*Geophys. J. R. astr. Soc.* (1970) **21**, 33–46.

# Diffraction of *SH*-Waves in a Spherically Isotropic Medium by a Rigid or Fluid Spherical Core

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(Received 1970 April 13)*




*Summary*

In this paper we have considered the diffraction of *SH*-waves from a point source by a rigid or fluid spherical core. The medium outside the core is assumed to be spherically isotropic about the centre of the core. It is found that the propagation of *SH*-waves in such a medium is characterized by two elastic parameters associated with the phase velocities $C_1$ and $C_2$ along and perpendicular to a radius, respectively. Depending on the ratio $1/\alpha$ of these two velocities and the distance of the source from the centre, the region of space outside the core can be separated into three different regions: (i) an illuminated zone which is reached either by the rays going left or right from the source and their reflections from the sphere; (ii) a shadow zone, which may be bounded; and (iii) if the shadow is bounded, then there is a third zone which may be reached by both rays going left and right and their reflections. This classification is true provided $\alpha$ is greater than one, which is true in most geophysical applications. Furthermore, if $\alpha$ is $\geqslant 2$, then there is no shadow when the distance $b$ of the source from the centre is large enough. We have discussed the solutions for $1 \leqslant \alpha \leqslant 2$ and have given simple geometrical interpretations to these solutions.


## 1. Introduction

In discussing the free non-axisymmetric vibrations of an elastic homogeneous sphere of spherically isotropic material we (Ramakrishnan, Shah & Datta 1969, unpubished) found that, if the centre of the sphere is also the centre of elastic symmetry, then the poloidal and toroidal modes of vibration are independent of each other. Thus, *SH*-waves, in which the displacement has no radial component, can propagate in such a medium independent of the other coupled *P*- and *S*- waves. In this paper we have studied the diffraction of *SH*-waves in such a medium by a rigid or also a fluid spherical core. The material is assumed to be transversely isotropic about a radius drawn from the centre of the spherical core. The simple harmonic *SH*-point-source is located at a distance $b$ from the centre. We have obtained the solution to this problem in the geometrical acoustics' limit of very short wave-lengths.

The analysis is quite similar to that by Nussenzveig (1965). So we shall keep the details to a minimum—these can be found in the paper by Nussenzveig.

In Section 2 we have derived the expression for the wave function in a spherically isotropic infinite medium without any core. In Sections 3–5 we have discussed the effect of a rigid core. Finally, in Section 6, we have indicated the relevant modification for a fluid core.

* Received in original form 1969 September 8.

## 2. Equations

Let the medium outside the core $r = a$ is spherically isotropic, the centre $r = o$ being the centre of symmetry. The physical properties of the medium are then governed by five elastic constants. The stress–strain relations are given by (see Backus (1967), Eq. 5.25)

$$\left.\begin{aligned}\tau_{rr} &= c_{13}(\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi})+c_{33}\,\varepsilon_{rr},\\ \tau_{\theta\theta} &= c_{11}\,\varepsilon_{\theta\theta}+c_{13}\,\varepsilon_{rr}+c_{12}\,\varepsilon_{\phi\phi},\\ \tau_{\phi\phi} &= c_{12}\,\varepsilon_{\theta\theta}+c_{13}\,\varepsilon_{rr}+c_{11}\,\varepsilon_{\phi\phi},\\ \tau_{r\theta} &= c_{44}\,\varepsilon_{r\theta},\ \tau_{r\phi} = c_{44}\,\varepsilon_{r\phi},\ \tau_{\theta\phi} = c_{55}\,\varepsilon_{\theta\phi},\\ c_{55} &= \tfrac{1}{2}(c_{11}-c_{12}).\end{aligned}\right\} \tag{1}$$

Here we have used notations for the elastic constants different from that used by Backus. $\tau_{ij}$ and $\varepsilon_{ij}$ are the stress and strain components, respectively, in spherical polar co-ordinates $(r, \theta, \phi)$.

For the motion of the *SH*-type the radial component of the displacement vanishes, and the displacement $\mathbf{u}$ is then given by

$$\mathbf{u} = \nabla \wedge (\hat{r} r \psi). \tag{2}$$

The equations of motion are

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho \mathbf{F} + \operatorname{div} \tau, \tag{3}$$

where $\mathbf{F}$ is the external force per unit mass, $\tau$ is the stress tensor. We shall assume that

$$\mathbf{F} = \nabla \wedge (\hat{r} r K). \tag{4}$$

The use of strain-displacement relations in (1) will express the stresses in terms of the displacement components and the substitution of these stress components in equation (3) will give the three equations for solving the three displacement components $u_r$, $u_\theta$ and $u_\phi$. Now, assumptions (2) and (4) imply that

$$F_r = u_r = 0, \text{ and } \Delta = 0, \tag{5}$$

where $\Delta$ is the dilatation. It can then be shown (Ramakrishnan *et al.* 1969 unpublished) that one of the equations of motion is automatically satisfied and the other two are satisfied if $\psi$ satisfies the equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{\alpha^2}{r^2}\left[\frac{\partial^2 \psi}{\partial \theta^2} + \cot\theta \frac{\partial \psi}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} + 2(1-\alpha^{-2})\psi\right] = \frac{1}{C_1^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{C_1^2} K, \tag{6}$$

where

$$\alpha^2 = c_{55}/c_{44},\ C_1^2 = c_{44}/\rho.$$

For isotropic medium, $\alpha^2 = 1$, and equation (6) reduces to that for the isotropic case.

Let $K$ be taken as

$$K = K_0 \frac{\delta(r-b)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)}{r^2 \sin^2\theta} e^{-i\omega t},\ b > a. \tag{7}$$

It can be shown that (See Singh & Ben-Menahem (1969, Eq. 2.18)

$$\frac{\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)}{\sin\theta} = \frac{1}{4\pi}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{(n-m)!}{(n+m)!}\overset{*}{Y}_{m,n}(\theta_0,\phi_0)\,Y_{m,n}(\theta,\phi), \tag{8}$$

where

$$Y_{m,n}(\theta,\phi) = P_n^m(\cos\theta)\,e^{im\phi},$$

and $\overset{*}{Y}_{m,n}$ is the complex conjugate of $Y_{m,n}$. Assuming harmonic time dependence, the solution to (6) can now be written as

$$\psi = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} W_{nm}(r)\,Y_{m,n}(\theta,\phi). \tag{9}$$

Here we have suppressed the time factor $e^{-i\omega t}$.

$W_{nm}(r)$ satisfies the equation

$$\frac{d^2W_{nm}}{dr^2}+\frac{2dW_{nm}}{rdr}+\left[\beta^2-\frac{\alpha^2}{r^2}\{n(n+1)-2(1-\alpha^{-2})\}\right]W_{nm}$$

$$= -\frac{K_0}{4\pi c_1{}^2}\frac{(n-m)!}{(n+m)!}\overset{*}{Y}_{m,n}(\theta_0,\phi_0)\frac{\delta(r-b)}{r^2}, \tag{10}$$

where

$$\beta^2 = \omega^2/C_1{}^2.$$

The solution to (10) (in the absence of the core) satisfying the boundedness condition at infinity is

$$W_{nm} = \frac{iK_0}{8C_1{}^2(rb)^{\frac{1}{2}}}(2n+1)\frac{(n-m)!}{(n+m)!}J_\nu(\beta b)H_\nu{}^{(1)}(\beta r)\overset{*}{Y}_{m,n}(\theta_0,\phi_0),\ r > b, \tag{11a}$$

$$= \frac{iK_0}{8C_1{}^2(rb)^{\frac{1}{2}}}(2n+1)\frac{(n-m)!}{(n+m)!}H_\nu{}^{(1)}(\beta b)J_\nu(\beta r)\overset{*}{Y}_{m,n}(\theta_0,\phi_0),\ 0 \leqslant r < b, \tag{11b}$$

$$\nu^2 = \alpha^2(n+\tfrac{1}{2})^2+9(1-\alpha^2)/4.$$

Substitution of (11) in (9) gives

$$\psi = \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{(n-m)!}{(n+m)!}J_\nu(\beta b)$$

$$\times H_\nu{}^{(1)}(\beta r)\,Y_{m,n}(\theta,\phi)\,\overset{*}{Y}_{m,n}(\theta_0,\phi_0),\ r > b, \tag{12a}$$

$$= \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{(n-m)!}{(n+m)!}H_\nu{}^{(1)}(\beta b)$$

$$\times J_\nu(\beta r)\,Y_{m,n}(\theta,\phi)\,\overset{*}{Y}_{m,n}(\theta_0,\phi_0),\ 0 \leqslant r < b. \tag{12b}$$

For the isotropic case, $\alpha = 1$, (12) goes to

$$\psi = \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{(n-m)!}{(n+m)!}J_{n+\frac{1}{2}}(\beta b)$$

$$\times H^{(1)}_{n+\frac{1}{2}}(\beta r)\,Y_{m,n}(\theta,\phi)\,\overset{*}{Y}_{m,n}(\theta_0,\phi_0),\ r > b. \tag{13a}$$

$$= \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) \frac{(n-m)!}{(n+m)!} H_{n+\frac{1}{2}}^{(1)}(\beta b)\, J_{n+\frac{1}{2}}(\beta r)$$

$$\times Y_{m,n}(\theta, \phi)\, \overset{*}{Y}_{m,n}(\theta_0, \phi_0),\ 0 \leqslant r < b. \quad (13b)$$

But these are well-known expansions (See Duff & Naylor (1966), Eq. 9.4.14) of

$$\frac{K_0/C_1{}^2}{4\pi R} e^{i\beta R} \text{ for } r > b \text{ or } < b,$$

respectively.
Here

$$R = \left[r^2 + b^2 - 2rb(\cos\theta \cos\theta_0 + \sin\theta \sin\theta_0 \cos(\phi - \phi_0))\right]^{\frac{1}{2}}.$$

## 3. Wave propagation in the presence of a rigid spherical core

In this section we shall discuss the effect of a rigid spherical core, $r = a(< b)$, on the solution (10). Without loss of generality we may assume the source to be located on the line $\theta = 0$. Then, from (12),

$$\psi_{\text{inc}} = \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (2n+1) H_\nu^{(1)}(\beta b)\, J_\nu(\beta r)\, P_n(\cos\theta),\ r < b, \quad (14a)$$

$$= \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (2n+1) H_\nu^{(1)}(\beta r)\, J_\nu(\beta b)\, P_n(\cos\theta),\ r > b. \quad (14b)$$

Let $\psi_t$ be the total field outside $r = a$. Then,

$$\psi_t = \psi_{\text{inc}} + \psi_r, \quad (15)$$

where $\psi_r$ is assumed to be

$$\psi_r = \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \sum_{n=0}^{\infty} A_n H_\nu^{(1)}(\beta r)\, P_n(\cos\theta),\ r > a. \quad (16)$$

The boundary condition

$$\mathbf{u} = 0, \text{ on } r = a, \quad (17)$$

is satisfied if

$$A_n = -(2n+1) \frac{H_\nu^{(1)}(\beta b)}{H_\nu^{(1)}(\beta a)} J_\nu(\beta a). \quad (18)$$

Using (18), (16) and (14) in (15), one obtains

$$\psi_t = \psi_{\text{inc}} - \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (2n+1)\, J_\nu(\beta a)\, H_\nu^{(1)}(\beta b) \frac{H_\nu^{(1)}(\beta r)}{H_\nu^{(1)}(\beta a)} P_n(\cos\theta),\ r \geqslant a, \quad (19)$$

$$= \frac{iK_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (2n+1)\, g_i(\nu, \beta a, \beta b, \beta r)\, P_n(\cos\theta), \quad (20)$$

where

$$g_i = \begin{cases} g_1 = \frac{1}{2}\left[H_\nu^{(2)}(\beta r) H_\nu^{(1)}(\beta b) - \dfrac{H_\nu^{(2)}(\beta a)}{H_\nu^{(1)}(\beta a)} H_\nu^{(1)}(\beta r) H_\nu^{(1)}(\beta b)\right], a \leqslant r < b, & (21a) \\ g_2 = \frac{1}{2}\left[H_\nu^{(2)}(\beta b) H_\nu^{(1)}(\beta r) - \dfrac{H_\nu^{(2)}(\beta a)}{H_\nu^{(1)}(\beta a)} H_\nu^{(1)}(\beta r) H_\nu^{(1)}(\beta b)\right], r > b. & (21b) \end{cases}$$

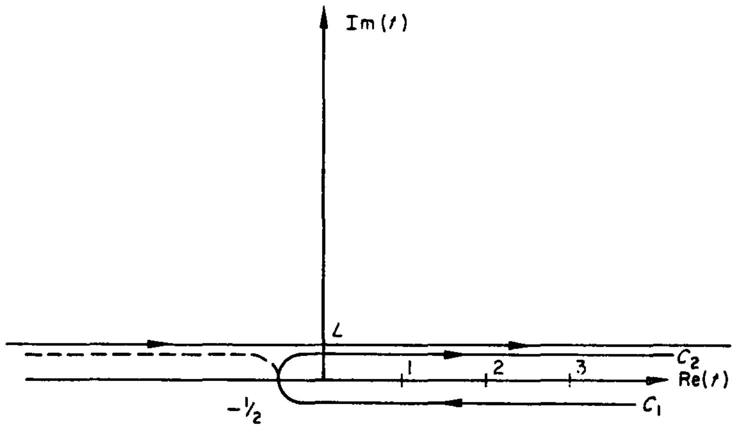


FIG. 1 Contour for Watson's transformation.

Using Watson's transformation equation (20) can be written as

$$\psi_t = -\frac{K_0/C_1^2}{16(rb)^{\frac{1}{2}}} \int_{c=c_1+c_2} \frac{t+\frac{1}{2}}{\sin \pi t} g_i(\nu(t), \beta a, \beta b, \beta r)\, P_t(-\cos\theta)\, dt, \tag{22}$$

where the contour $c$ is shown in Fig. 1.

Since $\nu(n) = \nu(-n-1)$, $H_{\nu(n)} = H_{\nu(-n-1)}$, on $C_1$ we shall replace $t$ by $-t-1$. Then

$$\psi_t = -\frac{K_0/C_1^2}{16(rb)^{\frac{1}{2}}} \int_L \frac{t+\frac{1}{2}}{\sin \pi t} g_i\, P_t(-\cos\theta)\, dt, \tag{23}$$

where $L$ is the contour shown in Fig. 1 and we have used the relation $P_t = P_{-t-1}$.

Equation (23) can alternatively be written as

$$\psi_t = \frac{K_0/C_1^2}{16(rb)^{\frac{1}{2}}} \int_L \frac{\lambda}{\cos \pi\lambda} g_i(\nu(\lambda), \beta a, \beta b, \beta r)\, P_{\lambda-\frac{1}{2}}(-\cos\theta)\, d\lambda. \tag{24}$$

Here

$$\nu(\lambda) = \alpha\lambda[1+q(1-\alpha^2)/4\alpha^2\lambda^2]^{\frac{1}{2}}. \tag{25}$$

But on $L$

$$\frac{1}{\cos \pi\lambda} = 2\sum_{m=0}^{\infty} (-1)^m \exp[i\pi\lambda(2m+1)]. \tag{26}$$

Therefore

$$\psi_t = \sum_{m=0}^{\infty} \psi_m(r, \theta), \tag{27}$$

with

$$\psi_m = \frac{(-1)^m K_0/C_1^2}{8(rb)^{\frac{1}{2}}} \int_L \lambda g_i\, P_{\lambda-\frac{1}{2}}(-\cos\theta) \exp[i\pi\lambda(2m+L)]\, d\lambda. \tag{28}$$

## 4. Expressions for $\psi_{in}$ in the shadow region

The integrand in equation (28) has poles at the zeros of the function $H_\nu^{(1)}(\beta a)$. If $\mathrm{Im}\,\nu > 0$, then these zeros are all in the first quadrant of the complex $\lambda$-plane. The zeros of greatest physical interest are those with small imaginary parts, which restricts $\nu$ in the immediate neighbourhood of $(\beta a, 0)$. These are (See Nussenzveig (1965), Eq. 3.5) given by

$$\alpha\lambda_n \approx \nu_n = \beta a + (\beta a/2)^{\frac{1}{3}} x_n e^{i\pi/3} + O\big(1/(\beta a)^{\frac{1}{3}}\big). \quad (29)$$

Here $-x_n$ is the $n$-th zero of the Airy function $Ai(x)$.

Now, if we close the contour $L$ in the upper half-plane by a large semicircle $C_n$: $|\lambda| = R_n$, passing between the poles, it can be shown that for $\varepsilon < \theta < \pi - \varepsilon$, $(\varepsilon > 0)$, the integral tends to zero as $R_n \to \infty$. Thus $\psi_m$ as given by equation (28) can be expressed as

$$\psi_m = \frac{i\pi K_0/C_1^{\,2}}{4(rb)^{\frac{1}{2}}} (-1)^m \sum_{n=1}^{\infty} \lambda_n r_n \exp[i\pi\lambda_n(2m+1)] \times H_{\nu_n}^{(1)}(\beta b) H_{\nu_n}^{(1)}(\beta r) P_{\lambda_n - \frac{1}{2}}(-\cos\theta), \quad (30)$$

$$r_n \approx \frac{1}{2\pi\alpha} \exp(-i\pi/6)(\beta a/2)^{\frac{1}{3}}/[Ai'(-x_n)]^2.$$

Note that equation (30) holds good whether $r >$ or $< b$. Furthermore, it may also be pointed out that the above results remain valid at $\theta = \pi$ when

$$P_{\lambda_n - \frac{1}{2}}(-\cos\theta) = P_{\lambda_n - \frac{1}{2}}(1).$$

However, this is no longer true at $\theta = 0$.

Now, if both $b-a$ and $r-a$ are $\gg (\beta a)^{\frac{1}{3}}/\beta$, (i.e. when the source and the receiver are not too close to the surface of the sphere), and if $|\lambda_n|(\pi-\theta) \gg 1$, then using the asymptotic expansions of $H_{\nu_n}^{(1)}(\beta b)$, $H_{\nu_n}^{(1)}(\beta r)$, $P_{\lambda_n-\frac{1}{2}}(-\cos\theta)$, one gets

$$\psi_m \approx \frac{(-1)^m K_0/C_1^{\,2}}{4\alpha^{\frac{1}{2}} \pi^{\frac{1}{2}} (rb)^{\frac{1}{2}}} \exp(i\pi/3)\, 2^{-\frac{2}{3}} a[(b^2-a^2)(r^2-a^2)]^{-\frac{1}{4}} (\beta a)^{\frac{1}{3}} \times \frac{\exp[i\beta\{(b^2-a^2)^{\frac{1}{2}} + (r^2-a^2)^{\frac{1}{2}}\}]}{(\beta a \sin\theta)^{\frac{1}{2}}} \sum_n \{\exp[i\lambda_n \delta_m - i\pi/4] \exp[i\lambda_n \gamma_m + i\pi/4]\}/[Ai'(-x_n)]^2, \quad (31)$$

$$\left.\begin{aligned} \gamma_m &= 2(m+1)\pi - \theta - \alpha\cos^{-1}(a/b) - \alpha\cos^{-1}(a/r), \\ \delta_m &= 2m\pi + \theta - \alpha\cos^{-1}(a/b) - \alpha\cos^{-1}(a/r). \end{aligned}\right\} \quad (32)$$

In order to interpret the results, let us note that

$$\left.\begin{aligned} \gamma_0 &= 2\pi - \theta - \alpha\cos^{-1}(a/b) - \alpha\cos^{-1}(a/r), \\ \delta_0 &= \theta - \alpha\cos^{-1}(a/b) - \cos^{-1}(a/r). \end{aligned}\right\} \quad (33)$$

Thus, the first term in the series (27) becomes exponentially increasing if $\delta_0 < 0$, i.e. $\theta < \alpha[\cos^{-1}(a/b) + \cos^{-1}(a/r)]$, so that the series is meaningful only when $\theta > \alpha[\cos^{-1}(a/b) + \cos^{-1}(a/r)]$. The line

$$\theta = \alpha[\cos^{-1}(a/b) + \cos^{-1}(a/r)] \quad (34)$$

has been drawn as the line $A_1 S'$ in Fig. 2. This line is tangent to the sphere at $A_1$. The series (27) is valid in the region bounded by the curves $A_1 S'$, $A_1' S'$ and $A_1 T_1 T_1' A_1'$ (Fig. 2). This region is then the geometric shadow region. Note that the line $A_1' S'$ is the image of $A_1 S'$ about the axis $SS'$ in the plane containing $SA_1 S'$ and $SOS'$. In Section 5 we shall derive the equation for the fundamental ray paths.

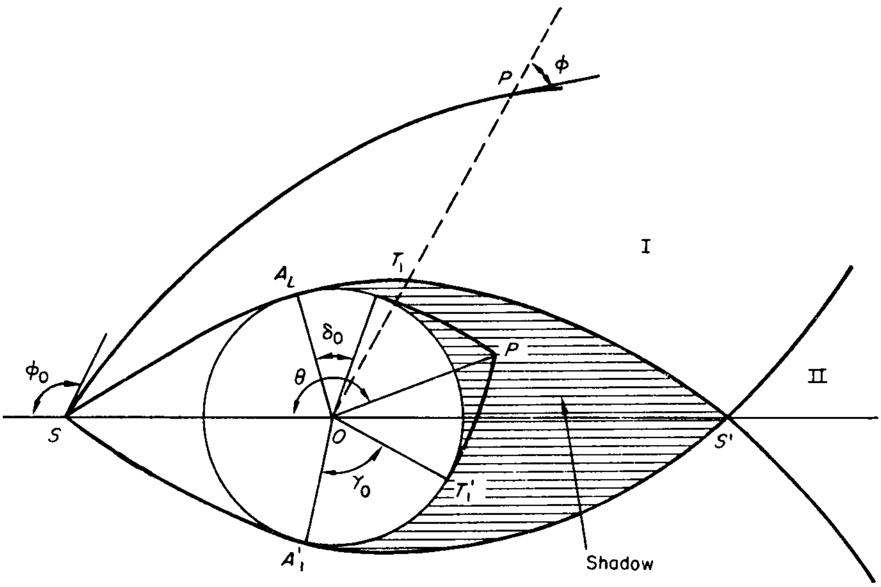


FIG. 2. Diffracted rays in the physical $(r, \theta)$-plane.

Let $SA_1S'$ and $SA_1'S'$ be the incident rays from $S$ that are tangent to the sphere. It will be shown in Section 5 that

$$\angle SOA_1(=\angle SOA_1') = \alpha\cos^{-1}(a/b),\ \angle POT_1(=\angle POT_1') = \alpha\cos^{-1}(a/r).$$

Here, lines $T_1P$ and $T_1'P$ are the rays emanating (tangentially) from $T_1$ and $T_1'$. Thus, $\angle A_1OT_1 = \delta_0$, $\angle A_1'OT_1' = \gamma_0$. Furthermore, the propagation times from $S$ to $A_1$ (or $S$ to $A_1'$) and $T_1$ to $P$ (or $T_1'$ to $P$) are $(b^2-a^2)^{1/2}/c_1$ and $(r^2-a^2)^{1/2}/c_1$, respectively. In fact, these results become apparent if one introduces the new variable $\vartheta = \theta/\alpha$. This transformation takes the $(r,\theta)$-plane into the image $(r,\vartheta)$-plane. In terms of these new variables, $\psi_m$ becomes

$$\psi_m \approx \frac{(-1)^m K_0/C_1{}^2}{4\alpha^{1/2}\pi^{3/2}(rb)^{1/2}}\, e^{i\pi/3}\, 2^{-1/3} a[(b^2-a^2)(r^2-a^2)]^{-1/4}$$

$$\times\frac{(\beta a)^{1/3}\exp[i\beta\{(b^2-a^2)^{1/2}+(r^2-a^2)^{1/2}\}]}{(\beta a\sin\theta)^{1/2}}\sum_n\{\exp[i\nu_n\delta_m'-i\pi/4]$$

$$+\exp[i\nu_n\gamma_m'+i\pi/4]\}/[Ai'(-x_n)]^2, \quad (31a)$$

where

$$\delta_m' = \frac{2m\pi}{\alpha}+\vartheta-\cos^{-1}(a/b)-\cos^{-1}(a/r),$$

$$\gamma_m' = \frac{2(m+1)\pi}{\alpha}-\vartheta-\cos^{-1}(a/b)-\cos^{-1}(a/r).$$

In Fig. 3 we have shown the shadow and the illuminated regions in the $(r,\vartheta)$-plane. In this plane the ray paths become straight lines and the straight lines $SA_1S'$ and $S_1'A_1'S'$ are the images of the curved incident rays $SA_1S'$ and $SA_1'S'$ that are tangent to the sphere. Here $S_1'$ is the image of $S$ on the line $\vartheta = \pi/\alpha$. In this image plane the disturbances move with constant speed $c_1$.

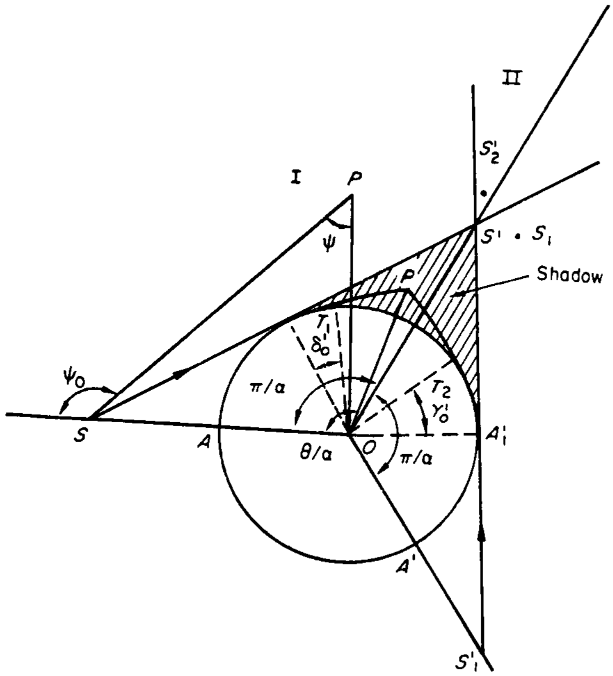


FIG. 3. Diffracted rays in the image $(r, \vartheta)$-plane.

So if one employs the concept of propagation along rays at short wavelengths, the physical interpretation of each $\psi_m$ in the image plane becomes clear. The geometric shadow is shown shaded in Fig. 3. The incident rays reaching $A_1$ and $A_1'$ excite a series of surface waves emanating from these points. These waves travel along the sphere with phase velocity $C_1/[1+(4a\beta)^{-\frac{2}{3}} x_n]$, which is slightly smaller than $C_1$. As they travel along the surface, they shade radiations along tangential directions, leading to the angular damping factor $\exp[-\text{Im}((\nu_n/a)L_m)]$, where $L_m = L_0+(2\pi am/\alpha)$, $m \geqslant 0$. Here $L_0$ is the arcual distance $A_1 T_1$ or $A_1' T_2$. The point $P$ within the geometrical shadow is reached by rays emanating from $T_1$ and $T_2$. These are the straight lines from $P$ that are tangent to the sphere. For $m = 0$, the first term within the bracket in the expression for $\psi_0$ (equation (31a)) corresponds to the ray that has gone along the surface through the distance $A_1 T_1$ before leaving the sphere and the second one corresponds to that which has gone along $A_1' T_2$.

Thus, the phase factor

$$\exp[i\beta\{\surd(b^2-a^2)+\surd(r^2-a^2)\}+i\beta\{1+x_n(4\beta a)^{-\frac{2}{3}}\}a\delta_0']$$

corresponds to the path $SA_1 T_1 P$. Similarly, the other term containing $\gamma_0'$ corresponds to the path $S_1' A_1' T_2 P$. For $m = 1$, the corresponding terms in $\psi_1$ can be interpreted as the rays from $S_1$ and $S_2'$ ,which are the images of $S$ and $S_1$ on the lines $\vartheta = -\pi/\alpha$ and $\pi/\alpha$, respectively. For $m > 1$, these correspond to the rays from $S_m$ and $S'_{m+1}$, which are the images of $S'_{m-1}$ and $S_m$ on the lines $\vartheta = -\pi/\alpha$ and $\vartheta = \pi/\alpha$, respectively. The rays emanating from $S_m$ go clockwise crossing the point $A$ $m$-times and those from $S'_{m+1}$ go anticlockwise crossing $A'$ $m$-times. For, note that

$$\delta_m' = 2m\pi+\delta_0'-\left(2m\pi-\frac{2m\pi}{\alpha}\right), m \geqslant 1,$$

$$\gamma_m' = 2m\pi+\gamma_0'-\left(2m\pi-\frac{2m\pi}{\alpha}\right), m \geqslant 1.$$

Returning now to the physical $(r, \theta)$-plane (Fig. 2), the interpretation of each $\psi_m$ is apparent. The incident rays reaching $A_1$ and $A_1'$ (tangentially) excite a series of surface waves emanating from these points. These waves travel along the sphere with phase velocity $C_2/[1+x_n(4a\beta)^{-\frac{1}{3}}]$, which is slightly smaller than $C_2$. As they travel along the sphere, they shed radiation along tangential directions, leading to the angular damping factor $\exp[-\text{Im}((\lambda_n/a)L_m)]$, where $L_m = L_0+2\pi am$. Here, $L_0$ is the arcual distance $A_1\,T_1$ or $A_1'\,T_1'$. For $m \geqslant 1$, $L_m$ correspond to the paths that have gone around the sphere $m$ times. Since $\text{Im}\,(\lambda_n) = (1/\alpha)x_n(\beta a/2)^{\frac{1}{3}}$, the damping constant varies as $\beta^{\frac{1}{3}}/\alpha$. Also, note that the amplitude of $\psi_m$ has a factor $\alpha^{-\frac{1}{2}}$. So, larger $\alpha$ is, faster the decay.

The shadow boundary given by equation (34) will meet the line $\theta = \pi$ after being tangent to the sphere $r = a$ provided

$$\pi/\alpha - \cos^{-1}(a/b) < \pi/2, \; 1 < \alpha < 2. \tag{35}$$

However, if $\alpha > 2$, then the relation (35) cannot be satisfied when

$$b \geqslant a \sec \pi/\alpha,$$

so that there is no shadow. If $\alpha = 2$, then there will be no shadow when the source recedes to infinity. Henceforth, we shall confine our attention to the case $1 \leqslant \alpha \leqslant 2$, which condition is satisfied for most materials. Returning to condition (34), it is found that the shadow extends to infinity provided $b < a \sec[(\pi/\alpha)-(\pi/2)]$. Otherwise, the shadow is confined into a finite region beyond the sphere. (See Fig. 2).

It can be seen from Figs 2 and 3 that the illuminated region can be divided into two sub-regions, I and II. In sub-region I a point $P$ can be reached only by one direct ray that is going left from $S$. In sub-region II, $P$ can be reached by both the direct rays going right and left from $S$. (See also Fig. 4).

## 5. Expressions for $\psi$ in the illuminated region

Let us consider the sub-region I in which

$$\frac{\theta}{\alpha} < \cos^{-1}(a/b)+\cos^{-1}(a/r) < \frac{2\pi-\theta}{\alpha}. \tag{36}$$

In this case the term containing $\delta_0$ will blow up, but the rest will be rapidly convergent. Hence, we shall re-examine the term

$$\psi_0 = \frac{K_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \int_L \lambda g_l\, P_{\lambda-\frac{1}{2}}(-\cos\theta)\exp(i\pi\lambda)\,d\lambda. \tag{37}$$

Using the relation

$$P_{\lambda-\frac{1}{2}} = Q^{(1)}_{\lambda-\frac{1}{2}}+Q^{(2)}_{\lambda-\frac{1}{2}}, \qquad \text{(Nussenzveig, Eq. C.2)}, \tag{38}$$

and noting that the term in $\delta_0$ arises from $Q^{(1)}_{\lambda-\frac{1}{2}}$ after the substitution of (38) in (37), we will have to re-examine

$$\psi_0{}^{(1)} = \frac{K_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \int_L \lambda g_l\, Q^{(1)}_{\lambda-\frac{1}{2}}(-\cos\theta)\exp(i\pi\lambda)\,d\lambda. \tag{39}$$

The saddle-point approximation of (39) yields (for details see Nussenzveig (1965), p. 42),

$$\psi_0{}^{(1)} = \psi_{\text{inc}}+\psi_{\text{refl}}, \tag{40}$$

where

$$\psi_{\text{inc}} \approx \frac{K_0/C_1{}^2}{4\pi\alpha^{\frac{1}{2}} R}\exp(i\beta R)\left(\frac{\sin\vartheta}{\sin\theta}\right)^{\frac{1}{2}}, \; R = r\cos\varphi - b\cos\varphi_0. \tag{41}$$

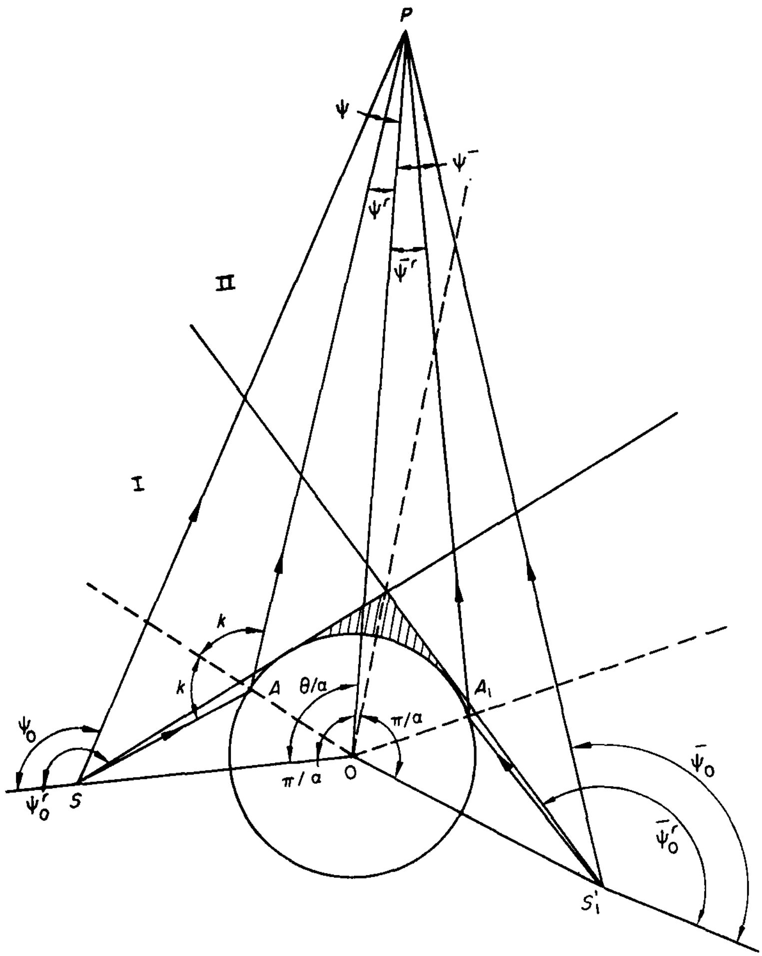


FIG. 4. Direct and reflected rays in the image $(r, \vartheta)$-plane.

This comes from the saddle-point

$$\nu = \beta r \sin\varphi = \beta b \sin(\pi - \varphi_0), \tag{42}$$

$$\varphi - \varphi_0 + \theta/\alpha = 0. \tag{43}$$

Since $\tan\phi = r(d\theta/dr)$, we get from equations (42) and (43)

$$\tan\phi = \alpha \tan\varphi.$$

Equation (42) is the Snell's law and (43) describes the direct ray from the source to the point $P$ in the image plane $(r, \vartheta)$ (See Fig. 3). Similar relations were also obtained by Sato & Lapwood (1968) in discussing $SH$-waves in a transradially isotropic cylinder. Equation (41) shows that the amplitude is changed by a factor of $\alpha^{-\frac{1}{2}}(\sin\vartheta/\sin\theta)^{\frac{1}{2}}$, due to the presence of anisotropy. Thus, the equation for a ray path from $S$ is, in the $(r, \theta)$-plane,

$$\frac{r\tan\phi}{\sqrt{(\alpha^2+\tan^2\phi)}} = k_0,$$

where $k_0 = b\tan\phi_0/[\alpha^2+\tan^2\phi_0]^{\frac{1}{2}}$ is the ray parameter, $\phi_0$ being the angle that this ray makes with the radial line to $S$ and $\phi$ is the angle made with the radial line to $P$. A ray path is shown in Fig. 2 as line $SP$. In the image $(r, \vartheta)$-plane this line becomes the straight line $SP$ in Fig. 3. It can be seen from equation (43) that if the ray from $S$ is tangent to the sphere at $A_1$, then the angle $\theta$ that the ray $SA_1$ subtends at the centre is

$$\angle SOA_1 = \alpha[\pi/2-(\pi-\varphi_0)] = \alpha \cos^{-1}(a/b),$$

since $\varphi = \pi/2$ at $A_1$ and $\pi-\varphi_0 = \sin^{-1}(a/b)$. (See Fig. 3). Also, if $(r, \theta)$ is a point on this ray, then

$$\theta = \alpha[\cos^{-1}(a/b)+\cos^{-1}(a/r)].$$

Thus, this gives the equation for the shadow boundary, which is consistent with the results in Section 4. To obtain the angle subtended by the ray $T_1 P$ that leaves (tangentially) the sphere at $T_1$ and reaches $P(r, \theta)$, we note that the equation for this ray is, by equations (42) and (43),

$$r\frac{d\theta}{dr} = \alpha \tan\varphi = \frac{\alpha a}{\sqrt{(r^2-a^2)}}.$$

Integrating, one obtains

$$\theta-\theta_0 = \alpha \cos^{-1}(a/r),$$

where $\theta_0 = \angle SOT_1$. Thus, $\angle T_1 OP = \alpha \cos^{-1}(a/r)$. These justify the results used in Section 4.

From (41) we have the equation for the wave front as

$$(r\cos\varphi - b\cos\varphi_0) - C_1 t = \text{const.}, \quad \tan\varphi = \alpha^{-1}\tan\phi.$$

It can be seen that the wave-front normal is not in general tangent to the ray. Using this equation, one derives the phase velocity $v$ at a point $(r, \phi)$ on the ray path $SP$ as

$$\frac{1}{v^2} = \frac{\cos^2\varphi}{C_1{}^2} + \frac{\sin^2\varphi}{C_2{}^2} = \frac{(C_2/C_1)^2+(C_1/C_2)^2\tan^2\phi}{C_2{}^2+C_1{}^2\tan^2\phi}.$$

Thus, if the wave is moving in the radial direction, it can be shown that $\phi = 0$ so that the phase velocity $v = C_1$. If it is moving perpendicular to the radius, then it can be shown that $\phi = \pi/2$ and so phase velocity $v = C_2$. So the parameter $\alpha = C_2/C_1$ may be interpreted as the ratio of the phase velocities in directions perpendicular and along the radius, respectively.

Also, using the equations for the wave-front and the ray, one can derive the group velocity $U$ at a point $(r, \phi)$ as

$$\frac{1}{U^2} = \frac{1}{C_1{}^2}\cos^2\phi \sec^2\varphi.$$

(See also Sato & Lapwood, Eq. 5.1)

The term $\psi_{\text{refl}}$ comes from the saddle-point between 0 and $\beta a$, and this can be obtained as

$$\psi_{\text{refl}} \approx -\frac{aK_0/C_1{}^2}{4\pi\alpha^{\frac{1}{2}}(rb)^{\frac{1}{2}}}\left[\frac{\sin k\cos k}{\sin\theta\,(SA.PM+SL.AP)}\right]^{\frac{1}{2}} \exp\left(i\beta(SA+AP)\right), \quad (44)$$

$$PM = r\cos\varphi^r, \quad SL = b\cos(\pi-\varphi_0{}^r).$$

Again, it can be seen that the amplitude is changed by the factor of $\alpha^{-\frac{1}{2}}(\sin\vartheta/\sin\theta)^{\frac{1}{2}}$, due to anisotropy. The contribution given by (44) comes from the saddle-point

$$\nu = \beta r\sin\varphi^r = \beta b\sin(\pi-\varphi_0{}^r) = \beta a\sin k, \quad (45)$$

$$\varphi^r + \pi - \varphi_0{}^r + \frac{\theta}{\alpha} - 2k = 0. \tag{46}$$

Equation (46) describes the ray reflected off the surface of the sphere and reaching the point $P$ (See Fig. 4).

Next we shall consider sub-region II of the illuminated region, given by

$$\frac{\theta}{\alpha} < \frac{2\pi-\theta}{\alpha} < \cos^{-1}\left(\frac{a}{b}\right) + \cos^{-1}\left(\frac{a}{r}\right). \tag{47}$$

In this case point $P$ can be reached by the direct rays going right and left from the points $S$ and $S_1'$, respectively, and also by their reflections from the sphere. In this case the terms containing $\gamma_0$ and $\delta_0$ both blow up. However, the rest of the terms in $\psi_t$ will still be rapidly convergent if $\alpha < 2$. So we shall have to re-examine the term $\psi_0$.

The contribution $\psi_0{}^{(1)}$ to $\psi_0$ is still given by equations (40), (41) and (44). Therefore, we evaluate

$$\psi_0{}^{(2)} = \frac{K_0/C_1{}^2}{8(rb)^{\frac{1}{2}}} \int_L \lambda g_i\, Q_{\lambda-\frac{1}{2}}^{(2)}(-\cos\theta) \exp(i\pi\lambda)\, d\lambda, \tag{48}$$

by the saddle-point method, and find

$$\bar{\psi}_{\text{inc}} \approx \frac{K_0/C_1{}^2}{4\pi\alpha^{\frac{1}{2}}} \left(\frac{\sin((2\pi-\theta)/\alpha)}{\sin\theta}\right)^{\frac{1}{2}} \frac{\exp(i\beta R_1)}{R_1},\ R_1 = r\cos\bar{\varphi} - b\cos\bar{\varphi}_0. \tag{49}$$

This contribution to (48) comes from the saddle-point

$$v = \beta r \sin\bar{\varphi} = \beta b \sin(\pi - \bar{\varphi}_0), \tag{50}$$

$$\bar{\varphi} - \bar{\varphi}_0 + \frac{2\pi-\theta}{\alpha} = 0. \tag{51}$$

This corresponds to the direct ray from $S_1'$ to $P$. As can be seen, this saddle-point exists only if $(2\pi-\theta)/\alpha < \cos^{-1}(a/b) + \cos^{-1}(a/r)$, i.e. in the region II.

Similarly, the contribution to (48) from the saddle-point

$$v = \beta r \sin\bar{\varphi}^r = \beta b \sin(\pi - \bar{\varphi}_0{}^r) = \beta b \sin\bar{k}, \tag{52}$$

$$\bar{\varphi}^r + \pi - \bar{\varphi}_0{}^r + \frac{2\pi-\theta}{\alpha} - 2\bar{k} = 0, \tag{53}$$

is

$$\bar{\psi}_{\text{refl}} \approx -\frac{aK_0/C_1{}^2}{4\pi\alpha^{\frac{1}{2}}(rb)^{\frac{1}{2}}} \left[\frac{\sin\bar{k}\cos\bar{k}}{\sin\theta(S_1'A_1.r\cos\bar{\varphi}^r + A_1P.b\cos(\pi-\bar{\varphi}_0{}^r))}\right]^{\frac{1}{2}} \exp[i\beta(S_1'A_1 + A_1P)]. \tag{54}$$

So the total field at $P$ in the region II is given by

$$\psi = \psi_{\text{inc}} + \psi_{\text{refl}} + \bar{\psi}_{\text{inc}} + \bar{\psi}_{\text{refl}},$$

where $\psi_{\text{inc}}$ etc. are given by equations (41), (44), (49) and (54).

In Table 1 we have given some values of $\alpha^{-\frac{1}{2}}(\sin\vartheta/\sin\theta)^{\frac{1}{2}}$ for different values of $\theta$ for two different materials. Note that for zinc this factor first increases, then decreases, and then increases again with increasing $\theta$.

**Table 1**

| $\alpha$ | $\theta$ 10° | 20° | 40° | 60° | 80° | 100° | 120° |
|---|---|---|---|---|---|---|---|
| 1·1 (Beryl) | 0·822 | 0·828 | 0·833 | 0·843 | 0·854 | 0·873 | 0·906 |
| 1·3 (Zinc) | 0·589 | 0·606 | 0·604 | 0·615 | 0·639 | 0·671 | 0·724 |

Values of $\alpha^{-3/2}(\sin\vartheta/\sin\theta)^{\frac{1}{2}}$ for different angles $\theta$ and for two materials.

## 6. Diffraction by a fluid sphere

In the case where the core is a fluid sphere of radius $a$ the shear stresses

$$\tau_{r\theta} = \tau_{r\phi} = 0 \text{ on } r = a. \tag{55}$$

The use of this boundary condition in place of (17) will give

$$\psi_t = \psi_{\text{inc}} - \frac{iK_0/C_1^{\,2}}{8(rb)^{\frac{1}{2}}} \int_{n=0}^{\infty} (2n+1) H_\nu^{(1)}(\beta b) H_\nu^{(1)}(\beta r) \frac{S_\nu(\beta a)}{S_\nu^{(1)}(\beta a)} P_n(\cos\theta). \tag{56}$$

Here

$$S_\nu(\beta a) = J_\nu'(\beta a) - \frac{3}{2\beta a} J_\nu(\beta a),\ S_\nu^{(1)}(\beta a) = H_\nu^{(1)\prime}(\beta a) - \frac{3}{2\beta a} H_\nu^{(1)}(\beta a).$$

Equation (56) can alternatively be written as

$$\psi_t = \frac{iK_0/C_1^{\,2}}{8(rb)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, h_i(\nu, \beta a, \beta b, \beta r)\, P_n(\cos\theta), \tag{57}$$

where

$$h_i = \begin{cases} h_1 = H_\nu^{(1)}(\beta b) H_\nu^{(2)}(\beta r) - H_\nu^{(1)}(\beta b) H_\nu^{(1)}(\beta r) \dfrac{S_\nu^{(2)}(\beta a)}{S_\nu^{(1)}(\beta a)},\ a \leqslant r < b, \\[2ex] h_2 = H_\nu^{(1)}(\beta r) H_\nu^{(2)}(\beta b) - H_\nu^{(1)}(\beta b) H_\nu^{(1)}(\beta r) \dfrac{S_\nu^{(2)}(\beta a)}{S_\nu^{(1)}(\beta a)},\ r > b, \end{cases} \tag{58}$$

$$S_\nu^{(2)}(\beta a) = H_\nu^{(2)\prime}(\beta a) - \frac{3}{2\beta a} H_\nu^{(2)}(\beta a).$$

Proceeding exactly in the same way as in Sections 3 and 4, it can be shown that

$$\psi_t = \sum_{m=0}^{\infty} \psi_m,$$

with

$$\psi_m = \frac{i\pi K_0/C_1^{\,2}}{4(rb)^{\frac{1}{2}}} (-1)^m \sum_n \lambda_n r_n H_{\nu_n}^{(1)}(\beta b) H_{\nu_n}^{(1)}(\beta r) \exp[i\pi\lambda_n(2m+1)]\, P_{\lambda_n - \frac{1}{2}}(-\cos\theta).$$

Here $\nu_n \approx \alpha\lambda_n$ is a root in the first quadrant of $S_\nu^{(1)}(\beta a) = 0$, and

$$r_n = -S_\nu^{(2)}(\beta a) \Big/ \left[\frac{\partial}{\partial\lambda} S_\nu^{(1)}(\beta a)\right]_{\lambda=\lambda_n}.$$

For large $\beta a$ the roots of $S_\nu^{(1)} = 0$ that are close to $\beta a$ (and with positive imaginary part) are approximately those of $H_\nu^{(1)\prime}(\beta a) = 0$ and these are given by (Keller, Rubinow & Goldstein (1963))

$$\nu_n = \beta a + 2^{-\frac{1}{3}} \exp(i\pi/3)(\beta a)^{\frac{1}{3}} y_n,$$

$-y_n$ being the $n$-th zero of $Ai'(x) = 0$.

Calculating the residues and using the asymptotic expansions, it can be shown, as before, that

$$\psi_m = \frac{(-1)^m K_0/C_1^2 . \beta^{-\frac{1}{3}}(a/2)^{\frac{2}{3}} \exp(i\pi/3)}{4\pi^{\frac{1}{2}} \alpha^{\frac{1}{2}}(rb)^{\frac{1}{2}}\sqrt{(\sin\theta)}} [(b^2-a^2)(r^2-a^2)]^{-\frac{1}{4}}$$

$$\times \exp\left[i\beta\left(\sqrt{(b^2-a^2)}+\sqrt{(r^2-a^2)}\right)\right] \sum_n [\exp(i\nu_n \gamma_m + i\pi/4)$$

$$+\exp(i\nu_n \delta_m - i\pi/4)]/y_n[Ai(-y_n)]^2.$$

Proceeding in the same way as in Section 5, one can derive expressions for the reflected field.

**Acknolwedgement**

We wish to express our sincere gratitude to the referee whose valuable comments led to the present version of this paper.

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