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DIFFRACTION OF ELECTROMAGNETIC WAVE BY DISK AND CIRCULAR HOLE IN A PERFECTLY CONDUCTING PLANE

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Abstract—The scattering of electromagnetic plane wave by a perfectly conducting disk is formulated rigorously in a form of the dual integral equations (abbreviated as DIE). The unknowns are the induced surface current (or magnetic field) and the tangential components of the electric field on the disk. The solution for the surface current is expanded in terms of a set of functions which satisfy Maxwell's equation for the magnetic field on the disk and the required edge condition. At this step we have used the method of the Kobayashi potential and the vector Hankel transform. Applying the projection solves the rest of a pair of equations. Thus the problem reduces to the matrix equations for the expansion coefficients. The matrix elements are given in terms of the infinite integrals with a single variable and these may be transformed into infinite series that are convenient for numerical computation. The numerical results are obtained for far field patterns, current densities induced on the disk, transmission coefficient through the circular aperture, and radar cross section. The results are compared with those obtained by other methods when they are available, and agreement among them is fairly well.

1. INTRODUCTION

The problem of scattering of electromagnetic plane wave by a circular disk of perfect conductor has attracted much attentions of the researchers both theoretically and practically and many methods of analysis have been developed. As widely used approximate methods we can mention high frequency techniques. A simple and relatively accurate method is the physical optics (PO) and it is useful to predict the far field pattern near the main lobe [1, 2]. As it is well known that the PO current approximation becomes inaccurate near the edge and the transition region. To circumvent this defect Ufimtsev [3] proposed physical theory of diffraction (PTD) which corrects the effect of the distortion of current distribution near the edge. This method has been improved and generalized so that it can be applied to objects having more general shape [4–8]. The second useful high frequency method is the geometrical theory of diffraction (GTD) proposed by Keller [9–11]. The original GTD has some drawbacks and various improvements have been attempted. The uniform theory of diffraction (UTD) by Kouyoumjian and Pathak, and uniform asymptotic theory (UAT) studied by Ahluwalia et al. [12], and Lee and Deschamps [13] contributed the proposals toward this aim. An equivalent current method (ECM) is also a useful method, which was originated by Miller [14, 15], Clemmow [16] and Braunbeck [17], independently. They used current of Sommerfeld's half-plane problem (actual current). A similar method was proposed again by Rian and Peters and current used by them is virtual and determined from the solution of the GTD [18, 19]. When the size of the disk is not large, different methods were devised. Levine and Schwinger derived vector integral equations and the field in the aperture or on the plane screen, and they showed how to calculate the far-zone diffracted field and transmission coefficient in terms of variational principles related to these integral equations. This method was applied to circular hole and more general apertures [20, 21]. The method of moment (MoM) developed by Harrington is very useful and is considered to be numerically exact [22]. This method was improved by Kim and Thiele, and applied to the disk problem successfully by Li et al. [23]. The works on the circular aperture and the related problems before 1953 were reviewed by Bouwkamp [24, 25].

Next we have mentioned about the exact solution to which our present paper belongs. Since the surface of the disk is the limiting case of the oblate spheroid, the direct approach is to find a series expansion for the field in terms of suitable characteristic functions. Mixner and Andrejewski [26, 27] used the three rectangular components of the Hertz vectors that describe the incident and scattered fields are expanded in terms of appropriate oblate spheroidal wave functions. The coefficients in those expansions are determined by the boundary conditions at the surface of the disk and the edge condition. In these papers they made clear the edge condition. The edge condition is necessary for uniqueness of the solution and it is stated that the tangential and normal components of electromagnetic field is finite (or zero) and singular ($\delta^{-\frac{1}{2}}$) near the edge of thin plate, where δ is the distance from the edge. The results of the complementary problem of a circular aperture can be readily obtained from the results of the disk problem via Babinet's principle. There is another characteristic functions which satisfy the boundary conditions. This can be constructed by applying the properties of the Weber-Schafheitlin's discontinuous integrals [28]. This idea was first proposed by Kobayashi to solve the electrostatic problem of the electrified conducting disk and this method was named by Sneddon as the Kobayashi potential. By using these characteristic functions Nomura and Katsura derived an exact solution for plane wave incidence [29] and Inawashiro derived for spherical wave incidence [30]. They used two rectangular components of the Hertz vector plus auxiliary scalar wave function which describes the surface field of the disk. They enforced the edge condition to the total field to determine the expansion coefficients of the auxiliary function. The summaries of the works by the above two groups, Meixner and Andrejeski, and Nomura and Katsura are reproduced in the handbook by Bowman et al. [31].

It is the purpose of the present paper to improve the analysis by Nomura and Katsura. The analytical procedure is described as follows. We have used two longitudinal components of the vector potentials of electric and magnetic types in the form of Fourier-Hankel transform. By introducing the boundary conditions we have derived the dual integral equations (DIE), one is for induced electric current densities and another is for the tangential components of the electric field. The equations may be written in the form of the vector Hankel transform given by Chew and Kong [32]. The expressions for the current densities are expanded in terms of a set of the functions with expansion coefficients. These functions are constructed by applying the discontinuous properties of the Weber-Schafheitlin's integrals and it is readily shown that these functions satisfy, the Maxwell's equations on the surface of the disk, the required edge conditions and the required equation for the current density in DIE. Applying the inverse vector Hankel transform derives corresponding spectral functions of the current densities. The derived results are substituted into the equation for the electric field in DIE and we have derived the solutions of the expansion coefficients by using the projection. Thus the problem reduces to the matrix equation for the expansion coefficients of the current densities. The matrix elements are given in the form of an infinite integrals which are similar to those in slit problem. These integrals can be transformed into infinite series which is convenient for numerical computation.

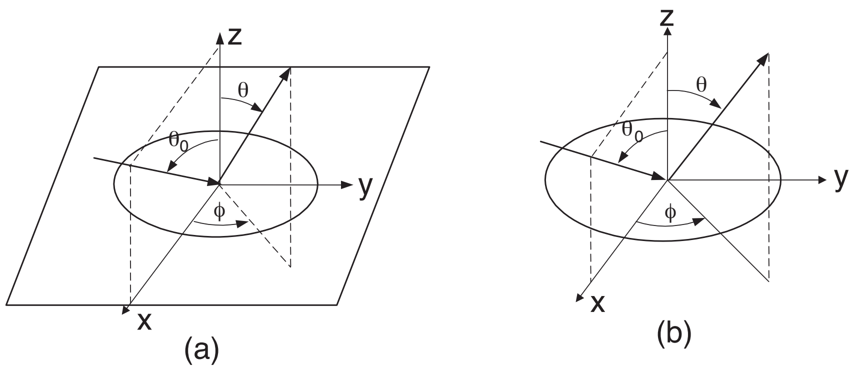

Figure 1. Diffraction of plane wave by a disk or a circular hole.

2. STATEMENT OF THE PROBLEM AND EXPRESSIONS FOR INCIDENT WAVE

The geometry of the problem and the associated coordinates are described in Fig. 1, where the radius of the hole and disk is a and thickness of the conducting plane is assumed to be negligibly small. Two kinds of incident plane wave are possible and these are expressed by

$$\mathbf{E}^i = (E_2\mathbf{i}_\theta + E_1\mathbf{i}_\phi)\exp[jk\Phi^i(\mathbf{r})], \quad \mathbf{H}^i = Y_0(-E_2\mathbf{i}_\phi + E_1\mathbf{i}_\theta)\exp[jk\Phi^i(\mathbf{r})] \tag{1}$$

where

$$\mathbf{i}_\theta = \cos\theta_0\cos\phi_0\mathbf{i}_x + \cos\theta_0\sin\phi_0\mathbf{i}_y - \sin\theta_0\mathbf{i}_z, \quad \mathbf{i}_\phi = -\sin\phi_0\mathbf{i}_x + \cos\phi_0\mathbf{i}_y \tag{2a}$$

$$\Phi^i(\mathbf{r}) = x\sin\theta_0\cos\phi_0 + y\sin\theta_0\sin\phi_0 + z\cos\theta_0 \tag{2b}$$

where (θ_0, ϕ_0) are the angles of incidence and $Y_0 = \sqrt{\frac{\epsilon_0}{\mu_0}}$ is the free space intrinsic admittance. Since a disk or a circular hole has rotational symmetry with respect to z-axis, we can assume without loss of generality that the plane of incidence lies in xz-plane ($\phi_0 = 0$). We may split field into two kinds of polarization, E-polarization specified by E_1 and H-polarization specified by E_2, and discuss both cases simultaneously.

The electric and magnetic vector potentials are defined by $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{D} = -\nabla\times\mathbf{F}$, respectively. Therefore, the z-components of the vector potentials F_z and A_z for the incident and reflected waves are obtained as follows:

$$\begin{aligned} A_z^i &= \frac{\mu_0 Y_0 E_2}{jk\sin\theta_0}\exp[jkx\sin\theta_0 + jkz\cos\theta_0], \\ F_z^i &= \frac{\epsilon_0 E_1}{jk\sin\theta_0}\exp[jkx\sin\theta_0 - jkz\cos\theta_0] \end{aligned} \tag{3a}$$

$$\begin{aligned} A_z^r &= \frac{\mu_0 Y_0 E_2}{jk\sin\theta_0}\exp[jkx\sin\theta_0 - jkz\cos\theta_0], \\ F_z^r &= -\frac{\epsilon_0 E_1}{jk\sin\theta_0}\exp[jkx\sin\theta_0 - jkz\cos\theta_0] \end{aligned} \tag{3b}$$

We express a plane waves given above in terms of cylindrical coordinates to facilitate the imposition of the boundary conditions. These are obtained by using the formulas of wave transformation given by

$$\begin{aligned} \exp[jk\rho\sin\theta_0\cos\phi] &= \sum_{m=-\infty}^{\infty} j^m J_m(k\rho\sin\theta_0)\exp(-jm\phi) \\ &= \sum_{m=0}^{\infty} \epsilon_m j^m J_m(k\rho\sin\theta_0)\cos m\phi \end{aligned} \tag{4}$$

where ϵ_m is Neumann's constant given by $\epsilon_m = 1$ for $m = 0$ and $\epsilon_m = 2$ for $m \geq 1$, and $x = \rho\cos\phi$, $y = \rho\sin\phi$. Then the incident and reflected field on the plane $z = 0$ may be represented as follows.

(1) : E-wave (Magnetic field is perpendicular to the plane of incidence)

The incident electromagnetic plane wave and the field reflected by a complete conductor over the $z = 0$ plane are given by

$$\begin{aligned} H_\rho^i &= H_\rho^r = Y_0 E_\phi^i \cos\theta_0 \\ &= -jY_0 E_1 \cos\theta_0 \sum_{m=0}^{\infty} \epsilon_m j^m J_m{}'(k\rho\sin\theta_0)\cos m\phi \end{aligned} \tag{5a}$$

$$\begin{aligned} H_\phi^i &= H_\phi^r = -Y_0 E_\rho^i \cos\theta_0 \\ &= jY_0 E_1 \cos\theta_0 \sum_{m=0}^{\infty} \epsilon_m j^m \frac{m}{k\rho\sin\theta_0} J_m(k\rho\sin\theta_0)\sin m\phi \end{aligned} \tag{5b}$$

where $J_m(x)$ and $J_m'(x)$ are the Bessel function of the first kind and its derivative with respect to the argument.

(2): H-wave (Electric field is perpendicular to the plane of incidence)

In this case, the incident and reflected fields are given by

$$\begin{aligned} H_\rho^i &= H_\rho^r = jY_0E_2\sum_{m=0}^{\infty}\epsilon_m j^m\frac{m}{k\rho\sin\theta_0}J_m(k\rho\sin\theta_0)\sin m\phi, \\ E_\phi^i &= Z_0\cos\theta_0 H_\rho^i \end{aligned} \tag{6a}$$

$$\begin{aligned} H_\phi^i &= H_\phi^r = jY_0E_2\sum_{m=0}^{\infty}\epsilon_m j^m J_m'(k\rho\sin\theta_0)\cos m\phi, \\ E_\rho^i &= -Z_0\cos\theta_0 H_\phi^i \end{aligned} \tag{6b}$$

3. THE EXPRESSIONS FOR THE FIELDS SCATTERED BY A DISK

We now discuss about our analytical method for predicting the field scattered by a perfectly conducting disk on the plane $z = 0$.

3.1. Spectrum Functions of the Current Density on the Disk

Since E_ρ^d and E_ϕ^d are continuous on the plane $z = 0$, we assume the vector potential corresponding to the diffracted field is expressed in the form

$$\begin{aligned} A_z^d(\rho,\phi,z) &= \pm\mu_0 a\kappa Y_0\sum_{m=0}^{\infty}\int_0^{\infty}\left[\widetilde{f}_{cm}(\xi)\cos m\phi+\widetilde{f}_{sm}(\xi)\sin m\phi\right] \\ &\quad \times J_m(\rho_a\xi)\exp\left[\mp\sqrt{\xi^2-\kappa^2}\,z_a\right]\xi^{-1}d\xi \end{aligned} \tag{7a}$$

$$\begin{aligned} F_z^d(\rho,\phi,z) &= \epsilon_0 a\sum_{m=0}^{\infty}\int_0^{\infty}\left[\widetilde{g}_{cm}(\xi)\cos m\phi+\widetilde{g}_{sm}(\xi)\sin m\phi\right] \\ &\quad \times J_m(\rho_a\xi)\exp\left[\mp\sqrt{\xi^2-\kappa^2}\,z_a\right]\xi^{-1}d\xi \end{aligned} \tag{7b}$$

where the upper and lower signs refer to the region $z > 0$ and $z < 0$, respectively, $\kappa = ka$ is the normalized radius of the disk, and $\rho_a = \frac{\rho}{a}$ and $z_a = \frac{z}{a}$ are the normalized variables with respect to the radius a of the disk. In the above equations $\widetilde{f}(\xi)$ and $\widetilde{g}(\xi)$ are the unknown spectrum functions and they are to be determined so that they satisfy all the required boundary conditions. Equations (7a) and (7b) are of the form of the Hankel transform for $z = 0$. First we consider the surface field at the plane $z = 0$ to derive the dual integral equations associated with them. By using the relation between the vector potentials and the electromagnetic field, the tangential components of the electric field and the current density on the disk become

$$E_\rho^d(\rho,\phi,0) = \sum_{m=0}^{\infty}\Big[E_{\rho c,m}(\rho_a)\cos m\phi + E_{\rho s,m}(\rho_a)\sin m\phi\Big] \quad (8a)$$

$$E_\phi^d(\rho,\phi,0) = \sum_{m=0}^{\infty}\Big[E_{\phi c,m}(\rho_a)\cos m\phi + E_{\phi s,m}(\rho_a)\sin m\phi\Big] \quad (8b)$$

$$\begin{aligned} K_\rho(\rho,\phi) &= -2H_\phi^d(\rho,\phi,0) \\ &= \sum_{m=0}^{\infty}\Big[K_{\rho c,m}(\rho_a)\cos m\phi + K_{\rho s,m}(\rho_a)\sin m\phi\Big] \end{aligned} \quad (8c)$$

$$\begin{aligned} K_\phi(\rho,\phi) &= 2H_\rho^d(\rho,\phi,0) \\ &= \sum_{m=0}^{\infty}\Big[K_{\phi c,m}(\rho_a)\cos m\phi + K_{\phi s,m}(\rho_a)\sin m\phi\Big] \end{aligned} \quad (8d)$$

where

$$\begin{bmatrix} E_{\rho c,m}(\rho_a) \\ E_{\phi s,m}(\rho_a) \end{bmatrix} = \int_0^\infty \Big[H^-(\xi\rho_a)\Big] \begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\,\widetilde{f}_{cm}(\xi)\xi^{-1} \\ \widetilde{g}_{sm}(\xi)\xi^{-1} \end{bmatrix} \xi d\xi \quad (9a)$$

$$\begin{bmatrix} E_{\rho s,m}(\rho_a) \\ E_{\phi c,m}(\rho_a) \end{bmatrix} = \int_0^\infty \Big[H^+(\xi\rho_a)\Big] \begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\,\widetilde{f}_{sm}(\xi)\xi^{-1} \\ \widetilde{g}_{cm}(\xi)\xi^{-1} \end{bmatrix} \xi d\xi \quad (9b)$$

$$\begin{aligned} \begin{bmatrix} K_{\rho c,m}(\rho_a) \\ K_{\phi s,m}(\rho_a) \end{bmatrix} &= 2Y_0\int_0^\infty \Big[H^-(\xi\rho_a)\Big] \begin{bmatrix} \kappa\widetilde{f}_{cm}(\xi)\xi^{-1} \\ j\sqrt{\xi^2-\kappa^2}\,\widetilde{g}_{sm}(\xi)(\kappa\xi)^{-1} \end{bmatrix} \xi d\xi \\ &= \int_0^\infty \Big[H^-(\xi\rho_a)\Big] \begin{bmatrix} \widetilde{K}_{\rho c,m}(\xi) \\ \widetilde{K}_{\phi s,m}(\xi) \end{bmatrix} \xi d\xi \end{aligned} \quad (9c)$$

$$\begin{aligned} \begin{bmatrix} K_{\rho s,m}(\rho_a) \\ K_{\phi c,m}(\rho_a) \end{bmatrix} &= 2Y_0\int_0^\infty \Big[H^+(\xi\rho_a)\Big] \begin{bmatrix} \kappa\widetilde{f}_{sm}(\xi)\xi^{-1} \\ j\sqrt{\xi^2-\kappa^2}\,\widetilde{g}_{cm}(\xi)(\kappa\xi)^{-1} \end{bmatrix} \xi d\xi \\ &= \int_0^\infty \Big[H^+(\xi\rho_a)\Big] \begin{bmatrix} \widetilde{K}_{\rho s,m}(\xi) \\ \widetilde{K}_{\phi c,m}(\xi) \end{bmatrix} \xi d\xi \end{aligned} \quad (9d)$$

where the vector Hankel transform pair is defined by [32]

$$\begin{bmatrix} F_m(\rho_a) \\ G_m(\rho_a) \end{bmatrix} = \int_0^\infty \Big[H^\pm(\xi\rho_a)\Big] \begin{bmatrix} \widetilde{f}_m(\xi) \\ \widetilde{g}_m(\xi) \end{bmatrix} \xi d\xi,$$

$$\begin{bmatrix} \widetilde{f}_m(\xi) \\ \widetilde{g}_m(\xi) \end{bmatrix} = \int_0^\infty \left[H^\pm(\xi\rho_a)\right] \begin{bmatrix} F_m(\rho_a) \\ G_m(\rho_a) \end{bmatrix} \rho_a d\rho_a \tag{10a}$$

where the kernel matrices $\left[H^+(\xi\rho_a)\right]$ and $\left[H^-(\xi\rho_a)\right]$ are given by

$$\left[H^\pm(\xi\rho_a)\right] = \begin{bmatrix} J'_m(\xi\rho_a) & \pm\dfrac{m}{\xi\rho_a}J_m(\xi\rho_a) \\ \pm\dfrac{m}{\xi\rho_a}J_m(\xi\rho_a) & J'_m(\xi\rho_a) \end{bmatrix} \tag{10b}$$

The required boundary conditions state that the current densities on the plane $z = 0$ are zero for $\rho_a \geq 1$ and the tangential components of the total electric field vanish on the disk. These are written as

$$\begin{aligned} \int_0^\infty \left[H^-(\xi\rho_a)\right] \begin{bmatrix} \widetilde{K}_{\rho c,m}(\xi) \\ \widetilde{K}_{\phi s,m}(\xi) \end{bmatrix} \xi d\xi &= 0, \\ \int_0^\infty \left[H^+(\xi\rho_a)\right] \begin{bmatrix} \widetilde{K}_{\rho s,m}(\xi) \\ \widetilde{K}_{\phi c,m}(\xi) \end{bmatrix} \xi d\xi &= 0, \quad \rho_a \geq 1 \end{aligned} \tag{11}$$

$$\begin{aligned} \begin{bmatrix} E^t_{\rho c,m}(\rho_a) \\ E^t_{\phi s,m}(\rho_a) \end{bmatrix} &= \int_0^\infty \left[H^-(\xi\rho_a)\right] \begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\,\widetilde{f}_{cm}(\xi)\xi^{-1} \\ \widetilde{g}_{sm}(\xi)\xi^{-1} \end{bmatrix} \xi d\xi + \begin{bmatrix} E^i_{\rho c,m}(\rho_a) \\ E^i_{\phi s,m}(\rho_a) \end{bmatrix} \\ &= 0, \qquad \rho_a \leq 1 \end{aligned} \tag{12a}$$

$$\begin{aligned} \begin{bmatrix} E^t_{\rho s,m}(\rho_a) \\ E^t_{\phi c,m}(\rho_a) \end{bmatrix} &= \int_0^\infty \left[H^+(\xi\rho_a)\right] \begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\,\widetilde{f}_{sm}(\xi)\xi^{-1} \\ \widetilde{g}_{cm}(\xi)\xi^{-1} \end{bmatrix} \xi d\xi + \begin{bmatrix} E^i_{\rho s,m}(\rho_a) \\ E^i_{\phi c,m}(\rho_a) \end{bmatrix} \\ &= 0, \qquad \rho_a \leq 1 \end{aligned} \tag{12b}$$

where the superscript "t" refer to the total field. In the above equations $E^i_{\rho c,m}$ and $E^i_{\rho s,m}$ denote the $\cos m\phi$ and $\sin m\phi$ parts of the incident wave E^i_ρ, respectively, and same is true for $E^i_{\phi c,m}$ and $E_{\phi s,m}$. The expressions for these factors are given by

$$\begin{aligned} E^i_{\rho c,m}(\rho_a) &= -jE_2\cos\theta_0\epsilon_m j^m J'_m(\kappa\rho_a\sin\theta_0), \\ E^i_{\rho s,m}(\rho_a) &= -jE_1\epsilon_m j^m \frac{m}{\kappa\rho_a\sin\theta_0} J_m(\kappa\rho_a\sin\theta_0) \\ E^i_{\phi c,m}(\rho_a) &= -jE_1\epsilon_m j^m J'_m(\kappa\rho_a\sin\theta_0), \\ E^i_{\phi s,m}(\rho_a) &= jE_2\cos\theta_0\epsilon_m j^m \frac{m}{\kappa\rho_a\sin\theta_0} J_m(\kappa\rho_a\sin\theta_0) \end{aligned} \tag{12c}$$

Equations (11) and (12) are the dual integral equations to determine the spectrum functions $\widetilde{f}_m(\xi)'s$ and $\widetilde{g}_m(\xi)'s$. To solve (11), we expand $\mathbf{K}(\rho_a)$ in terms of the functions which satisfy the Maxwell's equations and edge conditions. These functions can be found by taking into account the discontinuity property of the Weber-Schafheitlin's integrals. Once the expressions for $\mathbf{K}(\rho_a)$ are established, the corresponding spectrum functions can be derived by applying the vector Hankel transform introduced by Chew and Kong [32]. That is, from (9c) and (9d) we have

$$\begin{aligned}
\begin{bmatrix} \widetilde{K}_{\rho c,m}(\xi) \\ \widetilde{K}_{\phi s,m}(\xi) \end{bmatrix} &= \int_0^\infty \Big[H^-(\xi\rho_a)\Big] \begin{bmatrix} K_{\rho c,m}(\rho_a) \\ K_{\phi s,m}(\rho_a) \end{bmatrix} \rho_a d\rho_a, \\
\begin{bmatrix} \widetilde{K}_{\rho s,m}(\xi) \\ \widetilde{K}_{\phi c,m}(\xi) \end{bmatrix} &= \int_0^\infty \Big[H^+(\xi\rho_a)\Big] \begin{bmatrix} K_{\rho s,m}(\rho_a) \\ K_{\phi c,m}(\rho_a) \end{bmatrix} \rho_a d\rho_a
\end{aligned} \tag{13}$$

It is noted that (K_ρ, K_ϕ) satisfy the vector Helmholtz equation $\nabla^2\mathbf{K} + k^2\mathbf{K} = 0$ in circular cylindrical coordinates on the plane $z = 0$ since $\mathbf{K}$ and $\mathbf{H}$ are related by $\mathbf{K} = \mathbf{n} \times \mathbf{H}$ on the plane. Furthermore (K_ρ, K_ϕ) have the properties $K_\rho \sim (1-\rho_a^2)^{\frac{1}{2}}$ and $K_\phi \sim (1-\rho_a^2)^{-\frac{1}{2}}$ near the edge of the disk. By taking into these facts, we set $K_{\rho c,m}(\rho_a) \sim K_{\phi s,m}(\rho_a)$ defined in (8c) and (8d) as follows

$$\begin{aligned}
K_{\rho c,m}(\rho_a) &= \sum_{n=0}^{\infty} \Big[A_{mn}F_{mn}^-(\rho_a) - B_{mn}G_{mn}^+(\rho_a)\Big], \\
K_{\rho s,m}(\rho_a) &= \sum_{n=0}^{\infty} \Big[C_{mn}F_{mn}^-(\rho_a) + D_{mn}G_{mn}^+(\rho_a)\Big] \\
K_{\phi s,m}(\rho_a) &= \sum_{n=0}^{\infty} \Big[-A_{mn}F_{mn}^+(\rho_a) + B_{mn}G_{mn}^-(\rho_a)\Big], \\
K_{\phi c,m}(\rho_a) &= \sum_{n=0}^{\infty} \Big[C_{mn}F_{mn}^+(\rho_a) + D_{mn}G_{mn}^-(\rho_a)\Big]
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
F_{mn}^{\pm}(\rho_a) &= \int_0^\infty \Big[J_{|m-1|}(\eta\rho_a)J_{|m-1|+2n+\frac{1}{2}}(\eta) \\
&\quad \pm J_{m+1}(\eta\rho_a)J_{m+2n+\frac{3}{2}}(\eta)\Big]\eta^{\frac{1}{2}}d\eta
\end{aligned} \tag{15a}$$

$$F_{0n}^{+}(\rho_a) = 2\int_0^\infty J_1(\eta\rho_a)J_{2n+\frac{3}{2}}(\eta)\eta^{\frac{1}{2}}d\eta \tag{15b}$$

$$G_{mn}^{\pm}(\rho_a) = \int_0^\infty \Big[J_{|m-1|}(\eta\rho_a)J_{|m-1|+2n+\frac{3}{2}}(\eta)$$

$$\pm J_{m+1}(\eta\rho_a)J_{m+2n+\frac{5}{2}}(\eta)\Big]\eta^{-\frac{1}{2}}d\eta \qquad (15c)$$

$$G^{+}_{0n}(\rho_a) = 2\int_0^\infty J_1(\eta\rho_a)J_{2n+\frac{5}{2}}(\eta)\eta^{-\frac{1}{2}}d\eta \qquad (15d)$$

These integrals are of the form of the discontinuous Weber-Schafheitlin's integral [28] and they can be performed analytically and expressed in terms of the hypergeometric functions and explicit forms are given in Appendix A. It may readily be verified that $F^{\pm}_{mn}(\rho_a) = G^{\pm}_{mn}(\rho_a) = 0$ for $\rho_a \geq 1$, and $F^{+}_{mn}(\rho_a) \sim (1-\rho_a^2)^{-\frac{1}{2}}$, $F^{-}_{mn}(\rho_a) \sim (1-\rho_a^2)^{\frac{1}{2}}$, $G^{+}_{mn}(\rho_a) \sim (1-\rho_a^2)^{\frac{1}{2}}$, and $G^{-}_{mn}(\rho_a) \sim (1-\rho_a^2)^{\frac{3}{2}}$ near the edge $\rho_a \simeq 1$. Thus the expressions (14) satisfy one part of the dual integral equations (11) with the unknown expansion coefficients $A_{mn} \sim D_{mn}$. To derive the spectrum functions $\widetilde{f}(\xi)$ and $\widetilde{g}(\xi)$ of the vector potentials we first determine the spectrum functions of the current densities, since they are related each other. We substitute (14) into (13) and perform the integration, then the spectrum functions of the current density are determined. The result are

$$\begin{aligned}
\widetilde{K}_{\rho c,m}(\xi) &= \sum_{n=0}^{\infty}\Big[A_{mn}\Xi^{+}_{mn}(\xi) - B_{mn}\Gamma^{-}_{mn}(\xi)\Big] \\
\widetilde{K}_{\phi s,m}(\xi) &= \sum_{n=0}^{\infty}\Big[-A_{mn}\Xi^{-}_{mn}(\xi) + B_{mn}\Gamma^{+}_{mn}(\xi)\Big] \\
\widetilde{K}_{\rho s,m}(\xi) &= \sum_{n=0}^{\infty}\Big[C_{mn}\Xi^{+}_{mn}(\xi) + D_{mn}\Gamma^{-}_{mn}(\xi)\Big] \\
\widetilde{K}_{\phi c,m}(\xi) &= \sum_{n=0}^{\infty}\Big[C_{mn}\Xi^{-}_{mn}(\xi) + D_{mn}\Gamma^{+}_{mn}(\xi)\Big]
\end{aligned} \qquad (16a)$$

for $m \geq 1$ and

$$\begin{aligned}
\widetilde{K}_{\rho c,0}(\xi) &= 2\sum_{n=0}^{\infty} B_{0n}J_{2n+\frac{5}{2}}(\xi)\xi^{-\frac{3}{2}} \\
\widetilde{K}_{\phi s,0}(\xi) &= 2\sum_{n=0}^{\infty} A_{0n}J_{2n+\frac{3}{2}}(\xi)\xi^{-\frac{1}{2}} \\
\widetilde{K}_{\rho s,0}(\xi) &= -2\sum_{n=0}^{\infty} D_{0n}J_{2n+\frac{5}{2}}(\xi)\xi^{-\frac{3}{2}} \\
\widetilde{K}_{\phi c,0}(\xi) &= -2\sum_{n=0}^{\infty} C_{0n}J_{2n+\frac{3}{2}}(\xi)\xi^{-\frac{1}{2}}
\end{aligned} \qquad (16b)$$

for $m = 0$. In the above equations the functions $\Xi^{\pm}_{mn}(\xi)$ and $\Gamma^{\pm}_{mn}(\xi)$ are defined by

$$\begin{aligned} \Xi^{\pm}_{mn}(\xi) &= \left[J_{m+2n-\frac{1}{2}}(\xi) \pm J_{m+2n+\frac{3}{2}}(\xi)\right]\xi^{-\frac{1}{2}}, \\ \Gamma^{\pm}_{mn}(\xi) &= \left[J_{m+2n+\frac{1}{2}}(\xi) \pm J_{m+2n+\frac{5}{2}}(\xi)\right]\xi^{-\frac{3}{2}}, \quad m \geq 1 \end{aligned} \tag{16c}$$

In deriving (16a) and (16b), we use the formula of the Hankel transform given by

$$\int_0^{\infty} J_m(\alpha\rho)J_m(\beta\rho)\rho d\rho = \frac{\delta(\alpha-\beta)}{\alpha} \tag{17}$$

where $\delta(x)$ is the Dirac's delta function. We see from (9c) and (9d) the spectral functions $\widetilde{f}_{cm}(\xi) \sim \widetilde{g}_{sm}(\xi)$ are represented in terms of $\widetilde{K}_{\rho c}(\xi) \sim \widetilde{K}_{\phi sc}(\xi)$.

$$\begin{aligned} \widetilde{f}_{cm}(\xi) &= \frac{Z_0}{2\kappa}\widetilde{K}_{\rho c,m}(\xi)\xi & \widetilde{f}_{sm}(\xi) &= \frac{Z_0}{2\kappa}\widetilde{K}_{\rho s,m}(\xi)\xi \\ \widetilde{g}_{cm}(\xi) &= \frac{\kappa Z_0}{j2\sqrt{\xi^2-\kappa^2}}\widetilde{K}_{\phi c,m}(\xi)\xi & \widetilde{g}_{sm}(\xi) &= \frac{\kappa Z_0}{j2\sqrt{\xi^2-\kappa^2}}\widetilde{K}_{\phi s,m}(\xi)\xi \end{aligned} \tag{18}$$

3.2. Derivation of the Expansion Coefficients

The equations for the expansion coefficients can be obtained by applying the remaining boundary condition that the tangential components of the electric field vanish on the disk, which are given by (12). If we substitute (18) into (12a) and (12b), we have the relations

$$\begin{aligned} \begin{bmatrix} E^t_{\rho c,m}(\rho_a) \\ E^t_{\phi s,m}(\rho_a) \end{bmatrix} &= \frac{Z_0}{2}\int_0^{\infty}\left[H^-(\xi\rho_a)\right]\begin{bmatrix} \dfrac{j\sqrt{\xi^2-\kappa^2}}{\kappa}\widetilde{K}_{\rho c,m}(\xi) \\ \dfrac{\kappa}{j\sqrt{\xi^2-\kappa^2}}\widetilde{K}_{\phi s,m}(\xi) \end{bmatrix}\xi d\xi \\ &\quad + \begin{bmatrix} E^i_{\rho c,m}(\rho_a) \\ E^i_{\phi s,m}(\rho_a) \end{bmatrix} = 0 \qquad \rho_a \leq 1 \end{aligned} \tag{19a}$$

$$\begin{aligned} \begin{bmatrix} E^t_{\rho s,m}(\rho_a) \\ E^t_{\phi c,m}(\rho_a) \end{bmatrix} &= \frac{Z_0}{2}\int_0^{\infty}\left[H^+(\xi\rho_a)\right]\begin{bmatrix} \dfrac{j\sqrt{\xi^2-\kappa^2}}{\kappa}\widetilde{K}_{\rho s,m}(\xi) \\ \dfrac{\kappa}{j\sqrt{\xi^2-\kappa^2}}\widetilde{K}_{\phi c,m}(\xi) \end{bmatrix}\xi d\xi \\ &\quad + \begin{bmatrix} E^i_{\rho s,m}(\rho_a) \\ E^i_{\phi c,m}(\rho_a) \end{bmatrix} = 0 \qquad \rho_a \leq 1 \end{aligned} \tag{19b}$$

where $\widetilde{K}_{\rho c,m}(\xi) \sim \widetilde{K}_{\phi s,m}(\xi)$ are defined in (16). Equations (19a) and (19b) are projected into the functional space with elements $v_n^m(\rho_a^2)$ for E_ρ and $u_n^m(\rho_a^2)$ for E_ϕ, where $v_n^m(\rho_a^2)$ and $u_n^m(\rho_a^2)$ are the Jacobi's polynomials given in Appendix C. Then we obtain the matrix equations for the expansion coefficients $A_{mn} \sim D_{mn}$. The results are written as

$$\begin{aligned} \sum_{n=0}^{\infty}\left[A_{mn}Z_{mp,n}^{(1,1)} - B_{mn}Z_{mp,n}^{(1,2)}\right] &= H_{m,p}^{(1)}, \\ \sum_{n=0}^{\infty}\left[A_{mn}Z_{mp,n}^{(2,1)} - B_{mn}Z_{mp,n}^{(2,2)}\right] &= H_{m,p}^{(2)} \end{aligned} \tag{20a}$$

$$\begin{aligned} \sum_{n=0}^{\infty}\left[C_{mn}Z_{mp,n}^{(1,1)} + D_{mn}Z_{mp,n}^{(1,2)}\right] &= K_{m,p}^{(1)}, \\ \sum_{n=0}^{\infty}\left[C_{mn}Z_{mp,n}^{(2,1)} + D_{mn}Z_{mp,n}^{(2,2)}\right] &= K_{m,p}^{(2)} \end{aligned} \tag{20b}$$

$$m = 1, 2, 3, \cdots; \; p = 0, 1, 2, 3, \cdots$$

$$\sum_{n=0}^{\infty} B_{0n}Z_{0p,n}^{(1,2)} = H_{0,p}^{(1)}, \quad \sum_{n=0}^{\infty} C_{0n}Z_{0p,n}^{(2,1)} = K_{0,p}^{(2)} \quad p = 0, 1, 2, 3, \cdots \tag{20c}$$

where

$$\begin{aligned} Z_{mp,n}^{(1,1)} = & \frac{2(m+2n+\frac{1}{2})}{\kappa}\left[\alpha_p^m K\left(m+2p+\frac{1}{2}, m+2n+\frac{1}{2}\right)\right. \\ & \left. -(\alpha_p^m+3)K\left(m+2p+\frac{5}{2}, m+2n+\frac{1}{2}\right)\right] \\ & -\kappa m(2m+4p+3)\left[G_2\left(m+2p+\frac{3}{2}, m+2n-\frac{1}{2}\right)\right. \\ & \left. -G_2\left(m+2p+\frac{3}{2}, m+2n+\frac{3}{2}\right)\right] \end{aligned} \tag{21a}$$

$$\begin{aligned} Z_{mp,n}^{(1,2)} = & \frac{1}{\kappa}\left[\alpha_p^m K\left(m+2p+\frac{1}{2}, m+2n+\frac{1}{2}\right)\right. \\ & -\alpha_p^m K\left(m+2p+\frac{1}{2}, m+2n+\frac{5}{2}\right) \\ & -(\alpha_p^m+3)K\left(m+2p+\frac{5}{2}, m+2n+\frac{1}{2}\right) \\ & \left. +(\alpha_p^m+3)K\left(m+2p+\frac{5}{2}, m+2n+\frac{5}{2}\right)\right] \\ & -\kappa m\left[G_2\left(m+2p+\frac{1}{2}, m+2n+\frac{1}{2}\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+G_2\left(m+2p+\frac{5}{2},m+2n+\frac{1}{2}\right)\\
&+G_2\left(m+2p+\frac{1}{2},m+2n+\frac{5}{2}\right)\\
&+G_2\left(m+2p+\frac{5}{2},m+2n+\frac{5}{2}\right)\Bigg] \qquad (21b)
\end{aligned}
$$

$$
\begin{aligned}
Z^{(2,1)}_{mp,n} =\ & \frac{4m(m+2n+\frac{1}{2})(m+2p+\frac{1}{2})}{\kappa}K\left(m+2p+\frac{1}{2},m+2n+\frac{1}{2}\right)\\
&-\kappa\left[\alpha_p^m G\left(m+2p-\frac{1}{2},m+2n-\frac{1}{2}\right)\right.\\
&-\alpha_p^m G\left(m+2p-\frac{1}{2},m+2n+\frac{3}{2}\right)\\
&-(\alpha_p^m+1)G\left(m+2p+\frac{3}{2},m+2n-\frac{1}{2}\right)\\
&\left.+(\alpha_p^m+1)G\left(m+2p+\frac{3}{2},m+2n+\frac{3}{2}\right)\right] \qquad (21c)
\end{aligned}
$$

$$
\begin{aligned}
Z^{(2,2)}_{mp,n} =\ & \frac{2m(m+2p+\frac{1}{2})}{\kappa}\left[K\left(m+2p+\frac{1}{2},m+2n+\frac{1}{2}\right)\right.\\
&\left.-K\left(m+2p+\frac{1}{2},m+2n+\frac{5}{2}\right)\right]\\
&-2\kappa\left(m+2n+\frac{3}{2}\right)\left[\alpha_p^m G_2\left(m+2p-\frac{1}{2},m+2n+\frac{3}{2}\right)\right.\\
&\left.-(\alpha_p^m+1)G_2\left(m+2p+\frac{3}{2},m+2n+\frac{3}{2}\right)\right] \qquad (21d)
\end{aligned}
$$

$$
Z^{(1,2)}_{0p,n} = \frac{1}{\kappa}\left[-pK\left(2p+\frac{1}{2},2n+\frac{5}{2}\right)+(p+1.5)K\left(2p+\frac{5}{2},2n+\frac{5}{2}\right)\right] \qquad (21d)
$$

$$
Z^{(2,1)}_{0p,n} = \kappa\left[-pG\left(2p-\frac{1}{2},2n+\frac{3}{2}\right)+(p+0.5)G\left(2p+\frac{3}{2},2n+\frac{3}{2}\right)\right] \qquad (21e)
$$

and

$$
\begin{aligned}
H^{(1)}_{m,p} =\ & 4Y_0E_2\cos\theta_0 j^m\Big[\alpha_p^m J_{m+2p+\frac{1}{2}}(\kappa\sin\theta_0)\\
&-(\alpha_m^p+3)J_{m+2p+\frac{5}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{3}{2}} \qquad (22a)
\end{aligned}
$$

$$
H^{(2)}_{m,p} = 4Y_0E_2\cos\theta_0 j^m m\Big[J_{m+2p-\frac{1}{2}}(\kappa\sin\theta_0)
$$

$$+J_{m+2p+\frac{3}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{1}{2}} \tag{22b}$$

$$\begin{aligned} K^{(1)}_{m,p} &= 4Y_0E_1j^m m\Big[J_{m+2p+\frac{1}{2}}(\kappa\sin\theta_0) \\ &\quad +J_{m+2p+\frac{5}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{3}{2}} \end{aligned} \tag{22c}$$

$$\begin{aligned} K^{(2)}_{m,p} &= 4Y_0E_1j^m\Big[\alpha^m_p J_{m+2p-\frac{1}{2}}(\kappa\sin\theta_0) \\ &\quad -(\alpha^p_m+1)J_{m+2p+\frac{3}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{1}{2}} \end{aligned} \tag{22d}$$

$$m=1,2,3,\cdots,\qquad p=0,1,2,3,\cdots$$

$$\begin{aligned} H^{(1)}_{0,p} &= Y_0E_2\cos\theta_0\Big[-pJ_{2p+\frac{1}{2}}(\kappa\sin\theta_0) \\ &\quad +(p+1.5)J_{2p+\frac{5}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{3}{2}} \end{aligned} \tag{22e}$$

$$\begin{aligned} K^{(2)}_{0,p} &= Y_0E_1\Big[-pJ_{2p-\frac{1}{2}}(\kappa\sin\theta_0) \\ &\quad +(p+0.5)J_{2p+\frac{3}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{1}{2}} \end{aligned} \tag{22f}$$

$$p=0,1,2,3,\cdots$$

where $\alpha^m_p = m+2p$. The functions $K(\alpha,\beta,\lambda)$ and $G(m,n,\lambda)$ are defined by

$$\begin{aligned} K(\alpha,\beta) &= \int_0^\infty \frac{\sqrt{\xi^2-\kappa^2}}{\xi^2} J_\alpha(\xi)J_\beta(\xi)d\xi, \\ G(\alpha,\beta) &= \int_0^\infty \frac{1}{\sqrt{\xi^2-\kappa^2}} J_\alpha(\xi)J_\beta(\xi)d\xi, \\ G_2(\alpha,\beta) &= \int_0^\infty \frac{1}{\xi^2\sqrt{\xi^2-\kappa^2}} J_\alpha(\xi)J_\beta(\xi)d\xi \end{aligned} \tag{23}$$

These integrals converge when $\alpha+\beta>\lambda-1$ and $\lambda>1$ for $K(\alpha,\beta)$ and $\alpha+\beta>\lambda-1$ and $\lambda>-1$ for $G(\alpha,\beta)$. Taking into account the recurrence relation for the Bessel function $J_{\nu-1}(\xi)+J_{\nu+1}(\xi) = 2\nu J_\nu(\xi)\xi^{-1}$, we find that all the matrix elements contained in (21) converge for all indices. It is worthwhile to note that $K(\alpha,\beta)$, $G(\alpha,\beta)$ and $G_2(\alpha,\beta)$ can be integrated into infinite series which are more convenient for numerical computation.

Equations (20) can be rewritten in terms of the matrix equations

$$\begin{aligned} \Big[Z^{(1,1)}_m\Big]\Big[A_m\Big]-\Big[Z^{(1,2)}_m\Big]\Big[B_m\Big] &= \Big[H^{(1)}_m\Big], \\ \Big[Z^{(2,1)}_m\Big]\Big[A_m\Big]-\Big[Z^{(2,2)}_m\Big]\Big[B_m\Big] &= \Big[H^{(2)}_m\Big] \end{aligned} \tag{24a}$$

$$\begin{aligned} \left[Z_m^{(1,1)}\right]\left[C_m\right] + \left[Z_m^{(1,2)}\right]\left[D_m\right] &= \left[K_m^{(1)}\right], \\ \left[Z_m^{(2,1)}\right]\left[C_m\right] + \left[Z_m^{(2,2)}\right]\left[D_m\right] &= \left[K_m^{(1)}\right] \end{aligned} \tag{24b}$$

Therefore the expansion coefficients are derived from

$$\left\{\left[Z_m^{(1,1)}\right]^{-1}\left[Z_m^{(1,2)}\right] - \left[Z_m^{(2,1)}\right]^{-1}\left[Z_m^{(2,2)}\right]\right\}\left[B_m\right] = \left[Z_m^{(2,1)}\right]^{-1}\left[H_m^{(2)}\right] - \left[Z_m^{(1,1)}\right]^{-1}\left[H_m^{(1)}\right] \tag{25a}$$

$$\left[A_m\right] = \left[Z_m^{(1,1)}\right]^{-1}\left[Z_m^{(1,2)}\right]\left[B_m\right] + \left[Z_m^{(1,1)}\right]^{-1}\left[H_m^{(1)}\right] \tag{25b}$$

$$\left\{\left[Z_m^{(1,1)}\right]^{-1}\left[Z_m^{(1,2)}\right] - \left[Z_m^{(2,1)}\right]^{-1}\left[Z_m^{(2,2)}\right]\right\}\left[D_m\right] = \left[Z_m^{(1,1)}\right]^{-1}\left[K_m^{(1)}\right] - \left[Z_m^{(2,1)}\right]^{-1}\left[K_m^{(2)}\right] \tag{25c}$$

$$\left[C_m\right] = -\left[Z_m^{(1,1)}\right]^{-1}\left[Z_m^{(1,2)}\right]\left[D_m\right] + \left[Z_m^{(1,1)}\right]^{-1}\left[K_m^{(1)}\right] \tag{25d}$$

$$\left[B_0\right] = \left[Z_0^{(1,2)}\right]^{-1}\left[H_0^{(1)}\right], \qquad \left[C_0\right] = \left[Z_0^{(2,1)}\right]^{-1}\left[K_0^{(2)}\right] \tag{25e}$$

3.3. Far Field Expression

Here we derive the far field expressions of A_z^d and F_z^d given in (9) directly by applying the stationary phase method of integration. These integrals can be written in the form

$$I_{nt} = \int_0^\infty \widetilde{P}(\xi) J_m(\rho_a \xi) \exp\left[-\sqrt{\xi^2 - \kappa^2} z_a\right] \xi^{-1} d\xi \tag{26}$$

where we assume that $\widetilde{P}(\xi)$ is slowly varying function. To perform this integration asymptotically, we use the integral representation for $J_m(\rho_a \xi)$ given by

$$J_m(\rho_a \xi) = \frac{j^m}{2\pi} \int_{-\pi}^{\pi} \exp\left(-j\xi\rho_a \cos\alpha - jm\alpha\right) d\alpha \tag{27a}$$

We transform the cylindrical coordinate variables (ρ_a, z_a) into the polar coordinate variables (R_a, θ) with $R_a = \frac{R}{a}$ through $\rho_a = R_a \sin\theta$, $z_a = R_a \cos\theta$, and integration variable ξ into β by $\xi = \kappa \sin\beta$. Then I_{nt} changes into

$$\begin{aligned} I_{nt} &= \frac{j^m}{2\pi} \int_{-\pi}^{\pi} \exp\left(-j\kappa R_a \sin\beta \sin\theta \cos\alpha - jm\alpha\right) d\alpha \\ &\quad \times \int_C \widetilde{P}(\kappa \sin\beta) \exp\left[-j\kappa R_a \cos\beta \cos\theta\right] \frac{\cos\beta}{\sin\beta} d\beta \end{aligned} \tag{27b}$$

where the contour C is running along $(0,0) \to (\frac{\pi}{2},0) \to (\frac{\pi}{2},\infty)$ in the complex $\beta-$plane . Stationary points are loacted at (α_0, β_0) satisfying the equations

$$\sin\theta\cos\beta\cos\alpha_0 - \cos\theta\sin\beta_0 = 0, \qquad \kappa R_a \sin\theta\sin\beta_0\sin\alpha - m = 0 \tag{27c}$$

When the value of κR_a is sufficiently large, α_0 may be set to 0, so that the approximate stationary points are given by

$$\alpha_0 = 0, \qquad \beta = \theta \tag{27d}$$

Application of the standard process of the method yields the result given by

$$I_{nt} = \exp\left(j\frac{m+1}{2}\pi\right)\frac{\exp(-j\kappa R_a)}{\kappa R_a}\widetilde{P}(\kappa\sin\theta)\frac{\cos\theta}{\sin^2\theta} \tag{28a}$$

If we apply this formula to the vector potential given in (9) we have

$$\begin{aligned} A_z^d(\mathbf{r}) \;=\; & j\mu_0 a^2 \frac{\exp(-jkR)}{R}\frac{\cos\theta}{\sin\theta}\sum_{n=0}^{\infty} B_{0n} J_{2n+\frac{5}{2}}(\kappa\sin\theta)(\kappa\sin\theta)^{-\frac{3}{2}} \\ & +\mu_0 a^2 \frac{\exp(-jkR)}{2R}\frac{\cos\theta}{\sin\theta}\sum_{m=1}^{\infty} j^{m+1}\sum_{n=0}^{\infty} \\ & \times\Big\{\Big[A_{mn}\Xi_{mn}^{+}(\kappa\sin\theta) - B_{mn}\Gamma_{mn}^{-}(\kappa\sin\theta)\Big]\cos m\phi \\ & +\Big[C_{mn}\Xi_{mn}^{+}(\kappa\sin\theta) + D_{mn}\Gamma_{mn}^{-}(\kappa\sin\theta)\Big]\sin m\phi\Big\} \end{aligned} \tag{28b}$$

$$\begin{aligned} F_z^d(\mathbf{r}) \;=\; & j\epsilon_0 a^2 Z_0 \frac{\exp(-jkR)}{R}\frac{1}{\sin\theta}\sum_{n=0}^{\infty} C_{0n} J_{2n+\frac{3}{2}}(\kappa\sin\theta)(\kappa\sin\theta)^{-\frac{1}{2}} \\ & -\epsilon_0 a^2 Z_0 \frac{\exp(-jkR)}{2R}\frac{1}{\sin\theta}\sum_{m=1}^{\infty} j^{m+1}\sum_{n=0}^{\infty} \\ & \times\Big\{\Big[C_{mn}\Xi_{mn}^{-}(\kappa\sin\theta) + D_{mn}\Gamma_{mn}^{+}(\kappa\sin\theta)\Big]\cos m\phi \\ & +\Big[-A_{mn}\Xi_{mn}^{-}(\kappa\sin\theta) + B_{mn}\Gamma_{mn}^{+}(\kappa\sin\theta)\Big]\sin m\phi\Big\} \end{aligned} \tag{28c}$$

In the far region we have the relations

$$\begin{aligned} E_\theta &= -j\omega A_\theta = j\omega\sin\theta A_z, \\ H_\theta &= -j\omega F_\theta = j\omega\sin\theta F_z = -Y_0 E_\phi,\; A_\phi = Z_0\sin\theta F_z \end{aligned} \tag{29a}$$

or

$$E_\theta = Z_0\frac{\exp(-jR)}{kR}D_\theta(\theta,\phi), \qquad E_\phi = Z_0\frac{\exp(-jR)}{kR}D_\phi(\theta,\phi) \tag{29b}$$

where

$$D_\theta(\theta,\phi) = -k^2a^2\cos\theta\sum_{n=0}^{\infty}B_{0n}J_{2n+\frac{5}{2}}(\kappa\sin\theta)(\kappa\sin\theta)^{-\frac{3}{2}} + \frac{j}{2}k^2a^2\cos\theta$$
$$\sum_{m=1}^{\infty}j^{m+1}\sum_{n=0}^{\infty}\Big\{\Big[A_{mn}\Xi^+_{mn}(\kappa\sin\theta)-B_{mn}\Gamma^-_{mn}(\kappa\sin\theta)\Big]\cos m\phi$$
$$\times\Big[C_{mn}\Xi^+_{mn}(\kappa\sin\theta)+D_{mn}\Gamma^-_{mn}(\kappa\sin\theta)\Big]\sin m\phi\Big\} \quad (29c)$$

$$D_\phi(\theta,\phi) = k^2a^2\sum_{n=0}^{\infty}C_{0n}J_{2n+\frac{3}{2}}(\kappa\sin\theta)(\kappa\sin\theta)^{-\frac{1}{2}} + \frac{j}{2}k^2a^2$$
$$\sum_{m=1}^{\infty}j^{m+1}\sum_{n=0}^{\infty}\Big\{\Big[C_{mn}\Xi^-_{mn}(\kappa\sin\theta)+D_{mn}\Gamma^+_{mn}(\kappa\sin\theta)\Big]\cos m\phi$$
$$\times\Big[-A_{mn}\Xi^-_{mn}(\kappa\sin\theta)+B_{mn}\Gamma^+_{mn}(\kappa\sin\theta)\Big]\sin m\phi\Big\} \quad (29d)$$

4. THE EXPRESSIONS FOR THE FIELDS DIFFRACTED BY A CIRCULAR HOLE IN A PERFECTLY CONDUCTING PLATE

This is a complementary problem of the scattering by a disk discussed in the previous section and the solution is directly obtained by using the result of the disk problem via Babinet's principle. However we will give the summary of the analysis for convenience of later reference.

4.1. Electric Field Distribution

The vector potential for the diffracted wave is given by (7). Then the dual integral equations for the hole problem become as follows. One part is

$$\begin{aligned}&\int_0^\infty\Big[H^-(\xi\rho_a)\Big]\begin{bmatrix}\widetilde{E}_{\rho c,m}(\xi)\\ \widetilde{E}_{\phi s,m}(\xi)\end{bmatrix}\xi d\xi = 0,\\ &\int_0^\infty\Big[H^+(\xi\rho_a)\Big]\begin{bmatrix}\widetilde{E}_{\rho s,m}(\xi)\\ \widetilde{E}_{\phi c,m}(\xi)\end{bmatrix}\xi d\xi = 0,\ \rho_a \geq 1\end{aligned} \quad (30a)$$

where we set

$$\begin{aligned}\widetilde{f}_{cm}(\xi)\xi^{-1} &= \frac{1}{j\sqrt{\xi^2-\kappa^2}}\widetilde{E}_{\rho c,m}(\xi), \qquad \widetilde{g}_{sm}(\xi)\xi^{-1} = \widetilde{E}_{\phi s,m}(\xi)\\ \widetilde{f}_{sm}(\xi)\xi^{-1} &= \frac{1}{j\sqrt{\xi^2-\kappa^2}}\widetilde{E}_{\rho s,m}(\xi), \qquad \widetilde{g}_{cm}(\xi)\xi^{-1} = \widetilde{E}_{\phi c,m}(\xi)\end{aligned} \quad (30b)$$

An another part is given by

$$Y_0\int_0^\infty\Big[H^-(\xi\rho_a)\Big]\begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\widetilde{g}_{cm}(\xi)(\kappa\xi)^{-1} \\ -\kappa\widetilde{f}_{sm}(\xi)\xi^{-1}\end{bmatrix}\xi d\xi+\begin{bmatrix}H^i_{\rho c,m}(\rho_a)\\ H^i_{\phi s,m}(\rho_a)\end{bmatrix} = 0, \quad \rho_a\le 1 \tag{31a}$$

$$Y_0\int_0^\infty\Big[H^+(\xi\rho_a)\Big]\begin{bmatrix} j\sqrt{\xi^2-\kappa^2}\widetilde{g}_{sm}(\xi)(\kappa\xi)^{-1} \\ -\kappa\widetilde{f}_{cm}(\xi)\xi^{-1}\end{bmatrix}\xi d\xi+\begin{bmatrix}H^i_{\rho s,m}(\rho_a)\\ H^i_{\phi c,m}(\rho_a)\end{bmatrix} = 0, \quad \rho_a\le 1 \tag{31b}$$

where $H^i_{\rho c,m}$ and $H^i_{\rho s,m}$ denote the $\cos m\phi$ and $\sin m\phi$ parts of the incident wave H^i_ρ, respectively, and same is true for $H^i_{\phi c,m}$ and $H_{\phi s,m}$. The expressions for these factors are given by

$$\begin{aligned}
H^i_{\rho c,m}(\rho_a) &= -jY_0E_1\cos\theta_0\epsilon_m j^m J'_m(\kappa\rho_a\sin\theta_0),\\
H^i_{\rho s,m}(\rho_a) &= jY_0E_2\epsilon_m j^m\frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0)\\
H^i_{\phi c,m}(\rho_a) &= jY_0E_2\epsilon_m j^m J'_m(\kappa\rho_a\sin\theta_0),\\
H^i_{\phi s,m}(\rho_a) &= jY_0E_1\cos\theta_0\epsilon_m j^m\frac{m}{\kappa\rho_a\sin\theta_0}J_m(\kappa\rho_a\sin\theta_0)
\end{aligned} \tag{31c}$$

The aperture electric field can be expanded in a manner similar to the disk problem given in (16a) to (16c). It is noted that (E_ρ, E_ϕ) satisfy the vector Helmholtz equation $\nabla^2\mathbf{E}+k^2\mathbf{E}=0$ in circular cylindrical coordinates. Furthermore (E_ρ, E_ϕ) have the property $E_\rho\sim(1-\rho_a^2)^{-\frac{1}{2}}$ and $E_\phi\sim(1-\rho_a^2)^{\frac{1}{2}}$ near the edge of the hole. By taking into these facts, we set

$$\begin{aligned}
E_{\rho c,m}(\rho_a) &= \sum_{n=0}^\infty\Big[C_{mn}F^+_{mn}(\rho_a)+D_{mn}G^-_{mn}(\rho_a)\Big],\\
E_{\rho s,m}(\rho_a) &= \sum_{n=0}^\infty\Big[A_{mn}F^+_{mn}(\rho_a)-B_{mn}G^-_{mn}(\rho_a)\Big]\\
E_{\phi s,m}(\rho_a) &= -\sum_{n=0}^\infty\Big[C_{mn}F^-_{mn}(\rho_a)+D_{mn}G^+_{mn}(\rho_a)\Big],\\
E_{\phi c,m}(\rho_a) &= \sum_{n=0}^\infty\Big[A_{mn}F^-_{mn}(\rho_a)-B_{mn}G^+_{mn}(\rho_a)\Big]
\end{aligned} \tag{32}$$

where $A_{mn} \sim D_{mn}$ are the expansion coefficients and are to be determined from the remaining boundary condition that the tangential components of magnetic field are continuous on the aperture. We have allocate these coefficients so that they satisfy the same equations (23) since we know that the solution for circular hole can be obtained from that of disk problem. The functions $F^{\pm}_{mn}(\rho_a)$ and $G^{\pm}_{mn}(\rho_a)$ are defined in (15a)–(15d). Then the corresponding spectrum functions can be derived by applying the vector Hankel transform (13). The result is

$$\begin{aligned}
\widetilde{E}_{\rho c,m}(\xi) &= -\sum_{n=0}^{\infty}\left[C_{mn}\Xi^{-}_{mn}(\xi)+D_{mn}\Gamma^{+}_{mn}(\xi)\right],\\
\widetilde{E}_{\phi s,m}(\xi) &= \sum_{n=0}^{\infty}\left[C_{mn}\Xi^{+}_{mn}(\xi)+D_{mn}\Gamma^{-}_{mn}(\xi)\right]\\
\widetilde{E}_{\rho s,m}(\xi) &= \sum_{n=0}^{\infty}\left[A_{mn}\Xi^{-}_{mn}(\xi)-B_{mn}\Gamma^{+}_{mn}(\xi)\right],\\
\widetilde{E}_{\phi c,m}(\xi) &= \sum_{n=0}^{\infty}\left[A_{mn}\Xi^{+}_{mn}(\xi)-B_{mn}\Gamma^{-}_{mn}(\xi)\right]
\end{aligned} \tag{32a}$$

for $m \geq 1$ and

$$\begin{aligned}
\widetilde{E}_{\rho c,0}(\xi) &= -2\sum_{n=0}^{\infty}C_{0n}J_{2n+\frac{3}{2}}(\xi)\xi^{-\frac{1}{2}},\\
\widetilde{E}_{\phi s,0}(\xi) &= 2\sum_{n=0}^{\infty}D_{0n}J_{2n+\frac{5}{2}}(\xi)\xi^{-\frac{3}{2}}\\
\widetilde{E}_{\rho s,0}(\xi) &= -2\sum_{n=0}^{\infty}A_{0n}J_{2n+\frac{3}{2}}(\xi)\xi^{-\frac{1}{2}},\\
\widetilde{E}_{\phi c,0}(\xi) &= 2\sum_{n=0}^{\infty}B_{0n}J_{2n+\frac{5}{2}}(\xi)\xi^{-\frac{3}{2}}
\end{aligned} \tag{32b}$$

for $m = 0$, where $\Xi^{\pm}_{mn}$ and $\Gamma^{\pm}_{mn}$ are defined by (16c). From (30b) we can express the spectral functions $\widetilde{f}_{cm}(\xi) \sim \widetilde{g}_{sm}(\xi)$ of the vector potentials in terms of those of the aperture distribution. This means that the surface magnetic field can be expressed in terms of the spectrum functions of the surface electric field.

4.2. Derivation of the Expansion Coefficients

Equations (41a) and (41b) are projected into the functional space with elements $v^m_n(\rho^2_a)$ for H_ρ and $u^m_n(\rho^2_a)$ for H_ϕ, then we obtain the matrix equations for the expansion coefficients $A_{mn} \sim D_{mn}$. The result is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\Big[A_{mn}Z^{(1,1)}_{mp,n}-B_{mn}Z^{(1,2)}_{mp,n}\Big] &= H^{(1)}_{m,p},\\ \sum_{n=0}^{\infty}\Big[A_{mn}Z^{(2,1)}_{mp,n}-B_{mn}Z^{(2,2)}_{mp,n}\Big] &= H^{(2)}_{m,p}\end{aligned}\tag{33a}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\Big[C_{mn}Z^{(1,1)}_{mp,n}+D_{mn}Z^{(1,2)}_{mp,n}\Big] &= K^{(1)}_{m,p},\\ \sum_{n=0}^{\infty}\Big[C_{mn}Z^{(2,1)}_{mp,n}+D_{mn}Z^{(2,2)}_{mp,n}\Big] &= K^{(2)}_{m,p}\end{aligned}\tag{33b}$$

$$m=1,2,3,\cdots;\qquad p=0,1,2,3,\cdots$$

$$\sum_{n=0}^{\infty}C_{0n}Z^{(1,2)}_{0p,n}=H^{(1)}_{0,p},\qquad \sum_{n=0}^{\infty}B_{0n}Z^{(2,1)}_{0p,n}=K^{(2)}_{0,p}\qquad p=0,1,2,3,\cdots\tag{33c}$$

where $Z^{(1,1)}_{mp,n}\sim Z^{(2,2)}_{mp,n}$ are defined by (24) and $H^{(1)}_{m,p}\sim K^{(2)}_{m,p}$ are given by

$$\begin{aligned}H^{(1)}_{m,p} &= 2Y_0E_1\cos\theta_0 j^m\Big[\alpha^m_p J_{m+2p+\frac{1}{2}}(\kappa\sin\theta_0)\\ &\quad -(\alpha^p_m+3)J_{m+2p+\frac{5}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{3}{2}}\end{aligned}\tag{34a}$$

$$\begin{aligned}H^{(2)}_{m,p} &= 2Y_0E_1\cos\theta_0 j^m m\Big[J_{m+2p-\frac{1}{2}}(\kappa\sin\theta_0)\\ &\quad +J_{m+2p+\frac{3}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{1}{2}}\end{aligned}\tag{34b}$$

$$\begin{aligned}K^{(1)}_{m,p} &= -2Y_0E_2 j^m m\Big[J_{m+2p+\frac{1}{2}}(\kappa\sin\theta_0)\\ &\quad +J_{m+2p+\frac{5}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{3}{2}}\end{aligned}\tag{34c}$$

$$\begin{aligned}K^{(2)}_{m,p} &= -2Y_0E_2 j^m\Big[\alpha^m_p J_{m+2p-\frac{1}{2}}(\kappa\sin\theta_0)\\ &\quad -(\alpha^p_m+1)J_{m+2p+\frac{3}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{1}{2}}\end{aligned}\tag{34d}$$

$$m=1,2,3,\cdots,\quad p=0,1,2,3,\cdots$$

$$\begin{aligned}H^{(1)}_{0,p} &= Y_0E_1\cos\theta_0\Big[-pJ_{2p+\frac{1}{2}}(\kappa\sin\theta_0)\\ &\quad +(p+1.5)J_{2p+\frac{5}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{3}{2}}\end{aligned}\tag{34e}$$

$$K^{(2)}_{0,p} = -Y_0E_2\Big[-pJ_{2p-\frac{1}{2}}(\kappa\sin\theta_0)$$

$$+(p+0.5)J_{2p+\frac{3}{2}}(\kappa\sin\theta_0)\Big](\kappa\sin\theta_0)^{-\frac{1}{2}} \qquad (34f)$$
$$p = 0, 1, 2, 3, \cdots$$

It is readily seen that equations (33) to (34) reduce to (23) if we change $E_1 \to 2E_2$, and $E_2 \to 2E_1$. This is consquence of Babinet's principle.

4.3. Far Field Expression

Far field is obtained by applying formula (28a) to the expression of the vector potential (7) with (30) and (33). The result is given by

$$E_\theta = \frac{\exp(-jkR)}{kR} D_\theta(\theta,\phi), \qquad E_\phi = \frac{\exp(-jkR)}{kR} D_\phi(\theta,\phi) \qquad (35a)$$

where

$$\begin{aligned} D_\theta(\theta,\phi) = & -2k^2a^2 \sum_{n=0}^{\infty} C_{0n} J_{2n+\frac{3}{2}}(\kappa\sin\theta)(\kappa\sin\theta)^{-\frac{1}{2}} \\ & +k^2a^2 \sum_{m=1}^{\infty} j^m \sum_{n=0}^{\infty} \Big\{ \Big[C_{mn}\Xi^-_{mn}(\kappa\sin\theta) + D_{mn}\Gamma^+_{mn}(\kappa\sin\theta) \Big] \\ & \times \cos m\phi \; + \Big[A_{mn}\Xi^-_{mn}(\kappa\sin\theta) - B_{mn}\Gamma^+_{mn}(\kappa\sin\theta) \Big] \sin m\phi \Big\} \end{aligned} \qquad (35b)$$

$$\begin{aligned} D_\phi(\theta,\phi) = & \; 2k^2a^2\cos\theta \sum_{n=0}^{\infty} B_{0n} J_{2n+\frac{5}{2}}(\kappa\sin\theta)(\kappa\sin\theta)^{-\frac{3}{2}} \\ & +k^2a^2\cos\theta \sum_{m=0}^{\infty} j^m \sum_{n=0}^{\infty} \Big\{ \Big[A_{mn}\Xi^+_{mn}(\kappa\sin\theta) - B_{mn}\Gamma^-_{mn}(\kappa\sin\theta) \Big] \\ & \times \cos m\phi - \Big[C_{mn}\Xi^+_{mn}(\kappa\sin\theta) + D_{mn}\Gamma^-_{mn}(\kappa\sin\theta) \Big] \sin m\phi \Big\} \end{aligned} \qquad (35c)$$

where we have used the relations $E_\theta \sim -j\omega A_\theta = j\omega \sin\theta A_z$ and $H_\theta \sim -j\omega F_\theta$ or $E_\phi = -j\omega \sin\theta F_z$, which hold in a far region.

5. NUMERICAL COMPUTATION

With the formulation developed in the previous sections, we have performed the numerical computation for physical quantities such as, far field patterns, distribution of current density or aperture field, total scattering cross section of disk or transmission coefficient, and mono-static radar cross section. To perform the computation, we must determine the expansion coefficients $A_{mn} \sim D_{mn}$ for given values of radius $\kappa = ka$ and incident angle θ_0. These are derived from (25) by using the standard numerical code of matrix equation for complex coefficients. The matrix elements contain the functions $G(\alpha,\beta)$, $G_2(\alpha,\beta)$, and $K(\alpha,\beta)$ and these infinite integrals can be transformed into infinite series. These series are convenient for numerical computation. How to derive these expressions are firstly discussed by Nomura and Katsura [29], but we use a slightly different method as discussed in Appendix B. To verify the validity of these expressions numerically, we compare the results of series expressions with those by direct numerical integration. The agreement is complete. The required maximum size M of matrix equations to determine the expansion coefficients depends on the values of κ and we have chosen $M \simeq 1.6\kappa+5$ and it is found to be sufficient in the present computation.

5.1. Radiation Pattern

The theoretical expressions for the far field are given by (29) for the disk and (35) for aperture problem. The patterns for parallel ($E_2 = 0$) and perpendicular ($E_1 = 0$) polarizations are denoted by D_ϕ and D_θ, respectively. Fig. 2 and Fig. 3 show the far field patterns of circular disks in the $\phi-$cut plane $\phi = 0, \pi$. The normalized radii are $\kappa = ka = 3$ and $ka = 5$, respectively. The plane of incidence is $xz-$plane ($\phi_0 = 0, \pi$). In both these figures, part (a) corresponds to the normal incidence ($\theta_0 = 0$), (b) and (c) correspond to the incident angles $\theta_0 = 30°$ and $\theta_0 = 60°$. In these figures the results obtained using the physical optics (PO) method are also included for comparison. The PO expression for the far field is given by

$$E_\theta \sim -j\omega A_\theta = -j2\pi a\kappa G_0(R)\cos\theta\Big(E_2\cos\phi + E_1\cos\theta_0\sin\phi\Big)\frac{2J_1(\kappa\Theta)}{\kappa\Theta} \tag{36a}$$

$$E_\phi \sim -j\omega A_\phi = -j2\pi a\kappa G_0(R)\Big(-E_2\sin\phi + E_1\cos\theta_0\cos\phi\Big)\frac{2J_1(\kappa\Theta)}{\kappa\Theta} \tag{36b}$$

where $\Theta = \sqrt{(\sin\theta_0 + \sin\theta\cos\phi)^2 + \sin^2\sin^2\phi}$. The factor $2J_1(\kappa\Theta)/\kappa\Theta$ is known as the Airy pattern. As it is clear from (36), PO patterns have null points at some observation point, but the results based on the theoretical analysis do not become zero at any observation angle. It is seen from the comparison, the PO and present results agree well for normal incidence, but degree of the discrepancies increases

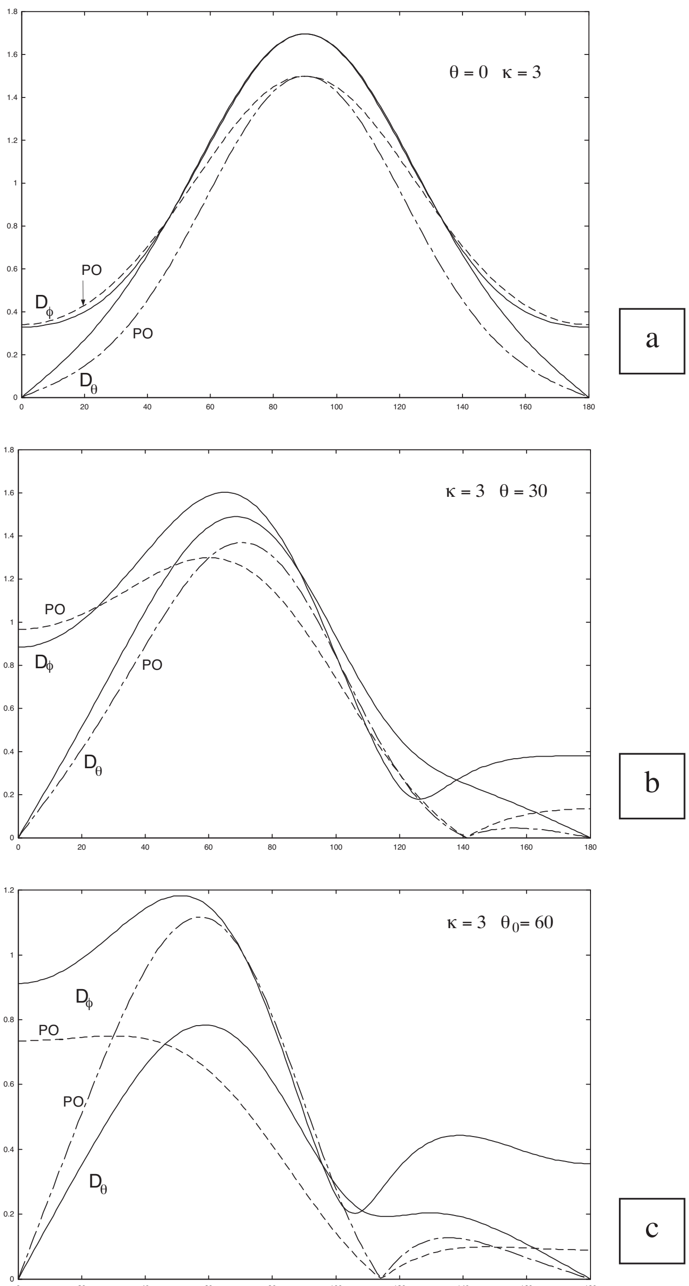

Figure 2. Far scattered field pattern of disk.

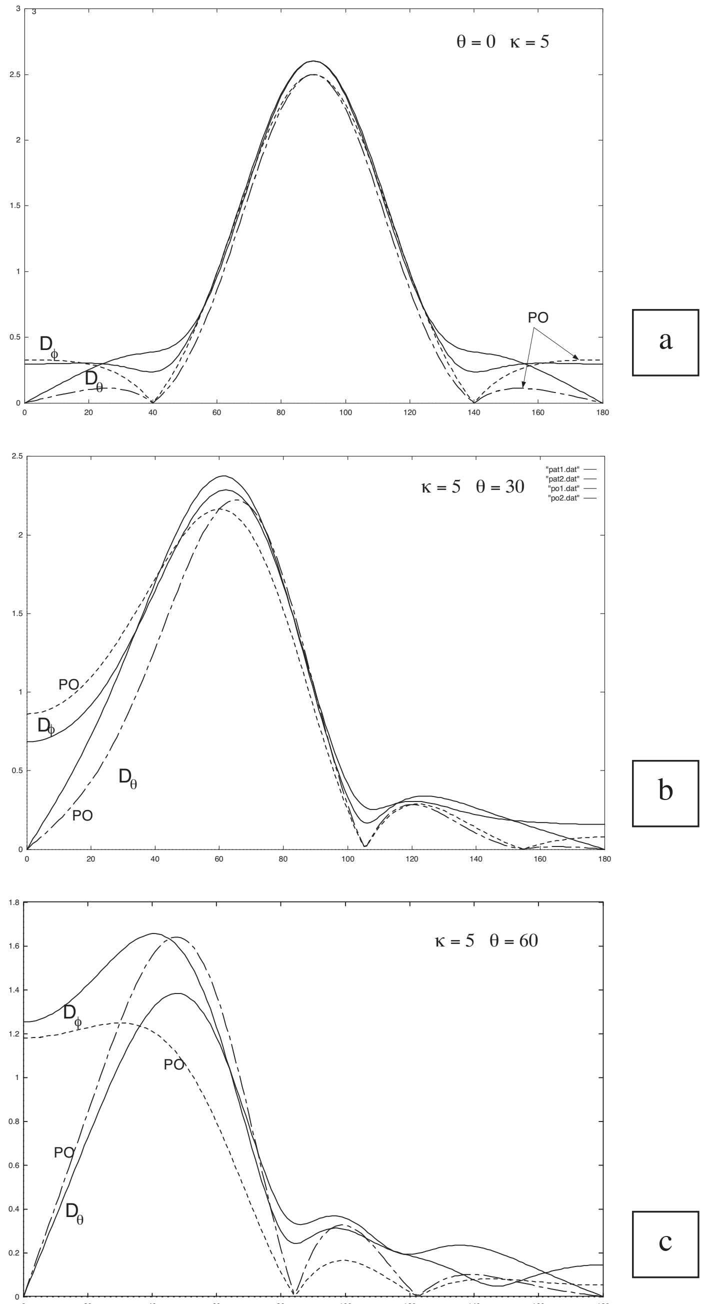

Figure 3. Far scattered field pattern of disk.

as the angle of incidence becomes large and the radius of the disk decreases, as expected.

5.2. Distribution of Current Density

The current density induced on the perfectly conducting disk is given by (8c) and (8d) with (14), while the electric field distribution on the aperture in a perfectly conducting plane is given by (32). The functions $F^{\pm}_{mn}(\rho_a)$ and $G^{\pm}_{mn}(\rho_a)$ can be expressed in terms of hypergeometric functions and their explicit expressions are given in Appendix A. Thus, once the expansion coefficients $A_{mn} \sim D_{mn}$ are determined, these quantities can be computed readily. Fig. 4 shows the distribution of the current densities induced on the disk with the normalized radius $ka = 5$. In these figures parts (a), (b) and (c) also correspond to the incident angles $\theta_0 = 0°$, $30°$ and $60°$, respectively. In the physical optics current approximation, we have $K_\rho = 2$ and $K_\phi = 2\cos\theta_0$ when unit amplitude plane wave is incident. Near the edge of the disk we readily find from (14) and (15) that K_ρ and K_ϕ have the properties $K_\rho \approx (a^2-\rho^2)^{\frac{1}{2}}$ and $K_\phi \approx (a^2-\rho^2)^{-\frac{1}{2}}$. From these figures it is seen that K_ρ undulates more strongly than that of K_ϕ. It seems to be due to that K_ρ is forced to zero at the edges.

5.3. Transmission Coefficient or Total Scattering Cross Section

The total scattering cross section σ_T is given by the forward scattering theorem

$$\sigma_T = \frac{4\pi}{k^2} S(\hat{\mathbf{r}}_0)(\mathbf{i}_e \cdot \mathbf{i}_i) \tag{37}$$

where $\hat{\mathbf{r}}_0$ is oriented in the direction of propagation of the incident wave $\mathbf{E}^i = \mathbf{i}_i E_0$ with amplitude E_0. The transmission coefficient of circular hole can be computed from

$$t = \frac{W^s}{W^i}, \qquad W^s = \int_S \Re\Big[E^s_\phi H^{s*}_\rho - E^s_\rho H^{s*}_\phi\Big]_{z=0-} dS \tag{38a}$$

On the aperture $\mathbf{n} \times (\mathbf{H}^s + \mathbf{H}^i) = 0$, therefore the scattered magnetic field on the lower side is $-\mathbf{H}^s = \mathbf{H}^i$. Using this fact W^s is evaluated and we have the following relatios

$$\begin{aligned} \sigma_\perp &= W^s(E_2 = 0) = \Im\left[\frac{2\pi}{k^2} D_\phi(\theta_0, \pi)\right], \\ \sigma_\parallel &= W^s(E_1 = 0) = \Im\left[\frac{2\pi}{k^2} D_\theta(\theta_0, \pi)\right] \end{aligned} \tag{38b}$$

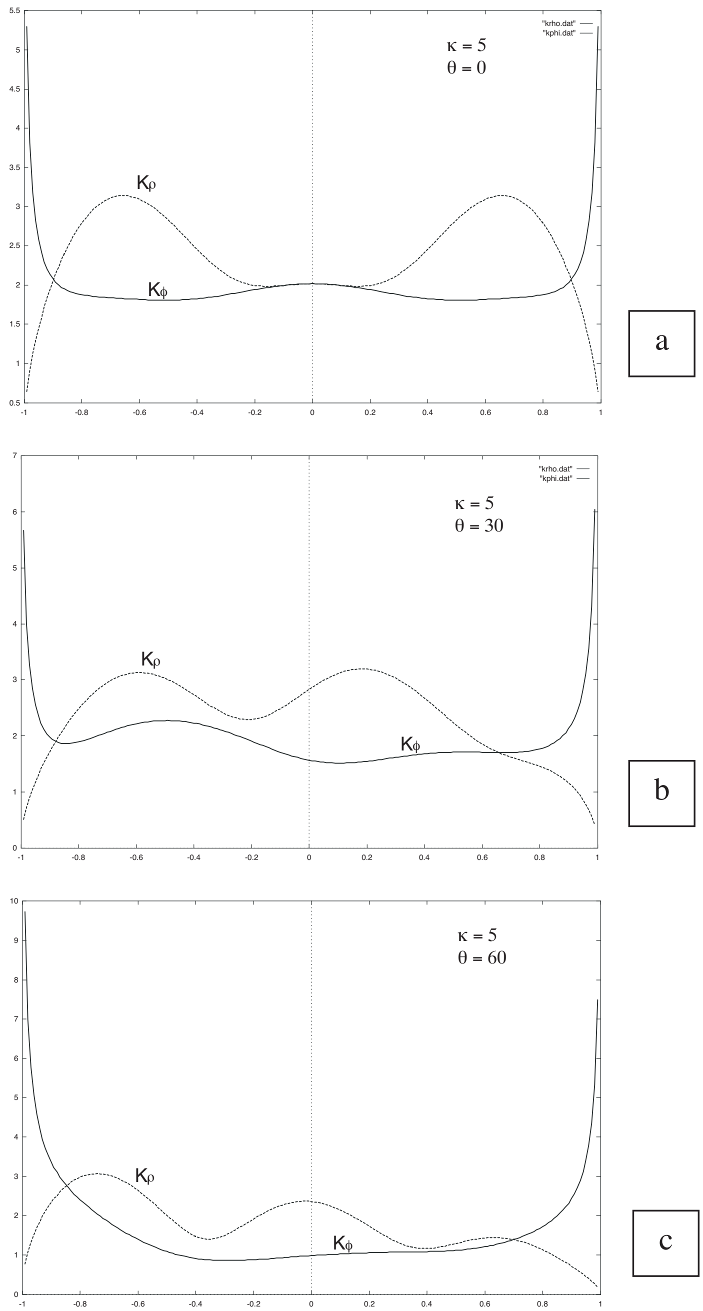

Figure 4. Distribution of current density on the disk.

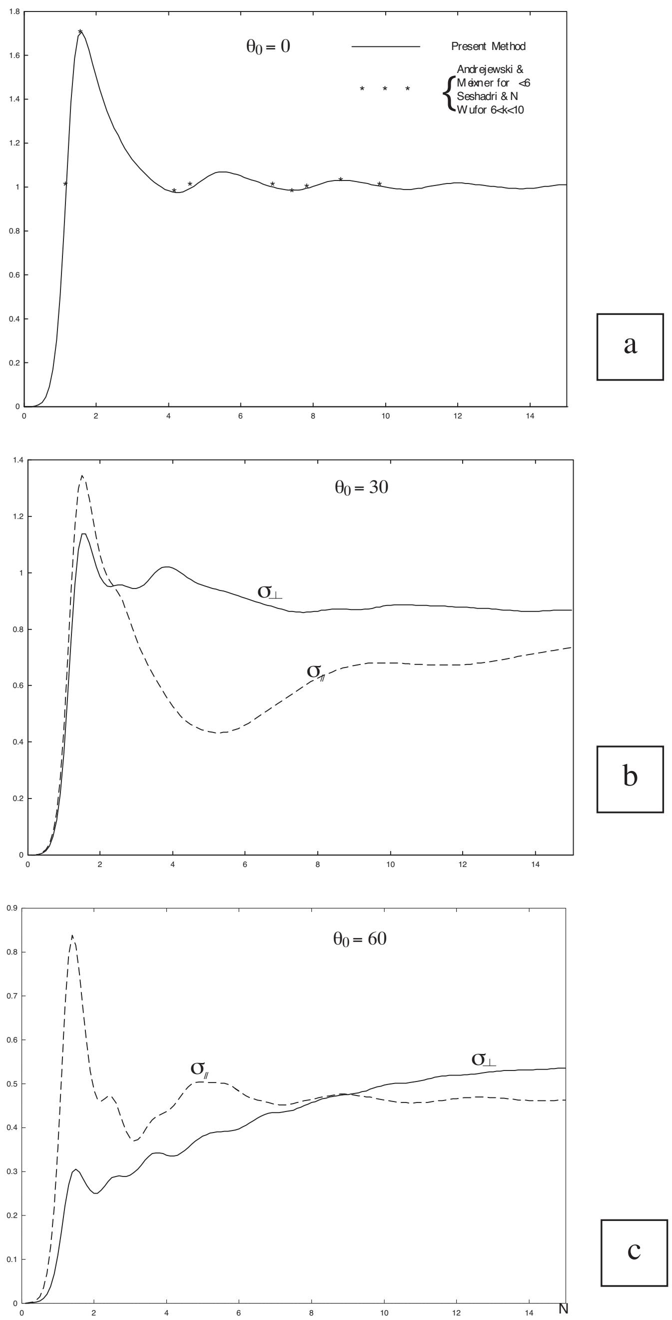

Figure 5. Transmission coefficient of the circular aperture.

Table 1. Numerical comparison of transmission coefficients.

κ	Andrejewski and Meixner	Seshadri and Wu	Jones	Present Method
1	...	1.00257	1.01487	0.50462
2	...	1.38364	1.37742	1.50369
3	1.127	1.13291	1.13400	1.12731
4	0.992	0.98092	0.98158	0.98322
5	1.039	1.03367	1.03306	1.04012
6	1.047	1.05359	1.05368	1.05136
7	0.995	0.99365	0.99386	0.99469
8	0.999	1.00239	1.00222	1.00333
9	1.030	1.03044	1.03043	1.02953
10	1.001	0.99915	0.99925	0.99970
11	...	0.99572	0.99566	0.99581
12	...	1.01930	1.01928	1.01893
13	...	1.00197	1.00203	1.00227
14	...	0.99400	0.99438	0.99434
15	...	1.01254	1.01251	1.01241

This represents the forward scattering theorem. The factor $\frac{2\pi}{k^2}$ instead of $\frac{4\pi}{k^2}$ is the consequence that the transmitted field exists in only half-space.

The transmission coefficient "t" for and oblique incidences is computed for the range $0 \leq \kappa \leq 15$. The results are normalized by area of disk πa^2 and shown in Fig. 5. Fig. 5a is for normal incidence, and results by Andrejewski and Meixner and results by Seshadri and Wu [33] are also shown for comparison. In the range of κ less than 10, our results agree well with those of Andrejewski and Meixner, and our results agree well with those of Seshadri and Wu for κ larger than 3. When the values of κ is very small, t is known to be proportional to $(ka)^4$, or more explicitly $t \simeq 64(ka)^4/27\pi$. When κ is very large, asymptotic exressions were derived by Seshadri and Wu, and Jones, the results are written as

$$\frac{t}{\pi a^2} = 1 - \frac{1}{\pi^{\frac{1}{2}}(ka)^{\frac{3}{2}}} \sin\left(2ka - \frac{\pi}{4}\right) + \frac{1}{(ka)^2}\left[\frac{3}{4} - \frac{1}{2\pi}\cos(4ka)\right]$$
$$-\frac{1}{\pi^{\frac{1}{2}}(ka)^{\frac{5}{2}}}\left[a_0 \cos\left(2ka - \frac{\pi}{4}\right) + \frac{1}{4\pi}\sin\left(6ka - \frac{3\pi}{4}\right)\right] + O[(ka)^{-3}] \quad (39)$$

where $q_0 = 7/4$ in Seshadri and Wu's result and $q_0 = 27/16$ for Jones's result. In table I we listed the results obtained by four different methods, Andrejewski and Meixner [26], Seshadri and Wu [33], Jones [34] and the present method in the range $1 \leq \kappa \leq 15$. In the paper of Seshadri and Wu, the numerical results are given only in the range $3 \leq \kappa \leq 10$, we computed other range from (39). Considering that (39) becomes more precise, we can see our method can cover to the extent of $\kappa = 15$ sufficiently. Fig. 5b and 5c are the results for oblique incidence. We could not find the data by other methods for comparison.

5.4. Radar Cross Section of Perfectly Conducting Disk

We consider the numerical computation of the mono-static back scattering cross section (RCS). This is readily obtained by using the expressions derived in this paper and the results are shown in Fig. 6. For comparison we have also the results due to Ufimtsev who obtained by applying the Sommerfeld-Macdonal method [35, p. 515] and experimental results extracted from the paper by Li et al. The experimental results are denoted by the symbols $*$ and $\bigoplus$. In a relatively narrow scanning angle, our results agree completely with the numerical results given by Li et al. [23], that they obtained by hybrid MoM method though these results are not shown in the figure. In a angle far from the broad-sight, the experimental results are more close to ours rather than to theirs.

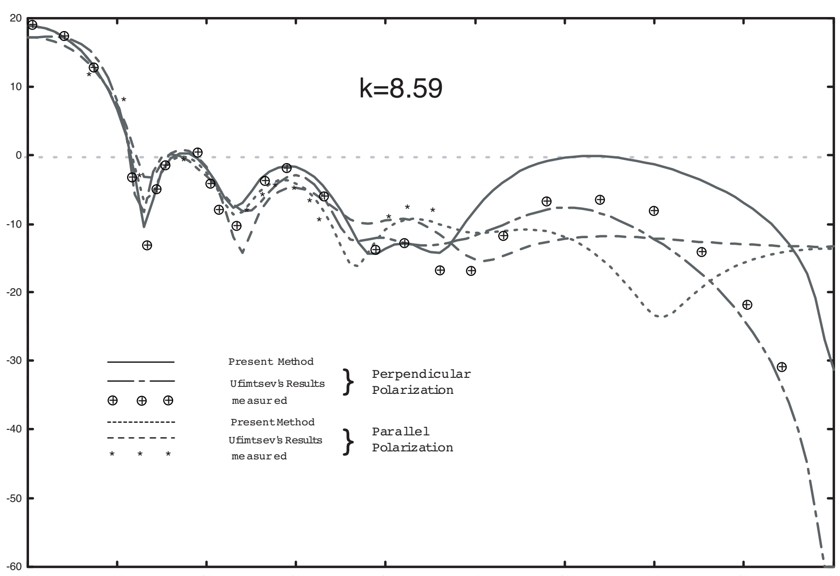

Figure 6. Transmission coefficient of the circular aperture.

6. CONCLUSION

We have formulated the plane wave field scattered by a perfectly conducting circular disk and its complementary circular hole in a perfectly conducting infinite plane. We derived dual integral equations for the induced current and the tangential components of the electric field on the disk. The equations for the current densities are solved by applying the discontinuous properties of the Weber-Schafheitlin's integrals and the vector Hankel transform. It is readily found that the solution satisfies Maxwell's equations and edge conditions. Therefore it may be considered as the eigen function expansion. The equations for the electric field are solved by applying the projection. We use the functional space of the Jacobi's polynomials. Thus the problem reduces to the matrix equations and their elements are given by infinite integrals of a single variables. These integrals are transformed into infinite series in terms of the normalized radius. Numerical computation for the far field patterns, distribution of the current densities, and transmission coefficients of the circular hole in a perfectly conducting screen for $ka = 0.1$ to $ka = 15$. The results for the transmission coefficient for normal incident case is compared with some published results and we have a good agreement.

APPENDIX A. THE EXPRESSIONS FOR $F^{\pm}_{MN}(\rho_A)$ AND $G^{\pm}_{MN}(\rho_A)$

$$\begin{aligned}
F^{\pm}_{mn}(\rho_a) &= \sqrt{2}\rho_a^{m-1}\frac{\Gamma(m+n)}{\Gamma(n+\frac{1}{2})\Gamma(m)}F\left(m+n,-n+\frac{1}{2},m,\rho_a^2\right)\\
&\quad\pm\sqrt{2}\rho_a^{m+1}\frac{\Gamma(m+n+2)}{\Gamma(n+\frac{1}{2})\Gamma(m+2)}F\left(m+n+2,-n+\frac{1}{2},m+2,\rho_a^2\right)\\
&= (-1)^n\sqrt{\frac{2}{\pi}}(1-\rho_a^2)^{-\frac{1}{2}}\left\{\rho_a^{m-1}F\left(-n,m+n-\frac{1}{2},\frac{1}{2},1-\rho_a^2\right)\right.\\
&\quad\left.\pm\rho_a^{m+1}F\left(-n,m+n+\frac{3}{2},\frac{1}{2},1-\rho_a^2\right)\right\}\qquad m\geq 1 \qquad \text{(A1)}\\
F^{+}_{0n}(\rho_a) &= 2\sqrt{2}\rho_a\frac{\Gamma(n+2)}{\Gamma(n+\frac{1}{2})\Gamma(2)}F\left(n+2,-n+\frac{1}{2},2,\rho_a^2\right)\\
&= (-1)^n\sqrt{\frac{8}{\pi}}\rho_a(1-\rho_a^2)^{-\frac{1}{2}}F\left(-n,n+\frac{3}{2},\frac{1}{2},1-\rho_a^2\right) \qquad \text{(A2)}\\
G^{\pm}_{mn}(\rho_a) &= \frac{\rho_a^{m-1}}{\sqrt{2}}\frac{\Gamma(m+n)}{\Gamma(n+\frac{3}{2})\Gamma(m)}F\left(m+n,-n-\frac{1}{2},m,\rho_a^2\right)
\end{aligned}$$

$$\pm\frac{\rho_a^{m+1}}{\sqrt{2}}\frac{\Gamma(m+n+2)}{\Gamma(n+\frac{3}{2})\Gamma(m+2)}F\left(m+n+2,-n-\frac{1}{2},m+2,\rho_a^2\right)$$
$$=(-1)^n\sqrt{\frac{2}{\pi}}(1-\rho_a^2)^{\frac{1}{2}}\left\{\rho_a^{m-1}F\left(-n,m+n+\frac{1}{2},\frac{3}{2},1-\rho_a^2\right)\right.$$
$$\left.\pm\rho_a^{m+1}F\left(-n,m+n+\frac{5}{2},\frac{3}{2},1-\rho_a^2\right)\right\}\qquad m\geq 1\qquad \text{(A3)}$$

$$G_{0n}^{+}(\rho_a)=\sqrt{2}\frac{\rho_a\Gamma(n+2)}{\Gamma(n+\frac{3}{2})\Gamma(2)}F\left(n+2,-n-\frac{1}{2},2,\rho_a^2\right)$$
$$=(-1)^n 2\sqrt{\frac{8}{\pi}}\rho_a(1-\rho_a^2)^{\frac{1}{2}}F\left(-n,n+\frac{5}{2},\frac{3}{2},1-\rho_a^2\right)\qquad \text{(A4)}$$

APPENDIX B. SERIES SOLUTIONS OF THE INTEGRALS $G(\alpha,\beta)$, $G_2(\alpha,\beta)$ AND $K(\alpha,\beta)$

B.1. The Results of $G(\alpha,\beta)$ for $\alpha=m+\frac{1}{2}$ and $\beta=n+\frac{1}{2}$

Since the integrals $G(\alpha,\beta)$, $G_2(\alpha,\beta)$ and $K(\alpha,\beta)$ can be performed in a similar way, we show here how to derive the expression of $G(\alpha,\beta)$. This integral was first evaluated by Nomura and Katsura [29]. Consider an evaluation of the integral defined by

$$G(\alpha,\beta;\kappa)=\lim_{v\to 0}\int_0^\infty\frac{J_\alpha(\xi)J_\beta(\xi)}{\sqrt{\xi^2-\kappa^2}}\exp[-\sqrt{\xi^2-\kappa^2}v]d\xi\qquad \text{(B1)}$$

In this section we will derive the series solution of this integral in a different way. As shown in (B1) we define his integral $G(\alpha,\beta)$ as the limitting value. In the first step, we derive the series representation for larger values of v ($v>1$), then the expression is converted into contour integral and derive the result which is valid for smaller value of v ($v<1$) [28]. Using the integral representation for the product of the Bessel functionsn and the integral representation for the resulting expression given by

$$J_\mu(\xi)J_\nu(\xi)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}J_{\mu+\nu}(2\xi\cos\theta)\cos([\mu-\nu)\theta]d\theta\qquad \text{(B2)}$$
$$J_{\alpha+\beta}(2\xi\cos\theta)=\frac{1}{2\pi j}\int_{-j\infty-\epsilon}^{j\infty-\epsilon}\frac{\Gamma(-t)}{\Gamma(\alpha+\beta+t+1)}(\xi\cos\theta)^{\alpha+\beta+2t}dt\ (\epsilon>0)\qquad \text{(B3)}$$

(B1) is transformed into

$$G(\alpha,\beta;\kappa)=\frac{1}{\pi^2 j}\int_{-j\infty-\epsilon}^{j\infty-\epsilon}\frac{\Gamma(-t)dt}{\Gamma(\alpha+\beta+t+1)}$$

$$\times \int_0^{\frac{\pi}{2}} \cos^{\alpha+\beta+2t}\theta \cos[(\alpha-\beta)\theta] d\theta$$
$$\times \int_0^{\infty} \frac{1}{\sqrt{\xi^2-\kappa^2}} \xi^{\alpha+\beta+2t} \exp[-\sqrt{\xi^2-\kappa^2}v] d\xi \quad \text{(B4)}$$

The integral with respect to θ and ξ may be carried out with the results

$$\int_0^{\frac{\pi}{2}} \cos^{\alpha+\beta+2t}\theta \cos[(\alpha-\beta)\theta] d\theta = \frac{\pi\Gamma(\alpha+\beta+2t+1)}{2^{\alpha+\beta+2t+1}\Gamma(\alpha+t+1)\Gamma(\beta+t+1)} \quad \text{(B5)}$$

$$\int_0^{\infty} \frac{(\xi/2)^{\alpha+\beta+2t}}{\sqrt{\xi^2-\kappa^2}} \exp[-\sqrt{\xi^2-\kappa^2}v] d\xi = \frac{\sqrt{\pi}}{2}\Gamma\left[\frac{1}{2}(\alpha+\beta)+t+\frac{1}{2}\right]$$
$$\times \frac{(\kappa/2)^{\alpha+\beta+2t}}{(\kappa v/2)^{(\alpha+\beta)/2+t}} \left[-Y_{\frac{1}{2}(\alpha+\beta)+t}(\kappa v) - jJ_{\frac{1}{2}(\alpha+\beta)+t}(\kappa v)\right] \quad \text{(B6)}$$

(B6) is valid for $v > 1$. Substituting these results into (B4) we have

$$\begin{aligned} G(\alpha,\beta;\kappa) \ = \ & \frac{1}{4\sqrt{\pi}j}\int_{-j\infty-\epsilon}^{j\infty-\epsilon} \frac{\Gamma(-t)}{\Gamma(\alpha+\beta+t+1)} \\ & \times \frac{\Gamma(\alpha+\beta+2t+1)\Gamma[\frac{1}{2}(\alpha+\beta)+t+\frac{1}{2}]}{\Gamma(\alpha+t+1)\Gamma(\beta+t+1)} \\ & \times \frac{(\kappa/2)^{\alpha+\beta+2t}}{(\kappa v/2)^{(\alpha+\beta)/2+t}} \left[-Y_{\frac{1}{2}(\alpha+\beta)+t}(\kappa v) - jJ_{\frac{1}{2}(\alpha+\beta)+t}(\kappa v)\right] dt \end{aligned} \quad \text{(B7)}$$

The Neumann function in the above equation is

$$Y_{\frac{1}{2}(\alpha+\beta)+t}(\kappa v) = \frac{1}{\sin\left\{\left[\frac{1}{2}(\alpha+\beta)+t\right]\pi\right\}}$$
$$\times \left[\cos\left\{\left(\frac{1}{2}(\alpha+\beta)+t\right)\pi\right\} J_{\frac{1}{2}(\alpha+\beta)+t}(\kappa v) - J_{-\left[\frac{1}{2}(\alpha+\beta)+t\right]}(\kappa v)\right] \quad \text{(B8)}$$

$G(\alpha,\beta;\kappa)$ is decomposed into three parts shown below.

$$I_1 \ = \ \frac{1}{4\sqrt{\pi}j}\int_{-j\infty-\epsilon}^{j\infty-\epsilon} Q(t)\text{cosec}\left[\left(\frac{\alpha+\beta}{2}+t\right)\pi\right] J_{-\left[\frac{1}{2}(\alpha+\beta)+t\right]}(\kappa v) dt \quad \text{(B9)}$$

$$I_2 \ = \ -\frac{1}{4\sqrt{\pi}}\int_{-j\infty-\epsilon}^{j\infty-\epsilon} Q(t) J_{\frac{1}{2}(\alpha+\beta)+t}(\kappa v) dt \quad \text{(B10)}$$

$$I_3 = -\frac{1}{4\sqrt{\pi}j}\int_{-j\infty-\epsilon}^{j\infty-\epsilon} Q(t)\cot\left[\left(\frac{\alpha+\beta}{2}+t\right)\pi\right] J_{\frac{1}{2}(\alpha+\beta)+t}(\kappa v)dt \quad \text{(B11)}$$

$$Q(t) = \frac{\Gamma(-t)\Gamma(\alpha+\beta+2t+1)\Gamma\left[\frac{1}{2}(\alpha+\beta)+t+\frac{1}{2}\right]}{\Gamma(\alpha+\beta+t+1)\Gamma(\alpha+t+1)\Gamma(\beta+t+1)}\left(\frac{\kappa}{2v}\right)^{\frac{1}{2}(\alpha+\beta)+t} \quad \text{(B12)}$$

In the limit $|t| \to \infty$, the integrand of I_1 approaches to $\exp[2t\ln(2/v)]$, while those of I_2, I_3 approach $\exp[-t\ln v]$. Therefore, we may close the contour of I_1 in the left half plane for $v < 2$, and those of I_2 and I_3 in the right half plane. Since the evaluation of the integrals depends on the indices α and β, we consider the following cases.

The integrand of I_1 has the singularities

(I) simple poles at $t = -\frac{1}{2} - p \quad \left(p = 0, 1, 2, 3, \cdots, \frac{\alpha+\beta-1}{2}\right)$

(II) double poles at $t = -\frac{\alpha+\beta+1}{2} - p \quad (p = 0, 1, 2, \cdots)$

(III) double poles at $t = -\frac{\alpha+\beta+1+p}{2} \quad (p = 1, 3, 5\cdots)$

We find that the contributions from the double poles vanish as $v \to 0$. The result is

$$I_1 = \frac{1}{2\sqrt{\pi}}\sum_{p=0}^{(\alpha+\beta-1)/2}\frac{\Gamma\left(\frac{1}{2}(\alpha+\beta)-p\right)\Gamma(2p+1)\Gamma(p+\frac{1}{2})(\kappa/2)^{2p}}{\left(\begin{array}{r}\Gamma\left(\frac{1}{2}(\alpha+\beta)+p+1\right)\Gamma\left(\frac{1}{2}(\alpha-\beta)+p+1\right)\\ \times\Gamma\left(\frac{1}{2}(\beta-\alpha)+p+1\right)\Gamma(p+1)\end{array}\right)} \quad \text{(B13)}$$

The integrand of I_2 has simple poles at $t = p$ $(p = 0, 1, 2, \cdots)$, while the poles of I_3 are at $t = p + \frac{1}{2}$, $(p = 0, 1, 2, \cdots)$. Then from the residue theorem we have

$$I_2 = -j\frac{\sqrt{\pi}}{2}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma(\alpha+\beta+2p+1)\Gamma\left(\frac{1}{2}(\alpha+\beta)+p+\frac{1}{2}\right)}{\left(\begin{array}{r}\Gamma(p+1)\Gamma(\alpha+\beta+p+1)\Gamma(\alpha+p+1)\\ \times\Gamma(\beta+p+1)\Gamma\left(\frac{1}{2}(\alpha+\beta)+p+1\right)\end{array}\right)} \times\left(\frac{\kappa}{2}\right)^{\alpha+\beta+2p} \quad \text{(B14)}$$

$$I_3 = \frac{\sqrt{\pi}}{2}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma(\alpha+\beta+2p+2)\Gamma\left(\frac{1}{2}(\alpha+\beta)+p+1\right)}{\left(\begin{array}{r}\Gamma\left(\alpha+\beta+p+\frac{3}{2}\right)\Gamma\left(\alpha+p+\frac{3}{2}\right)\Gamma\left(\beta+p+\frac{3}{2}\right)\\ \times\Gamma\left(p+\frac{3}{2}\right)\Gamma\left[\frac{1}{2}(\alpha+\beta)+p+\frac{3}{2}\right]\end{array}\right)}$$

$$\times\left(\frac{\kappa}{2}\right)^{\alpha+\beta+2p+1} \tag{B15}$$

B.2. The Resuls of $G_2(\alpha,\beta)$ for $\alpha=m+\frac{1}{2}$ and $\beta=n+\frac{1}{2}$

In a manner similar to $G(\alpha,\beta;\kappa)$ we have

$$\begin{aligned}
G_2(\alpha,\beta;\kappa)&=\int_0^\infty \frac{J_\alpha(\xi)J_\beta(\xi)}{\xi^2\sqrt{\xi^2-\kappa^2}}d\xi\\
&=\frac{1}{8\sqrt{\pi}}\sum_{p=0}^{(\alpha+\beta-3)/2}\frac{\Gamma\left(\frac{1}{2}(\alpha+\beta)-p-1\right)\Gamma(2p+3)\Gamma(p+\frac{1}{2})}{\left(\begin{array}{r}\Gamma\left[\frac{1}{2}(\alpha+\beta)+p+2\right]\Gamma\left(\frac{1}{2}(\alpha-\beta)+p+2\right)\\ \times\Gamma\left(\frac{1}{2}(\beta-\alpha)+p+2\right)\Gamma(p+1)\end{array}\right)}\\
&\times\left(\frac{\kappa}{2}\right)^{2p}-j\frac{\sqrt{\pi}}{8}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma(\alpha+\beta+2p+1)\Gamma\left(\frac{1}{2}(\alpha+\beta)+p-\frac{1}{2}\right)}{\left(\begin{array}{r}\Gamma(p+1)\Gamma(\alpha+\beta+p+1)\Gamma(\alpha+p+1)\\ \times\Gamma(\beta+p+1)\Gamma\left[\frac{1}{2}(\alpha+\beta)+p\right]\end{array}\right)}\\
&\times\left(\frac{\kappa}{2}\right)^{\alpha+\beta+2p-2}\\
&-\frac{\sqrt{\pi}}{8}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma(\alpha+\beta+2p+2)\Gamma\left(\frac{1}{2}(\alpha+\beta)+p\right)}{\left(\begin{array}{r}\Gamma\left(\alpha+\beta+p+\frac{3}{2}\right)\Gamma\left(\alpha+p+\frac{3}{2}\right)\Gamma\left(\beta+p+\frac{3}{2}\right)\\ \times\Gamma\left(p+\frac{3}{2}\right)\Gamma\left[\frac{1}{2}(\alpha+\beta)+p+\frac{1}{2}\right]\end{array}\right)}\\
&\times\left(\frac{\kappa}{2}\right)^{\alpha+\beta+2p-1}
\end{aligned} \tag{B16}$$

B.3. The Results of $K(\alpha,\beta)$ for $\alpha=m+\frac{1}{2}$ and $\beta=n+\frac{1}{2}$

$$\begin{aligned}
K(\alpha,\beta;\kappa)&=\lim_{v\to0}\int_0^\infty\sqrt{\xi^2-\kappa^2}\frac{J_\alpha(\xi)J_\beta(\xi)}{\xi^2}\exp[-\sqrt{\xi^2-\kappa^2}v]d\xi\\
&=-\frac{1}{4\sqrt{\pi}}\sum_{p=0}^{\frac{1}{2}(\alpha+\beta)-1}\frac{\Gamma\left(\frac{1}{2}(\alpha+\beta)-p\right)\Gamma(2p+1)\Gamma(p-\frac{1}{2})}{\left(\begin{array}{c}\Gamma(p+1)\Gamma\left(\frac{1}{2}(\alpha+\beta)+p+1\right)\\ \times\Gamma\left(\frac{\alpha-\beta}{2}+p+1\right)\Gamma\left(\frac{\beta-\alpha}{2}+p+1\right)\end{array}\right)}\left(\frac{\kappa}{2}\right)^{2p}
\end{aligned}$$

$$+j\frac{\sqrt{\pi}}{4}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma(\alpha+\beta+2p+1)\Gamma\left(\frac{\alpha+\beta}{2}+p-\frac{1}{2}\right)}{\left(\begin{array}{r}\Gamma(p+1)\Gamma(\alpha+\beta+p+1)\Gamma(\alpha+p+1)\\ \times\Gamma(\beta+p+1)\Gamma\left(\frac{\alpha+\beta}{2}+p+1\right)\end{array}\right)}$$

$$\times\left(\frac{\kappa}{2}\right)^{\alpha+\beta+2p}$$

$$+\frac{\sqrt{\pi}}{4}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma(\alpha+\beta+2p+2)\Gamma\left[\frac{1}{2}(\alpha+\beta)+p\right]}{\left(\begin{array}{r}\Gamma\left(\alpha+\beta+p+\frac{3}{2}\right)\Gamma\left(\alpha+p+\frac{3}{2}\right)\Gamma\left(\beta+p+\frac{3}{2}\right)\\ \times\Gamma\left(\frac{1}{2}(\alpha+\beta)+p+\frac{3}{2}\right)\Gamma\left(p+\frac{3}{2}\right)\end{array}\right)}$$

$$\times\left(\frac{\kappa}{2}\right)^{\alpha+\beta+2p+1} \qquad \text{(B17)}$$

It is noted that $K(\alpha,\beta,\kappa)$ is related to $G(\alpha,\beta,\kappa)$ and $G_2(\alpha,\beta,\kappa)$by

$$K(\alpha,\beta,\kappa)=G(\alpha,\beta,\kappa)-G_2(\alpha,\beta,\kappa) \qquad \text{(B18)}$$

APPENDIX C. SOME PROPERTIES OF THE JACOBI'S POLYNOMIALS $v_n^m(x)$ AND $u_n^m(x)$

C.1. Definition

$$\begin{aligned}u_n^m(x) &= G\left(m+\frac{1}{2},m+1,x\right)=F\left(n+m+\frac{1}{2},-n,m+1;x\right)\\ &= \frac{\sqrt{2}\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+\frac{1}{2})}x^{-m/2}\int_0^\infty\frac{J_m(\sqrt{x}\xi)J_{2n+m+\frac{1}{2}}(\xi)}{\sqrt{\xi}}d\xi\\ &= \frac{\Gamma(m+1)}{\Gamma(n+m+1)}x^{-m}(1-x)^{\frac{1}{2}}\frac{d^n}{dx^n}\left\{x^{n+m}(1-x)^{n-\frac{1}{2}}\right\} \qquad \text{(C1)}\end{aligned}$$

$$\begin{aligned}v_n^m(x) &= G\left(m+\frac{3}{2},m+1,x\right)=F\left(n+m+\frac{3}{2},-n,m+1;x\right)\\ &= \frac{\Gamma(n+1)\Gamma(m+1)}{\sqrt{2}\Gamma(n+m+\frac{3}{2})}x^{-m/2}\int_0^\infty\sqrt{\xi}J_m(\sqrt{x}\xi)J_{2n+m+\frac{3}{2}}(\xi)d\xi\\ &= \frac{\Gamma(m+1)}{\Gamma(n+m+1)}x^{-m}(1-x)^{-\frac{1}{2}}\frac{d^n}{dx^n}\left\{x^{n+m}(1-x)^{n+\frac{1}{2}}\right\} \qquad \text{(C2)}\end{aligned}$$

C.2. The Expansion Formulas of the Bessel Function

$$x^{-m/2}J_m(\xi\sqrt{x}) = \sum_{n=0}^{\infty}\frac{\sqrt{2}(2n+m+\frac{1}{2})\Gamma(n+m+\frac{1}{2})}{\Gamma(n+1)\Gamma(m+1)}\frac{J_{2n+m+\frac{1}{2}}(\xi)}{\sqrt{\xi}}u_n^m(x) \tag{C3}$$

$$= \sum_{n=0}^{\infty}\frac{\sqrt{8}(2n+m+\frac{3}{2})\Gamma(n+m+\frac{3}{2})}{\Gamma(n+1)\Gamma(m+1)}\frac{J_{2n+m+\frac{3}{2}}(\xi)}{\xi^{\frac{3}{2}}}v_n^m(x) \tag{C4}$$

C.3. The Orthogonal Property

$$\int_0^1 x^m(1-x)^{-\frac{1}{2}}u_n^m(x)u_{n'}^m(x)dx = \frac{\Gamma(n+1)\Gamma^2(m+1)\Gamma(n+\frac{1}{2})}{(2n+m+\frac{1}{2})\Gamma(n+m+1)\Gamma(n+m+\frac{1}{2})}\delta_{n,n'} \tag{C5}$$

$$\int_0^1 x^m(1-x)^{\frac{1}{2}}v_n^m(x)v_{n'}^m(x)dx = \frac{\Gamma(n+1)\Gamma^2(m+1)\Gamma(n+\frac{3}{2})}{(2n+m+\frac{3}{2})\Gamma(n+m+1)\Gamma(n+m+\frac{3}{2})}\delta_{n,n'} \tag{C6}$$

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