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Diffraction by Anomalous Regions in the Earth's Mantle

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Summary

Asymptotic results are derived for the travel times and amplitudes of waves diffracted by surfaces of discontinuity in the Earth's mantle. A canonical problem, defined to be that of diffraction by a uniform elastic or visco-elastic sphere in a uniform elastic medium, is first solved asymptotically and the result applied, by means of simple ray theory, to an Earth model which is spherically symmetric except for the diffracting discontinuity. Some calculations are carried out for comparison with the signals from the Bukhara explosion discussed by Douglas *et al.* and an attempt is made to identify their arrival P_{H_1} with a diffracted arrival from a mantle discontinuity. To explain the low amplitude of the direct wave a dissipative region in the mantle, which is avoided by the diffracted wave, is postulated and the approximate values of Q needed to equate the amplitudes of direct and diffracted waves are calculated.

Introduction

In a recent article concerning the analysis of the first few seconds of seismic records, Douglas *et al.* (1972) have stressed the importance of multiple paths for body waves from the source to the receiver. In that article the authors are concerned with *pP* arrivals which, after the initial reflection at the Earth's surface, travel to the receiver along a path close to that of *P*. In an earlier paper (Douglas, Marshall & Corbishley 1971) evidence is given for arrivals which have paths markedly different from that of *P* (that is, they arrive with different angles of emergence) and yet cannot be classified as standard arrivals *PP*, *PcP*, *pP* etc. Furthermore it is shown that the main pulse is sometimes greatly diminished so that the later arrivals, being relatively enhanced, give the seismogram apparent complexity. We examine in the present paper the possibility than an arrival following *P* might be a wave diffracted by a discontinuity in the mantle. If, in addition, the direct *P* wave has been attenuated in passing through a dissipative region in the mantle, which is avoided by the diffracted wave, then discontinuities of sufficient size may produce waves of comparable amplitude at a given station.

We wish to find the amplitudes of waves whose ray paths are shown in Fig. 1. For sufficiently short wavelengths ray theory (Jeffreys 1959; Karal & Keller 1959; Bullen 1963) may be applied along paths to and from the discontinuity, but this method fails to give the ' conversion factors ' for creeping waves around the inclusion so that a canonical problem must first be solved. This problem is defined to be that of finding the diffracted wave generated when a wave in a uniform elastic medium is incident on a spherical inclusion of different elastic and visco-elastic properties.

Several authors (Scholte 1956; Duwalo & Jacobs 1959; Nussenzveig 1965, 1969; Nagase 1956; Ansell 1969) have examined the problem of diffraction by a sphere. The wave type and the boundary conditions applied have been various, but the

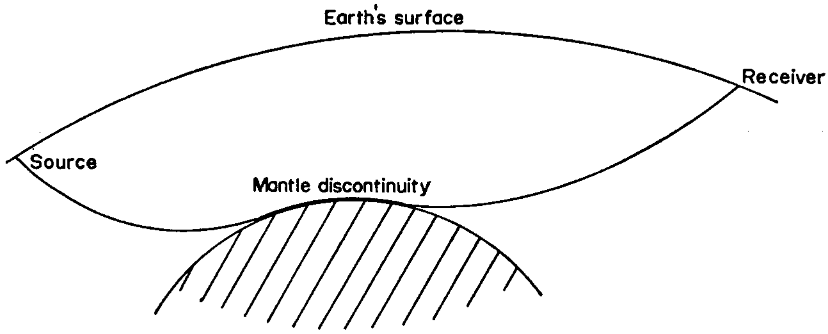

FIG. 1. A diffracted ray path through the mantle.

diffraction of elastic waves by an elastic sphere of different material from that of the surrounding elastic medium has not yet been investigated. The treatments given by the above authors follow the methods developed by Watson (1918) and van der Pol & Bremmer (1937) to obtain expressions valid at high frequencies for the diffracted waves in addition to terms corresponding to waves travelling by the ray paths of geometrical optics. A similar method for a pulse inside a liquid sphere with a free surface has been given by Jeffreys & Lapwood (1957), though in this case there is no ordinary diffracted wave. Diffraction by cylinders has been treated in a similar fashion (e.g. Gilbert & Knopoff 1959, who give further references).

The diffraction of elastic waves by a liquid sphere in an unbounded solid has been investigated by Scholte (1956), Duwalo & Jacobs (1959) and Ansell (1969), the last paying special attention to the regions of the shadow boundaries, and our analysis and results are similar to those given by these authors.

Sections 1 and 2 give an asymptotic solution to the canonical problem and Section 3 gives the ray theory necessary to apply this solution to a spherically symmetric Earth model. Finally the results of a calculation for particular waves observed in the Earth are given.

1. Formal solution of the canonical problem

1.1 *Equations of motion and their solution in series*

The equations of motion for elastic waves in a homogeneous, isotropic medium may be written in terms of the displacement $\mathbf{u}$ and in standard notation:

$$(\lambda+2\mu)\,\nabla(\nabla.\mathbf{u})-\mu\nabla\wedge(\nabla\wedge\mathbf{u}) = \rho\,\frac{\partial^2\mathbf{u}}{\partial t^2}.$$

For sufficiently regular solutions we may write (Morse & Feshbach 1953):

$$\mathbf{u} = \nabla\phi+\nabla\wedge(\mathbf{r}\chi)+\nabla\wedge(\nabla\wedge\mathbf{r}\psi)$$

where $\mathbf{r}$ is the position vector of the field point and the potentials satisfy:

$$\nabla^2\phi-\frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}=0$$

$$\nabla^2\psi-\frac{1}{\beta^2}\frac{\partial^2\psi}{\partial t^2}=0$$

$$\nabla^2\chi-\frac{1}{\beta^2}\frac{\partial^2\chi}{\partial t^2}=0$$

with $\alpha^2 = \lambda+2\mu/\rho$, $\beta^2 = \mu/\rho$.

ϕ corresponds to a dilatational disturbance with associated wave speed α, ψ and χ to a rotational disturbance with associated wave speed β. When there is an axis of symmetry through the point $\mathbf{r} = 0$ the potentials ϕ, ψ give rise to displacements lying in planes passing through the axis and χ gives displacements perpendicular to these planes, so that referred to a spherical surface centred at $\mathbf{r} = 0$, the pair of potentials ϕ, ψ represent a *P–SV* system of waves and χ represents the *SH* system. It will be shown, however, that even when there is no axis of symmetry the two-wave systems are independent, provided the boundary conditions are applied on surfaces $|\mathbf{r}|$ = constant.

Assuming a time dependence $\phi = \phi_0 \, e^{-i\omega t}$ (or alternatively, taking Fourier transforms in time) we obtain:

$$\left.\begin{aligned}(\nabla^2 + h^2)\phi &= 0\\ (\nabla^2 + k^2)\psi &= 0\\ (\nabla^2 + k^2)\chi &= 0\end{aligned}\right\} \tag{1}$$

with $h = \omega/\alpha$, $k = \omega/\beta$.

General solutions of these Helmholtz equations in terms of spherical polar co-ordinates r, θ, ϕ may be written (Morse & Feshbach 1953)

$$\phi = \sum_{q=1,2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,q}^{m} \, h_n^{(q)}(hr) \, P_n^{|m|}(\cos\theta) \, e^{im\phi}$$

$$\psi = \sum_{q=1,2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{n,q}^{m} \, h_n^{(q)}(kr) \, P_n^{|m|}(\cos\theta) \, e^{im\phi} \tag{2}$$

$$\chi = \sum_{q=1,2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_{n,q}^{m} \, h_n^{(q)}(kr) \, P_n^{|m|}(\cos\theta) \, e^{im\phi}$$

with

$$h_n^{(q)}(x) = \sqrt{\left(\frac{\pi}{2x}\right)} \, H_{n+\frac{1}{2}}^{(q)}(x), q = 1, 2. \tag{3}$$

$h_n^{(q)}$ are spherical Hankel functions of the first and second kinds and $a_{n,q}^{m}$, $b_{n,q}^{m}$, $d_{n,q}^{m}$ are constants.

1.2 *Stresses and strains*

Let us define the following operators:

$$L = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}$$

$$M = \frac{1}{r^2}\left(r\frac{\partial}{\partial r} - 1\right)$$

$$N = \nabla^2 - \frac{2}{r^2}\left(L + r\frac{\partial}{\partial r} + 1\right)$$

$$K = \left(r\frac{\partial}{\partial r} + 1\right).$$

The stresses and strains derivable from the potentials ϕ, χ, ψ may be written:

$$\left.\begin{aligned} u_r &= \frac{\partial\phi}{\partial r} - \frac{1}{r}L\psi \\ u_\theta &= \frac{1}{r}\frac{\partial}{\partial\theta}(\phi+K\psi) + \frac{1}{\sin\theta}\frac{\partial\chi}{\partial\phi} \\ u_\phi &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\phi+K\psi) - \frac{\partial\chi}{\partial\theta} \end{aligned}\right\} \tag{4}$$

$$\tau_{rr} = \lambda\nabla^2\phi + 2\mu\frac{\partial u_r}{\partial r}$$

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\tau_{r\phi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)$$

and hence:

$$\left.\begin{aligned} \tau_{rr} &= \left\{(\lambda+2\mu)\nabla^2 - \frac{2\mu}{r^2}\left(2r\frac{\partial}{\partial r}+L\right)\right\}\phi - 2\mu ML\psi \\ \tau_{r\theta} &= \mu\frac{\partial}{\partial\theta}(2M\phi+N\psi) + \frac{r\mu}{\sin\theta}\frac{\partial}{\partial\phi}M\chi \\ \tau_{r\phi} &= \frac{\mu}{\sin\theta}\frac{\partial}{\partial\phi}(2M\phi+N\psi) - r\mu\frac{\partial}{\partial\theta}M\chi. \end{aligned}\right\} \tag{5}$$

1.3 *Geometry and boundary conditions*

A sphere of radius a centred at the origin of spherical polar co-ordinates r, θ, ϕ contains an elastic (or linearly visco-elastic) material with parameters ρ', λ', μ' with λ', μ' possibly complex and frequency dependent. The sphere is in welded contact with an infinite elastic medium with parameters ρ, λ, μ. Although much of the analysis is more general we shall eventually consider a symmetric, harmonic point source of longitudinal waves, situated at the point $r = r_0$, $\theta = 0$, and we shall be primarily concerned with the diffracted wave observed in the deep shadow where $\theta_0 < \theta < \pi$ and

$$\theta_0 = \cos^{-1}\frac{a}{r} + \cos^{-1}\frac{a}{r_0}. \qquad \text{(See Fig. 2)}$$

We shall assume that the complete solution is given by the expansions (2) outside the sphere (and near its surface) and by the corresponding expansions with coefficients $a^{m'}_{n,q}$, $b^{m'}_{n,q}$, $d^{m'}_{n,q}$ inside. We assume also that the incident wave is given by a similar series with coefficients $A^m_{n,q}$, $B^m_{n,q}$, $D^m_{n,q}$, in the neighbourhood of the boundary. Since we anticipate a singularity in the incident field, different expansions will be valid at points far from the sphere.

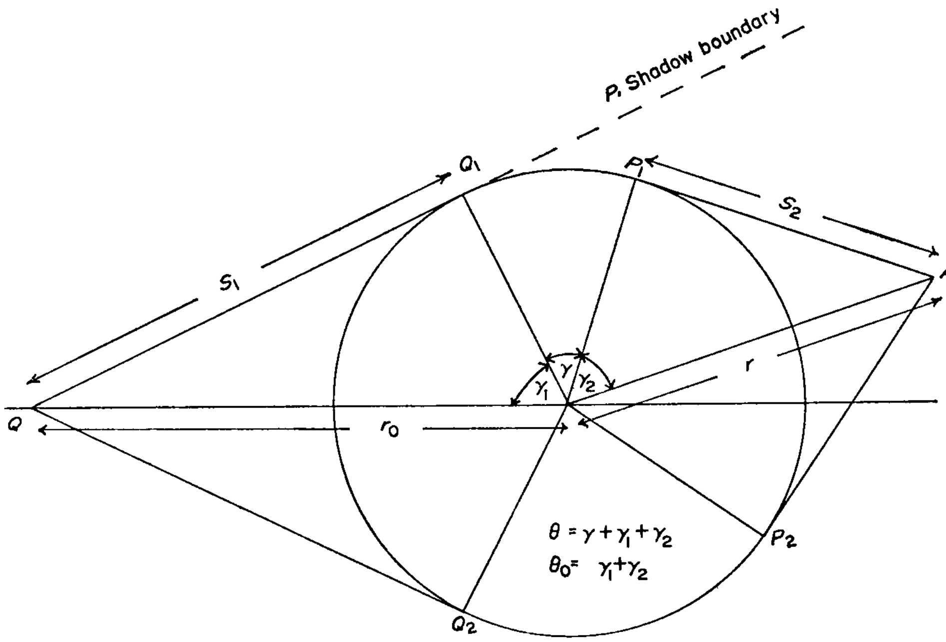

FIG. 2. The geometry of the canonical problem.

The boundary conditions to be applied at the surface of the sphere are that **u** is continuous and that τ_{rr}, $\tau_{r\theta}$, $\tau_{r\phi}$ are continuous. Using these boundary conditions together with equations (4) and (5) it is easily shown to be necessary that $L(\phi+K\psi)$, $L\chi$, $\mu L(2M\phi+N\psi)$, $\mu LM\chi$ be continuous, but since ϕ, χ, ψ are sums of eigenfunctions of L, and since L commutes with K, M, N this implies continuity of $\phi+K\psi$, χ, $\mu(2M\phi+N\psi)$, $\mu M\chi$. (In fact each of these quantities may have a constant discontinuity over the surface of the sphere since the eigenvalue of $P_0(\cos\theta)$ is zero. These constants, however, make no contribution to the stresses or strains and ψ, χ may be so chosen that they are zero.)

Thus the boundary conditions reduce to continuity of:

$$\frac{\partial\phi}{\partial r}-\frac{1}{r}L\psi$$

$$\phi+K\psi$$

$$\left\{(2\mu+\lambda)\nabla^2-\frac{2\mu}{r^2}\left(2r\frac{\partial}{\partial r}+L\right)\right\}\phi-2\mu ML\psi$$

$$\mu(2M\phi+N\psi)$$

$$\chi$$

$$\mu M\chi.$$

It can now be seen that the boundary conditions on χ are independent of ϕ and ψ so that the wave systems are completely independent.

We make the following definitions:

$$\left.\begin{aligned} X_n^{(q)}(x) &\equiv \frac{x\dfrac{d}{dx}h_n^{(q)}(x)}{h_n^{(q)}(x)} \\ X_n^{(q)} &\equiv X_n^{(q)}(ha) \\ Y_n^{(q)} &\equiv X_n^{(q)}(ka) \end{aligned}\right\} \qquad (6)$$

and note that

$$L P_n^{|m|}(\cos\theta)\,\mathrm{e}^{im\phi} = -n(n+1)\,P_n^{|m|}(\cos\theta)\,\mathrm{e}^{im\phi}.$$

We may now use equations (1) and (2) and the independence of the spherical harmonics to write the boundary conditions as follows:

$$\left.\begin{aligned}
&X_n^{(q)}\,h_n^{(q)}(ha)\,a_{n,q}^m + n(n+1)\,h_n^{(q)}(ka)\,b_{n,q}^m \\
&\quad = X_n^{(q)\prime}\,h_n^{(q)}(h'a)\,a_{n,q}^{m'} + n(n+1)\,h_n^{(q)}(k'a)\,b_{n,q}^{m'} \\
&h_n^{(q)}(ha)\,a_{n,q}^m + (Y_n^{(q)}+1)\,h_n^{(q)}(ka)\,b_{n,q}^m \\
&\quad = h_n^{(q)}(h'a)\,a_{n,q}^{m'} + (Y_n^{(q)\prime}+1)\,h_n^{(q)}(k'a)\,b_{n,q}^{m'} \\
&[k^2a^2 + 4X_n^{(q)} - 2n(n+1)]\,h_n^{(q)}(ha)\,a_{n,q}^m + 2\mu n(n+1)(Y_n^{(q)}-1)\,h_n^{(q)}(ka)\,b_{n,q}^m \\
&\quad = [k'^2a^2 + 4X_n^{(q)\prime} - 2n(n+1)]\,h_n^{(q)}(h'a)\,a_{n,q}^{m'} \\
&\qquad\qquad + 2\mu' n(n+1)(Y_n^{(q)\prime}-1)\,h_n^{(q)}(k'a)\,b_{n,q}^{m'} \\
&2\mu(X_n^{(q)}-1)\,a_{n,q}^m\,h_n^{(q)}(ha) - \mu[k^2a^2 - 2\{n(n+1) - Y_n^{(q)} - 1\}]\,h_n^{(q)}(ka)\,b_{n,q}^m \\
&\quad = 2\mu'(X_n^{(q)\prime}-1)\,a_{n,q}^{m'}\,h_n^{(q)}(h'a) \\
&\qquad\qquad - \mu'[k'^2a^2 - 2\{n(n+1) - Y_n^{(q)\prime} - 1\}]\,h_n^{(q)}(k'a)\,b_{n,q}^{m'}
\end{aligned}\right\}\quad (7)$$

$$\left.\begin{aligned}
d_{n,q}^m\,h_n^{(q)}(ka) &= d_{n,q}^{m'}\,h_n^{(q)}(k'a) \\
\mu(Y_n^{(q)}-1)\,d_{n,q}^m\,h_n^{(q)}(ka) &= \mu'(Y_n^{(q)\prime}-1)\,d_{n,q}^{m'}\,h_n^{(q)}(k'a).
\end{aligned}\right\}\quad (8)$$

Primes denote quantities relating to the interior of the sphere and summation over $q = 1, 2$ is understood.

Now the functions $h_n^{(q)}(kr)$ are singular at $r = 0$ but the combination

$$h_n^{(1)}(kr) + h_n^{(2)}(kr)$$

is finite there. Since we require the potentials to be finite at the origin we have the three relations:

$$\left.\begin{aligned} a^{m'}_{n,1} &= a^{m'}_{n,2} \\ b^{m'}_{n,1} &= b^{m'}_{n,2} \\ d^{m'}_{n,1} &= d^{m'}_{n,2}\,. \end{aligned}\right\} \qquad (9)$$

We also impose the radiation condition that no additional incoming wave is generated outside the sphere. Since potentials containing $h_n^{(2)}(hr)$, $h_n^{(2)}(kr)$ represent incoming waves (cf. asymptotic expansions for large r) this gives the three relations:

$$\left.\begin{aligned} a^{m}_{n,2} &= A^{m}_{n,2} \\ b^{m}_{n,2} &= B^{m}_{n,2} \\ d^{m}_{n,2} &= D^{m}_{n,2}\,. \end{aligned}\right\} \qquad (10)$$

For each n, m equations (7)–(10) give 12 equations for the 12 unknown coefficients $a^{m}_{n,q}$, $b^{m}_{n,q}$, $d^{m}_{n,q}$, $a^{m'}_{n,q}$, $b^{m'}_{n,q}$, $d^{m'}_{n,q}$ so that the formal solution given by (2) is determined.

1.4 *Decomposition into partial solutions*

The solution found in Section 1.3 involves direct waves, diffracted waves, and multiply reflected and refracted waves which have passed through the inclusion. Scholte (1956), Duwalo & Jacobs (1959), and Nussenzveig (1969) show that in the cases they consider, the total wave may be split into these separate parts and that the parts may be examined separately in order to evaluate the different effects in different regions. The expansion obtained is more complex than those obtained previously since refracted and reflected P and SV waves are produced each time a ray strikes the surface of the inclusion.

We begin by defining vectors $\mathbf{U}_q$, $\mathbf{V}_q$ which occur as columns in the determinants needed in the solution of equations (7)–(10):

$$\mathbf{U}_q = \begin{pmatrix} X_n^{(q)}\, h_n^{(q)}(ha) \\ h_n^{(q)}(ha) \\ -\mu(k^2 a^2 + 4X_n^{(q)} - 2n(n+1))\, h_n^{(q)}(ha) \\ 2\mu(X_n^{(q)} - 1)\, h_n^{(q)}(ha) \end{pmatrix}$$

$$\mathbf{V}_q = \begin{pmatrix} n(n+1)\, h_n^{(q)}(ka) \\ (Y_n^{(q)} + 1)\, h_n^{(q)}(ka) \\ 2\mu n(n+1)(Y_n^{(q)} - 1)\, h_n^{(q)}(ka) \\ -\mu[k^2 a^2 - 2n(n+1) + 2(Y_n^{(q)} + 1)]\, h_n^{(q)}(ka) \end{pmatrix}.$$

$\mathbf{U}_q'$, $\mathbf{V}_q'$ are defined in the obvious way. Equations (7)–(10) may now be written:

$$A\mathbf{a} = \mathbf{b}$$

$$B\mathbf{c} = \mathbf{d}$$

where:

$$A = (\mathbf{U}_1' + \mathbf{U}_2', \; -\mathbf{U}_1, \mathbf{V}_1' + \mathbf{V}_2', \; -\mathbf{V}_1)$$

$$\mathbf{b} = A^m_{n,\,2}\,\mathbf{U}_2 + B^m_{n,\,2}\,\mathbf{V}_2$$

$$\mathbf{a} = \begin{pmatrix} a^{m'}_{n,\,1} = a^{m'}_{n,\,2} \\ a^m_{n,\,1} \\ b^{m'}_{n,\,1} = b^{m'}_{n,\,2} \\ b^m_{n,\,1} \end{pmatrix}$$

$$B = \begin{pmatrix} h_n^{(1)}(k'a) + h_n^{(2)}(k'a) & -h_n^{(1)}(ka) \\ \mu'(Y_n^{(1)\prime} - 1)\,h_n^{(1)}(k'a) + \mu'(Y_n^{(2)\prime} - 1)\,h_n^{(2)}(k'a) & -\mu(Y_n^{(1)} - 1)\,h_n^{(1)}(ka) \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} D^m_{n,\,2}\,h_n^{(2)}(ka) \\ \mu(Y_n^{(2)} - 1)\,D^m_{n,\,2}\,h_n^{(2)}(ka) \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} d^{m'}_{n,\,1} = d^{m'}_{n,\,2} \\ d^m_{n,\,1} \end{pmatrix}.$$

We may solve the equations for $a^m_{n,\,1}$:

$$a^m_{n,\,1} = \frac{-A^m_{n,\,2}\,D(\mathbf{U}_1' + \mathbf{U}_2', \mathbf{U}_2, \mathbf{V}_1' + \mathbf{V}_2', \mathbf{V}_1)}{D(\mathbf{U}_1' + \mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_1' + \mathbf{V}_2', \mathbf{V}_1)} - B^m_{n,\,2}\cdot\frac{D(\mathbf{U}_1' + \mathbf{U}_2', \mathbf{V}_2, \mathbf{V}_1' + \mathbf{V}_2', \mathbf{V}_1)}{D(\mathbf{U}_1' + \mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_1' + \mathbf{V}_2', \mathbf{V}_1)} \tag{11}$$

where D denotes determinant.

We now define refraction and reflection coefficients for each spherical harmonic by considering waves approaching and leaving the interface. Now in defining these coefficients we are concerned with the local interaction of an incident wave with the interface; therefore instead of the condition that the potentials should be finite at $r = 0$, we impose the boundary condition that only waves locally emanating from the boundary should be generated (cf. the definition of plane wave refraction and reflection coefficients). Spherical Hankel functions of the first kind are associated with waves approaching the interface inside the sphere and with waves leaving the interface in the surrounding medium; those of the second kind are associated with waves leaving the interface inside the sphere and with waves approaching the interface in the surrounding medium. If, for example, a P wave with expansion coefficients $A^{m'}_{n,\,1}$ is incident on the boundary from inside the sphere the S wave outside the sphere, when the above boundary condition is applied, is given by:

$$b^m_{n,\,1} = (P'S)_{n,\,m}\,A^{m'}_{n,\,1}$$

where

$$(P'S)_{n,\,m} \equiv \frac{D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_2', \mathbf{U}_1')}{D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_2', \mathbf{V}_1)}.$$

We associate wave numbers h, h' with dilatational (P) waves and k, k' with shear (S) waves and thus $\mathbf{U}_1$ is associated with outgoing P waves (outgoing with respect to the centre of the sphere), $\mathbf{U}_2$ with incoming P waves, $\mathbf{V}_1$ with outgoing S waves and $\mathbf{V}_2$ with incoming S waves. $\mathbf{U}_1'$, $\mathbf{U}_2'$, $\mathbf{V}_1'$, $\mathbf{V}_2'$ are similarly associated but outgoing waves inside the sphere are approaching the interface and incoming waves are leaving it. With these associations we see that $(P'S)_{n,m}$ depends only upon the P wave approaching the interface inside the sphere (the incident wave) and upon the four waves emanating from the interface (cf. Scholte 1956). The complete set of coefficients is as follows:

$$(PP) = -\frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{U}_2, \mathbf{V}_2', \mathbf{V}_1)$$

$$(PS) = -\frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_2', \mathbf{U}_2)$$

$$(PP') = \frac{1}{D}\,.\,D(\mathbf{U}_2, \mathbf{U}_1, \mathbf{V}_2', \mathbf{V}_1)$$

$$(PS') = \frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{U}_2, \mathbf{V}_1)$$

$$(P'P') = -\frac{1}{D}\,.\,D(\mathbf{U}_1', \mathbf{U}_1, \mathbf{V}_2', \mathbf{V}_1)$$

$$(S'P') = -\frac{1}{D}\,.\,D(\mathbf{V}_1', \mathbf{U}_1, \mathbf{V}_2', \mathbf{V}_1)$$

$$(P'S') = -\frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{U}_1', \mathbf{V}_1)$$

$$(S'S') = -\frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_1', \mathbf{V}_1)$$

$$(P'P) = \frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{U}_1'. \mathbf{V}_2', \mathbf{V}_1)$$

$$(P'S) = \frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_2', \mathbf{U}_1')$$

$$(S'P) = \frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{V}_1', \mathbf{V}_2', \mathbf{V}_1)$$

$$(S'S) = \frac{1}{D}\,.\,D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_2'. \mathbf{V}_1)$$

where $D = D(\mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_2', \mathbf{V}_1)$ and dependence on n, m is understood.

We assume, for the moment, that $B^m_{n,2} = 0$ (that is, that there is no incident S wave) since the analysis for the second term of (11) is exactly analogous to the treatment of the first term. We have

$$a_{n,1} = -A^m_{n,2}\,.\,\frac{D(\mathbf{U}_1'+\mathbf{U}_2', \mathbf{U}_2, \mathbf{V}_1'+\mathbf{V}_2', \mathbf{V}_1)}{D(\mathbf{U}_1'+\mathbf{U}_2', \mathbf{U}_1, \mathbf{V}_1'+\mathbf{V}_2', \mathbf{V}_1)} \tag{12}$$

and also

$$\mathbf{U}_1'+\mathbf{U}_2' = [1-(P'P')]\,\mathbf{U}_2'+(P'P)\,\mathbf{U}_1-(P'S')\,\mathbf{V}_2'+(P'S)\,\mathbf{V}_1$$
$$\mathbf{V}_1'+\mathbf{V}_2' = -(S'P')\,\mathbf{U}_2'+(S'P)\,\mathbf{U}_1+[1-(S'S')]\,\mathbf{V}_2'+(S'S)\,\mathbf{V}_1$$
$$\mathbf{U}_2 = (P'P)\,\mathbf{U}_2'-(PP)\,\mathbf{U}_1+(PS')\,\mathbf{V}_2'-(PS)\,\mathbf{V}_1$$

so that (12) gives

$$a^m_{n,1} = -A^m_{n,2}\,\frac{\begin{vmatrix} 1-(P'P') & (P'P) & -(P'S') \\ (PP') & -(PP) & (PS') \\ -(S'P') & (S'P) & 1-(S'S') \end{vmatrix}}{\begin{vmatrix} 1-(P'P') & -(P'S') \\ -(S'P') & 1-(S'S') \end{vmatrix}}$$

$$= A^m_{n,2}\left((PP)+\frac{(PP')[1-(S'S')](P'P)+(PP')(P'S')(S'P)+(PS')[1-(P'P')](S'P)+(PS')(S'P)(P'P)}{[1-(P'P')][1-(S'S')]-(P'S')(S'P')}\right).$$

We may use the expansions

$$\frac{1-x}{(1-x)(1-w)-yz} = 1+w+yz+w^2+w^3+xyz+2wyz+\ldots$$

$$\frac{z}{(1-x)(1-w)-yz} = z+wz+xz+w^2z+wxz+yz^2+x^2z+\ldots$$

to obtain

$$\begin{aligned}\frac{a^m_{n,1}}{A^m_{n,2}} = (PP)&+(PP')[1+(P'P')+(P'S')(S'P')+(P'P')^2+(P'P')^3\\ &\qquad+(P'P')(P'S')(S'P')+\ldots](P'P)\\ &+(PP')[(P'S')+(P'S')(S'S')+(P'S')(S'S')^2+(P'P')^2(P'S')+\ldots](S'P)\\ &+\text{etc.}\ldots\end{aligned} \tag{13}$$

Each term is seen to have a clear association with a wave which has been reflected from or refracted through the spherical boundary and suffered multiple reflections (with splitting into P and S) inside the sphere. The coefficients $b^m_{n,1}$ may be treated in the same way. It is expected that further analysis along the lines of Scholte (1956) will confirm that by substituting equation (13) into (2) we get a series of terms which correspond to waves arriving by the ray paths indicated above.

Since we are not primarily concerned with waves which have been refracted into the sphere we shall deal only with the first term of this expansion, so that for an incident P wave we have the partial solution:

$$\left.\begin{aligned} a^m_{n,1} &= A^m_{n,2}\,(PP)_{n,m} \\ a^m_{n,2} &= A^m_{n,2} \end{aligned}\right\} \tag{14}$$

in the neighbourhood of the surface.

1.5 *The point source*

The incident wave will be taken to be that generated by a symmetric, harmonic point source of longitudinal waves, a distance r_0 from the centre of the inclusion (see Fig. 2)

$$\phi_0 = \frac{e^{ihR}}{ihR}$$

where $R^2 = r^2 + r_0^2 - 2rr_0 \cos\theta$. We have the following expansion (Sommerfeld 1949) for ϕ_0:

$$\phi_0 = \begin{cases} \sum_{n=0}^{\infty} (n+\frac{1}{2})[h_n^{(1)}(hr_0)\, h_n^{(1)}(hr) + h_n^{(2)}(hr_0)\, h_n^{(1)}(hr)]\, P_n(\cos\theta), \; r > r_0 \\ \sum_{n=0}^{\infty} (n+\frac{1}{2})[h_n^{(1)}(hr)\, h_n^{(1)}(hr_0) + h_n^{(2)}(hr)\, h_n^{(1)}(hr_0)]\, P_n(\cos\theta), \; r < r_0 \end{cases} \tag{15}$$

so that we may identify:

$$\left.\begin{aligned} A_{n,2}^{0} &= A_{n,1}^{0} = (n+\tfrac{1}{2})\, h_n^{(1)}(hr_0) \\ A_{n,1}^{m} &= A_{n,2}^{m} = 0 \qquad \text{for} \qquad m \neq 0. \end{aligned}\right\} \tag{16}$$

Using (14) to (16) and (2) and (3) we may write the partial solution:

$$\phi = \frac{e^{ihR}}{ihR} + \sum_{n=0}^{\infty} ((PP)_n - 1)(n+\tfrac{1}{2})\, h_n^{(1)}(hr_0)\, h_n^{(1)}(hr)\, P_n(\cos\theta),$$

i.e.

$$\phi = \begin{cases} \dfrac{\pi}{2h\sqrt{(rr_0)}} \sum_{\nu} \nu[(\bar{P}\bar{P})_\nu H_\nu^{(1)}(hr_0) + H_\nu^{(2)}(hr_0)]\, H_\nu^{(1)}(hr)\, P_{\nu-\frac{1}{2}}(\cos\theta) \\ \dfrac{\pi}{2h\sqrt{(rr_0)}} \sum_{\nu} \nu[(\bar{P}\bar{P})_\nu H_\nu^{(1)}(hr) + H_\nu^{(2)}(hr)]\, H_\nu^{(1)}(hr_0)\, P_{\nu-\frac{1}{2}}(\cos\theta) \end{cases} \tag{17}$$

where $\nu = n+\frac{1}{2}$, $(\bar{P}\bar{P})_\nu \equiv (PP)_n$ and the summation over ν is for $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots \infty$.

2. The transformation of the solution

2.1 *The Watson transformation*

From this point the analysis is similar to that given by Scholte (1956), Duwalo & Jacobs (1959), Nussenzveig (1965, 1969) and Ansell (1969). We wish to find the diffracted wave at high frequencies and it is found that the series solution (17) converges very slowly in this domain. Numerical computations with such series have shown that the number of terms which must be retained is (Nussenzveig 1969)

$$l \sim (ha) + c(ha)^{1/3}$$

with $c > 3$ and in our case we anticipate ha ($= 2\pi a/L$ where L is the wavelength) to be at least of the order of 100. Watson (1918) has given a method of transforming

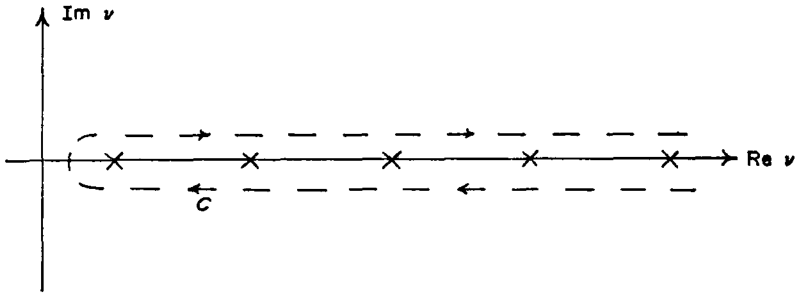

FIG. 3. The contour in the complex ν plane of the integration for the Watson transformation.

such series which has been used by all the above authors, and we shall now apply it to the present problem.

The method is based upon the formula

$$\sum_{n=0}^{\infty} (-1)^n f(n+\tfrac{1}{2}) = \frac{1}{2i} \int_C \frac{f(\nu)}{\cos \pi\nu} \, d\nu \tag{18}$$

where C is the contour shown in Fig. 3. The summation on the left-hand side is recovered as the sum of contributions to the integral at the poles of $\sec \pi\nu$ on the positive real axis. It is assumed that $f(\nu)$ has no poles on this axis.

Using (17) and (18) we have:

$$\phi = \frac{\pi}{4ih\sqrt{(rr_0)}} \int_C \nu [H_\nu^{(2)}(hr_0) + (\bar{P}\bar{P})_\nu H_\nu^{(1)}(hr_0)] \frac{H_\nu^{(1)}(hr)}{\cos \pi\nu} P_{(\nu-\frac{1}{2})}(-\cos\theta) \, d\nu. \tag{19}$$

We may write

$$(\bar{P}\bar{P})_\nu = -\frac{H_\nu^{(2)}(ha)}{H_\nu^{(1)}(ha)} \cdot \frac{C_2}{C_1} \tag{20}$$

with

$$C_q = \begin{vmatrix} X_\nu^{(2)'} - \frac{1}{2} & X_\nu^{(q)} - \frac{1}{2} & \nu^2 - \frac{1}{4} & \nu^2 - \frac{1}{4} \\ 1 & 1 & \bar{Y}_\nu^{(2)'} + \frac{1}{2} & \bar{Y}_\nu^{(1)} + \frac{1}{2} \\ -\mu'(4\bar{X}_\nu^{(2)} - \frac{3}{2} - 2\nu^2 + k'^2 a^2) & -\mu(\bar{X}_\nu^{(q)} - \frac{3}{2} - 2\nu^2 + k'^2 a^2) & 2\mu'(\nu^2 - \frac{1}{4})(\bar{Y}_\nu^{(2)'} - \frac{3}{2}) & 2\mu(\nu^2 - \frac{1}{4})(\bar{Y}_\nu^{(1)} - \frac{3}{2}) \\ \bar{X}_\nu^{(2)'} - \frac{3}{2}) & 2\mu(\bar{X}_\nu^{(q)} - \frac{3}{2}) & -\mu'(2\bar{Y}_\nu^{(2)'} + \frac{3}{2} - 2\nu^2 + k'^2 a^2) & -\mu(2\bar{Y}_\nu^{(1)} + \frac{3}{2} - 2\nu^2 + k^2 a^2) \end{vmatrix} \tag{21}$$

and

$$\bar{X}_\nu^{(q)} = \frac{ha \dfrac{d}{d(ha)} H_\nu^{(q)}(ha)}{H_\nu^{(q)}(ha)}$$

$$\bar{Y}_\nu^{(q)} = \frac{ka \dfrac{d}{d(ka)} H_\nu^{(q)}(ka)}{H_\nu^{(q)}(ka)}.$$

$\bar{X}_\nu^{(q)\prime}$, $\bar{Y}_\nu^{(q)\prime}$ are analogously defined, with arguments $h'a$, $k'a$ respectively.

2.2 *The analytic behaviour of I_ν*

We define I_ν as the integrand in (19). The analytic behaviour of $\bar{X}_\nu^{(q)}$, $\bar{Y}_\nu^{(q)}$ as functions of ν has been discussed by several authors (e.g. Nagase 1954; Nussenzveig 1965; Ansell 1969) and we shall give a brief summary of their results which are applicable when $ha \gg 1, ka \gg 1$.

Firstly $\bar{X}_\nu^{(q)}$, $\bar{Y}_\nu^{(q)}$ are even functions of ν and therefore we shall consider only the right half of the complex ν plane. The Debye asymptotic expansions of $H_\nu^{(q)}(ha)$ show that $\bar{X}_\nu^{(q)} \propto (\nu^2 - h^2 a^2)^{\frac{1}{2}}$ when $ha \gg 1$ or $\nu \gg 1$, provided ν is not in the neighbourhood of the family of zeros of the Hankel functions and their derivatives. The constant of proportionality is different in different regions since the asymptotic expansions exhibit Stokes' phenomenon across the regions in which the zeros of the Hankel

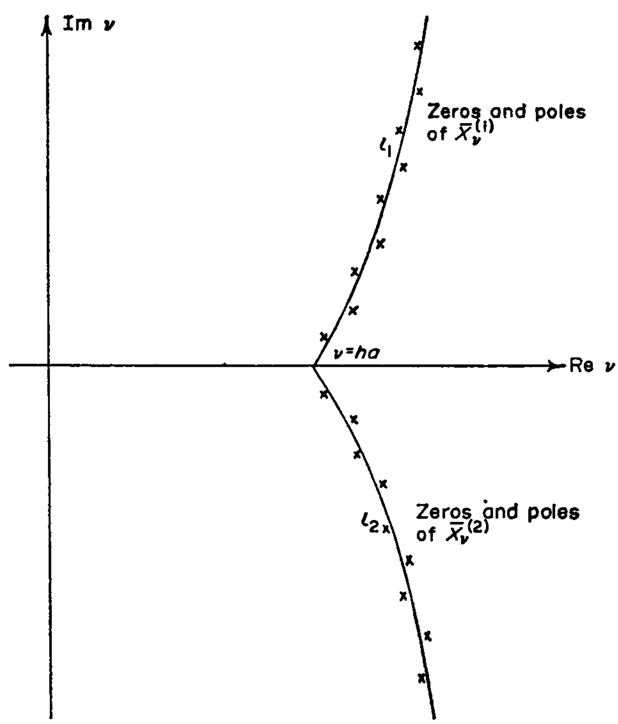

FIG. 4. The poles and zeros of $\bar{X}_\nu^{(1)}$, $\bar{X}_\nu^{(2)}$ in the right-hand half of the complex ν plane.

functions occur. These zeros are asymptotically close to the lines l_1 and l_2 (see Fig. 4) defined by

$$\mathrm{Re}(s) = 0 \qquad \text{with} \qquad s = (\nu^2 - h^2 a^2)^{\frac{1}{2}} - \nu \log\left(\frac{\nu}{ha} + \frac{(\nu^2 - h^2 a^2)^{\frac{1}{2}}}{ha}\right).$$

The zeros of $H_\nu^{(1)}(ha)$ are in the upper half plane and those of $H_\nu^{(2)}(ha)$ in the lower half plane. Along these lines $\bar{X}_\nu^{(q)}$ has zeros and poles placed alternately, corresponding to zeros of $d/d(ha)\, H_\nu^{(q)}(ha)$, $H_\nu^{(q)}(ha)$ respectively.

Now we may write:

$$\left.\begin{aligned} C_1 &= f_1 \bar{X}_\nu^{(2)'} - f_2 && \text{(a)} \\ &= f_3 \bar{X}_\nu^{(1)} - f_4 && \text{(b)} \\ &= f_5 \bar{Y}_\nu^{(2)'} - f_6 && \text{(c)} \\ &= f_7 \bar{Y}_\nu^{(1)} - f_8 && \text{(d)} \end{aligned}\right\} \qquad (22)$$

where the functions f_i may be evaluated from the determinant (21). We see that each of the functions $\bar{X}_\nu^{(2)'}$, $\bar{X}_\nu^{(1)}$, $\bar{Y}_\nu^{(2)'}$, $\bar{Y}_\nu^{(1)}$ has a region in which it is rapidly oscillatory and is slowly varying in the rest of the ν plane (see Fig. 5). In regions (5), (6), (7) and (8) different asymptotic expansions (involving Airy functions) are valid for $H_\nu^{(1)}(ha)$, $H_\nu^{(2)}(h'a)$, $H_\nu^{(1)}(ka)$, $H_\nu^{(2)}(k'a)$ respectively and these regions are given by:

$$|\nu - ha| = 0(ha)^{\frac{1}{3}}$$
$$|\nu - h'a| = 0(h'a)^{\frac{1}{3}}$$
$$|\nu - ka| = 0(ka)^{\frac{1}{3}}$$
$$|\nu - k'a| = 0(k'a)^{\frac{1}{3}}.$$

If the regions do not overlap C_1, for instance, is given by (21b) in region (5) and close to the line l_1 and f_3 and f_4 are slowly varying here. In this case it is clear that the equation:

$$C_1 = 0 \qquad (23)$$

will have solutions near the zero pole pairs of $\bar{X}_\nu^{(1)}$ since $\bar{X}_\nu^{(1)}$ takes any assigned value on some path joining the zero to the pole. Similarly (23) will have solutions in the regions of the poles and zeros of $\bar{X}_\nu^{(2)'}$, $\bar{Y}_\nu^{(2)'}$, $\bar{Y}_\nu^{(1)}$. Hence there are pole of I_ν along the lines l_1, l_2', l_3, l_4' (see Fig. 5). It will be shown that the first poles along the line l_1 are in region (5) and that these poles give the principal contribution to the diffracted P-wave. In order that the regions (5) and (6) should not overlap we demand that

$$|ha - h'a| \gg (ha)^{\frac{1}{3}}, \quad \text{i.e.} \left(\frac{2\pi a}{L}\right)^{\frac{1}{3}} \gg \left|\frac{\alpha'}{\alpha - \alpha'}\right| \qquad (24)$$

where L is the wavelength of the waves outside the sphere.

C_1 may have other isolated zeros in regions where the Debye asymptotic expansions are valid for all the $\bar{X}$, $\bar{Y}$ functions in the determinant; we shall not, however, consider these poles further. Ansell shows that the isolated poles for the liquid sphere give rise to negligible contributions far from the sphere. The first pole on each of the lines l_1, l_2', l_3, l_4' is a finite distance from the real ν axis. The factor $H_\nu^{(1)}(ha)$ in the denominator of (20) does not give rise to poles since C_1 has the same factor in the denominator; similarly C_2 does not contribute poles when $H_\nu^{(2)}(ha)$, $H_\nu^{(2)}(h'a)$, $H_\nu^{(2)}(k'a)$ or $H_\nu^{(1)}(ka)$ vanish, since for the first the pole is cancelled by the factor $H_\nu^{(2)}(ha)$ in the numerator (equation (20)) and for the remaining three C_1 posesses corresponding poles. The poles of I_ν, the integrand in (19), are precisely those of $1/C_1$.

The asymptotic behaviour of I_ν as $|\nu| \to \infty$ may be determined by means of the relations (Nussenzveig 1965):

$$\left.\begin{aligned} \frac{P_{\nu-\frac{1}{2}}(-\cos\theta)}{\cos\pi\nu} &\sim \sqrt{\left(\frac{2}{\pi\nu\sin\theta}\right)}\, e^{i(\nu\theta+\pi/4)}, \quad \mathrm{Im}\nu < 0, \quad |\nu| \to \infty \\ \frac{P_{\nu-\frac{1}{2}}(-\cos\theta)}{\cos\pi\nu} &\sim \sqrt{\left(\frac{2}{\pi\nu\sin\theta}\right)}\, e^{-i(\nu\theta+\pi/4)}, \quad \mathrm{Im}\nu > 0, \quad |\nu| \to \infty \end{aligned}\right\} \tag{25}$$

when $0 < \theta < \pi-\varepsilon$ and $|\nu|\varepsilon \gg 1$, together with the asymptotic expansions for the Hankel functions of large order. This behaviour is fully discussed by Ansell (1969) and since C_2/C_1 behaves as an algebraic functions of ν, away from the poles, his treatment needs very little modification. A similar treatment for an incident plane wave is given by Nussenzveig (1965). The conclusion is that I_ν is exponentially small as $|\nu| \to \infty$, except in the neighbourhood of the poles.

2.3 *The distortion of the contour and evaluation of potentials given by poles*

(*a*) *The distortion of the contour.* The contour C in Fig. 3 may be distorted as shown in Fig. 6 where the sections BC, DE, FG, GG′, G′F′, E′D′, C′B′ are 'at infinity', and therefore give no contribution to the integral (19). Using the relations:

$$P_{\nu-\frac{1}{2}}(-\cos\theta) = P_{-\nu-\frac{1}{2}}(-\cos\theta)$$

$$H^{(1)}_{-\nu}(z) = e^{i\pi\nu} H_\nu^{(1)}(z)$$

$$H^{(2)}_{-\nu}(z) = e^{-i\pi\nu} H_\nu^{(2)}(z)$$

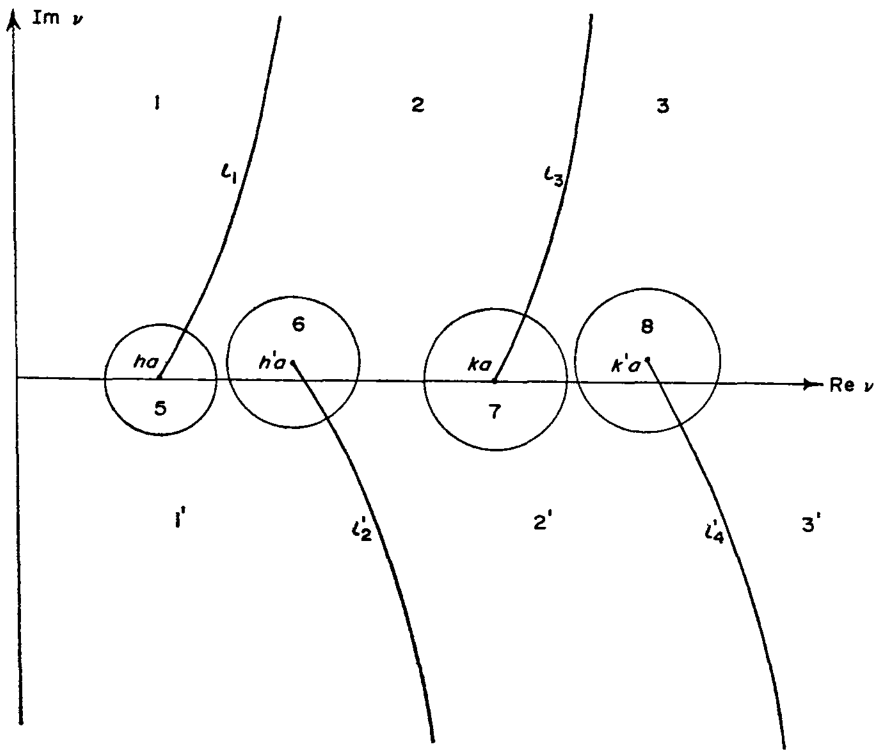

FIG. 5. Regions in the complex ν plane.

we see that I_ν is an odd function of ν so that the integral along B′B also vanishes. (19) may now be written:

$$\phi = \frac{\pi}{4ih\sqrt{(rr_0)}} \, .2\pi i . \sum_n R(p_n) \tag{26}$$

where $R(\rho_n)$ is the residue of:

$$\nu(\bar{P}\bar{P})_\nu H_\nu^{(1)}(hr_0) H_\nu^{(1)}(hr) \frac{P_{\nu-\frac{1}{2}}(-\cos\theta)}{\cos \pi\nu} \tag{27}$$

at the zeros $\nu = p_n$ of C_1 in the right half-plane.

(*b*) *Location of poles.* From the determinant (21) we can obtain

$$(\bar{P}\bar{P})_\nu = -\frac{f_3 \bar{X}_\nu^{(2)} - f_4}{f_3 \bar{X}_\nu^{(1)} - f_4} \cdot \frac{H_\nu^{(2)}(ha)}{H_\nu^{(1)}(ha)}$$

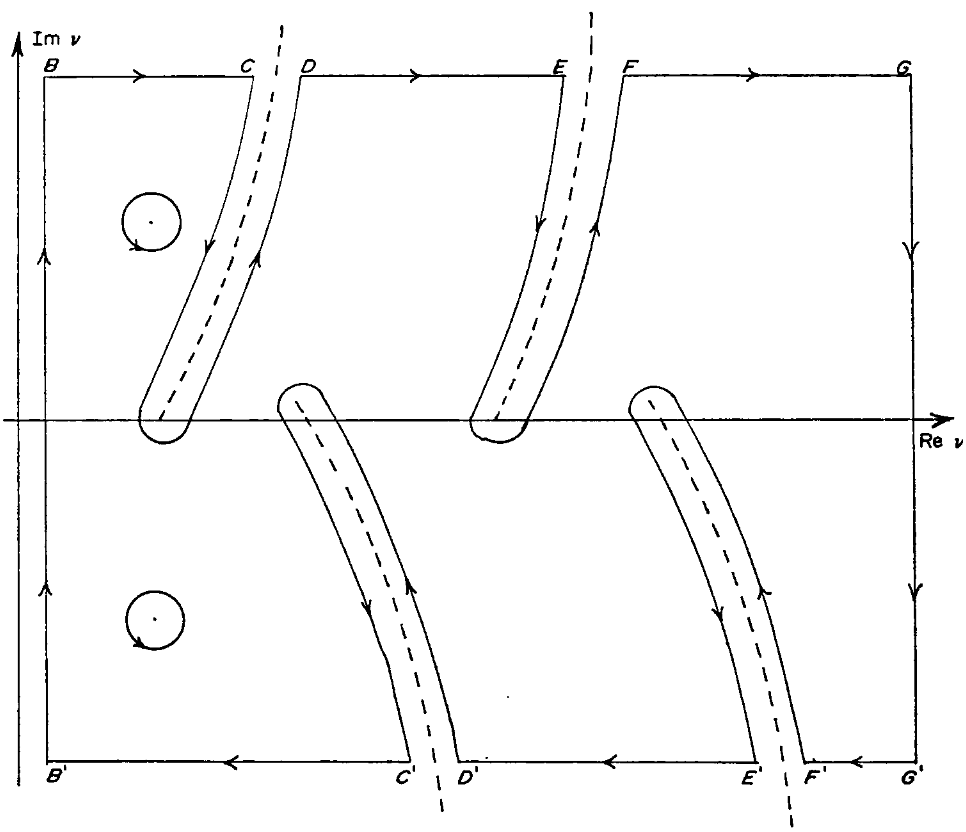

FIG. 6. The deformed contour for the Watson transformation. Isolated poles are representative of such poles which may exist but have not been located in the text.

with

$$
\begin{aligned}
f_3 = {} & 2\nu^2\eta[2\varepsilon-\eta\xi-\delta]+\tfrac{1}{2}\gamma\delta \\
& +\bar{X}_\nu^{(2)\prime}[\eta-2\gamma] \\
& +\bar{Y}_\nu^{(2)\prime}\{\eta[2\varepsilon-\eta\xi+\delta]+\varepsilon\delta\} \\
& +\bar{Y}_\nu^{(1)}\{-4\eta\nu^2[\eta\xi+\delta]-\delta^2\} \\
& +\bar{Y}_\nu^{(2)\prime}\,\bar{Y}_\nu^{(1)}.2\eta[\delta-\eta\xi] \\
& +\bar{Y}_\nu^{(1)}\,\bar{X}_\nu^{(2)\prime}.2\eta[\eta\xi+2\delta] \\
& +\bar{X}_\nu^{(2)\prime}\,\bar{Y}_\nu^{(2)\prime}.2\eta[\eta\xi-2\varepsilon] \\
& +\bar{X}_\nu^{(2)\prime}\,\bar{Y}_\nu^{(2)\prime}\,\bar{Y}_\nu^{(1)}.2\eta^2\xi \\
f_4 = {} & 4\eta\nu^4[\varepsilon-\delta-\eta\xi]+\gamma^2\nu^2 \\
& +\bar{X}_\nu^{(2)\prime}\{2\eta\nu^2[\eta\xi+2\delta-\varepsilon]+\tfrac{1}{2}\varepsilon\delta\} \\
& +\bar{Y}_\nu^{(2)\prime}\{2\eta\nu^2[2\delta+\varepsilon-\eta\xi]-\tfrac{1}{2}\gamma\varepsilon\} \\
& +\bar{Y}_\nu^{(1)}\{-2\eta\nu^2[\varepsilon+\eta\xi+\delta]-\varepsilon\delta\} \\
& +\bar{Y}_\nu^{(2)\prime}\,\bar{Y}_\nu^{(1)}\{-\eta[\eta\xi+\gamma]\} \\
& +\bar{Y}_\nu^{(1)}\,\bar{X}_\nu^{(2)\prime}\{\eta[\varepsilon+\eta\xi+2\delta]-\varepsilon\delta\} \\
& +\bar{X}_\nu^{(2)\prime}\,\bar{Y}_\nu^{(2)\prime}\{4\eta\nu^2[\eta\xi-\varepsilon]+\varepsilon^2\} \\
& +\bar{X}_\nu^{(2)\prime}\,\bar{Y}_\nu^{(2)\prime}\,\bar{Y}_\nu^{(1)}.2\eta[\varepsilon+\eta\xi]
\end{aligned}
$$

where

$$
\begin{aligned}
\xi &= \nu^2-9/4 \\
\delta &= \rho' a^2\omega^2 = \mu' k'^2 a^2 \\
\varepsilon &= \rho a^2\omega^2 = \mu k^2 a^2 \\
\gamma &= \varepsilon-\delta = (\rho-\rho')a^2\omega^2 \\
\eta &= \mu-\mu'.
\end{aligned}
$$

Hence

$$(\bar{P}\bar{P})_\nu = \frac{-(\bar{X}_\nu^{(2)}+f_\nu)}{(\bar{X}_\nu^{(1)}+f_\nu)}\cdot\frac{H_\nu^{(2)}(ha)}{H_\nu^{(1)}(ha)} \tag{28}$$

with

$$f_\nu \equiv \frac{-f_4}{f_3}.$$

We intend to find the zeros of $\bar{X}_\nu^{(1)}+f_\nu$ by using the fact that f_ν is slowly varying and of the same order of magnitude as ν. This is not true if regions (5) and (6) overlap (see Fig. 5 and Section 2.2) so that we need the condition (24). It is interesting to note that if $\rho' = \rho$ and $\beta' = \beta$ then $f_\nu = -\bar{X}_\nu^{(2)\prime}$ and

$$(\bar{P}\bar{P})_\nu = -\left[\frac{\dfrac{haH_\nu^{(2)\prime}(ha)}{H_\nu^{(2)}(ha)}-\dfrac{h'aH_\nu^{(2)\prime}(h'a)}{H_\nu^{(2)}(h'a)}}{\dfrac{haH_\nu^{(1)\prime}(ha)}{H_\nu^{(1)}(ha)}-\dfrac{h'aH_\nu^{(2)\prime}(h'a)}{H_\nu^{(2)}(ha)}}\right]\cdot\frac{H_\nu^{(2)}(ha)}{H_\nu^{(1)}(ha)}.$$

The zeros of the denominator here, which give rise to the poles of I_ν have been thoroughly discussed by Streifer & Kodis (1963) in connexion with diffraction by a dielectric cylinder, and they include the case when ha, $h'a$ are almost equal. It is clear that in this case the numerator becomes small and the denominator is proportional to the Wronskian, which has no zeros, so that there will be no diffracted wave when the two media are identical. The precise behaviour of the poles when $ha \to h'a$ is a matter of some difficulty, and Streifer and Kodis give a method for their calculation for real values of ha, $h'a$. On the other hand, for the case where (24) holds, we shall find that the diffracted wave is independent of the properties of the inner medium at sufficiently high frequencies.

Now from (27) and (28):

$$R(p_n) = \left(\nu H_\nu^{(1)}(hr_0)\, H_\nu^{(1)}(hr) \frac{P_{\nu-\frac{1}{2}}(-\cos\theta)}{\cos \pi\nu}\right)\Bigg|_{\nu=p_n} . P(p_n)$$

where

$$P(p_n) = \left(-\frac{\bar{X}_\nu^{(2)}+f_\nu}{\partial/\partial\nu(\bar{X}_\nu^{(1)}+f_\nu)} \cdot \frac{H_\nu^{(2)}(ha)}{H_\nu^{(1)}(ha)}\right)\Bigg|_{\nu=p_n} \tag{29}$$

$$= \frac{-4i}{\pi ha H_\nu^{(1)\prime}(ha)\partial/\partial\nu[(ha/f_\nu)\, H_\nu^{(1)\prime}(ha)+H_\nu^{(1)}(ha)]} \tag{30}$$

where we have used the value of the Wronskian:

$$x[H_\nu^{(1)\prime}(x)\, H_\nu^{(2)}(x) - H_\nu^{(1)}(x)\, H_\nu^{(2)\prime}(x)] = 4i/\pi$$

and also that $(\bar{X}_\nu^{(1)}+f_\nu)|_{\nu=pn} = 0$.

The poles with $|\nu - ha| \gg (ha)^{\frac{1}{3}}$ may be found by the methods of Scholte (1956) and Ansell (1969) but these poles can be shown to give a negligible contribution to the field in the shadow zone and we shall not evaluate them here. The main contribution will arise from the poles in the region $|\nu - ha| = 0(ha)^{\frac{1}{3}}$. Using the asymptotic approximations valid in this region (Abramowitz & Stegun 1965):

$$H_\nu^{(1)}(x) = 2e^{-i\pi/3}\left(\frac{2}{\nu}\right)^{\frac{1}{3}} Ai\left(e^{2\pi i/3}\left(\frac{2}{\nu}\right)^{\frac{1}{3}}(\nu - x)\right)(1+0(x^{-\frac{2}{3}}))$$

$$H_\nu^{(1)\prime}(x) = -2\,e^{i\pi/3}\left(\frac{2}{\nu}\right)^{\frac{2}{3}} Ai'\left(e^{2\pi i/3}\left(\frac{2}{\nu}\right)^{\frac{1}{3}}(\nu - x)\right)(1+0(x^{-\frac{2}{3}}))$$

we may find approximations to the positions of the poles ν_n near ha. (While p_n can be any pole of I_ν in the right half-plane we reserve the name ν_n for the poles along the line l_1; see Fig. 5).

We have $(x/f_{\nu n})\, H_{\nu n}^{(1)\prime}(x) + H_{\nu n}^{(1)}(x) = 0$ where $x = ha$; i.e.

$$\frac{-2x}{f_{\nu n}} e^{i\pi/3}\left(\frac{2}{\nu}\right)^{\frac{2}{3}} Ai'\left(e^{2\pi i/3}\left(\frac{2}{\nu_n}\right)^{\frac{1}{3}}(\nu_n - x)\right)$$

$$+2\,e^{-i\pi/3}\left(\frac{2}{\nu_n}\right)^{\frac{1}{3}} Ai\left(e^{2\pi i/3}\left(\frac{2}{\nu_n}\right)^{\frac{1}{3}}(\nu_n - x)\right) = 0(x^{-1}).$$

Since we seek poles with $|\nu_n - ha| = 0(ha)^{1/3}$ and since f_ν, $\nu^{1/3}$ are slowly varying functions in the neighbourhood of $\nu = ha$ we have:

$$Ai'(q_n(X)) + XAi(q_n(x)) = 0(x^{-2/3}X)$$

where

$$X = \frac{f_x e^{i\pi/3}}{x}\left(\frac{x}{2}\right)^{1/3}$$

and $\nu_n = x + e^{-2\pi i/3}(x/2)^{1/3} q_n(X)[1 + 0(x^{-2/3})]$.

$q_n(X)$ is defined as the nth solution of

$$Ai'(q) + XAi(q) = 0$$

and for large X it has been shown that (Keller, Rubinow & Goldstein 1963):

$$q_n(X) = q_n(\infty) - 1/X + 0(X^{-2}) \tag{31}$$

so that

$$\nu_n = x + e^{-2\pi i/3}(x/2)^{1/3}(q_n(\infty) - 1/X) \tag{32}$$

and $q_n(\infty)$ is the nth zero of $Ai(q)$.

(*c*) *Evaluation of residues.* Consider the form

$$Q_{\nu n} = \left(\frac{\partial}{\partial \nu}\left[a(\nu)\, Ai'(b(\nu)) + c(\nu)\, Ai(b(\nu))\right]\right)\Bigg|_{\nu = \nu n}$$

where

$$\left(a(\nu)\, Ai'(b(\nu)) + c(\nu)\, Ai(b(\nu))\right)\Bigg|_{\nu = \nu n} = 0. \tag{33}$$

Then

$$Q_{\nu n} = \left(cAi'(b)\left\{\frac{\partial}{\partial \nu}\left(\frac{a}{c}\right) - \frac{a^2}{c^2} b \frac{\partial b}{\partial \nu} + \frac{\partial b}{\partial \nu}\right\}\right)\Bigg|_{\nu = \nu n}$$

where we have used (33) and the Airy equation:

$$Ai''(b) - bAi(b) = 0.$$

With

$$a(\nu) \equiv \frac{2x}{f_\nu} e^{i\pi/3} . \left(\frac{2}{\nu}\right)^{2/3},$$

$$b(\nu) \equiv e^{2\pi i/3}\left(\frac{2}{\nu}\right)^{1/3}(\nu - x)$$

$$c(\nu) \equiv 2\, e^{i\pi/3}\left(\frac{2}{\nu}\right)^{1/3}$$

we have

$$\left(\frac{\partial}{\partial \nu}\left[\frac{x}{f_\nu} H_\nu^{(1)\prime}(x) + H_\nu^{(1)}(x)\right]\right)\Bigg|_{\nu = \nu n} \sim -4\, e^{2\pi i/3}\left(\frac{2}{\nu}\right)^{2/3}\left[1 - \frac{q_n(X)}{X^2}\right] Ai'(q_n(X)).$$

To derive this we have used the fact that $f_\nu = 0(\nu)$ and it will be seen that the properties of the inner medium enter only into the variable X of this expression. It will also be seen that $X \sim 0(x^{\frac{1}{3}})$ so that for very high frequencies X may be replaced by ∞ and the properties of the inner medium disappear completely from the solution. From (30) we have

$$P(\nu_n) = \frac{(x/2)^{\frac{1}{3}}\, e^{-i\pi/6}[1+0(x^{-\frac{1}{3}})]}{2\pi\left(1-\frac{q_n(X)}{X^2}\right)[Ai'(q_n(X))]^2} \quad \text{for} \quad |\nu_n - x| = 0(x^{\frac{1}{3}}). \tag{34}$$

The residues at poles with $|\nu_n - x| \gg x^{\frac{1}{3}}$ may be calculated in a similar way to those calculated above by using the asymptotic approximations to the Hankel functions valid in the neighbourhood of these poles. We quote the result from Ansell (1969), which is also true for our present problem:

$$P(\nu_n) = \frac{1+0(1/\nu_n)}{2x\log\left(\frac{\nu_n}{x}+\frac{(\nu_n^2-x^2)^{\frac{1}{2}}}{x}\right)} \quad \text{for} \quad |\nu_n - ha| \gg (ha)^{\frac{1}{3}}. \tag{35}$$

It will be shown that these poles give a negligible contribution to the potential in the shadow region.

Using equation (31) and the relation (Olver 1954)

$$Ai'(q_n(\infty)) \sim (-1)^{n-1}\,\pi^{-\frac{1}{2}}\left[\frac{3\pi}{2}(n-\tfrac{1}{4})\right]^{\frac{1}{6}}$$

it is clear that both (34) and (35) are slowly varying as n increases (i.e. as $\operatorname{Im}\nu_n$ and $|\nu_n|$ increase). For small values of n, numerical values are given in Table 1 (Abramowitz & Stegun 1965).

(*d*) *Evaluation of potential.* Using the expansions given by Ansell (1969):

$$H_\nu^{(1,2)}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(x^2-\nu^2)^{-\frac{1}{4}}\exp\left\{\pm i\left\{(x^2-\nu^2)-\nu\cos^{-1}\left(\frac{\nu}{x}\right)-\frac{\pi}{4}\right\}\right\}\left[1+0\left(\frac{1}{(x^2-\nu^2)^{\frac{1}{2}}}\right)\right]$$

together with expressions (25), (26) and (29), we find that:

$$\phi = \sum_{n=1}^{\infty}\sqrt{\left(\frac{2p_n\pi}{h^4\sin\theta\, rr_0}\right)}\cdot e^{-i\pi/4}P(p_n)\left(r_0^2-\frac{p_n^2}{h^2}\right)^{-\frac{1}{4}}\left(r^2-\frac{p_n^2}{h^2}\right)^{-\frac{1}{4}}$$
$$\times \exp\{i([h^2r_0^2-p_n^2]^{\frac{1}{2}}+[h^2r^2-p_n^2]^{\frac{1}{2}}+p_n[\theta-\cos^{-1}(p_n/hr_0)-\cos^{-1}(p_n/hr))\}$$

Table 1

Zeros of the Airy function and its derivative at its zeros. $q_n(0)$ is included to show the range of variation of $q_n(X)$ as X passes from 0 to ∞.

n	$q_n(\infty)$	$Ai'(q_n(\infty))$	$q_n(0)$
1	−2·33811	0·70121	−1·01879
2	−4·08795	−0·80311	−3·24820
3	−5·52056	0·86520	−4·82010
4	−6·78671	−0·91085	−6·16331
5	−7·94413	0·94734	−7·37218

As $\mathrm{Im}(p_n)$ increases with n the successive terms decrease exponentially provided that

$$\mathrm{Re}\left(\theta-\cos^{-1}\left(\frac{p_n}{hr_0}\right)-\cos^{-1}\left(\frac{p_n}{hr}\right)\right) > 0.$$

For poles near $\nu = x$ this condition is precisely the condition that the field point should lie inside the shadow zone (see Fig. 2) so that the potential ϕ is conveniently expressed as a rapidly decreasing series of residues in this zone. We may now use (32) and (34) to write:

$$\phi = \sum_{n=1}^{\infty}\left(\frac{ha}{2}\right)^{\frac{1}{3}}\exp(-5\pi i/12)\left(\frac{a}{2\pi rr_0\, s_1\, s_2\, h^3\sin\theta}\right)^{\frac{1}{2}}\exp\left(ih[s_1+s_2+a(\theta-\theta_0)]\right)$$

$$\frac{\exp\left\{i(\theta-\theta_0)\,e^{i\pi/3}\left(\frac{ha}{2}\right)^{\frac{1}{3}}\left(-q_n(\infty)+\frac{1}{X}\right)\right\}}{\left(1-\frac{q_n(X)}{X^2}\right)[Ai'(q_n(X))]^2}\,[1+0(x^{-\frac{2}{3}})] \quad (36)$$

with

$$\theta_0 = \cos^{-1}(a/r)+\cos^{-1}(a/r_0),$$
$$s_1 = (r^2-a^2)^{\frac{1}{2}},$$
$$s_2 = (r_0{}^2-a^2)^{\frac{1}{2}}.$$

This result is very similar to that given by Ansell for the liquid sphere, the only difference being in the expression for X. When $(ha)^{\frac{1}{3}} \gg 1$, X becomes large and the results for liquid and solid sphere become identical.

It can be shown (Waechter 1966) that

$$\frac{P_{\nu-\frac{1}{2}}(-\cos\theta)}{\cos\pi\nu} = \frac{2}{\pi}\sum_{m=0}^{\infty}\{E_{\nu-\frac{1}{2}}(\theta+2m\pi)-E_{\nu-\frac{1}{2}}(2\pi-\theta+2m\pi)\} \quad (37)$$

[$E_{\nu-\frac{1}{2}}$ is the Clemmow function (Clemmow 1960) given by

$$E_{\nu-\frac{1}{2}}(\theta) = \sqrt{(\pi/2)}\,e^{i\pi/4}\frac{(\nu-\frac{1}{2})!}{\nu!}\frac{e^{i\nu\theta}}{\sqrt{\sin\theta}}F\left(\tfrac{1}{2},\tfrac{1}{2};\nu+1;\frac{i\,e^{i\theta}}{2\sin\theta}\right)$$

and which satisfies:

$$E_{\nu-\frac{1}{2}}(\theta) = \frac{\pi}{2\cos\pi\nu}[i\,e^{i\pi\nu}P_{\nu-\frac{1}{2}}(\cos\theta)+P_{\nu-\frac{1}{2}}(-\cos\theta)]$$

$$E_{\nu-\frac{1}{2}}(\theta+m\pi) = i^m e^{i\pi m\nu}E_{\nu-\frac{1}{2}}(\theta)$$

$$P_{\nu-\frac{1}{2}}(\cos\theta) = -\frac{\cot\pi\nu}{\pi}[E_{\nu-\frac{1}{2}}(\theta)-E_{-\nu-\frac{1}{2}}(\theta)] \quad].$$

Now $E_{\nu-\frac{1}{2}}(\theta)$ is asymptotically given by

$$E_{\nu-\frac{1}{2}}(\theta) \sim \sqrt{(\pi/2)}\frac{e^{i\pi/4+i\nu\theta}}{\sqrt{(\nu\sin\theta)}},\ |\nu| \gg 1,\ -\pi < \arg\nu < \pi,\ \theta \text{ fixed}.$$

and since this form is valid for any θ ($\sin\theta \neq 0$) the please of the exponential shows that $E_{\nu-\frac{1}{2}}(\theta)$ is associated with a wave travelling round the sphere and passing repeatedly through its poles. Equation (37) has an interpretation as a sum of contributions corresponding to waves which have circled the sphere m times in one direction or the other. The asymptotic expansion (25) is the same as the asymptotic result for $(2/\pi)\, E_{\nu-\frac{1}{2}}(\theta)$ when $0 < \theta < \pi$, so that (36) represents the contribution to ϕ from the shortest diffracted path. This is in keeping with our aim, which is to calculate the diffracted wave produced by the discontinuity $Q_1\, P_1$ (Fig. 2) and not the waves which depend upon the complete geometry of the sphere.

The poles of I_ν in the lower half-plane (Fig. 5) may be shown to give rise to creeping waves travelling backwards from the point of incidence on the sphere, along paths such as $Q\, Q_2\, Q_1\, P_1\, P_2\, P$ (Fig. 2). They suffer great attenuation as they traverse an angle $\theta + \theta_0$ on the surface of the sphere and give a negligible contribution to the diffracted wave. They are transmitted, along the circular part of their path, with the wave speeds of the inner material and suffer further attenuation if this material is dissipative.

The contribution to the P-wave arising from poles in the neighbourhood of ka may also be evaluated and it is found that it corresponds to a wave which has traversed the spherical surface with speed close to the S-wave velocity, β, of the outer medium. It will have a later arrival time and will suffer slightly greater attenuation than the wave already calculated.

By differentiating (36) and retaining only the first term of the expansion, we find that $\mathbf{u}$ is directed principally along the ray and that its component in this direction is

$$U_0 \sim \frac{iF}{(4ha)^{\frac{1}{2}}} \cdot \frac{1}{(2\pi)^{\frac{1}{2}}}\, e^{ih[s_1+s_2+a(\theta-\theta_0)]}$$

$$\cdot \frac{\exp\left\{ i(\theta-\theta_0)e^{i\pi/3}\left(\dfrac{ha}{2}\right)^{\frac{1}{3}}\left(-q_n(\infty)+\dfrac{1}{X}\right)\right\}}{\left(1-q_n\dfrac{(X)}{X^2}\right)[Ai'(q_n(X))]^2} \tag{38}$$

where

$$F \equiv \left(\frac{a^2}{rr_0\, s_1\, s_2 \sin\theta}\right)^{\frac{1}{2}}. \tag{39}$$

We shall see in the next section that the factor F accounts for the geometrical spreading of the rays.

3. Simple ray theory

In the previous section we have calculated the dominant term contributing to the diffracted field in the shadow of a spherical obstacle in a uniform elastic medium. We shall now find expressions for the amplitudes of waves which have been diffracted by an obstacle within an otherwise spherically symmetric Earth model. With the assumption that the P-wave velocity varies only radially outside the obstacle, simple ray theory (Jeffreys 1959; Bullen 1963) may be applied. The dominant contribution to the P-wave can be evaluated by imposing the condition that energy is conserved within a ray tube, a condition which has been given a mathematical foundation by Karal & Keller (1959). They also show that the principal displacement in the P-wave is directed along the ray, which we have already shown to be the case for the diffracted wave calculated in Section 2. The ray paths are minimum time paths for a disturbance travelling with the local P-wave speed. Ray theory does not apply directly to the diffracted wave since neighbouring ray paths at the receiver correspond to the same ray path at the source, and because energy is radiated at each point by the creeping wave as it travels around the obstacle. Ray theory can, however, be applied along paths to and from the obstacle, and with the assumption that diffraction is a local phenomenon (Levy & Keller 1959), we may use the solution above to find the decay of the creeping wave.

3.1 *The geometrical coefficients for the canonical problem*

Following an approach similar to that of Levy & Keller (1959), we expect the following coefficient, accounting for the geometrical spreading of ray tubes, to appear in the expression for the diffracted wave (see Fig. 2):

$$F \equiv \left(\frac{d\omega_Q}{d\sigma_{Q_1}}\right)^{\frac{1}{2}} \left(\frac{dL_{Q_1}}{dL_{P_1}}\right)^{\frac{1}{2}} \left(\frac{dL_{P_1}}{dL_P}\right)^{\frac{1}{2}} \left(\frac{dl_{P_1}}{dM_P}\right)^{\frac{1}{2}}.$$

$d\omega_Q$ is an element of solid angle, measured at Q, about the line QQ_1.

$d\sigma_{Q_1}$ is an element of area of the wavefront at Q_1 (i.e. perpendicular to QQ_1), subtending solid angle $d\omega_Q$ at Q.

dL_{Q_1} is a line element at Q_1 perpendicular to the plane of the rays.

dL_{P_1} is a line element at P_1 perpendicular to the plane of the rays such that the endpoints of dL_{Q_1} and dL_{P_1} are connected by rays.

dl_{P_1} is a line element at P_1 lying within the surface and also within the plane of the rays.

dM_P is a line element at P lying within the plane of the rays and also within the wavefront, such that the endpoints of dl_{P_1} and dM_P are connected by rays.

dL_P is a line element at P, perpendicular to the plane of the rays, such that the endpoints of dL_{P_1} and dL_P are connected by rays.

We have:

$$\frac{d\omega_Q}{d\sigma_{Q_1}} = \frac{1}{s_1^2}$$

$$\frac{dL_{Q_1}}{dL_{P_1}} = \frac{\sin\gamma_1}{\sin\gamma_3}$$

$$\frac{dL_{P_1}}{dL_P} = \frac{a\sin\gamma_3}{r\sin\theta}$$

$$\frac{dl_{P_1}}{dM_P} = \frac{a}{s_2}.$$

Using

$$\sin\gamma_1 = \frac{s_1}{r_0}$$

we find

$$F = \left(\frac{a^2}{rr_0\, s_1\, s_2 \sin\theta}\right)^{\frac{1}{2}}$$

This is precisely the geometrical coefficient F appearing in equation (38) and defined in (39). The remaining terms in (38) give the decay along the path $Q_1 P_1$ (Fig. 2) not accounted for by the spreading of the surface rays. From simple ray theory we expect that in more complicated situations we may replace F by the corresponding factor, accounting for geometrical spreading, together with a factor $(\rho_Q \alpha_Q/\rho_P \alpha_P)^{\frac{1}{2}}$ to account for differing properties at P and Q. It is also possible to account for a non-circular diffraction path and for differing properties along such a path by the methods of Levy & Keller (1959).

3.2 *Ray theory for a spherically symmetric Earth model*

We now consider the problem of the diffraction of body waves by a spherical inclusion within an otherwise spherically symmetric Earth. For simplicity we deal only with a ray whose path lies in a plane with the centre of the inclusion.

For the geometry of the ray paths away from the inclusion we may use the methods of simple ray theory (Jeffreys 1959; Bullen 1963). A ray emanating from Q (see Fig. 7) is observed at P after traversing the path $Q\,Q_1\,P_1\,P$. O is the centre of the Earth, C the centre of the inclusion and r_1, r_2 are the lengths OQ_1, OP_1. We do not, for the moment, assume symmetry of the ray path. The radius of the Earth is R. The tangent to the surface of the obstacle at Q_1 is also tangent to the ray and meets QO in N_1. The line $N_1\,C$ meets the tangent to the ray at P_1 in X. The rays tangential to the sphere at Q_1 appear to be diverging from N_1 and, assuming that the Earth is uniform in properties along the path $Q_1\,P_1$, those at P_1 appear to be converging to X.

Now, by reciprocity,

$$\frac{d\omega_Q}{d\sigma_{Q_1}} = \frac{d\omega_{Q_1}}{d\sigma_Q}$$

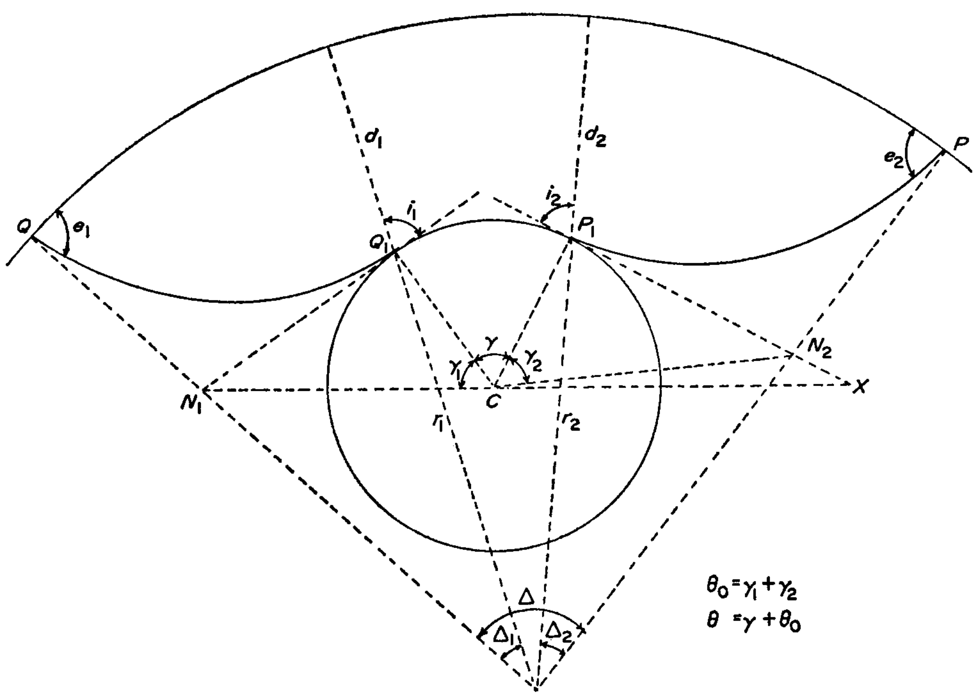

FIG. 7. Geometrical constructions associated with a diffracted ray in the mantle.

and therefore:

$$\frac{d\omega_Q}{d\sigma_{Q_1}} = \left|\frac{\sin i_1}{R^2 \sin e_1 \sin \Delta_1} \cdot \frac{di_1}{d\Delta_1}\right|.$$

Also

$$\frac{dL_{Q_1}}{dL_{P_1}} = \frac{\sin \gamma_1}{\sin(\gamma+\gamma_1)}, \quad \frac{dl_{P_1}}{dM_P} = a\left|\frac{1}{R \sin e_2} \cdot \frac{di_2}{d\Delta_2}\right|, \quad \frac{dL_{P_1}}{dL_P} = \frac{r_2 \sin \delta}{R \sin(\delta-\Delta_2)}.$$

Some rather involved geometry show that:

$$\frac{dL_{P_1}}{dL_P} \cdot \frac{dL_{Q_1}}{dL_{P_1}} = \frac{a \sin \gamma_1 \sin \gamma_2 \sin i_2}{R \sin \Delta_2 \sin \theta}$$

so that

$$F = \frac{a}{R^2}\left|\frac{\sin i_1 \sin i_2 \sin \gamma_1 \sin \gamma_2}{\sin e_1 \sin e_2 \sin \Delta_1 \sin \Delta_2 \sin \theta} \cdot \frac{di_1}{d\Delta_1} \cdot \frac{di_2}{d\Delta_2}\right|^{\frac{1}{2}} \tag{40}$$

and γ_1, γ_2 are given by:

$$\tan \gamma_1 = \frac{r_1}{a} \cdot \frac{\sin \Delta_1}{\sin(i_1+\Delta_1)}, \qquad \tan \gamma_2 = \frac{r_2}{a} \cdot \frac{\sin \Delta_2}{\sin(i_2+\Delta_2)}.$$

If C coincides with O we have a symmetrical situation and F reduces to:

$$F = \frac{a}{R^2 \sin e_1}\left|\frac{1}{\sin \Delta}\right|^{\frac{1}{2}}\left|\frac{di_1}{d\Delta_1}\right|. \tag{41}$$

The geometry for this case is shown in Fig. 8.

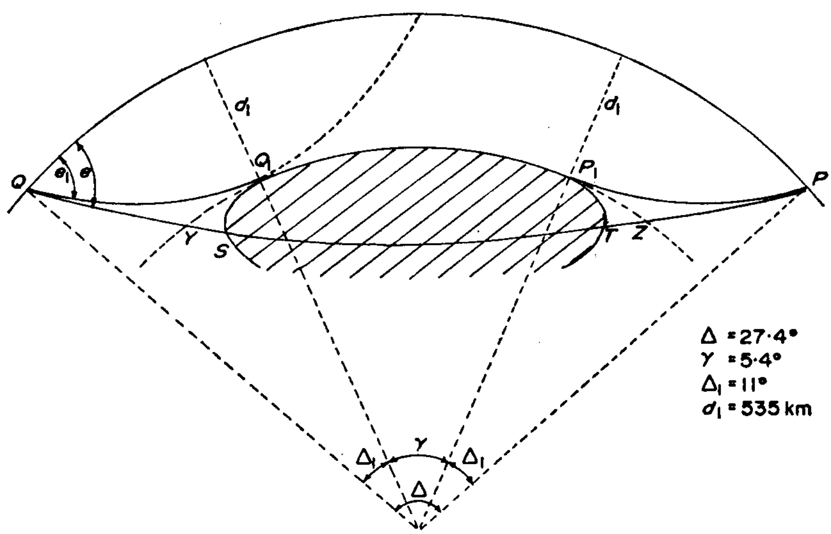

FIG. 8. A specialized case of an inclusion in the mantle, used as a model to explain the observations of the Bukhara explosion.

We now require expressions for $di_1/d\Delta_1$ etc. in terms of travel times. We have that $r \sin i/\alpha$ is constant along the ray and equal to $dT/d\Delta$ for the ray (i is the angle, at any point, between the ray and the radius of the Earth at the point and α is the local P-wave velocity). We find that

$$\frac{di_1}{d\Delta_1} = \frac{\alpha_1}{r_1 \cos i_1}\left[\frac{d^2 T_{d_1}}{d\Delta^2}\bigg|_{\Delta=\Delta_1}\right] \quad \text{when} \quad \cos i_1 \neq 0$$

where α_1 is the velocity at Q_1 and T_{d_1} is the travel time for an earthquake occuring at depth d_1, the depth of Q_1 beneath the Earth's surface. Similarly

$$\frac{di_2}{d\Delta_2} = \frac{\alpha_2}{r_2 \cos i_2}\left[\frac{d^2 T_{d_2}}{d\Delta^2}\bigg|_{\Delta=\Delta_2}\right] \quad \text{when} \quad \cos i_2 \neq 0.$$

These formulae are not appropriate for the case shown in Fig. 8 where the centre of curvature of the discontinuity $Q_1 P_1$ coincides with the centre of the Earth; in this case $i_1 = i_2 = \pi/2$ and the above formulae are replaced by:

$$\frac{di_1}{d\Delta_1} = \frac{di_2}{d\Delta_2} = \left(-\frac{\alpha_1}{r_1}\frac{d^3 T_{d_1}}{d\Delta^3}\bigg|_{\Delta=\Delta_1}\right)^{\frac{1}{2}}$$

The value of $|U_0|$ is now given by taking the modulus of equation (38) where F is now given by (40) or (41).

We may also calculate the amplitude $|U_1|$ of a ray traversing the path from Q to P in the absence of the inclusion. This is given by:

$$|U_1| = \frac{1}{\sin e} \cdot \left|\frac{\alpha_0 \cos e}{R^3 \sin \Delta}\left(\frac{d^2 T_0}{d\Delta^2}\right)\right|^{\frac{1}{2}}$$

where α_0 is the P-wave velocity at Q and the angle e is as shown in Fig. 8.

These results for $|U_0|$ and $|U_1|$ do not take into account dissipation in the mantle or inside the inclusion and to allow for such dissipation we assume average values of the quality factor, Q_0 in the upper mantle (outside the inclusion) and Q inside the inclusion. If the amplitudes, modified by dissipation, are denoted by $|U_0'|$, $|U_1'|$ we may estimate:

$$|U_1'| = \exp\left\{-\frac{\pi}{T}\left(\frac{t_1}{Q_0} + \frac{t_2}{Q}\right)\right\}|U_1|$$

$$|U_0'| = \exp\left\{-\frac{\pi}{T}\left(\frac{t_3}{Q_0} + \frac{t_4}{Q_0}\right)\right\}|U_0|.$$

Here t_1 is the combined travel time for the paths QS, TP

t_2 is the travel time for the path ST (Fig. 8)

t_3 is the combined travel time for the paths QQ_1, $P_1 P$

t_4 is the travel time for the diffracted path $Q_1 P_1$

T is the period of the wave under consideration.

Thus

$$\frac{|U_1'|}{|U_0'|} = \frac{|U_1|}{|U_0|}\exp\left\{\frac{\pi}{T}\left[\frac{t_d}{Q_0} - t_2\left(\frac{1}{Q} - \frac{1}{Q_0}\right)\right]\right\}$$

where $t_d \equiv t_3+t_4-t_1-t_2$ is the time delay of the diffracted wave with respect to the direct wave. Therefore

$$\frac{1}{Q} = \frac{1}{Q_0} + \frac{T}{A\pi t_4} \log_e \left[\frac{|U_1|}{|U_0|} \cdot \frac{|U_0'|}{|U_1'|} \right] + \frac{t_d}{AQ_0\, t_4}$$

where A is defined by

$$A = \frac{t_2}{t_4}$$

and depends upon the shape of the inclusion.

3.3 *Some calculations for the Bukhara explosion*

We now examine the possibility that certain aspects of complexity in seismic records may be produced by diffraction at a discontinuity in the Earth's mantle. The complexity of a signal received from an explosion near Bukhara, U.S.S.R., has been discussed by Douglas *et al.* (1971) in which they postulate that the direct wave has been attenuated in passing through a dissipative region. In this case the diffracted wave, travelling by a different path, may be comparable in amplitude with the direct wave. The arrival P_{H_1} discussed in the paper appears to come from a distance 22° and with a time delay of some 4 s relative to the direct P-wave, while the distance of the source is 27·4°.

A possible model for these observations is that shown in Fig. 8. We assume that there exists an inclusion whose upper surface has centre of curvature at the centre of the Earth and also that the inclusion is dissipative. We examine the possibility that the arrival which appears to come from 22° is a diffracted arrival from the upper surface of the inclusion and that the direct wave has been attenuated in passing through the inclusion. The model is somewhat arbitrary but we wish to show that it provides a possible explanation of the observations. We do not allow for reflection and refraction coefficients at S and T; nor do we take into account the crustal structure at P and Q.

The travel times used in this section are those given by Herrin (1968). The depth of the point Q_1 (Fig. 8) is determined by:

$$\frac{r_1}{\alpha_1} = \left.\frac{dT_0}{d\Delta}\right|_{22°} \simeq 593{\cdot}6\ \text{s.}$$

and using Herrin's velocity model we find that

$$d_1 = 535\ \text{km.}$$

We need the following derivatives:

$$D_1 = \left.\frac{dT_0}{d\Delta}\right|_{22°} \simeq 593{\cdot}6\ \text{s.}$$

$$D_2 = \left.\frac{d^3\, T_{535}}{d\Delta^3}\right|_{11°} \simeq 10870\ \text{s.}$$

$$D_3 = \left.\frac{d^2\, T_0}{d\Delta^3}\right|_{27.4°} \simeq -232{\cdot}5\ \text{s.}$$

$$D_4 = \left.\frac{dT_0}{d\Delta}\right|_{27.4°} \simeq 513{\cdot}9\ \text{s.}$$

and also take:

$$\alpha_1 = 9\cdot836\ \mathrm{km\,s^{-1}}$$

$$\alpha_0 = 8\ \mathrm{km\,s^{-1}}$$

$$R = 6371\ \mathrm{km}.$$

so that using the results of Section 3.2, we may write:

$$\frac{|U_1|}{|U_0|} = \frac{\left|\dfrac{\alpha_0^2}{\alpha_1}\dfrac{D_4 D_3}{D_2}\right|^{\frac{1}{2}}\left(1-\dfrac{\alpha_0^2 D_1^2}{R^2}\right)^{\frac{1}{2}}\exp\left(0\cdot05179\left(\dfrac{R-d_1}{\alpha_1 T}\right)^{\frac{1}{2}}\gamma\right)}{(0\cdot47407)[(R-d_1)^2 d_1 T]^{\frac{1}{2}}\left(1-\dfrac{\alpha_0^2}{R^2}D_4^2\right)^{\frac{1}{2}}}$$

where γ is now measured in degrees. A set of values of $|U_1|/|U_0|$ for different values of the period, T, is shown in Table 2; we see for instance, from the table that a wave of period 1 s is diminished in amplitude by a factor 6·2 in comparison with the wave we expect to observe at distance 27·4°. The derivatives used above are liable to considerable inaccuracy, especially D_2, and the calculations are therefore rather approximate The time delay is approximately 4·6 s.

Now using (42) we may estimate the values of Q needed inside the inclusion in order that the direct and diffracted waves should be approximately equal in amplitude (i.e. that $|U_0'| \simeq |U_1'|$). The calculated values of Q when $Q_0 = 450$ and $A = 1, 2, 3$ are given in Table 2 and these values will be increased if there is appreciable loss by reflection as the transmitted wave enters and leaves the inclusion. If the surface of the inclusion continues to Y and Z (Fig. 8) it may be estimated that $A \simeq 2\cdot9$.

Table 2

Values of $|U_1|/|U_0|$ and values of Q needed to equate the amplitudes of transmitted and diffracted waves for a range of frequencies. Q_1, Q_2, Q_3 were calculated with $A = 1, 2, 3$ respectively and with $Q_0 = 450$.

$T(s)$	$\dfrac{\lvert u_1\rvert}{\lvert u_0\rvert}$	Q_1	Q_2	Q_3
0·20	43·0	149	224	269
0·40	16·7	113	181	226
0·60	10·5	95·8	158	202
0·80	7·72	85·4	144	186
1·00	6·21	78·1	133	174
1·20	5·24	72·9	125	165
1·40	4·57	68·9	120	158
1·60	4·08	65·7	115	153
1·80	3·71	63·1	111	148
2·00	3·41	61·1	108	144
2·20	3·16	59·5	105	141
2·40	2·96	58·0	103	138
2·60	2·79	56·9	101	136
2·80	2·64	55·9	99·6	134
3·00	2·52	55·0	98·0	133

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