PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 4, August 1984

DIFFERENTIATION OF REAL-VALUED FUNCTIONS AND CONTINUITY OF METRIC PROJECTIONS

SIMON FITZPATRICK

ABSTRACT. We characterize the Fréchet differentiability of real-valued functions on certain real Banach spaces in terms of a directional derivative being equal to a modified version of the local Lipschitz constant. This yields the continuity of metric projections onto closed sets whose distance functions have directional derivatives equal to 1, provided the Banach space and its dual have Fréchet differentiable norms.

1. Introduction. Let E be a real Banach space and let M be a closed subset of E. We define the *distance function*

$$\varphi_M(x) = \inf\{\|y - x\|: y \in M\}$$

and the *metric projection*

$$P_M(x) = \{y \in M: \|y - x\| = \varphi_M(x)\},$$

which assigns to each $x \in E$ the set of *nearest points* in M to x. We call a sequence (y_n) from M a *minimizing sequence* for x provided $\|x - y_n\| \to \varphi_M(x)$ as $n \to \infty$ and we say that P_M is *continuous* at x provided $y_n \to y_0$ whenever $y_n \in P_M(x_n)$ for all $n \geq 0$ and $x_n \to x_0$. If every minimizing sequence for x converges then P_M is continuous at x; the converse holds in Banach spaces whose norms are sufficiently well behaved (see [**2**]).

For a real-valued function f on E, a point x of E and $u \in S(E) = \{y \in E: \|y\| = 1\}$ define the *directional derivative*

$$D_u f(x) = \lim_{t \to 0} t^{-1}[f(x + tu) - f(x)] \tag{1.1}$$

if it exists. If the limit in (1.1) exists uniformly for $u \in S(E)$ we say that f is *Fréchet differentiable* at x; that is equivalent to existence of $x^* \in E^*$ (the *Fréchet derivative* of f at x) such that

$$\lim_{\|y\| \to 0} \frac{f(x + y) - f(x) - x^*(y)}{\|y\|} = 0 \tag{1.2}$$

and we write $f'(x) = x^*$ and note that $f'(x)(u) = D_u f(x)$ for $u \in S(E)$.

In §2 we will show that a real-valued function f is Fréchet differentiable at a point x whenever there exists $u \in S(E)$ such that the norm is Fréchet differentiable at u and $D_u f(x)$ equals a number we define and call $N_f(x)$; it is dominated by the local Lipschitz constant of f at x. For reflexive Banach spaces whose norms are Fréchet differentiable (except at 0, of course) this characterizes the Fréchet derivative of a real-valued function.

Received by the editors September 20, 1983.
1980 *Mathematics Subject Classification*. Primary 46G05, 41A65.

Applying this in §3 we show that if $D_u\varphi_M(x) = 1$ for some $u \in S(E)$ and the norm of E has Fréchet derivative u^* at the point u then φ_M is Fréchet differentiable at x and $\varphi'_M(x) = u^*$. If, in addition, the norm of E^* is Fréchet differentiable at u^* then the corresponding metric projection P_M is continuous at x. This uses and improves on results from [**2**].

This continuity result when combined with a result of Vlasov [**6**] shows that if E and E^* have Fréchet differentiable norms and φ_M has a directional derivative equal to 1 at each point outside M, then M is convex.

2. Differentiability of real-valued functions. Suppose f is a real-valued function on a Banach space E and $x \in E$. We define the *local Lipschitz constant* of f at x by

$$L_f(x) = \lim_{\delta \to 0+} \omega_f(x, \delta),$$

where

$$\omega_f(x,\delta) = \sup\{|f(y) - f(z)|/\|y - z\| \colon \|y - x\| \le \delta, \|z - x\| \le \delta, y \ne z\}.$$

Similarly we define

$$N_f(x) = \lim_{\delta \to 0+} \nu_f(x, \delta),$$

where

$$\nu_f(x,\delta) = \sup\{|f(y) - f(z)|/\|y - z\| \colon 2\|z - x\| \le \|y - x\| \le \delta, y \ne z\}.$$

EXAMPLE. To see that L_f and N_f are not always equal, consider $f(x) = x^2\sin(x^{-1})$ for $x \in \mathbf{R}\backslash\{0\}$, taking $f(0) = 0$. Then $L_f(0) = 1$ while $N_f(0) = 0$ and $f'(0) = 0$. This suggests that N_f has more to do with the derivative than L_f has.

We will need the following trivial lemma.

2.1 LEMMA. *$N_f(x) \le L_f(x)$ and $N_f(x)$ is the least number N such that for each $\varepsilon > 0$ there is $\delta > 0$ with $|f(y) - f(z)| \le (N + \varepsilon)\|y - z\|$ whenever $2\|z - x\| \le \|y - x\| \le \delta$.*

Now we show why $N_f(x)$ is a good bound on the derivatives.

2.2 THEOREM. (i) *If $u \in S(E)$ and $D_uf(x)$ exists then $D_uf(x) \le N_f(x)$.*
(ii) *If f is Fréchet differentiable at x then $\|f'(x)\| = N_f(x)$.*

PROOF. (i) For $t > 0$ we have

$$t^{-1}(f(x + tu) - f(x)) = (f(x + tu) - f(x))/\|tu\| \le \nu_f(x, t),$$

so in the limit as $t \to 0+$ we obtain $D_uf(x) \le N_f(x)$.

(ii) Since $\|f'(x)\| = \sup\{f'(x)(u) \colon u \in S(E)\}$ and $f'(x)(u) = D_uf(x) \le N_f(x)$ for all $u \in S(E)$, we have $\|f'(x)\| \le N_f(x)$. Write $x^* = f'(x)$ and for each $\varepsilon > 0$ choose $\delta > 0$ such that $\|y - x\| < \delta$ implies that

$$|f(y) - f(x) - x^*(y - x)| \le \varepsilon\|y - x\|.$$

Then for $2\|z - x\| \le \|y - x\| \le \delta$ we have

$$|f(y) - f(x) - x^*(y - x) - f(z) + f(x) + x^*(z - x)| \le \varepsilon\|y - x\| + \varepsilon\|z - x\|$$

so that

$$|f(y) - f(z)| \le |x^*(y - z)| + \varepsilon(\|y - x\| + \|z - x\|).$$

Now $|x^*(y-z) \le \|f'(x)\| \cdot \|y-z\|$ and $2\|z-x\| \le \|y-x\|$ implies by the triangle inequality that $\|z-x\| \le \|y-z\|$ and that $\|y-x\| \le 2\|y-z\|$. Thus

$$|f(y)-f(z)| \le \|f'(x)\| \cdot \|y-z\| + 3\varepsilon\|y-z\|$$

so that $N_f(x) \le \|f'(x)\|$ by Lemma 2.1, which completes the proof.

2.3 COROLLARY. *If E is reflexive and f is Fréchet differentiable at x then there is $u \in S(E)$ with $N_f(x) = D_u f(x)$.*

PROOF. We have $N_f(x) = \|f'(x)\|$ and since E is reflexive there is $u \in S(E)$ such that $f'(x)(u) = \|f'(x)\|$. Thus $D_u f(x) = f'(x)(u) = N_f(x)$.

From the definiton we easily see that if $N_f(x) = 0$ then f is Fréchet differentiable at x and $f'(x) = 0$. Our main result is a partial converse to Corollary 2.3.

2.4 THEOREM. *Let E be a real Banach space and f a real-valued function on E. Suppose that $D_u f(x) = N_f(x)$ for some $x \in E$ and $u \in S(E)$. If the norm of E is Fréchet differentiable at u with derivative u^*, then f is Fréchet differentiable at x and $f'(x) = N_f(x)u^*$.*

PROOF. Let $0 < \varepsilon \le \frac{1}{2}$ and choose $0 < \gamma \le \varepsilon$ such that $\|u+z\| - \|u\| \le u^*(z) + \varepsilon\|z\|$ whenever $\|z\| \le \gamma$. Now let $\delta > 0$ be such that $|f(x+tu) - f(x) - tD_u f(x)| \le \gamma^2|t|$ whenever $|t| \le \delta$ and $|f(y)-f(z)| \le (N_f(x)+\gamma^2)\|y-z\|$ whenever $0 \le 2\|z-x\| \le \|y-x\| \le \delta$.

Suppose $0 < \|w\| \le \gamma\delta$ and let $t = \gamma^{-1}\|w\|$. Thus $0 < t \le \delta$, $\|t^{-1}w\| = \gamma$ and $2\|w\| = 2\gamma t \le t = \|tu\|$. Hence $\|u \pm t^{-1}w\| - \|u\| \le \pm u^*(t^{-1}w) + \varepsilon\|t^{-1}w\|$ and $|f(x+w) - f(x \pm tu)| \le (N_f(x) + \gamma^2)\|w \mp tu\|$. Now

$$\begin{aligned}
f(x+w) - f(x) &= \{f(x+w) - f(x-tu)\} + \{f(x-tu) - f(x)\} \\
&\le (N_f(x) + \gamma^2)\|w+tu\| - D_u f(x)t + \gamma^2 t \\
&= N_f(x)(\|w+tu\| - t) + \gamma^2(\|w+tu\| + t) \\
&\le tN_f(x)(\|u + t^{-1}w\| - \|u\|) + 3\gamma^2 t \\
&\le tN_f(x)(u^*(t^{-1}w) + \varepsilon\|t^{-1}w\|) + 3\gamma^2 t \\
&\le N_f(x)u^*(w) + \varepsilon N_f(x)\|w\| + 3\gamma\|w\| \\
&\le N_f(x)u^*(w) + \varepsilon(N_f(x) + 3)\|w\|.
\end{aligned}$$

For the reverse inequality

$$\begin{aligned}
f(x+w) - f(x) &= \{f(x+w) - f(x+tu)\} + \{f(x+tu) - f(x)\} \\
&\ge -(N_f(x) + \gamma^2)\|tu - w\| + D_u f(x)t - \gamma^2 t \\
&= -tN_f(x)(\|u - t^{-1}w\| - \|u\|) - \gamma^2(t + \|tu - w\|) \\
&\ge -tN_f(x)(u^*(-t^{-1}w) + \varepsilon\|t^{-1}w\|) - 3\gamma^2 t \\
&= N_f(x)u^*(w) - N_f(x)\varepsilon\|w\| - 3\gamma\|w\| \\
&\ge N_f(x)u^*(w) - \varepsilon(N_f(x) + 3)\|w\|.
\end{aligned}$$

Therefore $|f(x+w) - f(x) - N_f(x)u^*(w)| \le (N_f(x) + 3)\varepsilon\|w\|$ whenever $\|w\| \le \gamma\delta$ and hence $f'(x) = N_f(x)u^*$.

2.5 COROLLARY. *Let E be a reflexive Banach space with Fréchet differentiable norm. A real valued function f on E is Fréchet differentiable at a point x if and only if there is $u \in S(E)$ such that $D_u f(x)$ exists and equals $N_f(x)$.*

PROOF. If such a vector u exists then Theorem 2.4 shows that $f'(x)$ exists. Conversely if $f'(x)$ exists then Corollary 2.3 shows that there is such a vector u.

Since $L_f(x) \geq N_f(x) \geq D_u f(x)$ (if this exists) for all $u \in S(E)$ we have the following result.

2.6 COROLLARY. *If $D_u f(x) = L_f(x)$ for some $x \in E$ and $u \in S(E)$ such that the norm of E is Fréchet differentiable at u with derivative u^*, then f is Fréchet differentiable at x and $f'(x) = L_f(x)u^*$.*

3. Continuity of metric projections. In this section we apply the results of the previous section to the distance function $\varphi = \varphi_M$ from a closed subset M of the real Banach space E to get continuity results for the metric projection $P = P_M$.

Recall that $u^* \in E^*$ *strongly exposes* a subset C of E at a point z of C provided $\|y_n - z\| \to 0$ whenever $y_n \in C$ and $u^*(y_n) \to u^*(z)$. The following basic result dates back to Šmulian [**5**]; see also [**3**, §3.4 or **4**]. Let $B(E)$ denote the closed unit ball of E.

3.1 PROPOSITION. *The norm of E^* is Fréchet differentiable at a point u^* of E^* if and only if u^* strongly exposes $B(E)$. The norm of E is Fréchet differentiable at a point u with derivative u^* if and only if $B(E^*)$ is strongly exposed at u^* by $u \in E \subseteq E^{**}$.*

We will need the following result from [**2**].

3.2 PROPOSITION. *If $x \in E \backslash M$ is a point of Fréchet differentiability of φ then $\|\varphi'(x)\| = 1$. If $\varphi'(x)$ strongly exposes the unit ball of E at a point z then every minimizing sequence for x converges to $x - \varphi(x)z$.*

The following result improves Theorem 2.4(a) of [**2**].

3.3 THEOREM. *Suppose that $x \in E$ and $u \in S(E)$ with $D_u\varphi(x) = 1$. If u^* strongly exposes $B(E)$ at u and u strongly exposes $B(E^*)$ at u^*, then φ is Fréchet differentiable at x with derivative u^* and every minimizing sequence for x in M converges to $x - \varphi(x)u$.*

PROOF. Since $|\varphi(y) - \varphi(z)| \leq \|y - z\|$ for all y and z we have $1 = D_u\varphi(x) \leq N_\varphi(x) \leq L_\varphi(x) \leq 1$ so $D_u\varphi(x) = L_\varphi(x) = 1$ and the norm of E is Fréchet differentiable at u with derivative u^* by Proposition 3.1. Now Corollary 2.6 shows that φ is Fréchet differentiable at x with derivative u^* and the other statement follows from Proposition 3.2.

3.4 COROLLARY. *Suppose that the norms of E and E^* are Fréchet differentiable. If $D_u\varphi(x) = 1$ for some $u \in S(E)$ then P is continuous at x.*

PROOF. By Proposition 3.1, u strongly exposes $B(E^*)$ at a point u^* and u^* strongly exposes $B(E)$, necessarily at the point u. Now Theorem 3.3 shows that every minimizing sequence for x converges to $x - \varphi(x)u$ so P is continuous at x and $P(x) = x - \varphi(x)u$.

A Banach space E is *strictly convex* if $S(E)$ contains no line segments and M is a *Čebyšev set* in E provided every $x \in E$ has a unique nearest point in M. We need the following result of Vlasov [**6**]; see also [**3**, §4.2].

3.5 PROPOSITION. *If E^* is strictly convex, then every Čebyšev set in E with continuous metric projection is convex.*

Our final result improves Theorem 4.2 of [**2**].

3.6 THEOREM. *Let M be a closed subset of E such that for each $x \in E\backslash M$ there exists $u \in S(E)$ such that $D_u\varphi(x) = 1$. If the norms of E and E^* are Fréchet differentiable, then M is convex.*

PROOF. From Corollary 3.4 we see that M is a Čebyšev set and P is continuous at each $x \in E\backslash M$. Also E is reflexive [**1**, p. 34, Corollary 1] and E^* is strictly convex [**1**, p. 24, Corollary 1]. Applying Proposition 3.5 we find that M is convex.

REFERENCES

1. Joseph Diestel, *Geometry of Banach spaces–selected topics,* Springer Verlag, New York, 1975.
2. Simon Fitzpatrick, *Metric projections and the differentiability of distance functions,* Bull. Austral. Math. Soc. **22** (1980), 291–312.
3. J. R. Giles, *Convex analysis with application in differentiation of convex functions,* Pitman, London, 1982.
4. R. R. Phelps, *Differentiability of convex functions on Banach spaces,* lecture notes from a post-graduate course, University College, London, 1978.
5. V. Šmulian, *Sur la dérivabilité de la norme dans les espaces de Banach,* Rec. Math. [Mat. Sb.] N. S. **9(5)** (1941), 545–561.
6. L. P. Vlasov, *Almost convexity and Chebyshev sets,* Math. Notes. Acad. Sci. USSR **8** (1970), 776–779.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF AUCKLAND, PRIVATE BAG, AUCKLAND, NEW ZEALAND