\sqrt{\frac{\log d}{n}}) \ [4].$

Originating with work by Bittau et al. and Cheu et al. [5, 10], shuffle privacy has emerged as an appealing middle-ground. Here, we assume that there is a service called the shuffler that uniformly permutes user messages. The output of the shuffler must satisfy (ε, δ) -differential privacy. Intuitively, if each user generates a locally private message, then the anonymity provided by the shuffler "amplifies" the privacy guarantees.

But a user can send *multiple* messages to the shuffler. And a users does not need to produce these messages in a differentially private manner, since we only require that the output of the

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shuffler is differentially private. This flexibility is leveraged by the histogram protocol of Balcer & Cheu [2], where each user sends d + 1 messages and the maximum error is $O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta})$ for $\delta = O(1/n)$. Alternative histogram protocols in the shuffle model have been introduced by Ghazi, Golowich, Kumar, Pagh, and Velingker [16] and by Ghazi, Kumar, Manurangsi, and Pagh [17]. As shown in Table 1, these protocols demand much fewer messages from each user than the protocol from [2].

We will use message complexity to refer to the number of messages sent by each user and communication complexity to refer to total number of bits consumed by those messages. The message complexity is necessary to have a complete picture of a protocol's resource consumption. For starters, the amount of randomness needed to perform the shuffle is a function of the message complexity but not the length of each message. Furthermore, two protocols $\mathcal{P}, \mathcal{P}'$ with the same communication complexity can incur different costs, since the physical delivery of a message over a network in a secure fashion requires overhead. If \mathcal{P} sends more messages than \mathcal{P}' , the computing cost of transmitting messages is larger for \mathcal{P} , since it needs to perform cryptographic operations on each message. Also, the bandwidth overhead is larger for \mathcal{P} , due to both encryption and physical network protocols such as TCP/IP.

In light of the above, one can ask the following question:

Are there shuffle private protocols for histograms that have low message complexity but still provide estimates that are competitive with prior work?

Given the distributed nature of local and shuffle protocols, they are impacted by users who deviate from the intended behavior. In the local privacy literature, there is research on *manipulation attacks* where corrupted users aim to skew estimates and tests by sending carefully crafted messages. One baseline attack is to simply feed wrong inputs into the protocol, but the prior work has shown that there are attacks against locally private protocols that introduce significantly worse error (see e.g. Cao, Jia, and Gong [7] and Cheu, Smith, and Ullman [11] and citations within). Here, we investigate manipulation against shuffle private protocols. Specifically,

Are there shuffle private protocols for histograms that are robust to manipulation?

1.1 Our Contributions

Our primary contribution is a shuffle private protocol for histograms that answers both questions in the affirmative. For a large range of n, the communication complexity is the same as [2] up to a logarithmic factor but the message complexity can be as small as two. For a natural use case and set of parameters, experiments also show that the new protocol is more accurate than [2]. Finally, we show that one consequence of the low message complexity is robustness to manipulation by corrupt users.

Section 3 contains the full specification and analysis, but we give an overview of the main features in the theorem below.

Theorem 1.1 (Informal). For any privacy parameters $\varepsilon = O(1)$, $\delta < 1/100$, and number of users $n = \Omega(\log d + \frac{1}{\varepsilon^2} \log \frac{1}{\delta})$, there is an (ε, δ) -differentially private shuffle protocol that approximates *d*-bin histograms with the following properties

- i. The message complexity is k + 1, where k can be set to any positive integer. Each message is d bits.
- ii. The maximum error of any bin estimate is $f(k) \cdot O(\frac{\log d}{n} + \frac{1}{\varepsilon n}\sqrt{\log d \log \frac{1}{\delta}})$ with probability 9/10,¹ where f(k) monotonically approaches 1 from above.
- iii. m corrupted users can skew an estimate by at most $\frac{m}{n} \cdot (k+1) \cdot f(k)$.

We unpack this theorem. Parts i and ii show that the protocol allows for a tradeoff between message complexity and the measurement accuracy, since increasing k reduces the scaling factor

 $^{^{1}}$ We use 9/10 as a target success probability throughout this work, but it can be changed to any other constant without affecting the asymptotic analysis.

f(k). This may not be significant for large n, but it could be useful for smaller n (e.g. the target population of a health survey can consist of much fewer subjects than the dictionary-building example). Re-scaling k by a factor of c will naturally increase the transmission cost by c but the traffic remains feasible since n is small. Thus, we can improve accuracy without altering the privacy guarantee.

Meanwhile, Part *iii* bounds the impact of any manipulation attack. Each corrupt user in our protocol can introduce bias $O(\frac{1}{n})$ whenever $k \cdot f(k) = O(1)$. For comparison, we also prove that a protocol by Ghazi et al. [16] suffers bias $\Omega(\frac{1}{n} \cdot \frac{1}{\varepsilon^2} \log \frac{1}{\delta})$ per corrupt user.

Our other results build upon this protocol. In Section 4, we describe how to exponentially reduce the protocol's communication complexity. The price is an increased message complexity and a mildly increased error. In Section 5, we simulate our protocol on text sampled from Twitter. The error introduced by our protocol to the histogram is consistent with our theoretical bounds. We also show that the top-t items in the output of the protocol are consistent with those in the raw dataset, for several choices of t. The experimental results of our protocol compare favorably to that of [2].

Appendix C presents an analysis of our main protocol in the special case where k = 0. This is done by enhancing work by Ghazi et al. [16] with the state-of-the art amplification lemma by Feldman, McMillan, and Talwar [15]. The protocol's maximum error is now proportional to $1/n^{3/4}$ instead of 1/n.

Techniques Each user in our main protocol first encodes their data as a binary string with a single 1 bit. They then flip each bit independently with some fixed probability q. Next, they create k other zero vectors and repeat this bit flipping, which corresponds to introducing k fake users with null data. We show how to choose q so that the nk messages from these fake users provide differential privacy for the actual users. The privacy amplification lemma from [15] lets us analyze the case where k = 0. The analyzer simply de-biases and adjusts the scale of the sums over messages.

Our technique to reduce communication complexity proceeds in two stages. We first make the simple observation that a binary string with known length is equivalent to a list of the indices where the string has value 1. By construction, a message generated by our local randomizer is a binary string where the number of such indices has expectation O(dq). Our choice of q is proportional to 1/n, so this alternative representation is very effective when n approaches or exceeds d.

The small n regime motivates a second round of compression. We describe an adaptation of the count-min sketching technique. Given a uniformly random hash function, we can reduce the size of the domain d to some \hat{d} at the cost of some collisions. We repeatedly hash in order to reduce the likelihood of error due to collisions and run our histogram protocol on the hashed data. We remark that Ghazi et al. [16] build a specific histogram protocol out of count-min, while we use it as a tool that can improve the communication complexity of *arbitrary* histogram protocols.

1.2 Related Work

Cheu, Smith, Ullman, Zeber, and Zhilyaev [10] rigorously define the shuffle model and give a histogram protocol that requires d messages per user. Balcer & Cheu [2] give a different protocol with the same message complexity (up to constants) but with maximum error independent of d. Because the tradeoff between error and message complexity in [2] dominates that of [10], we omit the latter from Table 1.

Ghazi et al. [16] propose multi-message shuffle protocols for histograms. These adapt the Hadamard response and Count-Min techniques from the local privacy and sketching literature. [16] also presents a single-message shuffle protocol, using the amplification lemma from Balle, Bell, Gascón, and Nissim [3]. Unlike Theorem C.1, their result does not give explicit constants and holds for a narrower range of ε, δ .

In follow-up work Ghazi et al. [17] give a protocol where the message complexity shrinks as n increases. Our protocol has the same property but at a faster rate. Specifically, our message complexity is two when n is logarithmic in d while the prior work requires n to be linear in d.

Source	Bits per message	Messages per user	Max Error (90% Confidence)
[2]	$O(\log d)$	d+1	$O(\frac{1}{\varepsilon^2 n} \log \frac{1}{\delta})$
	$O(\log d)$	$O(d^{1/100})$	$O\left(\frac{1}{\varepsilon n}\sqrt{\log d \log \frac{1}{\delta}}\right)$
[16]	$O(\log n + \log \log d)$	$O\left(\frac{\log^3 d}{\varepsilon^2}\log\frac{\log d}{\delta}\right)$	$O\left(\frac{\log^{3/2} d}{\varepsilon n} \sqrt{\log \frac{\log d}{\delta}}\right)$
	$O(\log n \log d)$	$O\left(\frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon\delta}\right)$	$O\left(\frac{\log d}{n} + \frac{1}{\varepsilon n}\sqrt{\log d \log \frac{1}{\varepsilon \delta}}\right)$
[17]	$O(\log d)$	$1 + O(\tfrac{d}{n} \cdot u(\varepsilon, \delta)) \ast $	$O(\frac{1}{\varepsilon n} \log d)$
Thm 3.4	d	2	$O\left(\frac{\log d}{n} + \frac{1}{cn}\sqrt{\log d \log \frac{1}{\lambda}}\right)$
Thm 4.1	$O(\log d(1 + \tfrac{d}{n} \cdot v(\varepsilon, \delta, d))) \ast $		$(n en \gamma = 0)$
Thm 4.3	$O\left(\frac{1}{\varepsilon^2}\log d\log^3\frac{\log d}{\delta}\right)$ *	$O(\log d)$	$O\left(\frac{1}{\varepsilon n}\sqrt{\log d}\log^{3/2}\left(\frac{\log d}{\delta}\right)\right)$
Thm C.1,	d	1	$O\left(\frac{\log d}{2} + \frac{\sqrt{\log d}}{2} \left(\log \frac{1}{5}\right)^{\frac{1}{4}}\right)$

Table 1: Summary of shuffle protocols for histograms. To simplify presentation, we assume $\varepsilon = O(1), \ \delta = O(1/n), \ \text{and} \ n = \Omega(\frac{\log d}{\varepsilon^2} \log \frac{1}{\delta} \log \frac{\log d}{\delta})$. We also use $u(\varepsilon, \delta)$ as shorthand for $\frac{\log^2(1/\delta)}{\varepsilon^2}$ and $v(\varepsilon, \delta, d)$ for $\log d + \frac{\log(1/\delta)}{\varepsilon^2}$. * indicates bounds on expected values.

Manipulation attacks have previously been studied in the context of local privacy. Ambainis, Jakobsson, and Lipmaa [1] as well as Moran and Naor [19] study the vulnerability of randomized response to these attacks. Work by Cao, Jia, and Gong [7] also consider attacks against histogram and heavy hitter protocols. Cheu, Smith, and Ullman [11] show that powerful attacks are inevitable for any locally private protocol. In particular, these attacks are stronger when the privacy guarantee is stronger or the data dimension is larger.

2 Preliminaries

2.1 Differential Privacy

We define a dataset $\vec{x} \in \mathcal{X}^n$ to be an ordered tuple of n rows where each row is drawn from a data universe \mathcal{X} and corresponds to the data of one user. Two datasets $\vec{x}, \vec{x}' \in \mathcal{X}^n$ are considered *neighbors* (denoted as $\vec{x} \sim \vec{x}'$) if they differ in at most one row.

Definition 2.1 (Differential Privacy [12]). An algorithm $\mathcal{M} : \mathcal{X}^n \to \mathcal{Z}$ satisfies (ε, δ) -differential privacy if, for every pair of neighboring datasets \vec{x} and \vec{x}' and every subset $Z \subset \mathcal{Z}$,

$$\mathbb{P}[\mathcal{M}(\vec{x}) \in Z] \le e^{\varepsilon} \cdot \mathbb{P}[\mathcal{M}(\vec{x}') \in Z] + \delta.$$
(1)

We remark that an algorithm can be well-defined for a superset of the intended data universe $\overline{\mathcal{X}} \supset \mathcal{X}$ but (1) may not hold for every $\vec{x} \sim \vec{x}' \in \overline{\mathcal{X}}$; in these cases, we will disambiguate by saying it satisfies differential privacy for inputs from \mathcal{X} .

Because this definition assumes that the algorithm \mathcal{M} has "central" access to compute on the entire raw dataset, we sometimes call this *central* differential privacy. Two properties about differentially private algorithms will be useful. First, privacy is preserved under post-processing.

Fact 2.2. For (ε, δ) -differentially private algorithm $\mathcal{M} : \mathcal{X}^n \to \mathcal{Z}$ and randomized algorithm $f : \mathcal{Z} \to \mathcal{Z}', f \circ \mathcal{M}$ is (ε, δ) -differentially private.

This means that any computation based solely on the output of a differentially private function does not affect the privacy guarantee. Refer to Prop. 2.1 in the text by Dwork and Roth [13] for a proof. The second property is closure under composition.

Fact 2.3. For $(\varepsilon_1, \delta_1)$ -differentially private \mathcal{M}_1 and $(\varepsilon_2, \delta_2)$ -differentially private \mathcal{M}_2 , \mathcal{M}_3 defined by $\mathcal{M}_3(\vec{x}) = (\mathcal{M}_1(\vec{x}), \mathcal{M}_2(\vec{x}))$ is $(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$ -differentially private.

Fact 2.4. For (ε, δ) -differentially private algorithms $\mathcal{M}_1, \ldots, \mathcal{M}_d$, the algorithm $\hat{\mathcal{M}}_3$ defined by $\hat{\mathcal{M}}_3(\vec{x}) = (\mathcal{M}_1(\vec{x}), \ldots, \mathcal{M}_d(\vec{x}))$ is $\left(\varepsilon(e^{\varepsilon} - 1) \cdot d + \varepsilon \cdot \sqrt{2d \log \frac{1}{d\delta}}, 2d\delta\right)$ -differentially private.

Refer to Theorems 3.14 and 3.20 in [13] for proofs.

2.2 Local Model

In an extreme case, no user trusts any other party with protecting their data; here, we model the dataset as a distributed object where each of n users holds a single row. Each user i provides their data point as input to a randomizing function \mathcal{R} and publishes the outputs for some analyzer to compute on.

Definition 2.5 (Local Model [20, 14]). A protocol \mathcal{P} in the *local model* consists of two randomized algorithms:

- A randomizer $\mathcal{R}: \mathcal{X} \to \mathcal{Y}$ mapping data to a message.
- An analyzer $\mathcal{A}: \mathcal{Y}^n \to \mathcal{Z}$ that computes on a vector of messages.

We define its execution on input $\vec{x} \in \mathcal{X}^n$ as

$$\mathcal{P}(\vec{x}) := \mathcal{A}(\mathcal{R}(x_1), \dots, \mathcal{R}(x_n)).$$

We assume that \mathcal{R} and \mathcal{A} have access to an arbitrary amount of public randomness.

Definition 2.6 (Local Differential Privacy [12, 18]). A local protocol $\mathcal{P} = (\mathcal{R}, \mathcal{A})$ is (ε, δ) differentially private if \mathcal{R} is (ε, δ) -differentially private. The privacy guarantee is over the internal randomness of the users' randomizers and not the public randomness of the protocol.

For brevity, we typically call these protocols "locally private."

2.3 Shuffle Model

We focus on differentially private protocols in the shuffle model, which we define below.

Definition 2.7 (Shuffle Model [5, 10]). A protocol \mathcal{P} in the *shuffle model* consists of three randomized algorithms:

- A randomizer $\mathcal{R}: \mathcal{X} \to \mathcal{Y}^*$ mapping a datum to a vector of messages.
- A shuffler $S: \mathcal{Y}^* \to \mathcal{Y}^*$ that applies a uniformly random permutation to the messages in its input.
- An analyzer $\mathcal{A}: \mathcal{Y}^* \to \mathcal{Z}$ that computes on a permutation of messages.

As S is the same in every protocol, we identify each shuffle protocol by $\mathcal{P} = (\mathcal{R}, \mathcal{A})$. We define its execution by n users on input $\vec{x} \in \mathcal{X}^n$ as

$$\mathcal{P}(\vec{x}) := \mathcal{A}(\mathcal{S}(R(x_1), \dots, R(x_n))).$$

Importantly, we allow \mathcal{R} and \mathcal{A} to have parameters that depend on n.

The following is a definition of differential privacy in this model.

Definition 2.8 (Shuffle Differential Privacy [10]). A protocol $\mathcal{P} = (\mathcal{R}, \mathcal{A})$ is (ε, δ) -shuffle differentially private for n users if the algorithm $(\mathcal{S} \circ \mathcal{R}^n)(\vec{x}) := \mathcal{S}(\mathcal{R}(x_1), \ldots, \mathcal{R}(x_n))$ is (ε, δ) -differentially private. The privacy guarantee is over the internal randomness of the users' randomizers and not the public randomness of the shuffle protocol.

For brevity, we typically call these protocols "shuffle private."

2.4 Notation for Histogram and Top-t Selection Problems

We assume each user *i* has some private value belonging to the finite set [d] but encodes them as "one-hot" binary strings. That is, for any $j \in [d]$, let $e_{j,d}$ be the binary string of length *d* with zeroes in all entries except for coordinate *j*; user *i* has data $x_i = e_{j,d}$ for some *j*. Let \mathcal{X}_d denote the set $\{e_{1,d}, \ldots, e_{d,d}\}$ and let 0^d denote the binary string of all zeroes.

For any $j \in [d]$, let $\operatorname{hist}_j(\vec{x})$ be the function that takes the vector of one-hot values $\vec{x} \in \{e_{1,d}, \ldots, e_{d,d}\}^n$ and reports $\frac{1}{n} \sum_{i=1}^n x_{i,j}$, which is the frequency of $e_{j,d}$ in \vec{x} . Let $\operatorname{hist}(\vec{x})$ be shorthand for the vector $(\operatorname{hist}_1(\vec{x}), \ldots, \operatorname{hist}_d(\vec{x}))$.

We will use ℓ_{∞} error to quantify how well a vector $\vec{z} \in \mathbb{R}^d$ estimates the histogram hist (\vec{x}) . Specifically, $\|\vec{z} - \text{hist}(\vec{x})\|_{\infty} := \max_j |z_j - \text{hist}_j(\vec{x})|$.

Having defined histograms, we move on to defining the top-t items. For any vector $\vec{h} \in \mathbb{R}^d$ and value $j \in [d]$, let $\operatorname{rank}_j(\vec{h})$ be the relative magnitude of h_j : the index of h_j after sorting \vec{h} in descending order. For any $t \in [d]$, let $\operatorname{top}_t(\vec{h})$ denote the set of j such that $\operatorname{rank}_j(\vec{h}) \leq t$.

We now establish notation to quantify how well a set approximates the top-t items. Let $\text{hist}_{[t]}(\vec{x})$ denote the frequency of the t-th largest item: the quantity $\text{hist}_i(\vec{x})$ where $\text{rank}_i(\text{hist}(\vec{x})) = t$.

Definition 2.9. For any $\vec{x} \in \mathcal{X}_d^n$, a set of candidates $C \subset [d] \alpha$ -approximates the top-*t* items in \vec{x} if |C| = t and $\text{hist}_j(\vec{x}) > \text{hist}_{[t]}(\vec{x}) - \alpha$ for all $j \in C$.

Other metrics include precision p (the fraction of items in candidate set C that are actually in the top t) and recall r (the fraction of items in the top t that are in C). Note that when |C| = t, p = r so that the F1 score—the quantity $2 \cdot \frac{p \cdot r}{p+r}$ —is exactly p = r.

3 Our Histogram Protocol

A user who executes our protocol's local randomizer $\mathcal{R}_{\text{FLIP}}$ (Algorithm 1) reports k+1 messages. They make their first message by running $\mathcal{R}_{d,q}$ (Algorithm 2) on their one-hot string. An instance of randomized response, $\mathcal{R}_{d,q}$ flips each bit of with probability q. The user makes the k other messages by running $\mathcal{R}_{d,q}$ k times on the string 0^d , with fresh randomness in each execution. This effectively inserts k fake users into the protocol. We will show that the messages from these fake users are sufficiently noisy for differential privacy.

Stacking the nk + n messages results in a $(nk + n) \times d$ binary matrix; to estimate the frequency of j, our analyzer $\mathcal{A}_{\text{FLIP}}$ (Algorithm 3) simply de-biases and re-scales the sum of the j-th column.

Algorithm 1: \mathcal{R}_{FLIP} , a randomizer for histograms		
Input: $x \in \mathcal{X}_d$; implicit parameters d, k, q		
Output: $\vec{y} \in (\{0,1\}^d)^{k+1}$		
Initialize \vec{y} as an empty message vector.		
Append message generated by $\mathcal{R}_{d,q}(x)$ to \vec{y}		
$\mathbf{For} j\in [k]$		
Append message generated by $\mathcal{R}_{d,q}(0^d)$ to \vec{y}		
${f Return}~ec y$		

Our analysis of the protocol will be built upon two technical claims. The first gives a bound on the size of any confidence interval in terms of parameters q, k.

Claim 3.1. Fix any $n \in \mathbb{N}$ and $\beta \in (0,1)$. If $\frac{1}{nk+n} \ln \frac{2}{\beta} \leq q < 1/2$, then the protocol $\mathcal{P}_{FLIP} = (\mathcal{R}_{FLIP}, \mathcal{A}_{FLIP})$ reports approximate histograms with error behaving as follows:

$$\forall \vec{x} \in \mathcal{X}_d^n, \ j \in [d] \ \mathbb{P}\left[|z_j - \text{hist}_j(\vec{x})| > 2\sqrt{\frac{k+1}{n}q(1-q)\ln\frac{2}{\beta}} \cdot \left(\frac{1}{1-2q}\right) \right] \le \beta$$

We will prove this claim in Section 3.1. The second claim is a sufficient condition on q, k for (ε, δ) shuffle privacy.

Algorithm 2: $\mathcal{R}_{d,q}$, applies randomized response to a binary string

Input: $x \in \{0, 1\}^d$ Output: $y \in \{0, 1\}^d$ For $j \in [d]$ If $flip_j \sim Ber(q)$ If $flip_j = 1$: $\downarrow y_j \leftarrow 1 - x_j$ Else $\downarrow y_j \leftarrow x_j$ Return y

Algorithm 3: \mathcal{A}_{FLIP} , an analyzer for histograms

Input: $\vec{y} \in (\{0,1\}^d)^{nk+n}$; implicit parameters d, k, q **Output:** $\vec{z} \in \mathbb{R}^d$ **For** $j \in [d]$ $\lfloor z_j \leftarrow \frac{1}{n} \sum_{i=1}^{nk+n} \frac{1}{1-2q} \cdot (y_{i,j} - q)$ **Return** $\vec{z} \leftarrow (z_1, \dots, z_d)$

Claim 3.2. Fix any $\varepsilon > 0$, $\delta < 1/100$, and $n \in \mathbb{N}$. If parameters q < 1/2 and $k \in \mathbb{N}$ are chosen such that $q(1-q) \geq \frac{33}{5nk} (\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1})^2 \ln \frac{4}{\delta}$, then $\mathcal{P}_{FLIP} = (\mathcal{R}_{FLIP}, \mathcal{A}_{FLIP})$ is (ε, δ) -shuffle private.

We will prove this claim in Section 3.2. Combining the two claims yields the following confidence interval for the error of any single frequency estimate.

Theorem 3.3. Fix any $\varepsilon > 0$, $\delta < 1/100$, and $n \in \mathbb{N}$. For any choice of parameter $k > \frac{132}{5n} (\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1})^2 \ln \frac{4}{\delta}$, there is a choice of parameter q < 1/2 such that the protocol $\mathcal{P}_{FLIP} = (\mathcal{R}_{FLIP}, \mathcal{A}_{FLIP})$ has the following properties

- a. \mathcal{P}_{FLIP} is (ε, δ) -shuffle private for inputs from \mathcal{X}_d .
- b. For any $j \in [d]$ and $\vec{x} \in \mathcal{X}_d^n$, $\mathcal{P}_{FLIP}(\vec{x})$ reports frequency estimate z_j such that

$$|z_j - \operatorname{hist}_j(\vec{x})| < \frac{1}{n} \cdot \frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \cdot \sqrt{\frac{264}{5} \ln \frac{4}{\delta} \ln 20} \cdot g(k)$$

with probability 9/10, where g(k) monotonically approaches 1 from above. Refer to Figure 1.

Proof. Because k is sufficiently large, there is a solution \hat{q} to the quadratic equation $q(1-q) = \frac{33}{5nk} (\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1})^2 \ln \frac{4}{\delta}$ that lies in the interval (0, 1/2). Also, let $\tilde{q} \leftarrow \frac{1}{nk+n} \ln \frac{2}{\beta} < 1/2$.



Figure 1: Effect of bandwidth parameter k on scaling term g(k) and error of a single estimate.



Figure 2: Effect of bandwidth parameter k on scaling term f(k) and maximum error.

When we set $q \leftarrow \max(\hat{q}, \tilde{q})$, Part *a* follows immediately from Claim 3.2 and the error $|z_j - \text{hist}_j(\vec{x})|$ is at most

$$\max\left(\frac{1}{n} \cdot \frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \cdot \sqrt{\frac{264}{5} \ln \frac{4}{\delta} \ln 20}, \ \frac{2}{n} \cdot \ln 20\right) \cdot \frac{1}{1 - 2q}$$

with probability 9/10 via Claim 3.1 and the bound $(k+1)/k \leq 2$. Note that both forms of q approach zero as k increases, so 1/(1-2q) is a monotonically decreasing function of k as desired. Finally, the term $\frac{2}{n} \cdot \ln 20$ must be the smaller of the two due to our bound on δ .

We now iterate on our analysis to derive a bound on the maximum error. Parts i and ii in Theorem 1.1 are immediate corollaries.

Theorem 3.4. Fix any $\varepsilon > 0$, $\delta < 1/100$, and $n \in \mathbb{N}$. For any choice of parameter $k > \max\left(\frac{132}{5n}\left(\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1}\right)^2\ln\frac{4}{\delta}, \frac{2}{n}\ln 20d-1\right)$, there is a choice of parameter q < 1/2 such that the protocol $\mathcal{P}_{FLIP} = (\mathcal{R}_{FLIP}, \mathcal{A}_{FLIP})$ has the following properties

- a. \mathcal{P}_{FLIP} is (ε, δ) -shuffle private for inputs from \mathcal{X}_d .
- b. For any $\vec{x} \in \mathcal{X}_d^n$, $\mathcal{P}_{FLIP}(\vec{x})$ reports approximate histogram \vec{z} such that the maximum error is

$$\|\vec{z} - \operatorname{hist}(\vec{x})\|_{\infty} < \max\left(\frac{1}{n} \cdot \frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \cdot \sqrt{\frac{264}{5} \ln \frac{4}{\delta} \ln 20d}, \ \frac{2}{n} \cdot \ln 20d\right) \cdot f(k)$$

with probability 9/10, where f(k) monotonically approaches 1 from above. Refer to Figure 2.

Proof. The following is immediate from setting $\beta = 1/10d$ in Claim 3.1 and a union bound:

Corollary 3.5. Fix any $n \in \mathbb{N}$. If $\frac{1}{nk+n} \ln 20d \leq q < 1/2$, then \mathcal{P}_{FLIP} reports a histogram with maximum error behaving as follows:

$$\forall \vec{x} \in \mathcal{X}_d^n \mathbb{P}\left[\left\|\mathcal{P}_{FLIP}(\vec{x}) - \operatorname{hist}(\vec{x})\right\|_{\infty} > 2\sqrt{\frac{k+1}{n}q(1-q)\ln 20d} \cdot \left(\frac{1}{1-2q}\right)\right] \le 1/10$$

Because k is sufficiently large, there is a solution \hat{q} to $q(1-q) = \frac{33}{5nk} (\frac{e^{\epsilon}+1}{e^{\epsilon}-1})^2 \ln \frac{4}{\delta}$ such that $\hat{q} \in (0, 1/2)$. Similar to before, we will choose q to be the maximum of \hat{q} and $\tilde{q} \leftarrow \frac{1}{nk+n} \ln 20d$. The theorem follows from Claim 3.2 and Corollary 3.5 (where 1/1-2q is again the desired monotonically decreasing function of k)

We will prove Claims 3.1 and 3.2 in the following two subsections. In Subsection 3.3, we will bound the impact of corrupt users (Part *iii* of Theorem 1.1).

3.1 Accuracy of \mathcal{P}_{FLIP}

In this section, we show how to obtain confidence intervals of the per-bin error of \mathcal{P}_{FLIP} (Claim 3.1). To prove the claim, we will first analyze the bias and variance of each estimate.

Claim 3.6. Fix any q < 1/2 and $n, k \in \mathbb{N}$. For any index $j \in [d]$ and data $\vec{x} \in \mathcal{X}_d^n$, the protocol $\mathcal{P}_{FLIP} = (\mathcal{R}_{FLIP}, \mathcal{A}_{FLIP})$ reports an unbiased estimate of $\operatorname{hist}_j(\vec{x})$ with variance $\frac{k+1}{n} \cdot q(1-q) \cdot (\frac{1}{1-2q})^2$.

Proof. For $n < i \le nk + n$, we take $x_i = 0^d$. This will correspond to the empty data vector of a fabricated user. Recall that $z_j = \frac{1}{n} \sum_{i=1}^{nk+n} \frac{1}{1-2q} \cdot (y_{i,j} - q)$ is the protocol's estimate of $\operatorname{hist}_j(\vec{x}) = \frac{1}{n} \sum_{i=1}^n x_{i,j}$. We will first show each term $\frac{1}{1-2q} \cdot (y_{i,j} - q)$ is an unbiased estimate of the bit $x_{i,j}$.²

$$\mathbb{E}\left[\frac{1}{1-2q} \cdot (y_{i,j}-q)\right] = \frac{1}{1-2q} \cdot (\mathbb{E}[y_{i,j}]-q)$$

= $\frac{1}{1-2q} \cdot (((1-x_{i,j}) \cdot q + x_{i,j} \cdot (1-q)) - q)$ (see $\mathcal{R}_{d,q}$)
= $\frac{1}{1-2q} \cdot (q - x_{i,j}q + x_{i,j} - x_{i,j}q - q)$
= $x_{i,j}$

Next, we derive the variance of the term:

$$\operatorname{Var}\left[\frac{1}{1-2q} \cdot (y_{i,j}-q)\right] = \left(\frac{1}{1-2q}\right)^2 \cdot \operatorname{Var}[y_{i,j}]$$
$$= \left(\frac{1}{1-2q}\right)^2 \cdot q(1-q)$$

The second equality comes from the fact that $y_{i,j}$ is drawn from either $\mathbf{Ber}(q)$ or $\mathbf{Ber}(1-q)$, which have the same variance.

Because z_j is the summation over nk + n terms (normalized by $\frac{1}{n}$), the variance of z_j is $\frac{k+1}{n} \cdot q(1-q) \cdot (\frac{1}{1-2q})^2$ by independence. Finally, $\mathbb{E}[z_j] = \text{hist}_j(\vec{x})$ due to linearity of expectation and the fact that we normalize by n.

To arrive at Claim 3.1, we will show that the protocol's estimates are sums of bounded random variables. This will allow us to deploy a concentration inequality.

Proof of Claim 3.1. We first expand the random variable in question as

$$\frac{1}{n}\sum_{i=1}^{nk+n}\frac{1}{1-2q}\cdot(y_{i,j}-q)-x_{i,j}$$
(2)

where we again use $x_{i,j}$ (resp. $y_{i,j}$) to denote the *j*-th bit in the data (resp. message) sent by user *i*. When i > n, *i* corresponds to the index of a fabricated user; in this case, $x_{i,j} = 0$.

In the proof of Claim 3.6, we saw that each term in $\sum_{i=1}^{nk+n} \frac{1}{1-2q} \cdot (y_{i,j}-q)$ is an independent random variable with mean $x_{i,j}$ and variance $\frac{1}{(1-2q)^2} \cdot q \cdot (1-q)$. Naturally, this means each term in (2) is an independent random variable with mean zero and variance $\frac{1}{(1-2q)^2} \cdot q \cdot (1-q)$.

We now add the observation that each term in (2) has maximum magnitude $m = \frac{1-q}{1-2q}$. This follows from the fact that $x_{i,j}, y_{i,j} \in \{0,1\}$ and q < 1/2. We show that the variance of the

²Technically, shuffling means that the *i*-th message does not correspond to the *i*-th user. But summation is symmetric so we simply avoid inserting permutation notation for neatness.

summation is at least $m^2 \ln \frac{2}{\beta}$:

$$\operatorname{Var}\left[\sum_{i=1}^{nk+n} \frac{1}{1-2q} \cdot (y_{i,j}-q)\right] = \frac{nk+n}{(1-2q)^2} \cdot q \cdot (1-q)$$
$$= \left(\frac{1-q}{1-2q}\right)^2 \cdot \frac{q}{1-q} \cdot (nk+n)$$
$$\ge \left(\frac{1-q}{1-2q}\right)^2 \ln \frac{2}{\beta}$$
$$= m^2 \ln \frac{2}{\beta}$$

Because we have lower bounded the variance of the sum of bounded independent variables, the claim follows from an additive Chernoff bound. $\hfill \Box$

3.2 Privacy of \mathcal{P}_{FLIP}

In this section, we derive the range of q, k for which (ε, δ) -privacy will hold (Claim 3.2). The proof will proceed as follows: design a series of algorithms $\mathcal{M}_1, \mathcal{M}_2, \ldots$ such that $\mathcal{P}_{\mathsf{FLIP}}$ is private whenever \mathcal{M}_1 is private, \mathcal{M}_1 is private whenever \mathcal{M}_2 is private, and so on. Then we study the privacy of the final algorithm.

3.2.1 Step One

We first consider $C_{m,d,q}$ (Algorithm 4). It takes one user's data as input, constructs m copies of 0^d and executes the randomization algorithm $\mathcal{R}_{d,q}$ on all m + 1 strings. When m = nk, this algorithm simulates the set of messages produced by any single user and the fabricated users in our protocol \mathcal{P}_{FLIP} .

Algorithm 4: $C_{m,d,q}$		
Input: $x \in \{0^d\} \cup \mathcal{X}_d$		
Output: $\vec{y} \in (\{0,1\}^d)^{m+1}$		
Construct $\vec{x} \leftarrow (x, \underbrace{0^d, \dots, 0^d})$		
m copies		
$\textbf{Return } \vec{y} \leftarrow (\mathcal{S} \circ \mathcal{R}_{d,q}^{m+1})(\vec{x})$		

We claim that privacy of our protocol follows from privacy of this new algorithm.

Claim 3.7. If $C_{nk,d,q}$ is (ε, δ) -differentially private for inputs from \mathcal{X}_d , then $\mathcal{P}_{FLIP} = (\mathcal{R}_{FLIP}, \mathcal{A}_{FLIP})$ is (ε, δ) -shuffle private for inputs from \mathcal{X}_d .

Proof. In an execution of $(\mathcal{S} \circ \mathcal{R}^n_{\text{FLIP}})(\vec{x})$, $\mathcal{R}_{d,q}$ gets run on the values $x_1, \ldots, x_n, x_{n+1}, \ldots, x_{nk+n} - where <math>x_{n+1} = 0^d, \ldots, x_{nk+n} = 0^d$ and all nk + n messages are shuffled together. For any user i, we can decompose it into two stages: (1) run $\mathcal{R}_{d,q}$ on the values $x_i, x_{n+1} = 0^d, \ldots, x_{nk+n} = 0^d$ and shuffle the output then (2) run $\mathcal{R}_{d,q}$ on the values $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ and shuffle all nk + n messages. The first stage is precisely $\mathcal{C}_{nk,d,q}$ and the second is a post-processing of its output. Thus, privacy follows from post-processing (Fact 2.2).

3.2.2 Step Two

In this step, we argue that we only need to concern ourselves with the d = 2 case. Consider any $j, j' \in [d]$ where j < j'. Changing user data from $e_{j,d}$ to $e_{j',d}$ only affects the one-hot strings in positions j and j'. Because $\mathcal{C}_{m,d,q}$ operates by performing independent bit-flipping on the one-hot strings (via $\mathcal{R}_{d,q}$), it can essentially be decomposed into two phases: bit-flip positions j, j'(which depend on the user data) and then bit-flip on the rest of the bits (a post-processing that is independent of the user's data). We make this decomposition explicit in Algorithm 5.

Algorithm 5: $C_{m,d,q,j,j'}$
Input: $x \in \{e_{j,d}, e_{j',d}\}$
Output: $\vec{y} \in (\{0,1\}^d)^*$
If $x = e_{j,d}$:
$\perp u \leftarrow 10$
Else
$\vec{v} \leftarrow \mathcal{C}_{m,2,q}(u)$
$\vec{w} \leftarrow \mathcal{C}_{m,d-2,q}(0^{d-2})$
$\vec{y} \leftarrow \text{empty list}$
For $i \in [m+1]$
$y_i \leftarrow (\dots, w_{i,j-1}, v_{i,1}, w_{i,j+1}, \dots, w_{i,j'-1}, v_{i,2}, w_{i,j'+1}, \dots)$
$\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ $
Return \vec{y}

Claim 3.8. If, for every $j, j' \in [d]$ where j < j', $C_{m,d,q,j,j'}$ is (ε, δ) -differentially private for inputs from $\{e_{j,d}, e_{j',d}\}$, then $C_{m,d,q}$ is (ε, δ) -differentially private for inputs from \mathcal{X}_d .

Proof. To prove $C_{m,d,q}$ is (ε, δ) -differentially private for \mathcal{X}_d , it suffices to show the inequalities below are true for every j, j', Y:

$$\mathbb{P}[\mathcal{C}_{m,d,q}(e_{j,d}) \in Y] \le e^{\varepsilon} \cdot \mathbb{P}[\mathcal{C}_{m,d,q}(e_{j',d}) \in Y] + \delta$$
(3)

$$\mathbb{P}[\mathcal{C}_{m,d,q}(e_{j',d}) \in Y] \le e^{\varepsilon} \cdot \mathbb{P}[\mathcal{C}_{m,d,q}(e_{j,d}) \in Y] + \delta \tag{4}$$

If $\mathcal{C}_{m,d,q,j,j'}$ is (ε, δ) -differentially private, we have that

$$\mathbb{P}[\mathcal{C}_{m,d,q,j,j'}(e_{j,d}) \in Y] \le e^{\varepsilon} \cdot \mathbb{P}[\mathcal{C}_{m,d,q,j,j'}(e_{j',d}) \in Y] + \delta$$
(5)

$$\mathbb{P}[\mathcal{C}_{m,d,q,j,j'}(e_{j',d}) \in Y] \le e^{\varepsilon} \cdot \mathbb{P}[\mathcal{C}_{m,d,q,j,j'}(e_{j,d}) \in Y] + \delta$$
(6)

In the remainder of the proof, we will argue that $C_{m,d,q,j,j'}(e_{j,d})$ has the same distribution as $C_{m,d,q}(e_{j,d})$; a completely symmetric argument holds for the equivalence between $C_{m,d,q,j,j'}(e_{j',d})$ and $C_{m,d,q}(e_{j',d})$. Inequalities (3) and (4) will therefore hold by substitution.

Pick any $k \notin \{j, j'\}$ and any message index $i \in [m+1]$. Notice that when we obtain \vec{y} from $\mathcal{C}_{m,d,q}(e_{j,d}), y_{i,k}$ is an independent bit that has value 1 with probability q (since bit k has to flip from 0 to 1). But this is exactly the same distribution as in $\mathcal{C}_{m,d,q,j,j'}(e_{j,d})$.

Now consider $(y_{1,j}, \ldots, y_{m+1,j}, y_{1,j'}, \ldots, y_{m+1,j'})$ when obtained from $\mathcal{C}_{m,d,q}(e_{j,d})$. By construction, we know that there exists one uniformly random index i such that $y_{i,j} \sim \operatorname{Ber}(1-q), y_{i,j'} \sim \operatorname{Ber}(q)$ and, for every other index $\hat{i}, y_{\hat{i},j}$ and $y_{\hat{i},j'}$ are independent samples from $\operatorname{Ber}(q)$. But again this is the same as $\mathcal{C}_{m,d,q,j,j'}(e_{j,d})$.

Claim 3.9. If $C_{m,2,q}$ is (ε, δ) -differentially private for inputs from \mathcal{X}_2 , then $C_{m,d,q,j,j'}$ is (ε, δ) -differentially private for inputs from $\{e_{j,d}, e_{j',d}\}$.

Proof. The claim is immediate from the fact that $C_{m,d,q,j,j'}$ is executing $C_{m,2,q}$ on a value that is obtained from the user input and then post-processing the algorithm's output.

3.2.3 Step Three

In this section, we reduce the privacy of $\mathcal{C}_{m,2,q}$ to that of $\mathcal{B}_{m,q}$ (Algorithm 7). This algorithm generates a vector of four randomized integers via $\mathcal{M}(m,q)$ (Algorithm 6). Then it computes a binary string $j \leftarrow \mathcal{R}_{2,q}(x)$ where x is the input to $\mathcal{B}_{m,q}$. Finally it increments the integer at the position encoded by j.

We design $\mathcal{M}(m,q)$ to generate the histogram of the messages produced by m fabricated users. This means $\mathcal{B}_{m,q}(x)$ is sufficient to simulate $\mathcal{C}_{m,2,q}(x)$. In turn, it suffices to prove that $\mathcal{B}_{m,q}$ is private. Then we argue that, whenever m, q lie in a particular range, the noise produced by $\mathcal{M}(m,q)$ is enough to ensure $\mathcal{B}_{m,q}$ satisfies (ε, δ) -differential privacy.

Algorithm 6: \mathcal{M} , a multinomial noise generator

Algorithm 7: $\mathcal{B}_{m,q}$ Input: $x \in \{0,1\}^2$ Output: $\vec{y} \in \mathbb{Z}^4_{\geq 0}$ Initialize $\vec{y} = (y_1, y_2, y_3, y_4)$ with noise from $\mathcal{M}(m,q)$ Sample $j \sim \mathcal{R}_{2,q}(x)$ $y_{j+1} \leftarrow y_{j+1} + 1$ /* binary string j maps to a number between 0 and 3 */ Return \vec{y}

Claim 3.10. If $\mathcal{B}_{m,q}$ is (ε, δ) -differentially private for inputs from \mathcal{X}_2 , then $\mathcal{C}_{m,2,q}$ is (ε, δ) -differentially private for inputs from \mathcal{X}_2 .

Proof. Consider the post-processing algorithm which takes \vec{y} produced by $\mathcal{B}_{m,q}(x)$ and generates a uniformly random vector $\vec{w} \in \mathcal{X}_2^{m+1}$ such that y_j describes the frequency of the binary string corresponding to j in \vec{w} . This is exactly the distribution of $\mathcal{C}_{m,2,q}$ so privacy follows from post-processing.

Claim 3.11. Fix any $\varepsilon > 0$ and $\delta < 1/100$. If q < 1/2 and $mq(1-q) \ge \frac{33}{5} \left(\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1}\right)^2 \ln(4/\delta)$, then $\mathcal{B}_{m,q}$ is (ε, δ) -differentially private for inputs from \mathcal{X}_2 .

Our proof makes formal the following steps. Recall that the algorithm encodes the user's value x via the randomized algorithm $\mathcal{R}_{2,q}$. Changing x from 01 to 10 will affect the probability mass function (PMF) of this encoding, but we note that the PMF only changes at two elements of the support, 01 and 10 (see Table 2). This means we need only focus on how the noise produced by $\mathcal{M}(m,q)$ behaves on those elements.

We essentially argue that a noise vector $\vec{f} = (f_1, \ldots, f_4)$ produced by $\mathcal{M}(m, q)$ has properties that are in line with the binomial and Gaussian distributions: we show that a sample from $\mathcal{M}(m, q)$ is very likely to be in a set F and any outcome in F has the property that its probability is within e^{ε} of a "neighboring" outcome's probability. We formalize this in the two claims below, proven in Appendix A.

Claim 3.12. Fix $m \in \mathbb{N}$ and $q, \delta \in (0, 1)$. Define

$$\Delta := \sqrt{3mq(1-q)\ln\frac{4}{\delta}} \cdot \frac{q(1-q)}{1-q(1-q)}$$
$$U := mq(1-q) + \Delta + \sqrt{3(mq(1-q)+\Delta)\ln\frac{4}{\delta}}$$
$$L := mq(1-q) - \Delta - \sqrt{3(mq(1-q)+\Delta)\ln\frac{4}{\delta}}$$

Let $F \subset \mathbb{Z}^4$ denote the set of vectors where $\vec{t} \in F$ if and only if $t_2, t_3 \in [L, U]$. If $mq(1-q) > \frac{9}{2}\ln(4/\delta)$, then

$$\mathbb{P}_{\vec{f} \sim \mathcal{M}(m,q)} \left[\vec{f} \notin F \right] \le \delta \tag{7}$$

	$\mathcal{R}_{2,q}(10)$	$\mathcal{R}_{2,q}(01)$	$\mathcal{R}_{2,q}(00)$
0=00	q(1-q)	q(1-q)	$(1-q)^2$
1=01	q^2	$(1-q)^2$	q(1-q)
2=10	$(1-q)^2$	q^2	q(1-q)
3=11	q(1-q)	q(1-q)	q^2

Table 2: Probability mass functions of three distributions $\mathcal{R}_{2,q}(10), \mathcal{R}_{2,q}(01), \mathcal{R}_{2,q}(00)$

Claim 3.13. Fix any $\varepsilon > 0$ and $\delta < 1/100$. Define F as in Claim 3.12. If $mq(1-q) \geq \frac{33}{5} \left(\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1}\right)^2 \ln(4/\delta)$, then for any $\vec{y} = (y_1, \ldots, y_4)$,

$$\mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4), \vec{f} \in F \right] \leq e^{\varepsilon} \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3 - 1, y_4) \right] \\
\mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3 - 1, y_4), \vec{f} \in F \right] \leq e^{\varepsilon} \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4) \right]$$

The rest of this section is dedicated to proving Claim 3.11.

Proof of Claim 3.11. For any $Y \subset \mathbb{Z}^4$, we will prove

$$\mathbb{P}[\mathcal{B}_{m,q}(01) \in Y] \le e^{\varepsilon} \cdot \mathbb{P}[\mathcal{B}_{m,q}(10) \in Y] + \delta.$$

The inequality $\mathbb{P}[\mathcal{B}_{m,q}(10) \in Y] \leq e^{\varepsilon} \cdot \mathbb{P}[\mathcal{B}_{m,q}(01) \in Y] + \delta$ will hold by completely symmetric arguments.

We begin by using Claim 3.12 to rewrite $\mathbb{P}[\mathcal{B}_{m,q}(01) \in Y]$:

$$\mathbb{P}[\mathcal{B}_{m,q}(01) \in Y] = \sum_{\vec{y} \in Y} \mathbb{P}[\mathcal{B}_{m,q}(01) = \vec{y}] = \sum_{\vec{y} \in Y} \sum_{j \in \{0,1,2,3\}} \mathbb{P}[\mathcal{R}_{2,q}(01) = j] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = \vec{y} - e_{j+1,4}\right] = \sum_{\vec{y} \in Y} \sum_{j \in \{0,1,2,3\}} \mathbb{P}[\mathcal{R}_{2,q}(01) = j] \cdot \left(\prod_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = \vec{y} - e_{j+1,4}, \vec{f} \in F\right] + \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = \vec{y} - e_{j+1,4}, \vec{f} \notin F\right] \right) \\ \leq \left(\sum_{\vec{y} \in Y} \sum_{j \in \{0,1,2,3\}} \mathbb{P}[\mathcal{R}_{2,q}(01) = j] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = \vec{y} - e_{j+1,4}, \vec{f} \notin F\right] \right) + \delta \tag{8}$$

We will upper bound the inner summation. Expanding out the terms, we have

$$\sum_{j \in \{0,1,2,3\}} \mathbb{P}[\mathcal{R}_{2,q}(01) = j] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = \vec{y} - e_{j+1,4}, \vec{f} \in F \right]$$

$$= \mathbb{P}[\mathcal{R}_{2,q}(01) = 0] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1 - 1, y_2, y_3, y_4), \vec{f} \in F \right]$$

$$+ \mathbb{P}[\mathcal{R}_{2,q}(01) = 1] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4), \vec{f} \in F \right]$$

$$+ \mathbb{P}[\mathcal{R}_{2,q}(01) = 2] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3 - 1, y_4), \vec{f} \in F \right]$$

$$+ \mathbb{P}[\mathcal{R}_{2,q}(01) = 3] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3, y_4 - 1), \vec{f} \in F \right]$$
(9)

As displayed in Table 2, note that

$$\mathbb{P}[\mathcal{R}_{2,q}(01) = 0] = \mathbb{P}[\mathcal{R}_{2,q}(10) = 0] \text{ and } \mathbb{P}[\mathcal{R}_{2,q}(01) = 3] = \mathbb{P}[\mathcal{R}_{2,q}(10) = 3]$$

Also,

$$\mathbb{P}[\mathcal{R}_{2,q}(01) = 1] = \mathbb{P}[\mathcal{R}_{2,q}(10) = 2]$$
 and $\mathbb{P}[\mathcal{R}_{2,q}(01) = 2] = \mathbb{P}[\mathcal{R}_{2,q}(10) = 1]$.

By substitution,

$$(9) = \mathbb{P}[\mathcal{R}_{2,q}(10) = 0] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1 - 1, y_2, y_3, y_4), \vec{f} \in F \right] \\ + \mathbb{P}[\mathcal{R}_{2,q}(10) = 2] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4), \vec{f} \in F \right] \\ + \mathbb{P}[\mathcal{R}_{2,q}(10) = 1] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3 - 1, y_4), \vec{f} \in F \right] \\ + \mathbb{P}[\mathcal{R}_{2,q}(10) = 3] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3, y_4 - 1), \vec{f} \in F \right]$$
(10)

By combining (8) through (10) with Claim 3.13, we have

$$(8) \leq e^{\varepsilon} \cdot \left(\sum_{\vec{y} \in Y} \sum_{j \in \{0,1,2,3\}} \mathbb{P}[\mathcal{R}_{2,q}(10) = j] \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = \vec{y} - e_{j+1,4} \right] \right) + \delta$$
$$\leq e^{\varepsilon} \cdot \left(\sum_{\vec{y} \in Y} \mathbb{P}[\mathcal{B}_{m,q}(10) = \vec{y}] \right) + \delta$$
$$= e^{\varepsilon} \cdot \mathbb{P}[\mathcal{B}_{m,q}(10) \in Y] + \delta$$

which completes the proof.

3.3 Robustness of \mathcal{P}_{FLIP} to Manipulation

As mentioned in the Introduction, manipulation attacks have been studied in the local model of privacy [1, 19, 7, 11]. We assume there is a coalition of m users are corrupted who send specially crafted messages to skew the output of the protocol. In this section, we adapt this definition to the shuffle model and we upper bound the impact of corrupt users on the estimates produced by $\mathcal{P}_{\text{FLIP}}$. We also show that our protocol's robustness compares favorably with prior work.

A baseline attack against any histogram protocol is to simply run the randomizer on incorrect input. This introduces m/n bias to a single frequency estimate. But we can in fact bound the error of any attack against $\mathcal{P}_{\text{FLIP}}$.

Theorem 3.14. For any $\varepsilon > 0$, $\delta < 1/100$, and $n \in \mathbb{N}$, choose q, k as in Theorem 3.3. For any input $\vec{x} \in \mathcal{X}_d^n$, any value $j \in [d]$, and any coalition of m corrupt users $M \subset [n]$, the error of \mathcal{P}_{FLIP} on hist_j is

$$\frac{1}{n} \cdot \frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \cdot \sqrt{\frac{264}{5} \ln \frac{4}{\delta} \ln 20} \cdot g(k) + \frac{m}{n} \cdot (k+1) \cdot g(k)$$

with 90% probability.

Note that k = O(1) whenever $n = \Omega(\frac{1}{\varepsilon^2} \log \frac{1}{\delta})$ and recall g(k) approaches 1. In this regime, the above theorem implies that any attack launched by corrupt users is only a constant factor worse than the baseline attack.

Proof. Define the function $u(i, n) := ((i - 1) \mod n) + 1$. For $i \in [n]$, let $y_{i,j}$ be the *j*-th bit of the first message produced by user *i* (the output of $\mathcal{R}_{d,q}(x_i)$). For i > n, let $y_{i,j}$ be the *j*-th bit of the [i/n]-th message produced by user u(i, n) (the output of $\mathcal{R}_{d,q}(0^d)$).

Recall that the analyzer computes $z_j \leftarrow \frac{1}{n} \sum_{i=1}^{nk+n} \frac{1}{1-2q} \cdot (y_{i,j}-q)$. Let $z_j^{\text{hon}}, y_{i,j}^{\text{hon}}$ (resp. $z_j^{\text{cor}}, y_{i,j}^{\text{cor}}$) denote the random variables from an honest (resp. corrupted) execution of the protocol. Theorem 3.4 ensures that

$$|z_j^{\text{hon}} - \text{hist}_j(\vec{x})| < \frac{1}{n} \cdot \frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \cdot \sqrt{\frac{264}{5} \ln \frac{4}{\delta} \ln 20} \cdot g(k)$$

with probability 9/10. Via the triangle inequality, will suffice to show that

$$|z_j^{\text{hon}} - z_j^{\text{cor}}| < \frac{m}{n} \cdot (k+1) \cdot g(k)$$

with probability 1.

By construction, $|y_{i,j}^{\text{hon}} - y_{i,j}^{\text{cor}}| \in \{0, 1\}$ for any $j \in [d]$ and $i \in M$. Also, for all $i \notin M$, $y_{i,j}^{\text{hon}}$ is identically distributed with $y_{i,j}^{\text{cor}}$. This means

$$\begin{aligned} |z_{j}^{\text{hon}} - z_{j}^{\text{cor}}| &= \left| \frac{1}{n} \sum_{\{i \mid u(i,n) \in M\}} \frac{1}{1 - 2q} \cdot (y_{i,j}^{\text{hon}} - y_{i,j}^{\text{cor}}) \right| \\ &\leq \frac{1}{n} \cdot \sum_{\{i \mid u(i,n) \in M\}} \frac{1}{1 - 2q} \\ &= \frac{m}{n} \cdot (k+1) \cdot \frac{1}{1 - 2q} \end{aligned}$$

This concludes the proof, since g(k) is precisely 1/(1-2q) where q depends on k.

Now we bound the maximum error. Because the proof essentially generalizes the prior one, we omit it for brevity.

Theorem 3.15. For any $\varepsilon > 0$, $\delta < 1/100$, and $n \in \mathbb{N}$, choose q, k as in Theorem 3.4. For any input $\vec{x} \in \mathcal{X}_d^n$ and any coalition of m corrupt users, the maximum error of \mathcal{P}_{FLIP} is

$$\max\left(\frac{1}{n} \cdot \frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \cdot \sqrt{\frac{264}{5} \ln \frac{4}{\delta} \ln 20d}, \frac{2}{n} \cdot \ln 20d\right) \cdot f(k) + \frac{m}{n} \cdot (k+1) \cdot f(k)$$

with 90% probability.

We note that the resilience of the protocol stems from an implicit assumption that every user—both honest and corrupt—sends exactly k + 1 messages to the shuffler. In principle, a corrupt user could flood the network with misleading messages (for example, a thousand messages that increment each of the analyzer's *d* counters). But in practice, the analyzer can enforce³ the communication constraint via a blind signature scheme [8]: in a setup stage, each user interacts with the analyzer to sign exactly k+1 random strings. Each of the signed strings will serve as a tag of a message sent to the shuffler. The analyzer can limit its computation to messages with signed tags. This extra layer of security will only increase the communication cost by a small factor.

3.3.1 Comparison with \mathcal{P}_{HAD}

A highlight of [11] is that two locally private protocols for mean estimation can have the same accuracy absent manipulation but one can be more robust to manipulation than the other. In the same spirit, we show that another shuffle protocol for histograms has roughly the same accuracy as $\mathcal{P}_{\text{FLIP}}$ absent manipulation but is less robust to manipulation.

We will consider $\mathcal{P}_{HAD} = (\mathcal{R}_{HAD}, \mathcal{A}_{HAD})$ from Ghazi et al. [16]. We provide formal pseudocode in Appendix B but sketch the ideas here. Each of the k + 1 messages sent by a user is either an encoding of some value $j \in [d]$ or a sample from a distribution that serves to hide the encodings of user values. The encodings are based upon a public Hadamard matrix to optimize communication complexity (total number of bits sent by a user). When it encounters an encoding of j, the analyzer

 $^{^{3}}$ By placing the verification responsibility on the analyzer, we keep the shuffler lightweight. In particular, the shuffler does not have to keep track of the number of messages sent by users, which can be a sensitive attribute.

increments a counter for j. The approximate histogram is constructed by applying a linear function to the counters.

Ghazi et al. give the following result concerning accuracy and privacy. When $\varepsilon = \Theta(1)$, the bound on maximum error is asymptotically identical with that of $\mathcal{P}_{\text{FLIP}}$.

Theorem 3.16 (From [16]). Fix any $\varepsilon, \delta < 1$ and $n \in \mathbb{N}$. There exist parameter choices $k = \Theta(\frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon\delta})$ and $\tau = \Theta(\log n)$ such that \mathcal{P}_{HAD} is (ε, δ) -shuffle private for inputs from \mathcal{X}_d and reports an approximate histogram with maximum error $O(\frac{\log d}{n} + \frac{1}{\varepsilon n}\sqrt{\log\frac{1}{\varepsilon\delta}\log d})$ with probability 9/10.

In both $\mathcal{P}_{\text{FLIP}}$ and \mathcal{P}_{HAD} , *m* corrupt users can only shift counters by an additive factor of m(k+1). But because *k* is larger in \mathcal{P}_{HAD} than in $\mathcal{P}_{\text{FLIP}}$, each corrupted user has a greater impact in the former protocol than the latter.

Claim 3.17. Choose k, τ as in Theorem 3.16. If there is a coalition of m < n corrupt users $M \subset [n]$, then for any target value $j \in [d]$ there is an input \vec{x} such that \mathcal{P}_{HAD} produces an estimate of $\text{hist}_j(\vec{x})$ with $\text{bias } \frac{m}{n} \cdot (k+1) = \Omega(\frac{m}{n} \cdot \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon \delta}).$

We defer the proof to Appendix B for space.

3.4 Approximating Top-t from \mathcal{P}_{FLIP}

Given an approximate histogram—as guaranteed by $\mathcal{P}_{\text{FLIP}}$ —one can easily approximate the top-*t* items: output the top-*t* in the approximate histogram. If the maximum error is $\alpha/2$, then the rank of elements with frequency < hist_[t](\vec{x}) – α in the approximate histogram cannot exceed *t*. Thus, the following is immediate from our earlier results.

Corollary 3.18. For any $\vec{x} \in \mathcal{X}_d^n$, if we compute $\vec{z} \leftarrow \mathcal{P}_{FLIP}(\vec{x})$ (using the same parameters k, q as in Theorem 3.4), then $\operatorname{top}_t(\vec{z}) \alpha$ -approximates the top-t items in \vec{x} with probability $\geq 9/10$, where

$$\alpha = \max\left(\frac{1}{n} \cdot \frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \cdot \sqrt{\frac{1056}{5} \ln \frac{4}{\delta} \ln 20d}, \frac{4}{n} \cdot \ln 20d\right) \cdot f(k).$$

Corollary 3.19. For any $\vec{x} \in \mathcal{X}_d^n$ and parameters k, q such that $\frac{1}{nk+n} \ln 20d \leq q < 1/2$, if we compute $\vec{z} \leftarrow \mathcal{P}_{FLIP}(\vec{x})$, then $\operatorname{top}_t(\vec{z}) \alpha$ -approximates the top-t items in \vec{x} with probability $\geq 9/10$, where

$$\alpha = 4\sqrt{\frac{k+1}{n}q(1-q)\ln 20d} \cdot \left(\frac{1}{1-2q}\right)$$

4 Reducing Communication Complexity

Although $\mathcal{P}_{\text{FLIP}}$ has a constant message complexity for a large range of n, each message is a binary string of length d. The communication complexity therefore grows with the dimension. In this section, we describe how to mitigate the impact of large dimension.

4.1 Replacing Binary Strings with Lists

In this subsection, we use the observation that the messages are binary strings that are likely sparse so that they can be equated with a short list of indices. More precisely, let \mathcal{R}_{FLIP2} be the local randomizer that, on input x, computes messages from $\mathcal{R}_{FLIP}(x)$ but replaces each binary string it creates with a list of the indices that contain bit 1. Let \mathcal{A}_{FLIP2} be the analyzer that converts each of the messages output by the shuffler back into a binary string and then runs \mathcal{A}_{FLIP} .

Theorem 4.1. If parameters k, q are chosen in the same manner as in Theorem 3.4, then $\mathcal{P}_{FLIP3} = (\mathcal{R}_{FLIP3}, \mathcal{A}_{FLIP3})$ has the same number of messages and accuracy as \mathcal{P}_{FLIP} but now the expected length of each message is $\leq \log_2 d \cdot (1 + dq) = O(\log d(1 + \frac{d}{\varepsilon^2 n} \log \frac{1}{\delta} + \frac{d}{nk} \log d))$ bits.

Proof. A message is generated from either $\mathcal{R}_{d,q}(0^d)$ or $\mathcal{R}_{d,q}(x \in \mathcal{X}_d)$. By construction, $\mathcal{R}_{d,q}(0^d)$ produces a string where each bit is drawn from $\mathbf{Ber}(q)$. This means the number of 1s is drawn from $\mathbf{Bin}(d,q)$. Meanwhile, the number of 1s generated from executing $\mathcal{R}_{d,q}(x \in \mathcal{X}_d)$ is a sample from $\mathbf{Ber}(1-q) + \mathbf{Bin}(d-1,q)$.

Recall we set q to be $q = O(\frac{1}{\varepsilon^2 n} \log \frac{1}{\delta} + \frac{1}{nk} \log d)$. This means the expected number of 1s in any message is $O(\frac{d}{\varepsilon^2 n} \log \frac{1}{\delta} + \frac{d}{nk} \log d)$. And we need $\log_2 d$ bits to represent each index.

4.2 An Adaptation of Count-Min

The change-of-representation in the preceding section is powerful when n approaches (or exceeds) d. This subsection describes a method to reduce the communication complexity when n is not so large, at the price of logarithmic message complexity. The new protocol, which we call \mathcal{P}_{FLIP3} , uses the randomizer and analyzer of \mathcal{P}_{FLIP2} as black boxes. Based upon the Count-Min technique from the sketching literature, \mathcal{P}_{FLIP3} is an instance of a general method of transforming any shuffle protocol for histograms.

The pseudocode for the randomizer and analyzer is given in Algorithms 8 and 9, respectively. The heart of the transformation is hashing the universe [d] to $[\hat{d}]$. If an element j experiences no collisions, note that its frequency in the hashed dataset is the same as in the original dataset. Otherwise, the frequency is an overestimate. When there are many hash functions, it is likely that there is some hash function where j experiences no collisions; taking the minimum over the frequencies in the hashed datasets would recover the original frequency. We execute \mathcal{P}_{FLIP2} once for each hashed dataset to obtain estimates of the frequencies.

 Algorithm 8: \mathcal{R}_{FLIP3} a local randomizer for histograms

 Input: $x \in \mathcal{X}_d$; parameters $V, \hat{d}, k \in \mathbb{N}, q \in (0, 1/2)$

 Output: $\vec{y} \in ([V] \times [\hat{d}]^*)^*$

 Obtain hash functions $\{h^{(v)} : \mathcal{X}_d \to \mathcal{X}_{\hat{d}}\}$ from public randomness.

 Initialize \vec{y} to the empty vector

 For $v \in [V]$

 Compute messages $\vec{y}^{(v)} \leftarrow \mathcal{R}_{FLIP2}(h^{(v)}(x))$ using dimension \hat{d}

 For $y \in \vec{y}^{(v)}$
 $\[\] Append labeled message (v, y) to \vec{y} \]$

 Return \vec{y}

Algorithm 9: \mathcal{A}_{FLIP3} an analyzer for histograms

Input: $\vec{y} \in ([V] \times [\hat{d}]^*)^*$; parameters $V, \hat{d}, k \in \mathbb{N}, q \in (0, 1/2)$ Output: $\vec{z} \in \mathbb{R}^d$ Obtain hash functions $\{h^{(v)} : \mathcal{X}_d \to \mathcal{X}_{\hat{d}}\}$ from public randomness. For $j \in [d]$ $\[z_j \leftarrow \infty \]$ For $v \in [V]$ Initialize $\vec{y}^{(v)} \leftarrow \emptyset$ For $(v', y) \in \vec{y}$ $\[Append message y to <math>\vec{y}^{(v)}$ if label v' matches vCompute $\hat{z}^{(v)} \leftarrow \mathcal{A}_{FLIP2}(\vec{y}^{(v)})$ using dimension \hat{d} For $j \in [d]$ $\[\hat{j} \leftarrow h^{(v)}(j) \]$ $\[z_j \leftarrow \min(z_j, \hat{z}_{\hat{j}}^{(v)}) \]$ Return \vec{z}

We first analyze the protocol in terms of the parameter V, which determines the number of

hash functions and protocol repetitions. We will choose a value for V later in the section.

Claim 4.2. Fix any $\varepsilon = O(1)$, $\delta < 1/100$, and number of users $n \in \mathbb{N}$. If $\hat{d} \leftarrow \lceil n \cdot (100d)^{1/V} \rceil$ and $k > \max\left(\frac{134}{5n} \left(\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1}\right)^2 \ln \frac{4}{\delta}, \frac{2}{n} \ln 20 \hat{d}V - 1\right)$, then there is a choice of parameter q < 1/2 such that \mathcal{P}_{FLIP2} has the following properties:

- a. Each user sends $V \cdot (k+1)$ messages, each consisting of $O(\log V + \log \hat{d}(1 + \frac{\hat{d}}{\varepsilon^2 n} \log \frac{1}{\delta} + \frac{\hat{d}}{nk} \log \hat{d}))$ bits in expectation.
- b. \mathcal{P}_{FLIP2} is $\left(\varepsilon(e^{\varepsilon}-1)\cdot V + \varepsilon \cdot \sqrt{2V\log\frac{1}{V\delta}}, 2V\delta\right)$ -shuffle private for inputs from \mathcal{X}_d
- c. For any $\vec{x} \in \mathcal{X}_d^n$, $\mathcal{P}_{FLIP2}(\vec{x})$ reports approximate histogram \vec{z} such that the maximum error is

$$\|\vec{z} - \operatorname{hist}(\vec{x})\|_{\infty} = O\left(\frac{1}{\varepsilon n}\sqrt{\log\frac{1}{\delta}\log\hat{d}V} + \frac{\log\hat{d}V}{n}\right)$$

with probability $\ge 9/10 - (1/100)^V$.

Proof. We will choose q in much the same way as Theorems 3.4 and 4.1. The sole modification is that we change the $\ln 20d$ term to $\ln 20dV$.

The protocol executes $\mathcal{P}_{\text{FLIP2}}$ exactly V times using the hashed dimension \hat{d} . So the number of messages is simply $V \cdot (k+1)$. Each message is generated via $\mathcal{R}_{\text{FLIP2}}$ and then labeled by the execution number v, so Part a is immediate from Theorem 4.1. Meanwhile, Part b follows directly from advanced composition (Fact 2.4).

To prove Part c, we first analyze the randomness from hashing and consider privacy noise later. For any $j \in [d]$, let E_j denote the event that there is at least one hash function where j experiences no collisions with a user value $j' \neq j$. Formally, $\exists v^* \ \forall j' \in \vec{x}, j' \neq j \ h^{(v^*)}(j) \neq h^{(v^*)}(j')$. We will now bound the probability that E_j does not occur.

$$\mathbb{P}_{\vec{h}}[\neg E_j] = \mathbb{P}_{\vec{h}}\left[\forall v \; \exists j' \in \vec{x} \; h^{(v)}(j) = h^{(v)}(j')\right]$$
$$= \mathbb{P}_{\vec{h}}\left[\exists j' \in \vec{x} \; h^{(v)}(j) = h^{(t)}(j')\right]^V$$
$$\leq \left(n \cdot \mathbb{P}_{\vec{h}}\left[h^{(v)}(j) = h^{(v)}(j')\right]\right)^V$$
$$= (n/\hat{d})^V = (1/100)^V \cdot \frac{1}{d}$$

By a union bound, the probability that there is some j where E_j does not occur is at most $(1/100)^V$

The remainder of the proof conditions on E_j occurring for all j. In this event, each j can be paired with some v^* where the count of $h^{(v^*)}(j)$ in the hashed dataset is exactly the count of j in the original dataset. For any $v \neq v^*$, observe that the count of $h^{(v)}(j)$ in the hashed dataset is must be either (1) an overestimate due to collision or (2) also equal. Thus, the minimum over the counts yields the correct value.

Now we incorporate the fact that $\mathcal{A}_{\text{FLIP3}}$ only has private estimates of the counts. When we set $\beta = 1/10 dV$, Claim 3.1 and a union bound imply each protocol execution has ℓ_{∞} error

$$2\sqrt{\frac{k+1}{n}q(1-q)\ln 20\hat{d}V}\cdot\left(\frac{1}{1-2q}\right)$$

except with probability $\leq 1/10V$ A second union bound over the V executions ensures that the privatized count of $h^{(v^*)}(j)$ in the hashed dataset is the minimum of all privatized counts. Substitution of q completes the proof.

We now show that there is a choice of V where the expected communication complexity has only a polylogarithmic dependence on d and n. **Theorem 4.3.** Fix any $\varepsilon = O(1)$. If $n = \Omega(\frac{\log d}{\varepsilon^2} \log \frac{1}{\delta} \log \frac{\log d}{\delta})$ and $\delta = O(1/n)$, there are choices of parameters $V, k \in \mathbb{N}$ and q < 1/2 such that \mathcal{P}_{FLIP3} has the following properties:

- a. Each user sends $2\log_2 d$ messages, each consisting of $O\left(\frac{1}{\varepsilon^2}\log d\log^3\frac{\log d}{\delta}\right)$ bits in expectation.
- b. \mathcal{P}_{FLIP3} is (ε, δ) -shuffle private for inputs from \mathcal{X}_d
- c. For any $\vec{x} \in \mathcal{X}_d^n$, $\mathcal{P}_{FLIP3}(\vec{x})$ reports approximate histogram \vec{z} such that the maximum error is

$$\|\vec{z} - \operatorname{hist}(\vec{x})\|_{\infty} = O\left(\frac{1}{\varepsilon n}\sqrt{\log d}\log^{3/2}\left(\frac{\log d}{\delta}\right)\right)$$

with probability $\geq 9/10 - (1/100)^{\log_2 d}$.

Proof. We will set $V \leftarrow \log_2 d$. Let $\overline{\varepsilon} = \varepsilon/4\sqrt{V\log(1/\delta)}$ and $\overline{\delta} = \delta/2V$. Because n is sufficiently large, it is possible to set k = 1 and set $q \in (0, 1/2)$ such that each execution of $\mathcal{P}_{\text{FLIP2}}$ satisfies $(\bar{\varepsilon}, \delta)$ -shuffle privacy. By substitution into Claim 4.2, the expected length of a message is

$$O\left(\log V + \log \hat{d}\left(1 + \frac{\hat{d}}{\varepsilon^2 n}\log\frac{1}{\delta} + \frac{\hat{d}}{nk}\log\hat{d}\right)\right)$$
$$= O\left(\log V + \log n\left(\frac{1}{\varepsilon^2}\log\frac{1}{\delta} + \log n\right)\right)$$
(Choice of \hat{d}, k)

$$= O\left(\log V + \log n\left(\frac{1}{\varepsilon^2}V\log\frac{1}{\delta}\log\frac{V}{\delta} + \log n\right)\right)$$
(Choice of $\overline{\varepsilon}, \overline{\delta}$)

$$= O\left(\log\log d + \log n\left(\frac{1}{\varepsilon^2}\log d\log\frac{1}{\delta}\log\frac{\log d}{\delta} + \log n\right)\right)$$
(Choice of V)

$$= O\left(\frac{1}{\varepsilon^2}\log d\log^3 \frac{\log d}{\delta}\right) \qquad (\delta = O(1/n))$$

Meanwhile, the maximum error is

$$O\left(\frac{1}{\overline{\varepsilon}n}\sqrt{\log\frac{1}{\overline{\delta}}\log\hat{d}V} + \frac{\log\hat{d}V}{n}\right)$$

= $O\left(\frac{1}{\overline{\varepsilon}n}\sqrt{\log\frac{1}{\overline{\delta}}(\log n + \log\log d)} + \frac{\log n + \log\log d}{n}\right)$ (Choice of \hat{d}, V)
 $\left(\frac{1}{\sqrt{\log\frac{1}{\delta}(\log n + \log\log d)}} + \frac{\log n + \log\log d}{\log n}\right)$ (Choice of \hat{d}, V)

$$= O\left(\frac{1}{\varepsilon n}\sqrt{\log d \log \frac{1}{\delta} \log \frac{\log d}{\delta}}(\log n + \log \log d) + \frac{\log n + \log \log d}{n}\right)$$
(Choice of $\overline{\varepsilon}, \overline{\delta}$)

$$= O\left(\frac{1}{\varepsilon n}\sqrt{\log d}\log^{3/2}\left(\frac{\log d}{\delta}\right)\right) \qquad (\delta = O(1/n))$$
ompletes the proof.

This completes the proof.

Finally, we study the impact that corrupt users can have on the estimates generated by \mathcal{P}_{FLIP3} .

Claim 4.4. Fix any $\varepsilon = O(1)$, $\delta < 1/100$, and n, q, V as in Theorem 4.3. For any input $\vec{x} \in \mathcal{X}_d^n$, any coalition of *m* corrupt users can introduce $O(\frac{m}{n} \cdot \log d)$ error to \mathcal{P}_{FLIP3} .

Proof Sketch. Each honest user transmits O(V) messages, where each message is an output of $\mathcal{R}_{\text{FLIP}} \circ h^{(v)}$ labeled by v. A corrupt user is therefore limited a "budget" of O(V) messages each with the same structure. But all of these messages could share the same label v. In this case, we can adapt the analysis of $\mathcal{P}_{\mathsf{FLIP}}$ to show that *m* corrupt users will add $O(\frac{m}{n} \cdot V)$ bias to the *v*-th execution of $\mathcal{P}_{\text{FLIP}}$.

5 Experiments

In this section, we evaluate the accuracy of our protocol on natural language data. To give context for these results, we repeat the experiment on the histogram protocol by Balcer & Cheu [2]. It has essentially the same communication complexity as $\mathcal{P}_{\text{FLIP}}$, but its message complexity is O(d).

We acquire a list of $d \approx 4.7 \cdot 10^5$ English words from a publicly accessible repository and $\approx 3.7 \cdot 10^6$ tweets on Twitter in the United States previously used in work by Cheng, Caverlee, and Lee [9].⁴ We sampled one recognized word from each tweet, so that $n \approx 3.7 \cdot 10^6$. Fixing privacy parameters $\varepsilon = 1$ and $\delta = 10^{-7}$, we simulated our protocol on the dataset a hundred times for four choices of k.

As an aside, our method of sampling data ensures tweet-level privacy rather than user-level privacy. That is, we could have sampled one word per user instead of one word per tweet. But our goal was to evaluate the protocol when applied to large-scale data analysis and our dataset consists of only $\approx 10^4$ users.

5.1 Evaluation of Maximum Error

In Figure 3, we visualize the error introduced by \mathcal{P}_{FLIP} for varying choices of k. We also plot corresponding confidence bounds derived from Corollary 3.5. As predicted, the error decreases with larger k. And, at least for this particular dataset, our bounds are loose by only a small multiplicative factor.



Figure 3: Maximum error of frequency estimates in experiments, as a (decreasing) function of k. Confidence bounds (red filled circles) are derived from Corollary 3.5.

In Figure 4, we compare the max error of $\mathcal{P}_{\text{FLIP}}$ with the histogram protocol by Balcer & Cheu [2]. The primary advantage of [2] over $\mathcal{P}_{\text{FLIP}}$ is that the error introduced to any bin does not scale with d. Specifically, the maximum error is $O(\frac{1}{\varepsilon^2 n} \log \frac{1}{\delta})$. In contrast $\mathcal{P}_{\text{FLIP}}$ only ensures $O(\frac{\log d}{n} + \frac{1}{\varepsilon n} \sqrt{\log d \log \frac{1}{\delta}})$ error. But in our application, d is actually orders of magnitude smaller than $1/\delta$ so $\mathcal{P}_{\text{FLIP}}$ is in fact more accurate.

 $^{^4 \}rm The word list was downloaded from https://github.com/dwyl/english-words while the tweets were downloaded from https://archive.org/details/twitter_cikm_2010$



Figure 4: Comparison between the maximum error of $\mathcal{P}_{\text{FLIP}}$ and that of the protocol in [2]. The latter protocol would only have better error than the former if the dimension of the data were much larger.

5.2 Evaluation of Top-t selection

Recall the simple top-t selection strategy from Section 3.4: report the top-t items of a private version of the histogram. In Table 3, we fix t = 6000 and present our bound from Corollary 3.19 alongside the maximum observed value in our experiments. We remark that the frequency of the rank-t word is $1.33 \cdot 10^{-5}$. This is an upper bound on α that holds with probability 1, since the worst that can happen is that a word with frequency 0 displaces the t-th most common word.

	$\mathcal{P}_{\text{FLIP}}$'s α -approximation of top-6000			
k	Bound from Corollary 3.19	Maximum observed		
1	$1.43 \cdot 10^{-4}$			
2	$1.24 \cdot 10^{-4}$	1.22 10-5		
3	$1.17 \cdot 10^{-4}$	1.55 · 10 *		
4	$1.13\cdot 10^{-4}$			

Table 3: Comparing the bound on the error of top-t selection with experimental results.

In Figure 5, we plot the F1 score of the report-top-t strategy for both \mathcal{P}_{FLIP} and the Balcer-Cheu protocol. \mathcal{P}_{FLIP} preserves $\approx 95\%$ of the top-2000 words in the dataset and is consistently more accurate than the alternative protocol. Increasing t decreases the F1 score for both protocols because infrequent words are easily evicted from the top-t and natural language heavily favors a small set of words.

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Figure 5: F1 scores for top-t word selection, for both tested protocols. The lines connect medians while the error bars represent the complete range of observed values.

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A Technical Claims for \mathcal{P}_{FLIP}

Claim (Restatement of Claim 3.12). Fix $m \in \mathbb{N}$ and $q, \delta \in (0, 1)$. Define

$$\Delta := \sqrt{3mq(1-q)\ln\frac{4}{\delta}} \cdot \frac{q(1-q)}{1-q(1-q)}$$
$$U := mq(1-q) + \Delta + \sqrt{3(mq(1-q)+\Delta)\ln\frac{4}{\delta}}$$
$$L := mq(1-q) - \Delta - \sqrt{3(mq(1-q)+\Delta)\ln\frac{4}{\delta}}$$

Let $F \subset \mathbb{Z}^4$ denote the set of vectors where $\vec{t} \in F$ if and only if $t_2, t_3 \in [L, U]$. If $mq(1-q) > \frac{9}{2}\ln(4/\delta)$, then

$$\mathbb{P}_{\vec{f} \sim \mathcal{M}(m,q)} \left[\vec{f} \notin F \right] \le \delta$$

Proof. We will use a Chernoff bound to argue that the marginal distribution of f_3 is likely to be in some interval [L', U']. Then we will use a Chernoff bound to argue that the distribution of f_2 conditioned on $f_3 \in [L', U']$ is likely to be in [L, U]. The claim follows from the fact that $[L', U'] \subset [L, U]$.

By construction, f_3 is the random variable that counts the number of times the message 10 (2 in binary) is produced by m executions of $\mathcal{R}_{2,q}(00)$. Referring to Table 2, this means f_3 is distributed as $\operatorname{Bin}(m, q(1-q))$. Using μ_3 as shorthand for the mean mq(1-q), multiplicative Chernoff bounds imply the following for all $z \in (0, 1)$:

$$\mathbb{P}[|f_3 - \mu_3| > z\mu_3] \le 2\exp(-z^2\mu_3/3).$$

Because $\mu_3 \ge 3 \ln \frac{4}{\delta}$, we can assign $z \leftarrow \sqrt{\frac{3}{\mu_3} ln \frac{4}{\delta}}$ so that

$$\mathbb{P}\left[|f_3 - \mu_3| > \sqrt{3\mu_3 \cdot \ln \frac{4}{\delta}}\right] \le \delta/2$$

So if we define $L' \leftarrow \mu_3 - \sqrt{3\mu_3 \cdot \ln \frac{4}{\delta}}$ and $U' \leftarrow \mu_3 + \sqrt{3\mu_3 \cdot \ln \frac{4}{\delta}}$, $f_3 \in [L', U']$ except with probability $\delta/2$; the remainder of the proof conditions on this event.

Specifically, we assume that the random variable f_3 takes on some value $r \in [L, U]$. This means that f_2 is the random variable that counts the number of times the message 01 (1 in binary) is produced by m - r executions of $\mathcal{R}_{2,q}(00)$ conditioned on the output not being 10 (2 in binary). Referring to Table 2, this means f_2 is distributed as $\operatorname{Bin}\left(m - r, \frac{q(1-q)}{1-q(1-q)}\right)$. The mean of this distribution is $\mu_2 \leftarrow (m-r) \cdot \frac{q(1-q)}{1-q(1-q)}$. If we could show $\mu_2 \geq 3 \ln \frac{4}{\delta}$, we could again invoke multiplicative Chernoff bounds to argue

$$\mathbb{P}\left[|f_2 - \mu_2| > \sqrt{3\mu_2 \cdot \ln\frac{4}{\delta}}\right] \le \delta/2.$$

Notice that $r \in [L', U']$ implies

$$\begin{aligned} \mu_2 &= (m-r) \cdot \frac{q(1-q)}{1-q(1-q)} \\ &\geq \left(m - \mu_3 - \sqrt{3\mu_3 \cdot \ln\frac{4}{\delta}}\right) \cdot \frac{q(1-q)}{1-q(1-q)} \\ &= \left(m - mq(1-q) - \sqrt{3mq(1-q)\ln\frac{4}{\delta}}\right) \cdot \frac{q(1-q)}{1-q(1-q)} \\ &= mq(1-q) - \sqrt{3mq(1-q)\ln\frac{4}{\delta}} \cdot \frac{q(1-q)}{1-q(1-q)} \\ &= mq(1-q) - \Delta \end{aligned}$$

By symmetric arguments,

$$\mu_2 \le mq(1-q) + \Delta.$$

The claim follows by substitution.

We now argue that $\mu_2 \ge 3 \ln \frac{4}{\delta}$.

$$\mu_2 \ge mq(1-q) - \sqrt{\frac{1}{3} \cdot mq(1-q) \ln \frac{4}{\delta}} \qquad (q(1-q) < 1/4)$$
$$\ge \frac{2}{3}mq(1-q) \qquad (mq(1-q) > 3\ln(4/\delta))$$
$$\ge 3\ln \frac{4}{\delta}$$

Claim (Restatement of Claim 3.13). Fix any $\varepsilon > 0$ and $\delta < 1/100$. Define F as in Claim 3.12. If q < 1/2 and $mq(1-q) \ge \frac{33}{5} \left(\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1}\right)^2 \ln(4/\delta)$, then for any $\vec{y} = (y_1, \ldots, y_4)$,

$$\mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \Big[\vec{f} = (y_1, y_2 - 1, y_3, y_4), \vec{f} \in F \Big] \le e^{\varepsilon} \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \Big[\vec{f} = (y_1, y_2, y_3 - 1, y_4) \Big]$$
(11)

$$\mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3 - 1, y_4), \vec{f} \in F \right] \le e^{\varepsilon} \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4) \right]$$
(12)

Proof. It remains to prove (11); the proof of (12) will be completely symmetric. Let $F' \subset \mathbb{Z}^4$ denote the set of vectors where $\vec{f} \in F'$ if and only if $f_2, f_3 \in [L-1, U+1]$.

$$\mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4), \vec{f} \in F \right] \\
= \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4) \right] \cdot \mathbb{1}[(y_1, y_2 - 1, y_3, y_4) \in F] \\
= \frac{m!}{y_1!(y_2 - 1)!y_3!y_4!} \cdot (1 - q)^{2y_1}(q(1 - q))^{y_2 - 1}(q(1 - q))^{y_3}q^{2y_4} \\
\cdot \mathbb{1}[(y_1, y_2 - 1, y_3, y_4) \in F] \quad (\text{Defn. of } \mathcal{M}) \\
\leq \frac{m!}{y_1!(y_2 - 1)!y_3!y_4!} \cdot (1 - q)^{2y_1}(q(1 - q))^{y_2 - 1}(q(1 - q))^{y_3}q^{2y_4} \\
\cdot \mathbb{1}[(y_1, y_2, y_3 - 1, y_4) \in F'] \quad (13)$$

(13) comes from the fact that when $y_2 - 1 \in [L, U]$ and $y_3 \in [L, U]$, it must be the case that $y_2 \in [L - 1, U + 1]$ and $y_3 - 1 \in [L - 1, U + 1]$. We can also derive

$$\mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3 - 1, y_4), \vec{f} \in F' \right] \\
= \frac{m!}{y_1! y_2! (y_3 - 1)! y_4!} \cdot (1 - q)^{2y_1} (q(1 - q))^{y_2} (q(1 - q))^{y_3 - 1} q^{2y_4} \\
\cdot \mathbb{I}[(y_1, y_2, y_3 - 1, y_4) \in F']$$
(14)

By combining (13) and (14),

$$\mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4), \vec{f} \in F \right] \\
\leq \frac{y_2}{y_3} \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3 - 1, y_4) \right] \cdot \mathbb{1}[(y_1, y_2, y_3 - 1, y_4) \in F'].$$

In the case where $(y_1, y_2, y_3 - 1, y_4) \notin F'$, the right hand side is zero so that (11) trivially holds. Otherwise, $y_2/y_3 \leq (U+1)/(L-1)$ by definition of F'. This means

$$\mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2 - 1, y_3, y_4), \vec{f} \in F \right] \le \frac{U+1}{L-1} \cdot \mathbb{P}_{\vec{f} \leftarrow \mathcal{M}(m,q)} \left[\vec{f} = (y_1, y_2, y_3 - 1, y_4) \right]$$

so it simply remains to show $(U+1)/(L-1) \leq e^{\varepsilon}$. We rewrite this target inequality as

$$\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + 1} \ge \frac{\Delta + \sqrt{3(mq(1-q) + \Delta)\ln(4/\delta)} + 1}{mq(1-q)} = \underbrace{\sqrt{\frac{3\ln(4/\delta)}{mq(1-q)}} \cdot \frac{q(1-q)}{1-q(1-q)}}_{A} + \underbrace{\frac{\sqrt{3(mq(1-q) + \Delta)\ln(4/\delta)}}{mq(1-q)}}_{B} + \underbrace{\frac{1}{mq(1-q)}}_{C}$$
(15)

We will upper bound each term, beginning with A:

$$A = \sqrt{\frac{3\ln(4/\delta)}{mq(1-q)}} \cdot \frac{q(1-q)}{1-q(1-q)} < \sqrt{\frac{3\ln(4/\delta)}{mq(1-q)}} \cdot \frac{1}{3} < \frac{1}{3} \cdot \sqrt{\frac{5}{11}} \cdot \frac{e^{\varepsilon} - 1}{e^{\varepsilon} + 1}$$
 (q < $\frac{1}{2}$)

Now we bound B:

$$B = \frac{\sqrt{3(mq(1-q) + \Delta)\ln(4/\delta)}}{mq(1-q)}$$

$$= \sqrt{\frac{3\ln(4/\delta)}{mq(1-q)} + \frac{3\Delta\ln(4/\delta)}{(mq(1-q))^2}}$$

$$= \sqrt{\frac{3\ln(4/\delta)}{mq(1-q)} + \frac{3\ln(4/\delta)}{(mq(1-q))^2} \cdot \left(\sqrt{3mq(1-q)\ln\frac{4}{\delta}} \cdot \frac{q(1-q)}{1-q(1-q)}\right)}{1-q(1-q)}}$$
(Value of Δ)
$$= \sqrt{\frac{3\ln(4/\delta)}{mq(1-q)} + \left(\frac{3\ln(4/\delta)}{mq(1-q)}\right)^{3/2}} \cdot \frac{q(1-q)}{1-q(1-q)}}{1-q(1-q)}}$$

$$< \sqrt{\frac{5}{11} + \left(\frac{5}{11}\right)^{3/2}} \cdot \frac{1}{3} \cdot \frac{e^{\varepsilon} - 1}{e^{\varepsilon} + 1}}$$

Finally we bound C:

$$C = \frac{1}{mq(1-q)} \le \frac{5}{33\ln(4/\delta)} \cdot \frac{e^{\varepsilon} - 1}{e^{\varepsilon} + 1}$$
$$\le \frac{5}{33\ln(400)} \cdot \frac{e^{\varepsilon} - 1}{e^{\varepsilon} + 1} \qquad (\delta \le 1/100)$$

(15) follows by substitution.

B Manipulation Attack Against \mathcal{P}_{HAD}

In this section, we describe the Hadamard response protocol by Ghazi et al. [16] and a manipulation attack against it. For a wide range of n, the protocol's estimates are less robust (at least in the worst case) than $\mathcal{P}_{\text{FLIP}}$.

We present pseudocode for the randomizer and analyzer in Algorithms 10 and 11, which use parameters $k, \tau \in \mathbb{N}$. We remark that we have adjusted the algorithm and notation to be more consistent with our protocol and the problem it solves. Specifically, parameter ρ is renamed k to match $\mathcal{P}_{\text{FLIP}}$ and we limit user data to \mathcal{X}_d .⁵

Claim (Restatement of 3.17). Choose k, τ as in Theorem 3.16. If there is a coalition of m < n corrupt users $M \subset [n]$, then for any target value $j \in [d]$ there is an input \vec{x} such that $\mathcal{P}_{\mathsf{FLIP}}$ produces an estimate of $\operatorname{hist}_j(\vec{x})$ with bias $\frac{m}{n} \cdot (k+1) = \Omega(\frac{m}{n} \cdot \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon \delta})$.

Proof. The attack is simple: given the target j, each corrupt user samples k + 1 values i.i.d. from $\{\hat{j} \mid h_{j,\hat{j}} = 1\}$ in lieu of running \mathcal{R}_{HAD} . Now consider an input \vec{x} such that $\text{hist}_j(\vec{x}) = 0$. For z_j^{cor} computed by \mathcal{P}_{HAD} under attack by m corrupt users, we will argue that $\mathbb{E}[z_j^{\text{cor}}] = \frac{m}{n} \cdot (k+1)$.

computed by \mathcal{P}_{HAD} under attack by *m* corrupt users, we will argue that $\mathbb{E}[z_j^{\text{cor}}] = \frac{m}{n} \cdot (k+1)$. We require some notation. Let $b_{i,r}$ be the bit that indicates if the *r*-th message produced by user *i* will have the property that each element belongs to $\{\hat{j} \mid h_{j,\hat{j}} = 1\}$. Note that $c_j =$

⁵As originally written, \mathcal{P}_{HAD} solved the more general problem where users can have more than one item $\in [d]$. In principle, we could augment \mathcal{P}_{FLIP} to solve the same generalization, but we focus on the simplest case for clarity.

Algorithm 10: \mathcal{R}_{HAD} , local randomizer for histograms

Input: $x \in \mathcal{X}_d$ **Output:** $\vec{y} \in ([2d]^{\tau})^{k+1}$ Initialize \vec{y} to the empty vector. Let j(x) be the integer j such that $e_{j,d} = x$ Let $h_{j(x)}$ be the j(x) + 1-th row of the $2d \times 2d$ Hadamard matrix Sample a_1, \ldots, a_{τ} uniformly and independently from $\{\hat{j} \mid h_{j(x),\hat{j}} = 1\}$ Append the tuple (a_1, \ldots, a_{τ}) to \vec{y} **For** $i \in [k]$ $\[$ Sample a_1, \ldots, a_{τ} uniformly and independently from [2d] $\[$ Append the tuple (a_1, \ldots, a_{τ}) to \vec{y} **Return** \vec{y}

Algorithm 11: \mathcal{A}_{HAD} , analyzer for histograms

 $\sum_{i \in [n]} \sum_{r \in [k+1]} b_{i,r}$. We will use the superscripts "hon" and "cor" to denote random variables from honest and corrupted executions, respectively.

$$\begin{split} \mathbb{E}\left[z_{j}^{\text{cor}}\right] &= \frac{1}{n} \cdot \frac{1}{1 - 2^{-\tau}} \cdot \left(\mathbb{E}\left[c_{j}^{\text{cor}}\right] - n(k+1) \cdot 2^{-\tau}\right) \\ &= \frac{1}{n} \cdot \frac{1}{1 - 2^{-\tau}} \cdot \left(\sum_{i \in [n]} \sum_{r \in [k+1]} \mathbb{E}\left[b_{i,r}^{\text{cor}}\right] - n(k+1) \cdot 2^{-\tau}\right) \\ &= \frac{1}{n} \cdot \frac{1}{1 - 2^{-\tau}} \cdot \left(\sum_{i \notin M} \sum_{r \in [k+1]} \mathbb{E}\left[b_{i,r}^{\text{cor}}\right] + \sum_{i \in M} \sum_{r \in [k+1]} \mathbb{E}\left[b_{i,r}^{\text{cor}}\right] - n(k+1) \cdot 2^{-\tau}\right) \\ &= \frac{1}{n} \cdot \frac{1}{1 - 2^{-\tau}} \cdot \left(\sum_{i \notin M} \sum_{r \in [k+1]} \mathbb{E}\left[b_{i,r}^{\text{hon}}\right] + \sum_{i \in M} \sum_{r \in [k+1]} \mathbb{E}\left[b_{i,r}^{\text{cor}}\right] - n(k+1) \cdot 2^{-\tau}\right) \\ &= \frac{1}{n} \cdot \frac{1}{1 - 2^{-\tau}} \cdot \left((n - m)(k+1) \cdot 2^{-\tau} + \sum_{i \in M} \sum_{r \in [k+1]} \mathbb{E}\left[b_{i,r}^{\text{cor}}\right] - n(k+1) \cdot 2^{-\tau}\right) \\ &= \frac{1}{n} \cdot \frac{1}{1 - 2^{-\tau}} \cdot \left((n - m)(k+1) \cdot 2^{-\tau} + m(k+1) - n(k+1) \cdot 2^{-\tau}\right) \\ &= \frac{m}{n} \cdot (k+1) \end{split}$$

(16) comes from analysis done in [16]. (17) is immediate from the definition of the attack. \Box

C Histogram Protocol via Privacy Amplification

In this appendix, we will consider the variant of \mathcal{R}_{FLIP} where there are no messages from fabricated users. The privacy analysis is performed using the amplification-by-shuffling result by Feldman et al. [15].

Theorem C.1. Fix any $\varepsilon \leq 4$, $\delta < 1$, and k = 0. For any $n > \max(\frac{1024}{\varepsilon^2} \ln \frac{4}{\delta}, 6 \ln 20d)$, there is a choice of parameter q < 1/3 such that the protocol $\mathcal{P}_{FLIP} = (\mathcal{R}_{FLIP}, \mathcal{A}_{FLIP})$ has the following properties

- a. \mathcal{P}_{FLIP} is (ε, δ) -shuffle private
- b. For any $\vec{x} \in \mathcal{X}_d^n$, $\mathcal{P}_{FLIP}(\vec{x})$ reports a vector \vec{z} such that the maximum error with respect to hist (\vec{x}) is

$$\|\vec{z} - \operatorname{hist}(\vec{x})\|_{\infty} < \max\left(\frac{24}{n^{3/4}\sqrt{\varepsilon}} \left(\ln\frac{4}{\delta}\right)^{1/4} \sqrt{\ln 20d}, \ \frac{6}{n} \ln 20d\right)$$

with 90% probability.

We first restate the amplification lemma from [15] using our notation and variant of the model.

Lemma C.2. Fix any $\delta \in (0,1)$, $n \in \mathbb{N}$, and $\varepsilon_L \leq \ln(n/16\ln(2/\delta))$. If $\mathcal{R} : \mathcal{X} \to \mathcal{Y}$ is ε_L differentially private then $(\mathcal{S} \circ \mathcal{R}^n)$ is (ε_S, δ) -differentially private, where

$$\varepsilon_S = 8 \cdot \frac{e^{\varepsilon_L} - 1}{e^{\varepsilon_L} + 1} \cdot \left(\sqrt{\frac{e^{\varepsilon_L} \ln(4/\delta)}{n}} + \frac{e^{\varepsilon_L}}{n} \right).$$

A corollary of this lemma is that when the target privacy parameter ε_S is sufficiently small, there is always some choice of privacy parameter ε_L for \mathcal{R} and some threshold for n above which the shuffle protocol is (ε_S, δ) -shuffle private. More precisely,

Corollary C.3. Fix any $\varepsilon_S \leq 4$ and $\delta \in (0,1)$. If $n > \frac{256}{\varepsilon_S^2} \ln(4/\delta)$ and $\mathcal{R} : \mathcal{X} \to \mathcal{Y}$ is ε_L -differentially private for $\varepsilon_L \leq \ln(\varepsilon_S^2 n/256 \ln(4/\delta))$, then $(\mathcal{S} \circ \mathcal{R}^n)$ is (ε_S, δ) -differentially private.

Proof. Because ε_S is sufficiently small, ε_L satisfies the condition under which Lemma C.2 holds: $(\mathcal{S} \circ \mathcal{R}^n)$ is (ε, δ) -differentially private, where

$$\varepsilon = 8 \cdot \frac{e^{\varepsilon_L} - 1}{e^{\varepsilon_L} + 1} \cdot \left(\sqrt{\frac{e^{\varepsilon_L} \ln(4/\delta)}{n}} + \frac{e^{\varepsilon_L}}{n} \right)$$

$$\leq 8 \cdot \left(\sqrt{\frac{e^{\varepsilon_L} \ln(4/\delta)}{n}} + \frac{e^{\varepsilon_L}}{n} \right)$$

$$\leq 8 \cdot \left(\frac{\varepsilon_S}{16} + \frac{\varepsilon_S^2}{256 \ln(4/\delta)} \right)$$

$$= \frac{\varepsilon_S}{2} + \frac{\varepsilon_S}{2} \cdot \frac{\varepsilon_S}{16 \ln(4/\delta)}$$

$$\leq \varepsilon_S$$
(Bound on ε_L)

The final inequality follows from our bound on ε_s .

Now we find a value of q to ensure $\mathcal{P}_{\mathsf{FLIP}}$ satisfies ε_L -local privacy:

Claim C.4. For any $\varepsilon_L > 0$, if $k \leftarrow 0$ and $q \leftarrow 1/(e^{\varepsilon_L/2} + 1)$ then the randomizer \mathcal{R}_{FLIP} is ε_L -differentially private.

Theorem C.1 follows from Corollary C.3, Corollary 3.5, and Claim C.4.

Proof of Theorem C.1. We choose $\varepsilon_L \leftarrow \ln(\varepsilon^2 n/256 \ln(4/\delta))$ and $q \leftarrow \max(1/(e^{\varepsilon_L/2}+1), \frac{1}{n} \ln 20d)$. By substitution, we have that the following holds with probability 9/10:

$$\begin{split} \|\mathcal{P}_{\mathsf{FLIP}}(\vec{x}) - \operatorname{hist}(\vec{x})\|_{\infty} &< 2\sqrt{\frac{1}{n} \cdot q \ln 20d} \cdot \left(\frac{1}{1-2q}\right) \qquad (k = 0, q > 0) \\ &= 2\sqrt{\frac{1}{n} \cdot \max\left(\frac{1}{e^{\varepsilon_L/2} + 1}, \ \frac{1}{n} \ln 20d\right) \ln 20d} \cdot \left(\frac{1}{1-2q}\right) \\ &< 2\sqrt{\frac{1}{n} \cdot \max\left(\frac{16\sqrt{\ln(4/\delta)}}{\varepsilon\sqrt{n}}, \ \frac{1}{n} \ln 20d\right) \ln 20d} \cdot \left(\frac{1}{1-2q}\right) \\ &= \max\left(\frac{8}{n^{3/4}\sqrt{\varepsilon}} \left(\ln\frac{4}{\delta}\right)^{1/4} \sqrt{\ln 20d}, \ \frac{2}{n} \ln 20d\right) \cdot \left(\frac{1}{1-2q}\right) \end{split}$$

Given that n is sufficiently large, we conclude q < 1/3. The Theorem follows by substitution. \Box