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DIFFRACTION THEORY OF THE KNIFE-EDGE TEST AND ITS IMPROVED FORM, THE PHASE-CONTRAST METHOD.

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(Communicated by Professor F. J. M. Stratton)

1. The well-known knife-edge test was introduced by Foucault in 1856, and its principle explained by geometrical optics.* It is clear, however, that for a mirror having deviations comparable with the wave-length, the wave theory must necessarily be used, and it thus provides the only possible way to find the limit of sensibility.† A simple example will show at once the peculiarities of the wave treatment, and suggest an improved form of test.

Let a spherical mirror be tested in the usual way by placing an artificial star near its centre of curvature, the mirror-surface being observed with the eye just behind the star-image. Suppose the mirror-surface to be slightly wavy, *i.e.* to deviate from the ideal spherical form by a train of sine waves having an amplitude of a fraction of a wave-length. For the present, these will be taken as depending on one co-ordinate only, and the whole mirror be treated as one-dimensional. This mirror will evidently act as a diffraction grating, giving only first-order spectra on both sides of the central image of the star. Let it be called a *phase grating*. It is clear that the three light cones emerging from the three star-images will give two systems of interference fringes, the first one resulting from the left-hand cone with part of the central one, the second in the same way from the right-hand cone. On a screen some distance behind the star-image these cones will give an illuminated circle that may roughly be considered as an image of the mirror-surface. Strictly speaking, a lens is used to produce a sharp image, on the retina or otherwise, but this is not essential.

The effect of the wavy surface may be compared with the case of a mirror made into a diffraction grating by waves of a different kind, viz. of periodically changing reflecting power. Let this be called an *amplitude grating*. Evidently we shall not need any knife-edge or other device to see the bright and dark stripes on this mirror. In agreement with this, the diffraction theory shows that in this case the three images of the light-source have equal phase, causing the two fringe systems mentioned above to coincide. Now the phase grating gives small differences of phase to the

* *Annal. Obs. Paris*, **5**, 203, 1859; *M.N.*, **19**, 284, 1859.

† Lord Rayleigh, *Phil. Mag.*, **33**, 161, 1917; S. Banerji, *Ap. J.*, **48**, 50, 1918, applied the wave theory to the knife-edge test in the case of discontinuities of the surface.

reflected vibrations, but this is equivalent to adding small vibrations 90° in advance of the main vibration. Therefore the diffraction images will be 90° in advance of the central star-image. This 90° change of phase will shift the first fringe system, caused by the left-hand spectrum, a quarter fringe to the right, the second system the same distance to the left. If no knife-edge is used, both systems appear, with the result that their differences of intensity cancel exactly. Therefore the wavy deviations of the mirror-surface remain invisible.

The simplest way to detect the deviations will be to intercept one of the lateral diffraction images by a knife-edge, the other one remaining to form interference fringes with the central image. Because of the quarter-fringe displacement, however, these fringes, as seen projected on the mirror-surface, will not coincide with the surface waves that cause them, but the maxima and minima will fall on the slopes of the wave, which thus appear alternately brightened and darkened.

If the knife-edge is brought in from the other side, the other system remains, and the waves on the mirror are seen shaded on the other side. This explanation of the well-known relief effect of the knife-edge is clearly quite different from the ordinary geometrical treatment.

The surface waves will also appear when one of the systems of interference fringes is shifted by half a fringe with respect to the other. We may attain this by reversing the phase of one of the lateral spectra by means of a glass plate with two halves of slightly different thicknesses. Such a "phase reversing plate" will evidently have the same effect as a knife-edge, as it leaves one of the systems of fringes unchanged.

Clearly it will be much better, however, to make a symmetrical arrangement, changing the phase of the central image by 90°. This is the newly proposed *method of phase contrast*. By this both systems of fringes are shifted a quarter fringe in opposite directions, bringing them into coincidence with each other and with the maxima and minima of the waves on the surface. Indeed, the diffraction images are in this way brought into phase with the central image, and the eye will see the phase grating as if it were an amplitude grating. The same phase-changing device will clearly make an amplitude grating appear as a phase grating. Indeed, as an experimental test of our diffraction theory, a wire grating may be put in front of a mirror, and the phase of the central image changed by 90°. The opaque bars of the grating are then hardly visible.

Changes of phase in adjacent parts of the light are in all cases easily attained by glass plates, of which one surface has been partly etched away. As Lord Rayleigh indicated long ago, dilute hydrofluoric acid removes the glass surface very evenly, leaving a flat surface lying a fraction of a wave-length, or even a few wave-lengths, below the original surface. The depth attained is easily tested by means of Newton's rings.

The test case of a mirror with a single sine wave on its surface having shown that the phase-contrast method renders the deviations of the surface as an exactly corresponding sine wave of intensity, it is to be expected that the same method will render small deviations of arbitrary arrangement as proportional changes of intensity. By the following quite different treatment we shall find the effect of arbitrarily large deviations of the surface.

2. The optical arrangement for testing mirrors, or optical systems generally, by observing the surface of the mirror, or generally the aperture of the system, may be considered as a case of optical image formation of a non-luminous object—the surface of the mirror, etc.—with the aid of waves from a point-source. The exact theoretical treatment of the case is provided by the diffraction theory which Abbe proposed for microscopic images. As a general result of this theory, Lummer states that "if all the diffraction spectra of the light source can pass unmodified, the image will be exactly similar to the object in structure and phase." *

The principle of the phase-contrast method will now be elucidated by a vector diagram of the vibrations as they leave the mirror. Let each vibration be represented in the usual way by a vector, the length of the vector representing the amplitude, its angle with a fixed direction the phase angle of the vibration. The advantage of this kind of representation lies in the fact that composition of vibrations (interference) corresponds to composition (addition) of the vectors in the diagram.

Imagine the mirror to be divided into a large number N of surface elements of equal size. Let N vectors be drawn from the origin to represent the vibrations from all the elements. In the figure their terminal points only have been inserted. For an ideal mirror these points would all coincide, *e.g.* in M differences of reflecting power would give points on OM, while in our case—differences of height, changing the phases only—the N terminal points will lie on the circle through M, higher and lower parts of the mirror-surface corresponding respectively to points like P and P'. According to the statement of Lummer, quoted above, the same diagram will represent the amplitudes and phases in the image of the mirror, when no modification of the diffraction images has been introduced as yet. Now if the light from the central star-image only were admitted to form the image, this would be uniformly illuminated. As the light-paths to the central image are all equal, its representing vector will be found by adding all N vectors. In order to find the vector representing the vibration in the uniform image, the resulting sum must be divided by a certain factor. From the case of the perfect mirror which forms no lateral diffraction spectra, it is clear that this factor must be equal to N. This means that the effect of the central image alone is represented in the image of the mirror by a vector OG, where G is the centre of gravity of the N terminal points. Interception of the central image will therefore result in a subtraction of the vector OG from every vibration in the image, *i.e.* the origin of the vector diagram is shifted to G. In the phase-contrast method, however, the central pencil is not removed, but changed in phase, *e.g.* by 90°. The effect of this will be found by subtracting OG and adding the vector O_1G, if the phase has been advanced, or O_2G if the phase has been retarded. As a result the origin of the diagram is shifted to O_1 or to O_2. As the eye or the photographic plate can merely perceive the intensity and not the phase, observation

* *Handwörterbuch d. Naturw.*, 1, 35, 1912.

will yield the squares of the distances from the new origin. Therefore higher parts of the mirror will appear brighter, and lower parts darker than the mean illumination in the first case (O_1), and reversely in the second case (O_2).

The diagram shows at once that this method will be very sensitive. Thus if MP' be one-tenth of the radius, the change of intensity will be about 20 per cent. and still visible. The retardation corresponding to this is only one-tenth radian or $\lambda/60$. The method will indeed be specially suitable to make very accurate surfaces, for which the centre of gravity G will lie near M. On the other hand, a surface with large deviations will give terminal points all around the circle, and this may result in a position of G near O. In such a case the method will not be of much use. It is different, however, if only the smaller part of the mirror deviates largely from the

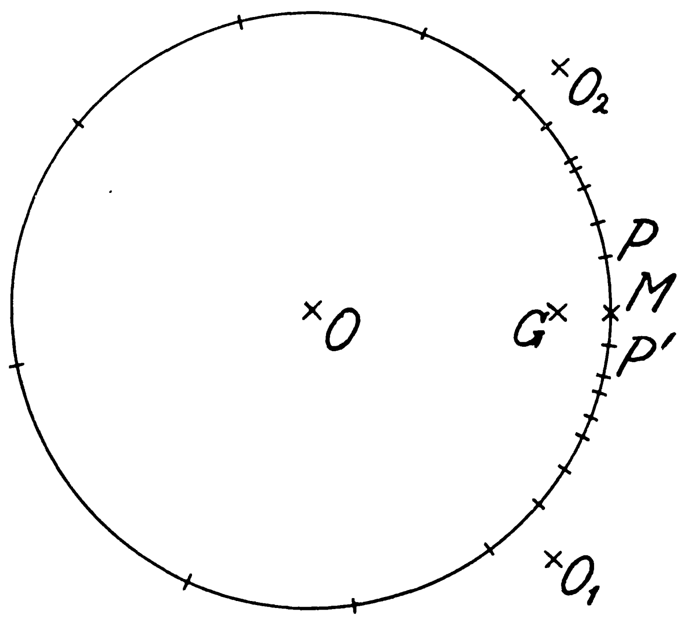

FIG. 1.

perfect form. The larger part will then assure that G and therefore O_1 lie far enough from O. Points on the mirror corresponding to the point on the circle nearest to O_1 will then appear with minimum brightness. The smaller, deviating part of the mirror will therefore be seen crossed by dark lines of equal deviation, the error in light-path increasing by one wave-length on going from one line to the next. This is exactly the same effect as Newton's rings that would be seen by applying a test glass to the mirror.

3. In the foregoing exposition of the principles of the phase-contrast method we have tacitly assumed that the central star-image can be changed in phase without influencing the diffraction spectra. This cannot be strictly true, as the star-image is not a sharply limited disc, but a diffraction pattern with concentric rings. This must especially be taken into account when the spectra lie near the central image, *i.e.* in the case of deviations of long period. An exact mathematical treatment of the problem has been worked out in the case of deviations small compared with the wave-length. We shall here give a short account of this treatment and of the relatively simple resulting

formulæ, referring for the proofs to the elaborate paper shortly to appear in *Physica*, where also a fuller treatment of the knife-edge test will be found.

In the following we shall denote by x, y, or r, ϕ the rectangular or polar co-ordinates on the surface of the mirror or in its image, by ξ, η, or ρ, ψ the co-ordinates in the plane of the star-image, and by J_n the Bessel function of nth order.

The diffraction image of a point-source formed by a perfect mirror was calculated by Airy with the well-known result that the amplitude is given by $J_1(\rho)/\rho$ if the unit of ρ is adequately chosen. Our first problem is to determine the small changes of this function caused by small deviations of the mirror-surface that are given as an arbitrary function of x and y. The second problem will be to find the diffraction effect in the image plane caused by the vibrations in the focal plane.

We develop the arbitrary function that determines the errors of the mirror-surface in a series of functions that are adapted to our problem. There can be little doubt that these functions should be polynomials in x and y. Indeed, the simplest errors are polynomials of low degree, *e.g.* astigmatism (2nd degree), spherical aberration (4th degree), etc. Instead of using a power series, however, it will prove more expedient to develop in a series of *orthogonal* polynomials. Clearly these should be orthogonal on the circular area of the mirror. Expressed in terms of r and ϕ, these *circle polynomials* were found to have the form $R_n{}^m(r) \sin m(\phi - \alpha)$ where $n-m$ is even, $m \leq n$, and R is a finite hypergeometric series (Jacobi polynomial):

$$R_n{}^m(r) = (-)^{\frac{n-m}{2}} \begin{pmatrix} \frac{n+m}{2} \\ m \end{pmatrix} r^m F\left(\frac{n+m+2}{2}, -\frac{n-m}{2}, m+1, r^2\right)$$

$$= \frac{r^{-m}}{\left(\frac{n-m}{2}\right)!} \left(\frac{d}{d(r^2)}\right)^{\frac{n-m}{2}} \left\{ r^{n+m}(r^2-1)^{\frac{n-m}{2}} \right\}$$

the constant being chosen in such a way that $R_n{}^m(1) = 1$.*

For the lowest degrees we give the polynomials explicitly :

$m \backslash n$	0	1	2	3	4	5	6
0	1	...	$2r^2-1$	...	$6r^4-6r^2+1$	...	$20r^6-30r^4+12r^2-1$
1	...	r	...	$3r^3-2r$	...	$10r^5-12r^3+3$	...
2	...	...	r^2	...	$4r^4-3r^2$	...	$15r^6-20r^4+6r^2$
3	...	...	...	r^3	...	$5r^5-4r^3$	...
4	...	...	...	...	r^4	...	$6r^6-5r^4$
5	...	...	...	...	...	r^5	...
6	...	...	...	...	...	...	r^6

* These formulæ and the properties of the circle polynomials as used in the calculations would seem to be new. The polynomials were found to be a kind of zonal harmonics in four dimensions. A different set of polynomials orthogonal on a circle was investigated by F. Didon, *Ann. Ec. Norm.*, **7**, 247, 1870.

If now the arbitrary deviation of the mirror-surface is represented by a series of these polynomials, it will be found that the terms with $r \sin \phi$, $r \cos \phi$, and $2r^2 - 1$ do not represent real errors, but can be made to vanish by adjusting the mirror. Except these three, every term of the general

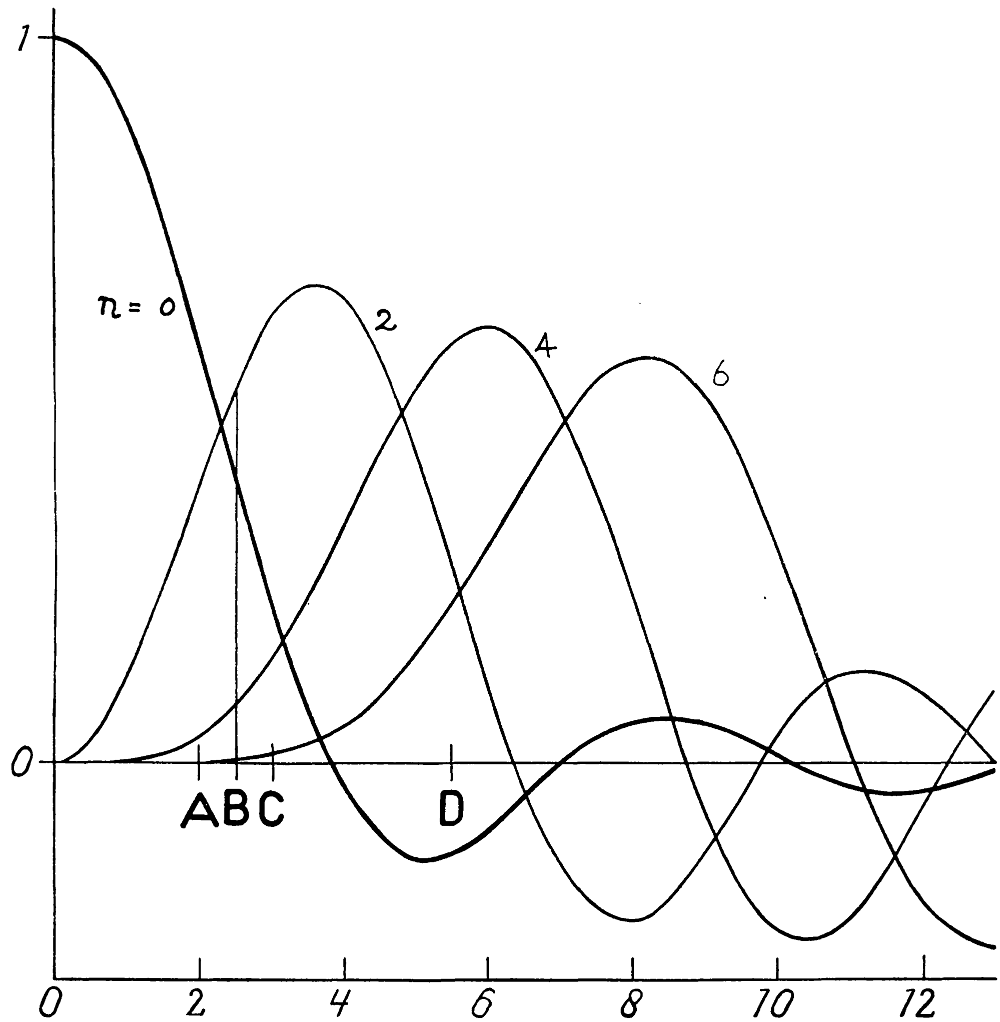

FIG. 2.

form $C_{nm} R_n{}^m(r) \begin{matrix} \sin m\phi \\ \cos m\phi \end{matrix}$ may be said to represent an "error" of nth order, the magnitude of which is given by the coefficient C_{nm}.

The diffracted amplitude caused by the error $C_{nm} R_n{}^m \cos m\phi$ is found for n even, by calculating

$$C_{nm}\int_0^1\int_0^{2\pi} \cos\{r\rho\cos(\phi-\psi)\} R_n{}^m(r) \cos m\phi \, r dr d\phi = (-)^{\frac{n}{2}} 2\pi C_{nm} \rho^{-1} J_{n+1}(\rho) \cos m\psi$$

and similarly for n odd and for $\sin \phi$. In this result the radial factor is independent of m. In fig. 2 the functions $(2n+2)\rho^{-1} J_{n+1}(\rho)$ are represented for $n = 2$, 4, 6 together with the pattern for the perfect mirror ($n = 0$).

The curves of fig. 2 show that the central part—the spurious disk—of the diffraction pattern of the perfect mirror is not changed by small errors of any order, and that such changes as occur lie farther and farther from the centre as the order is increased.

Clearly the pattern for a perfect mirror corresponds to the central star-image in the test case of § 1, while the changes of the pattern caused by deviations of the surface correspond to the diffraction spectra. The phase-contrast method, which requires us to change the phase of the central image only, can therefore be used without difficulty in the case of errors of higher orders. For the lowest orders it will be necessary to use a glass plate with a minute circular disk of differing thickness—a phase disk—of a size adapted

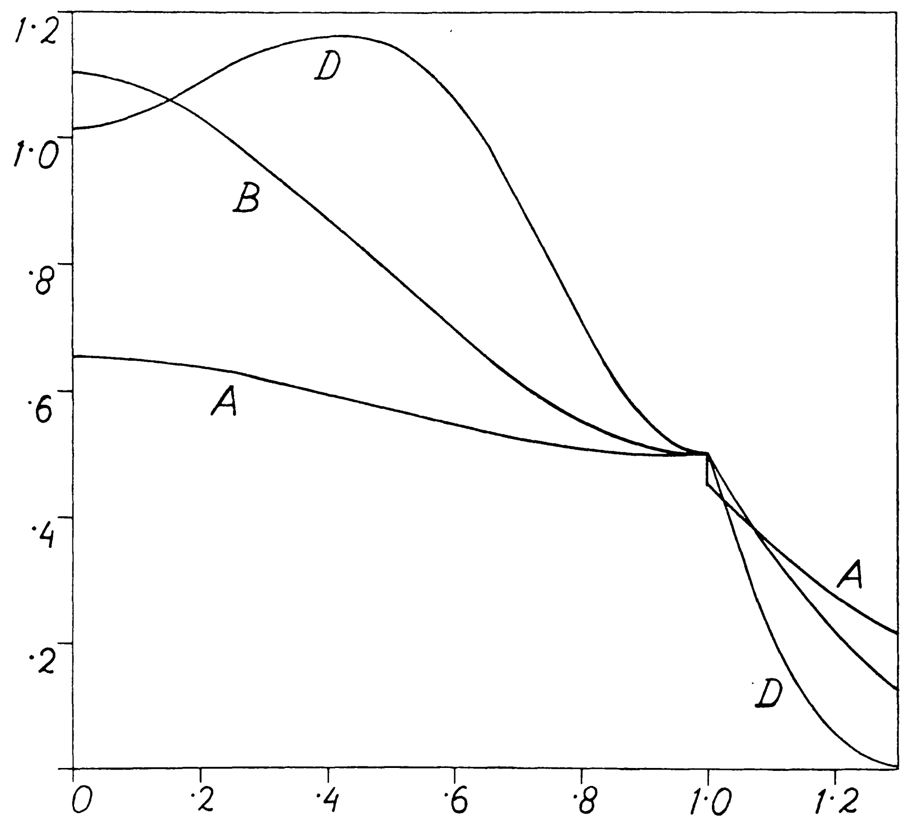

Fig. 3.

to the case. To detect astigmatism, the radius of the phase disk should be taken as 2·5 in the units of fig. 2 (point B); for spherical aberration, and any higher orders as well, the radius might be as large as 5·5 (point D). Of course the central image is more or less disturbed by such phase disks. This means that even a perfect mirror will no longer be seen as a uniformly illuminated circle, as it is without the disk. This effect of the change of phase can be calculated by a second integration of much the same kind as the first one. Fig. 3 shows the resulting intensity for phase disks of different radii. Any radius between A and C in fig. 2 will give practically the same uniform intensity, the increase of intensity towards the centre (curve B) being automatically compensated by a slightly different adjustment of focus. Larger phase disks, as D, give a somewhat higher intensity, but would seem less good in all other respects. The unit in fig. 2 being $\lambda F/\pi D$ for a relative aperture D/F, the diameter of the phase disk should be less than $2\lambda F/D$, and preferably from 60 to 90 per cent. of this value (range $A-C$). In all cases part of the light is thrown outside the image of the mirror, forming broad diffraction rings round it. Their appearance gives a sensitive criterion for the exact centring of the phase disk on the star-image.

ON THE PHASE-CONTRAST TEST OF F. ZERNIKE.

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Summary.—The Zernike test has the great advantage over the knife-edge test, as a null test for mirror systems, that it shows the absolute error, and not the differential of the absolute error in a particular direction. This makes the process of correcting irregular errors by local figuring much easier. A method of carrying out the test, which will detect 0·1 fringe of error, is described, and the practical limitations of this method are discussed, with special reference to the testing of large reflectors.

Acknowledgment.—When I learnt, in a private communication in 1933, of the principle of the Zernike test, I immediately resolved to apply the method to testing mirrors which I am figuring. I communicated the substance of my results to Professor Zernike, and the present paper is published with his approval.

§ 1. *The Nature of the Test*.—In the form of Zernike test which I have used, an artificial star of small diameter is placed at one of the stigmatic points of a mirror system, and at the other stigmatic point there is placed the small circular phase-retarding disk, which I shall refer to as a Zernike disk, or, for short, a Z-disk. The star and Z-disk are both preferably somewhat smaller than the Airy disks of the system at the points at which they are respectively placed. The eye of the observer is placed close behind the Z-disk, and he looks either directly at the mirror under test, or at its image in any auxiliary mirrors which may come between it and him. He then sees interference fringes formed between the wave leaving the mirror and the supplementary wave to which its passage through the Z-disk gives rise. If the Z-disk is not larger than the Airy disk produced by the system, that part of the supplementary wave lying within the cone of light proceeding from the mirror will be substantially spherical, so that since the wave-front leaving the mirror system departs from sphericity only by the errors of the mirror under test, the fringe system seen (in a given colour) will be substantially the Newton-ring system which would be seen if a true-figure test-plate were laid on, or very close to, the mirror under test. The colour-sequence of the fringes will be determined by the chromatic dispersion of phase retardation of the Z-disk, and can be ascertained from the colour