Journal of Statistical Physics, Vol. 86, Nos. 1/2, 1997

Diffusion Approximation for Billiards with Totally Accommodating Scatterers

Claude Bardos,[1] Laurent Dumas,[2] and François Golse[3]

Received November 27, 1995; final February 24, 1996

We study the 2D motion of independent point particles colliding with a periodic array of circular obstacles. The interaction between the particles and the obstacles is described by a total accommodation reflection law. Assuming that the array of scatterers has finite horizon, the density of particles is approximated by the solution of a diffusion equation in the long-time and large-scale regime. The proof relies on a multiscale asymptotics and gives the order of approximation.

KEY WORDS: Periodic Lorentz gas; dispersive billiards; hydrodynamic limit; diffusion coefficient; homogenization; multiscale asymptotic expansion.

1. INTRODUCTION

The problem of "hydrodynamic limits" lies at the heart of nonequilibrium statistical mechanics. In order to briefly revisit this notion, let us recall that there are essentially three types of mathematical models describing the evolution of $\sim 10^{23}$ molecules of a perfect monatomic gas:

(a) The equations of classical mechanics, once the interaction between the molecules is known and encoded in the $\sim 10^{23}$-body Hamiltonian.

(b) The kinetic theory of gases, based on the Boltzmann equation (here the interaction between the molecules enters the equation through the collision cross section).

[1] Université Paris VII and École Normale Supérieure de Cachan, CMLA, 94235 Cachan, France.

[2] Université Paris VII and École Normale Supérieure, DMI, 75005 Paris, France.

[3] Université Paris VII, UFR de Mathématiques, 75251 Paris Cedex 05, France.

0022-4715/97/0100-0351$12.50/0

(c) The equations of classical hydrodynamics (such as the Euler or Navier–Stokes systems); in the latter models, the only remaining information concerning the interaction between molecules is contained in the so-called "transport coefficients" such as viscosity and heat conduction.

Studying the relations between the above models from the point of view of a mathematician is still a mostly open problem, which has long been under scrutiny since the work of Hilbert, Chapman, and Enskog. Although the formal asymptotics relating (a) to (b) and (b) to (c) are known to a very large extent (see refs. 2 and 11, for example) complete convergence proofs are missing in all but favorable particular cases. We refer the interested reader to refs. 3, 4, 10, 11, 13, and 17 for such proofs.

One can argue that one of the main difficulties in the above program is the fact that the partial differential equations involved are mostly nonlinear. Another difficulty is that the models in (a) are "reversible" while those in (b) and (c) contain in some sense Carnot's "second principle of thermodynamics" and as such are "irreversible." We shall not discuss these notions in more detail here; we instead refer the interested reader to refs. 15 and 16 which describe in the simplest possible terms what has given birth to one of the greatest controversies in mathematical physics.

In a remarkable series of papers, Bunimovich, Sinai, and then joined by Chernov[6–9] (BSC) investigated the problem of relating directly a description of the type (a) above to one of the type (c) on a simplified model known as "the periodic Lorentz gas." The Lorentz gas is a gas of mutually independent point particles colliding with a periodic array of convex scatterers. The scatterers are supposed fixed (or infinitely heavy) and the collisions are elastic; in other words, the particles are specularly reflected from the surface of the scatterers. Using fairly sophisticated tools from the ergodic theory of billiards (and in particular the construction of Markov "sieves"), BSC were able to show that in the long-time, large-scale limit, and under an assumption that they called "finite horizon" that we shall recall later, the dynamics of the particles was described by a diffusion equation. One can understand from their proof how it is possible to relate a model of type (a), which is "reversible," to the diffusion equation, which is the foremost example of an "irreversible" dynamics.

In the present paper, we take again the same model as considered by BSC except that we change the interaction between the particles and the scatterers from specular reflection to diffuse (or fully accommodating) reflection. We then show that, in the same asymptotic regime as considered by BSC, that is, the long-time, large-scale limit, the dynamics of the particles is also described by a diffusion equation. However, our proof is much simpler and uses only elementary techniques of multiscale expansions for partial differential equations (PDEs). Note, however, that we achieve less than BSC: with the modification on the law of reflection, the model of type (a) (that is, the Liouville equation for the number density of particles) is no longer a "reversible" model. Hence, our result cannot serve the purpose of understanding the "appearance of irreversibility" in nonequilibrium statistical mechanics. However, it gives a strategy of proof which is based on the fact that a certain operator is Fredholm, and thus circumscribes the use of more sophisticated tools (as in the work of BSC) to situations where this Fredholm property is missing.

After these generalities aimed at placing the problem in its natural context, we come to the specifics of the model that we shall study. Let us begin with a description of the phase space.

The space of positions, denoted by X, is the complement in $\mathbf{R}^2$ of a periodic array of disks of radius r at the nodes of a regular triangular lattice

$$\mathscr{L} = \mathbf{Z}(a, 0) \oplus \mathbf{Z}\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$$

We impose that $r < a/2$, so that the scatterers do not overlap. The domain left free for the particles is then

$$X = \{x \in \mathbf{R}^2 / d(x, \mathscr{L}) > r\}$$

Let Y be the fundamental domain associated to X, that is, the image of $\bar{X}$ (the closure of X) under the canonical projection p, of $\mathbf{R}^2$ onto $\mathbf{R}^2/\mathscr{L}$. As a topological space, Y can be viewed as two circles linked at one common point. More precisely, Y is the compact manifold with boundary defined on Fig. 1 by gluing AA' with $C'C$ and BB' with $D'D$.

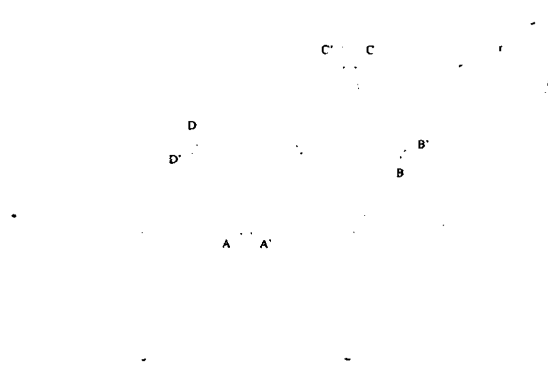

Fig. 1. The geometry of the fundamental domain.

On these two sets X and Y, n_x (respectively n_y) denotes the unit normal vector at $x \in \partial X$ (resp. ∂Y) directed toward X (resp. Y).

Particles considered here move with unit speed. Their number density is denoted by $f \equiv f(t, x, v)$: this notation means that $f(t, x, v)$ is the density of particles which, at time t, occupy position x and move with velocity v. In other words, $f(t, x, v)\, dx\, dv$ is the number of particles in an infinitesimal volume $dx\, dv$ of the phase space $X \times S^1$ centered at (x, v).

In $X \times S^1$, due to the mutual independence of the particles, the number density satisfies the free transport equation:

$$\partial_t f + v \cdot \nabla_x f = 0, \qquad t \geqslant 0, \quad x \in X, v \in S^1 \tag{1.1}$$

Let $\Gamma_\pm = \{(x, v) \in \partial X \times S^1,\ 0 < \pm v \cdot n_x\}$. The symbol $f_\pm$ refers to the trace of f on (i.e., restriction to) the set $\Gamma_\pm$. When this induces no confusion, the same notations are kept with ∂Y instead of ∂X. The boundary condition considered here is a diffuse reflection law with total accommodation. We refer to ref. 11 for a discussion of the physical relevance of this condition, as well as of many of its variants. The diffuse reflection law is

$$f_+(t, x, v) = \tfrac{1}{2} \int_{v' \cdot n_x < 0} |v' \cdot n_x|\, f_-(t, x, v')\, dv', \qquad (x, v) \in \Gamma_+ \tag{1.2}$$

This condition means that the sines of the angles of reflection of the particles are equidistributed after each collision. The normalizing factor 1/2 is chosen so that functions independent of the variable v satisfy (1.2): indeed,

$$1 = \tfrac{1}{2} \int_{v' \cdot \omega < 0} |v' \cdot \omega|\, dv', \qquad \forall \omega \in S^1 \tag{1.3}$$

Finally, the initial density is prescribed:

$$f(0, x, v) = \Phi(x, v), \qquad x \in X, v \in S^1 \tag{1.4}$$

The Scaling

Three length scales are present in the model above: a, r, and the characteristic scale of variation of Φ, denoted by L. The mean free path of particles between two collisions will tend to zero if a and r are infinitely small and of the same order of smallness compared to L.

This suggests that we introduce a small parameter ε such that $a = \hat{a}\varepsilon$, $r = \hat{r}\varepsilon$, where $\hat{a}$, $\hat{r}$, and L are of the same order of magnitude. Time will also have to be rescaled consistently with the limiting diffusion dynamics.

Define $\hat{f}_\varepsilon(t, x, v) = f_\varepsilon(t/\varepsilon, x, v)$, where f_ε satisfies (1.1)–(1.2)–(1.4) with $\hat{a}\varepsilon$ and $\hat{r}\varepsilon$ instead of a and r. Dropping the carets, one can see that f_ε is a solution of the rescaled system:

$$\varepsilon\,\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = 0, \qquad t \geqslant 0, \quad x \in X_\varepsilon, v \in S^1 \tag{1.5}$$

$$f_{\varepsilon+}(t, x, v) = \tfrac{1}{2} \int_{v' \cdot n_x < 0} |v' \cdot n_x| \, f_{\varepsilon-}(t, x, v') \, dv', \qquad (x, v) \in \Gamma^\varepsilon_+ \tag{1.6}$$

$$f_\varepsilon(0, x, v) = \Phi(x) \tag{1.7}$$

where

$$\begin{cases} \mathscr{L}_\varepsilon = \mathbf{Z}(a\varepsilon, 0) \oplus \mathbf{Z}\left(\dfrac{a\varepsilon}{2}, \dfrac{a\varepsilon\sqrt{3}}{2}\right) \\ X_\varepsilon = \{x \in \mathbf{R}^2 \mid d(x, \mathscr{L}_\varepsilon) > r\varepsilon\} \\ \Gamma^\varepsilon_\pm = \{(x, v) \in \partial X_\varepsilon \times S^1 \mid 0 < \pm v \cdot n_x\} \end{cases}$$

Observe that, in the initial condition (1.7), Φ is defined on $\mathbf{R}^2 \times S^1$ and is independent of v to avoid initial layers.

The main result of this paper, stated in the next section as Theorem 1, gives the asymptotic behavior of f_ε as ε goes to 0.

2. DIFFUSION APPROXIMATION

With the scalings and boundary conditions above, the particles will undergo a large number of collisions per unit time and will forget their individual velocity. This will result in the limiting Brownian dynamics. The convergence proof presented here relies on the fact that the space X has the finite horizon property. This assumption, introduced by Bunimovich and Sinai,[7] means precisely that

$$\sup\{|x - y|/[x, y] \subset X\} < +\infty \tag{2.1}$$

A sufficient condition for the billiard considered above to have the finite horizon property is

$$r > (a\sqrt{3})/4 \tag{2.2}$$

which implies that the maximum free path of particles in X_ε is of order ε.

The main result of this paper is:

Theorem 1. Assume that condition (2.2) holds. Let $\Phi \equiv \Phi(x)$ be a smooth function defined on $\mathbf{R}^2$ with bounded derivatives up to order 4 and let $F_0 \equiv F_0(t, x)$ be the solution of the diffusion equation

$$\partial_t F_0 - d\,\Delta_x F_0 = 0, \qquad t \geqslant 0, x \in \mathbf{R}^2 \tag{2.3}$$

with initial condition

$$F_0(0, x) = \Phi(x), \qquad x \in \mathbf{R}^2 \tag{2.4}$$

where $d > 0$ is given in (2.10) below. Then, for any $T > 0$, there exists a positive constant C_T such that

$$\sup_{t \in [0, T]} \|f_\varepsilon(t, \cdot) - F_0(t, \cdot)\|_{L^\infty(X_\varepsilon \times S^1)} \leqslant C_T \varepsilon$$

As in ref. 5, the proof is based on an asymptotic expansion for f_ε of the form

$$f_\varepsilon(t, x, v) = f^{(0)}(t, x, y, v) + \varepsilon f^{(1)}(t, x, y, v) + \varepsilon^2 f^{(2)}(t, x, y, v) + \cdots |_{y = x/\varepsilon}$$

where $f^{(k)}(t, x, \cdot, v)$ is defined on Y. The key point in the proof is a variant of the Fredholm alternative for the advection operator $v \cdot \nabla_y$ with the accommodation boundary condition in the function space $L^\infty(Y \times S^1)$, stated in the following lemma.

Lemma 2. Let $S \in L^\infty(Y \times S^1)$ and consider the problem

$$v \cdot \nabla_y \Theta = S, \qquad y \in Y, \quad v \in S^1 \tag{2.5}$$

$$\Theta_+(y, v) = \tfrac{1}{2} \int_{v' \cdot n_y < 0} |v' \cdot n_y|\, \Theta_-(y, v')\, dv', \qquad y \in \partial Y, \ v \cdot n_y > 0 \tag{2.6}$$

The following statements are equivalent:

(i) S satisfies the orthogonality condition

$$\iint_{Y \times S^1} S(y, v)\, dy\, dv = 0$$

(ii) There exists a unique solution $\Theta \in L^\infty(Y \times S^1)$ of (2.5)–(2.6) such that

$$\iint_{Y \times S^1} \Theta(y, v)\, dy\, dv = 0$$

Once, this lemma is proven, we can introduce the diffusion matrix by analogy with the classical Green–Kubo formula:

$$D = \iint_{Y \times S^1} (v \otimes \gamma(y, v) + \gamma(y, v) \otimes v) \, dv \, dy \tag{2.7}$$

where $\gamma_j(y, v) \in L^\infty(Y \times S^1)$, $j \in \{1, 2\}$, is the unique solution of

$$v \cdot \nabla_y \gamma_j = v_j, \qquad y \in Y, \quad v = (v_1, v_2) \in S^1 \tag{2.8}$$

$$\gamma_{j+}(y, v) = \tfrac{1}{2} \int_{v' \cdot n_y < 0} |v' \cdot n_y| \, \gamma_{j-}(y, v') \, dv', \qquad y \in \partial Y, \quad v \cdot n_y > 0 \tag{2.9}$$

defined according to Lemma 2. In the particular case of a regular triangular array, the diffusion matrix reduces to a diffusion coefficient.

Proposition 3. The diffusion matrix D is scalar positive, that is,

$$D = dI, \qquad \text{where} \quad d = \tfrac{1}{2} \iint_{Y \times S^1} (v_1 \gamma_1(y, v) + v_2 \gamma_2(y, v)) \, dv \, dy > 0 \tag{2.10}$$

3. A CHANGE OF VARIABLES

We begin with a change of variables in a certain integral. Although this computation might seem overly technical and with little conceptual interest, we insist that it is a fundamental step for the results that we have in mind; it will indeed be used repeatedly throughout the paper.

We first define the backward exit time. Note that this definition is valid for the transport of particles in both the compact set Y and its fundamental cover X. The backward exit time is defined as

$$t_{y, v} = \inf\{t > 0 \mid y - tv \in \partial Y\} \qquad \text{for any} \quad (y, v) \in Y \times S^1 \tag{3.1}$$

In other words, $t_{y, v}$ is the time spent inside the domain by a particle starting from point y with velocity $-v$ before its first reflection at the boundary.

With the notation above, the integral operator to be studied in this section is

$$(K\phi)(y) = \tfrac{1}{2} \int_{v \cdot n_y < 0} \phi(y - t_{y, v} v) \, |v \cdot n_y| \, dv \tag{3.2}$$

where $\phi \in C(\partial Y)$. We want to express $K\phi$ in the form

$$(K\phi)(y) = \tfrac{1}{2} \int_{\partial Y} k(y, y') \, \phi(y') \, dy' \tag{3.3}$$

This will result from the change of variables $v \mapsto y - t_{y,v} v = y'$ in (3.2). This change of variables is by no means elementary and the particular geometry of the billiard system considered is reflected in the kernel k.

The physical meaning of the integral operator K can be understood as follows: it is the number density of particles leaving the boundary ∂Y in terms of its former value at the previous time of collision with ∂Y.

Before embarking on the detailed calculation of k, we need some geometrical preparations.

Definition. Let $y \in \partial X$; y belongs to one of the circles in ∂X, denoted by C_y.

1. A neighbor of C_y visible from y is a circle C' included in ∂X having the following property: there exists an open arc $I' \subset C'$ such that, for all $x \in I'$, the open segment $]x, y[\subset X$. The set of neighbors of C_y visible from y is denoted by VN_y; it has at most three elements (see the shaded disks in Fig. 2).

2. Let $C' \in VN_y$; one denotes by $V_y(C')$ the maximal open arc I' included in C' such that, for all $x \in I'$ the open segment $]x, y[\subset X$. One denotes the corresponding arc in the space of directions at y by

$$W_y(C') = \left\{ \frac{y' - y}{|y' - y|} \;\middle|\; y' \in V_y(C') \right\}$$

We make the two following important observations:

1. If $z' \in V_y(C')$ and $z'' \in V_y(C'')$ for some C' and $C'' \in VN_y$, it is impossible that $z'' - z' \in \mathscr{L}$ because of assumption (2.2).

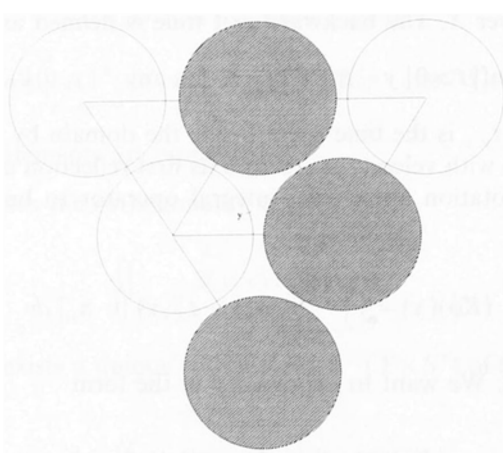

Fig. 2. Visible neighbors from point y.

2. One should remember that functions $\phi \in C(\partial Y)$ can be identified with $\mathscr{L}$-periodic functions of $C(\partial X)$.

With these definitions and remarks, one arrives at the following expression for the integral operator K:

$$
\begin{aligned}
(K\phi)(y) &= \sum_{C' \in VN_y} \tfrac{1}{2} \int_{v \cdot n_y < 0} \mathbf{1}_{V_y(C')}(y - t_{y,v}v)\, \phi(y - t_{y,v}v)\, |v \cdot n_y|\, dv \\
&= \sum_{C' \in VN_y} \tfrac{1}{2} \int_{W_y(C')} \phi(y - t_{y,v}v)\, |v \cdot n_y|\, dv
\end{aligned} \tag{3.4}
$$

(In the above expression, we denote by $\mathbf{1}_A$ the indicator function of a set A.)

Proposition 4. There exists a function k bounded on $\partial Y \times \partial Y$ such that, for all $\phi \in C(\partial Y)$,

$$(K\phi)(y) = \int_{\partial Y} k(y, y')\, \phi(y')\, dy'$$

One has $k(y, y') \geqslant 0$ on $\partial Y \times \partial Y$ and $k(y, y') > 0$ if $y' \in V_y(C')$ with $C' \in VN_y$.

Let us now consider a particular visible neighbor for y, and perform the change of variables $v \mapsto y' = y - t_{y,v}v$.

Consider a segment of trajectory between two scatterers C_1 and C_2 centered at O_1 and O_2 respectively, as indicated on Fig. 3. We have used

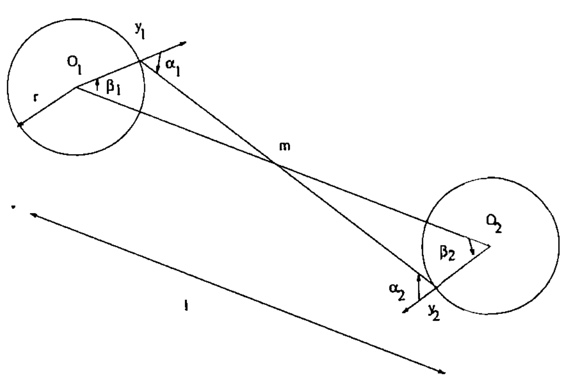

Fig. 3. The collision parameters.

the notations y_1 for y and y_2 for y' in Fig. 3 to emphasize the symmetrical role played by y and y' in this calculation. With these notations, $C_2 \in VN_{y_1}$. Denote by m the intersection point of the line $O_1 O_2$ with the line $y_1 y_2$. The point y_l on C_1 is parametrized by the angle $\beta_1 = (\widehat{mO_1\, y_1})$ and likewise y_2 is parametrized by $\beta_2 = (\widehat{mO_2\, y_2})$. The velocity v is parametrized either by the angle $\alpha_1 = (\widehat{n_{y_1}, v})$ or the angle $\alpha_2 = (\widehat{-v, n_{y_2}})$ (see Fig. 3).

For a single pair of such scatterers, we first study the change of variables $(\alpha_1, \beta_1) \mapsto (\alpha_2, \beta_2)$. Four different cases may occur as m moves on the fixed line $O_1 O_2$ according to the signs of $\overrightarrow{mO_1} \cdot \overrightarrow{mO_2}$ and $\overrightarrow{my_1} \cdot \overrightarrow{my_2}$. In all cases the same final result is established with the same arguments. We shall therefore restrict our attention to the case considered in Fig. 3 (that is, $\overrightarrow{mO_l} \cdot \overrightarrow{mO_2} < 0$ and $\overrightarrow{my_1} \cdot \overrightarrow{my_2} < 0$).

One has to write the classical relations for the triangles $O_1 m y_1$ and $O_2 m y_2$. Define the two angles

$$\delta_1 = (\widehat{O_1 m y_1}) \qquad \text{and} \qquad \delta_2 = (\widehat{O_2 m y_2}) \tag{3.5}$$

Clearly, one has $\delta_1 = \delta_2$, which means that

$$-\pi + (\beta_1 + \pi + \alpha_1) = -\pi + (\pi + \beta_2 + \pi - \alpha_2) \tag{3.6}$$

or equivalently

$$\alpha_1 + \beta_1 = -\alpha_2 + \beta_2 + \pi \tag{3.7}$$

Moreover, in the triangles $O_1 m y_1$ and $O_2 m y_2$ the following classical relations hold:

$$\begin{aligned} \frac{r}{\sin(-\delta_1)} &= \frac{d(O_1, m)}{\sin(\pi + \alpha_1)} \\ \frac{r}{\sin(-\delta_2)} &= \frac{d(m, O_2)}{\sin(\pi - \alpha_2)} \end{aligned} \tag{3.8}$$

Thus,

$$r \sin \alpha_1 - r \sin \alpha_2 = l \sin(\alpha_1 + \beta_1) \tag{3.9}$$

with the notation $l = d(O_1, m) + d(m, O_2)$. Therefore, the relations defining (α_2, β_2) in terms of (α_1, β_1) are

$$\begin{cases} \alpha_2 = \operatorname{Arcsin}\left(\sin\alpha_1 - \frac{l}{r}\sin(\alpha_1+\beta_1)\right) \\ \beta_2 = \operatorname{Arcsin}\left(\sin\alpha_1 - \frac{l}{r}\sin(\alpha_1+\beta_1)\right) + \alpha_1 + \beta_1 + \pi \end{cases} \tag{3.10}$$

and the Jacobian of the transformation $(\alpha_1, \beta_1) \mapsto (\alpha_2, \beta_2)$ is

$$J(\alpha_1, \beta_1) = \frac{1}{\cos\alpha_2} \times \begin{pmatrix} \cos\alpha_1 - \frac{l}{r}\cos(\alpha_1+\beta_1) & -\frac{l}{r}\cos(\alpha_1+\beta_1) \\ \cos\alpha_1 - \frac{l}{r}\cos(\alpha_1+\beta_1) + \cos\alpha_2 & -\frac{l}{r}\cos(\alpha_1+\beta_1) + \cos\alpha_2 \end{pmatrix} \tag{3.11}$$

We isolate the crucial property that we shall use to justify the change of variables below:

Lemma 5. The four entries of the matrix $(\cos\alpha_2)\,J$ belong to an interval of the form $[M_1, M_2]$ with $M_1 < M_2 < 0$. In particular one has

$$M_1 \leqslant \cos\alpha_1 - \frac{l}{r}\cos(\alpha_1+\beta_1) + \cos\alpha_2 \leqslant M_2 < 0 \tag{3.12}$$

Proof of Lemma 5. Since it is obvious that the absolute values of these entries are all bounded above, the only thing to show is that these entries are bounded above by a negative constant M_2. To verify this, it is enough to prove that the upper bound in (3.12) holds for any admissible (α_1, β_1) and l. Indeed, since both α_1 and α_2 belong to $[-\pi/2, \pi/2]$ one has $\cos\alpha_1$ and $\cos\alpha_2 \geqslant 0$: hence the largest entry in $(\cos\alpha_2)\,J$ is the one appearing in (3.12).

Before doing this, we discuss more precisely what we mean by "the admissible (α_1, β_1) and l." Keeping the point y_1 fixed, we draw the two lines tangent to C_2 passing through y_1; the directions of these lines is parametrized by the angle with the normal n_{y_1}, that they define. We shall call these angles $\alpha_{1\min}$ and $\alpha_{1\max}$; one has obviously

$$-\frac{\pi}{2} < \alpha_{1\min} < \alpha_{1\max} < \frac{\pi}{2}$$

On the other hand, the distance l between the centers of C_1 and C_2 cannot be smaller than a.

In order to prove (3.12), we need to study for any fixed $\beta_1 \in [-\pi/2, \pi/2]$ the function defined on $[\alpha_{1\min}; \alpha_{1\max}] \subset [-\pi/2, \pi/2]$ by

$$f(\alpha_1) = \cos \alpha_1 - \frac{l}{r} \cos(\alpha_1 + \beta_1) + \cos \alpha_2$$

where

$$\alpha_2 = \operatorname{Arcsin}\left(\sin \alpha_1 - \frac{l}{r} \sin(\alpha_1 + \beta_1)\right)$$

It is immediatly seen that

$$\begin{aligned} f'(\alpha_1) &= -\sin \alpha_1 + \frac{l}{r} \sin(\alpha_1 + \beta_1) - \sin \alpha_2 \frac{\partial \alpha_2}{\partial \alpha_1} \\ &= -\sin \alpha_2 \left(1 + \frac{\partial \alpha_2}{\partial \alpha_1}\right) \end{aligned}$$

where

$$\frac{\partial \alpha_2}{\partial \alpha_1} = \cos \alpha_1 - \frac{l}{r} \cos(\alpha_1 + \beta_1)$$

To prove that f is negative, it suffices to prove that f is negative when evaluated at its critical points as well as at $\alpha_1 = \alpha_{1\min}$, $\alpha_1 = \alpha_{1\max}$. The critical points are such that $\alpha_2 = 0$ or $1 + \partial \alpha_2 / \partial \alpha_1 = 0$.

- On $\alpha_1 = \alpha_{1\min}$: This case clearly corresponds to $\alpha_2 = \pi/2$ (see Fig. 3). Therefore

$$f(\alpha_{1\min}) = \cos \alpha_{1\min} - \frac{l}{r} \cos(\alpha_{1\min} + \beta_1)$$

Notice that $\widehat{O_1 m y_1} = \alpha_{1\min} + \beta_1$, which is an acute angle and hence

$$\begin{aligned} f(\alpha_{1\min}) &= \cos \alpha_{1\min} - \frac{l}{r} [1 - \sin^2(\alpha_{1\min} + \beta_1)]^{1/2} \\ &= \cos \alpha_{1\min} - \left[\frac{l^2}{r^2} - (\sin \alpha_{1\min} - 1)^2\right]^{1/2} \\ &= \cos \alpha_{1\min} - \left(\frac{l^2}{r^2} - 2 + 2 \sin \alpha_{1\min} + \cos^2 \alpha_{1\min}\right)^{1/2} \end{aligned}$$

Therefore, the fact that $f(\alpha_{1\min})$ is negative is clearly equivalent to the condition

$$\cos \alpha_{1\min} < \left(\frac{l^2}{r^2} - 2 + 2 \sin \alpha_{1\min} + \cos^2 \alpha_{1\min}\right)^{1/2}$$

or in other words

$$-2 \sin \alpha_{1\min} + 2 < l^2/r^2$$

But this last inequality is implied by the admissibility condition on l: $l > a > 2r$.

- On α_1 such that $1 + \partial\alpha_2/\partial\alpha_1 = 0$: This relation is rewritten as

$$1 + \cos \alpha_1 - \frac{l}{r} \cos(\alpha_1 + \beta_1) = 0$$

An alternative expression for $f(\alpha_1)$ is then

$$f(\alpha_1) = -1 + \cos \alpha_2$$

Thus $f(\alpha_1)$ is negative if $\alpha_2 \neq 0$. This case is precisely dealt with just below.

- On α_1 such that $\alpha_2 = 0$ (see Fig. 3): Here,

$$f(\alpha_1) = \cos \alpha_1 - \frac{l}{r} \cos(\alpha_1 + \beta_1) + 1$$

As above, $\cos(\alpha_1 + \beta_1)$ is positive because $\widehat{O_1 O_2 y_1}$ is an acute angle. Moreover, $\alpha_2 = 0$ implies that $r \sin \alpha_1 = l \sin(\alpha_1 + \beta_1)$. Thus

$$\begin{aligned} f(\alpha_1) &= \cos \alpha_1 + 1 - \frac{l}{r} [1 - \sin^2(\alpha_1 + \beta_1)]^{1/2} \\ &= \cos \alpha_1 + 1 - \left(\frac{l^2}{r^2} - \sin^2 \alpha_1\right)^{1/2} \\ &= \cos \alpha_1 + 1 - \left(\frac{l^2}{r^2} - 1 + \cos^2 \alpha_1\right)^{1/2} \end{aligned}$$

Again because of the admissibility condition $l > 2r$, $f(\alpha_1)$ is negative.

- On $\alpha_1 = \alpha_{1\max}$: Same proof as in the the first case $\alpha_1 = \alpha_{1\min}$.

This completes the proof of Lemma 5. ∎

An immediate consequence of Lemma 5 is the following.

Corollary 6. Let $y \in C \subset \partial X$ and $C' \in VN_{y}$. Then the map $v \mapsto y - t_{y,v} v = y'$ defines a C^1-diffieomorphism from the arc $W_y(C')$ to $V_y(C')$.

Proof of Proposition 4. Consider any of the integrals appearing in the finite sum (3.4). One has

$$\int_{v \in W_y(C')} \phi(y - t_{y,v} v)\, |v \cdot n_y|\, dv = \int_{y' \in V_y(C')} \phi(y') \left| \frac{y - y'}{|y - y'|} \cdot n_y \right| \cdot \left| \frac{\partial \beta_2}{\partial \alpha_1} \right|^{-1} dy'$$

with the convention that (y, v) are parametrized by β_1 and α_1 and that $y' = y - t_{y,v} v$ is parametrized by α_2 as explained above. With these notations, it follows precisely from (3.12) that

$$\begin{aligned}
-\frac{1}{M_1} & \left| \frac{y - y'}{|y - y'|} \cdot n_y \right| \cdot \left| \frac{y - y'}{|y - y'|} \cdot n'_y \right| \\
& \leqslant \left| \frac{y - y'}{|y - y'|} \cdot n_y \right| \cdot \left| \frac{\partial \beta_2}{\partial \alpha_1} \right|^{-1} \\
& = \left| \frac{y - y'}{|y - y'|} \cdot n_y \right| \frac{|\cos \alpha_2|}{|\cos \alpha_1 - (l/r) \cos(\alpha_1 + \beta_1) + \cos \alpha_2|} \\
& \leqslant -\frac{1}{M_2}
\end{aligned}$$

This shows that the kernel $k \in L^{\infty}(\partial Y \times \partial Y)$. On the other hand, the inequality above shows that $k(y, y') > 0$ if $y' \in V_y(C')$ with $C' \in VN_y$. This completes the proof of Proposition 4. ∎

4. PROOF OF THE FREDHOLM ALTERNATIVE

This section is devoted to the proof of Lemma 2. First, we check that the orthogonality condition is necessary for the existence of a solution Θ to (2.5)–(2.6). Indeed, let Θ be such a solution: integrating (2.5) over $Y \times S^1$ and using Green's formula gives

$$\iint_{Y \times S^1} v \cdot \nabla_y \Theta \, dy\, dv = \iint_{\partial Y \times S^1} (v \cdot n_y)\, \Theta(y, v)\, dy\, dv = \iint_{Y \times S^1} S(y, v)\, dy\, dv$$

where we have abused the notation dy to designate the length element on ∂Y. Then, using (1.3) and the boundary condition (2.6), we have

$$\begin{aligned}\iint_{Y\times S^1} &S(y, v)\, dy\, dv \\ &= \iint_{\Gamma_-} \left(\tfrac{1}{2} \int_{v'\cdot n_y<0} |v'\cdot n_y|\, \Theta(y, v')\, dv' - \Theta(y, v) \right) |v\cdot n_y|\, dy\, dv \\ &= \iint_{\Gamma_-} \left(\tfrac{1}{2} \int_{v\cdot n_y<0} |v'\cdot n_y| \cdot |v\cdot n_y|\, dv - |v'\cdot n_y| \right) \Theta(y, v')\, dy\, dv' \\ &= 0\end{aligned}$$

Uniqueness. Let $\Theta_0 \in L^\infty(Y\times S^1)$ be a solution of

$$v\cdot\nabla_y\Theta_0 = 0, \qquad (y, v)\in Y\times S^1 \tag{4.1}$$

$$\Theta_{0+}(y, v) = \tfrac{1}{2}\int_{v'\cdot n_y<0} |v'\cdot n_y|\, \Theta_{0-}(y, v')\, dv', \qquad (y, v)\in \Gamma_+ \tag{4.2}$$

The method of characteristics tells us that, for almost every $(y, v)\in Y\times S^1$, the map $t\mapsto \Theta_0(y+tv, v)$ is a constant on the interval $[-t_{y,v}; t_{y,-v}]$. Hence, for almost every $(y, v)\in\Gamma_+$,

$$\Theta_{0+}(y, v) \equiv \Theta_{0+}(y) = \tfrac{1}{2}\int_{v'\cdot n_y<0} |v'\cdot n_y|\, \Theta_{0+}(y - t_{y,v'}v')\, dv', \qquad (y, v)\in\Gamma_+ \tag{4.3}$$

Another way of writing Eq. (4.3) is

$$\Theta_{0+} = K\Theta_{0+} \tag{4.4}$$

At this point, we shall need some more information on the integral operator K defined in Section 3. Let $(k_n)_{n\geqslant l}$ be the sequence of bounded functions defined on $\partial Y\times\partial Y$ by the inductive relation

$$k_1(y, y') = k(y, y'), \qquad k_n(y, y') = \int_{\partial Y} k(y, y'')\, k_{n-1}(y'', y')\, dy'' \tag{4.5}$$

The function k_n is the integral kernel of the nth power K^n of $K\colon L^\infty(\partial Y)\to L^\infty(\partial Y)$. One has the following result.

Lemma 7. There exists $n^*>0$ such that $k_{n^*}(y, y')>0$ for all y and $y'\in\partial Y$.

Proof. We refer to the notations of Fig. 4. The middle points m_i of the arc C_i are of particular importance. Let $y\in C_1$; one can see that there exists $C'\in VN_y \subset VN_{m_1}$ such that $V_y(C')\cap V_{m_1}(C')\neq\varnothing$. Since

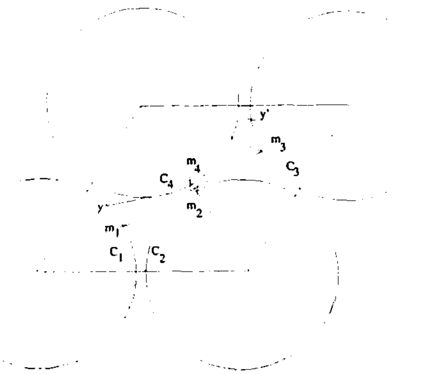

Fig. 4. Broken trajectory linking y and y'.

$m_i \in V_{m_j}(C_i)$ for all $i \neq j$ in $\{1, 2, 3, 4\}$, one deduces immediately the following geometric property:

There exists $n^* > 0$ such that, for all y and $y' \in \partial Y$, one can construct a sequence $(y_j)_{1 \leqslant j \leqslant n^*-1}$ of points in ∂Y and a sequence of nonempty open arcs A_j centered at y_j such that, for all sequences of points $(z_j)_{1 \leqslant j \leqslant n^*-1}$ with $z_j \in A_j$ for all $1 \leqslant j \leqslant n^* - 1$, the open segments $]y, z_1[$, $]z_j, z_{j+1}[$ for $1 \leqslant j \leqslant n^* - 2$ and $]z_{n^*-1}, y'[$ are all included in $\bar{Y}$.

One might wonder why the construction above (with the open arcs as above) is necessary: the reason is that a given broken trajectory connecting y to y' might fall in sets of measure zero on which the solution of the transport equation is not defined. Hence it is important to deal with an open neighborhood of one such trajectory.

With the characterization of the set of couples (y, y') such that $k(y, y') = 0$ provided by Proposition 4, the construction above proves our claim. A rapid inspection proves that it suffices to take $n^* \leqslant 8$. ∎

Lemma 8. The function k defined in Proposition 4 has the following properties:

$$\int_{\partial Y} k(y, y')\, dy' = \int_{\partial Y} k(y, y')\, dy = 1 \qquad \text{for a.e. } y, y' \in \partial Y \tag{4.6}$$

Proof. First observe that $K1 = \frac{1}{2} \int_{v \cdot n_y < 0} |v \cdot n_y|\, dv = 1$ for all $y \in \partial Y$. To prove the second equality, let $\phi \in C(\partial Y)$ be arbitrary and consider F the solution of the boundary value problem for the transport equation in Y:

$$v \cdot \nabla_y F = 0, \qquad y \in Y, \quad v \in S^1 \tag{4.7}$$

$$F(y, v) = \phi(y), \qquad (y, v) \in \Gamma_+ \tag{4.8}$$

Integrating both sides of the transport equation above over $Y \times S^1$, one finds

$$\begin{aligned}
&\iint_{Y\times S^1} \nabla_y \cdot (vF(y, v))\, dy\, dv \\
&\quad = \iint_{\partial Y\times S^1} F(y, v)\, v\cdot n_y\, dy\, dv \\
&\quad = \iint_{\Gamma_+} \phi(y)\, v\cdot n_y\, dy\, dv - \iint_{\Gamma_-} \phi(y - t_{y,v} v)\, |v\cdot n_y|\, dy\, dv = 0
\end{aligned}$$

which can, be recast as

$$\int_{\partial Y} \phi(y)\, dy - \int_{\partial Y} (K\phi)(y)\, dy = 0$$

Since this last relation holds for any $\phi \in C(\partial Y)$, it proves Lemma 8. ▮

To finish the proof of uniqueness, one proceeds as follows. First, (4.4) implies that

$$\Theta_{0_+} = K^{n^*}\Theta_{0_+} \tag{4.9}$$

It follows from Lemma 8 that

$$\int_{\partial Y} k_{n^*}(y, y')\, dy = \int_{\partial Y} k_{n^*}(y, y')\, dy' = 1$$

Hence, multiplying (4.6) by Θ_{0_+} (seen as a function on ∂Y) and integrating over ∂Y gives

$$\begin{aligned}
&\int_{\partial Y} \Theta_{0_+}(y)^2\, dy - \iint_{\partial Y\times\partial Y} k_{n^*}(y, y')\, \Theta_{0_+}(y)\, \Theta_{0_+}(y')\, dy\, dy' \\
&\quad = \iint_{\partial Y\times\partial Y} k_{n^*}(y, y')\, \Theta_{0_+}(y)^2\, dy\, dy' \\
&\qquad - \iint_{\partial Y\times\partial Y} k_{n^*}(y, y')\, \Theta_{0_+}(y)\, \Theta_{0_+}(y')\, dy\, dy' \\
&\quad = \tfrac{1}{2} \iint_{\partial Y\times\partial Y} k_{n^*}(y, y')(\Theta_{0_+}(y) - \Theta_{0_+}(y'))^2\, dy\, dy' = 0
\end{aligned}$$

Since, by Lemma 7, one knows that $k_{n^*} > 0$ on $\partial Y \times \partial Y$, the equality above shows that Θ_{0_+} is almost everywhere equal to a constant, say C. Then Θ_0

is the unique solution of the boundary value problem for the transport equation in Y, (4.7)–(4.8), with ϕ replaced with the constant C, that is, $\Theta_0 = C$ a.e. Since on the other hand one knows that

$$\iint_{Y \times S^1} \Theta_0(y, v)\, dy\, dv = 0$$

one deduces that $\Theta_0 = C = 0$ a.e. This completes the proof of the uniqueness part of Lemma 2.

Existence. Let $S \in L^\infty$ satisfy the orthogonality relation

$$\iint_{Y \times S^1} S(y, v)\, dy\, dv = 0 \tag{4.10}$$

Let $\lambda > 0$ and Θ_λ be the unique solution in $L^\infty(Y \times S^1)$ of

$$\lambda \Theta_\lambda + v \cdot \nabla_y \Theta_\lambda = S, \qquad y \in Y, v \in S^1 \tag{4.11}$$

$$(\Theta_\lambda)_+ (y, v) = \tfrac{1}{2} \int_{v' \cdot n_y < 0} |v' \cdot n_y| \, (\Theta_\lambda)_- (y, v')\, dv', \qquad (y, v) \in \Gamma_+ \tag{4.12}$$

By the maximum principle,

$$\|\Theta_\lambda\|_{L^\infty(Y \times S^1)} \leqslant \frac{\|S\|_{L^\infty(Y \times S^1)}}{\lambda} \tag{4.13}$$

In order to take the limit of Θ_λ in (4.11)–(4.12) as λ goes to 0, we first prove that $\|\Theta_\lambda\|_{L^\infty(Y \times S^1)}$ is bounded as λ goes to 0. Otherwise there would exist a sequence $\lambda_n \to 0$ such that

$$\lim_{n \to +\infty} \|\Theta_{\lambda_n}\|_{L^\infty(Y \times S^1)} = +\infty$$

Let

$$\Phi_\lambda = \frac{\Theta_\lambda}{\|\Theta_\lambda\|_{L^\infty(Y \times S^1)}}$$

so that Φ_λ satisfies

$$\lambda \Phi_\lambda + v \cdot \nabla_y \Phi_\lambda = \frac{S}{\|\Theta_\lambda\|_{L^\infty(Y \times S^1)}}, \qquad (y, v) \in Y \times S^1 \tag{4.14}$$

and

$$(\Phi_\lambda)_+ (y, v) = \tfrac{1}{2} \int_{v' \cdot n_y < 0} |v' \cdot n_y| (\Phi_\lambda)_- (y, v')\, dv', \qquad y \in \partial Y, \quad v \cdot n_y > 0 \tag{4.15}$$

After extraction of a subsequence if necessary, the Banach–Alaoglu theorem implies the existence of $\Phi \in L^\infty(Y \times S^1)$ such that

$$\Phi_{\lambda_n} \to \Phi \text{ weakly in } L^p(Y \times S^1, dy\, dv) \qquad \text{for all} \quad 1 < p < +\infty \tag{4.16}$$

Cessenat's trace theorem[12] and the boundedness of $t_{y,v}$ [resulting from the finite horizon hypothesis (2.2)] show that

$$(\Phi_{\lambda_n})_\pm \to \Phi \text{ weakly in } L^p(\Gamma_\pm, |v \cdot n_y|\, dy\, dv) \qquad \text{for all} \quad 1 < p < +\infty \tag{4.17}$$

Hence, taking the limit as $\lambda_n \to +\infty$ in (4.14)–(4.15) leads to

$$v \cdot \nabla_y \Phi = 0, \qquad (y, v) \in Y \times S^1 \tag{4.14'}$$

$$\Phi_+(y, v) = \tfrac{1}{2} \int_{v' \cdot n_y < 0} |v' \cdot n_y|\, \Phi_-(y, v')\, dv', \qquad y \in \partial Y, \quad v \cdot n_y > 0 \tag{4.15'}$$

The uniqueness part of Lemma 2 shows that Φ is a constant; since one also has for all n, because of the orthogonality relation,

$$\iint_{Y \times S^1} \Phi_{\lambda_n}(y, v)\, dy\, dv = 0$$

it follows from the weak convergence (4.16) that

$$\Phi = 0 \tag{4.18}$$

On the other hand, for any $\lambda > 0$ and $(y, v) \in \Gamma_+$,

$$(\Phi_{\lambda_n})_+ (y, v) = \mu_n(y) = A_{\lambda_n}(y) + B_{\lambda_n}(y) + \tfrac{1}{2}(K(\Phi_{\lambda_n})_+)(y) \tag{4.19}$$

where

$$\left\{ \begin{aligned} A_{\lambda_n}(y) &= \frac{1}{2} \int_{v' \cdot n_y < 0} |v' \cdot n_y| \left(\int_0^{t_{y,v'}} e^{-\lambda_n t} \frac{S(y - tv', v')}{\|\Theta_{\lambda_n}\|_{L^2}}\, dt \right) dv' \\ B_{\lambda_n}(y) &= \frac{1}{2} \int_{v' \cdot n_y < 0} |v' \cdot n_y| \left[\exp^{(-\lambda_n t_{y,v'})} - 1 \right] \\ &\quad \times (\Phi_{\lambda_n})_+ (y - t_{y,v'} v', v')\, dv' \end{aligned} \right. \tag{4.20}$$

To derive (4.19)–(4.20), it suffices to use relation (4.15) and to integrate (4.14) backward on each particle trajectory (or characteristic line) until the previous point where the trajectory meets ∂Y.

Proposition 4 implies that

$$\|K(\Phi_{\lambda_n})_+\|_{L^\infty(\partial Y)} \leqslant C' \, \|\partial(\Phi_{\lambda_n})\|_{L^p(\partial Y)}$$

with

$$C' = \|k\|_{L^\infty(\partial Y \times \partial Y)} \text{ meas } (\partial Y)^{(p-1)/p} \tag{4.21}$$

Then, the following bounds on A_{λ_n} and B_{λ_n} hold:

$$\begin{cases} \|A_{\lambda_n}\|_{L^\infty(\Gamma_+)} \leqslant \dfrac{C}{\|\Theta_{\lambda_n}\|_{L^\infty(Y \times S^1)}} \dfrac{1}{\lambda_n}(1 - e^{-\lambda_n T}) \\ \|B_{\lambda_n}\|_{L^\infty(\Gamma_+)} \leqslant C'(1 - e^{-\lambda_n T}) \, \|(\Phi_{\lambda_n})\|_{L^p(\partial Y)} \end{cases} \tag{4.22}$$

where C' is defined in (4.21), C and T being other positive constants. One can take, for instance,

$$T = \sup\{t_{y,v} \text{ s.t. } (y, v) \in \Gamma_-\}$$

It follows in particular from (4.22) that A_{λ_n} and B_{λ_n} converge to 0 uniformly on ∂Y as $n \to +\infty$.

Next, (4.17)–(4.18) show that

$$(\Phi_{\lambda_n})_+ \to 0 \text{ weakly in } L^p(\partial Y), \qquad \text{for all} \quad 1 < p < +\infty \tag{4.23}$$

and, by using Proposition 4, (4.21), and the dominated convergence theorem, one gets

$$K(\Phi_{\lambda_n})_+ \to 0 \quad \text{in the norm topology of } L^p(\partial Y), \qquad \text{for all} \quad 1 < p < +\infty \tag{4.24}$$

The relations (4.19) and (4.24) prove that the convergence in (4.23) holds in the norm topology. Then, using (4.21) and (4.19) again shows that

$$(\Phi_{\lambda_n})_+ \to 0 \qquad \text{in} \quad L^\infty(\partial Y) \tag{4.25}$$

The maximum principle for the limiting transport equation (4.14′) in $Y \times S^1$ shows that $\Phi_{\lambda_n} \to 0$ in $L^\infty(Y \times S^1)$ as $n \to +\infty$. But this contradicts the property $\|\Phi_{\lambda_n}\|_{L^\infty} = 1$, which follows directly from the construction of Φ_λ.

The discussion above shows that the existence of a subsequence $\lambda_n \to +\infty$ such that

$$\|\Phi_{\lambda_n}\|_{L^\infty} \to +\infty$$

is contradictory. Hence, there exists a constant $C>0$ such that

$$\|\Phi_\lambda\|_{L^\infty} \leqslant C$$

It follows again from the Banach–Alaoglu theorem that one can extract from this family a subsequence converging to a solution of (2.5)–(2.6) for the weak-* topology of $L^\infty(Y\times S^1)$.

This concludes the proof of Lemma 2. ∎

5. THE DIFFUSION COEFFICIENT

In this section, we shall prove Proposition 3.

First, let us show that D is a scalar matrix. To begin with, D is symmetric by construction. Moreover, observe that since the lattice $\mathscr{L}$ is invariant under the rotation of angle $\pi/3$, the matrix D must commute with

$$\mathscr{R} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

One can check with a direct calculation that the only symmetric matrices commuting with $\mathscr{R}$ are scalar, that is, of the form dI. Necessarily, d is then given by (2.10).

Next, it is also relatively easy to show that $d\geqslant 0$, or equivalently, that D is a nonnegative matrix. Indeed, multiplying (2.8) with $j=1$ by γ_1 and using (2.9) leads to

$$\begin{aligned}
&\iint_{Y\times S^1} v_1\gamma_1(y, v)\, dy\, dv \\
&\quad = -\frac{1}{2}\iint_{\partial Y\times S^1} \gamma_1(y, v)^2\, v\cdot n_y\, dy\, dv \\
&\quad = \frac{1}{2}\iint_{\Gamma_-} \gamma_1(y, v)^2\, |v\cdot n_y|\, dy\, dv - \frac{1}{2}\iint_{\Gamma_+} \gamma_1(y, v)^2\, v\cdot n_y\, dy\, dv
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2} \iint_{\Gamma_-} \gamma_1(y, v)^2 \, |v \cdot n_y| \, dy \, dv \\
&\quad - \tfrac{1}{2} \iint_{\Gamma_+} \left(\tfrac{1}{2} \int_{v' \cdot n_y < 0} \gamma_1(y, v') \, |v' \cdot n_y| \, dv' \right)^2 |v \cdot n_y| \, dy \, dv \\
&= \int_{\partial Y} \left(\tfrac{1}{2} \int_{v \cdot n_y < 0} \gamma_1(y, v)^2 \, |v \cdot n_y| \, dv \right) dy \\
&\quad - \int_{\partial Y} \left(\tfrac{1}{2} \int_{v \cdot n_y < 0} \gamma_1(y, v) \, |v \cdot n_y| \, dv \right)^2 dy \\
&\geqslant 0 \qquad (5.1)
\end{aligned}
$$

by the Cauchy–Schwarz inequality. Likewise, one shows that

$$\iint_{Y \times S^1} v_2 \gamma_2(y, v) \, dy \, dv \geqslant 0$$

and this proves that $d \geqslant 0$.

Finally, let us prove that $d > 0$, which is slightly more delicate. If $d = 0$, the last inequality in (5.1) would be an equality, which means that there exists a function $g \in L^\infty(\partial Y)$ such that

$$(\gamma_1)_- (y, v) = g(y), \qquad \text{for a.e.} \quad (y, v) \in \partial Y \times S^1 \tag{5.2}$$

[First, one obtains this equality on Γ_- and then extends it to $\partial Y \times S^1$ by using (2.9)]. On the other hand, if one uses the method of characteristics to solve (2.8), one finds that

$$\gamma_1(y, v) = \gamma_1(y - t_{y,v} v) + t_{y,v} v_1, \qquad \text{for a.e.} \quad (y, v) \in Y \times S^1 \tag{5.3}$$

and in particular, applying this last equality to (y, v) and $(y, -v)$, one sees that (5.3) holds for almost every $(y, v) \in \Gamma_-$. Therefore, applying (5.2), one has

$$g(y) = g(y - t_{y,v} v) + t_{y,v} v_1, \qquad \text{for a.e.} \quad (y, v) \in \partial Y \times S^1 \tag{5.4}$$

Using, the same argument as in the proof of Lemma 7, we can construct a finite sequence $z_0, ..., z_n \in \partial X$ such that:

- The open segment $]z_j, z_{j+1}[\subset X$ for all $0 \leqslant j \leqslant n$ and $z_n = z_0 + (1, 0)$.

- One has the relations

$$g(z_0) = g(z_n), \qquad g(z_{j+1}) = g(z_j) + (z_{j+1} - z_j) \cdot (1, 0), \qquad 0 \leqslant j \leqslant n \quad (5.5)$$

Summing all these equalities leads to the contradiction

$$g(z_0) = g(z_n) = g(z_0) + (z_n - z_0) \cdot (1, 0) = g(z_0) + 1$$

Hence the relation (5.2) cannot hold, which means that $d > 0$. ∎

6. PROOF OF THEOREM 1

Consider the following multiscale asymptotic expansion of f_ε:

$$f_\varepsilon(t, x, v) = f^{(0)}(t, x, y, v) + \varepsilon f^{(1)}(t, x, y, v) + \varepsilon^2 f^{(2)}(t, x, y, v) + \cdots \big|_{y = x/\varepsilon}$$

where $f^{(k)}(t, \cdot, v)$ is defined on $\mathbf{R}^2 \times Y$. Identifying the coefficients of the successive powers of ε in (1.5) leads to the following hierarchy of equations:

$$v \cdot \nabla_y f^{(0)} = 0 \quad (6.1)$$

$$v \cdot \nabla_y f^{(1)} + v \cdot \nabla_x f^{(0)} = 0 \quad (6.2)$$

$$v \cdot \nabla_y f^{(2)} + v \cdot \nabla_x f^{(1)} + \partial_t f^{(0)} = 0 \quad (6.3)$$

for any $(t, x, y, v) \in \mathbf{R}_+ \times \mathbf{R}^2 \times Y \times S^1$, with the boundary condition

$$f^{(k)}(t, x, y, v) = \tfrac{1}{2} \int_{v' \cdot n_y < 0} |v' \cdot n_y| \, f^{(k)}(t, x, y, v') \, dv', \qquad (y, v) \in \Gamma_+ \quad (6.4)$$

Equations (6.1) and (6.4) and Lemma 2 show that

$$f^{(0)}(t, x, y, v) \equiv F^{(0)}(t, x) \quad (6.5)$$

is independent of (y, v). Equations (6.2) and (6.4) and Lemma 2 show that

$$f^{(1)}(t, x, y, v) = -\gamma(y, v) \cdot \nabla_x F^{(0)}(t, x) \quad (6.6)$$

where the vector $\gamma = (\gamma_1, \gamma_2)$ is defined in (2.8)–(2.9). Moreover, (6.3)–(6.4) has a unique solution if and only if

$$\iint_{Y \times S^1} (v \cdot \nabla_x f^{(1)} + \partial_t f^{(0)}) \, dy \, dv = 0 \quad (6.7)$$

If one substitutes the expressions (6.5)–(6.6) in (6.7), one finds exactly the diffusion equation (2.3) for $F^{(0)}$. In the sequel, $F^{(0)}$ is assumed to satisfy also the initial condition (2.4) (a prescription which defines a unique $F^{(0)}$).

Introduce now the remainder $R_\varepsilon(t, x, v)$ for any $(t, x, v) \in \mathbf{R}_+ \times X_\varepsilon \times S^1$ defined as

$$R_\varepsilon(t, x, v) = f_\varepsilon(t, x, v) - F^{(0)}(t, x) - \varepsilon f^{(1)}\left(t, x, \frac{x}{\varepsilon}, v\right) - \varepsilon^2 f^{(2)}\left(t, x, \frac{x}{\varepsilon}, v\right) \tag{6.8}$$

Then

$$\partial_t R_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x R_\varepsilon = -\varepsilon\, \partial_t f^{(1)} - \varepsilon^2\, \partial_t f^{(2)} - \varepsilon v \cdot \nabla_x f^{(2)} \tag{6.9}$$

$$R_\varepsilon(0, x, v) = -\varepsilon f^{(1)}\left(0, x, \frac{x}{\varepsilon}, v\right) - \varepsilon^2 f^{(2)}\left(0, x, \frac{x}{\varepsilon}, v\right) \tag{6.10}$$

$$R_\varepsilon(t, x, v) = \tfrac{1}{2} \int_{v' \cdot n_x < 0} |v' \cdot n_x|\, R_\varepsilon(t, x, v')\, dv' \qquad (x, v) \in \Gamma_+ \tag{6.11}$$

Applying the maximum principle to the diffusion equation (2.3) with initial data $\nabla^m \Phi \in L^\infty(\mathbf{R}^2)$, $m \in \{0, 1, 2, 3, 4\}$, we first show that $\partial_t f^{(1)}(t, \cdot)$, $\partial_t f^{(2)}(t, \cdot)$, and $v \cdot \nabla_x f^{(2)}(t, \cdot)$ are all bounded in $L^\infty(\mathbf{R}^2 \times S^1)$ uniformly on every compact t-set. Notice that we use here the fact that $\gamma_j \in L^\infty(Y \times S^1)$ for $j = 1, 2$. Since $R_\varepsilon(0, x, v) = O(\varepsilon)$ in $L^\infty(X_\varepsilon \times S^1)$, the maximum principle for the transport equation (6.9) with the boundary condition (6.11) gives the desired order of approximation. ▮

7. CONCLUSION

We conclude this paper with some remarks on the proof given above as well as indications of future work and possible open problems.

First, it is fairly clear that the above strategy would apply to any such billiard system with finite horizon, independently of the dimension. We chose the two-dimensional case only to simplify the change of variables in Section 3.

With some nontrivial modifications, the same strategy as above can be used when the total accommodation condition (1.2) is replaced by the partial accommodation condition defined as[11]

$$\begin{aligned} f_+(t, x, v) &= (1 - \alpha)\, f_-(t, x, v - 2(v \cdot n_x)\, n_x) \\ &\quad + \alpha \tfrac{1}{2} \int_{v' \cdot n_x < 0} f_-(t, x, v')\, |v' \cdot n_x|\, dv \end{aligned} \tag{7.1}$$

with $\alpha \in]0, 1[$. The case $\alpha = 0$ corresponds to the work of Bunimovich *et al.* and cannot be treated by the above method. This type of result will be discussed in a forthcoming article.[14]

Once this result is established, the following question seems extremely natural: denote by D_α the diffusion coefficient corresponding to the boundary condition (7.1) for $\alpha \in]0, 1[$ and by D_0 the diffusion coefficient found by Bunimovich *et al.*[7,9] Does one have $D_\alpha \to D_0$ as $\alpha \to 0$? This seems likely, but such a result would probably require using part of the machinery in refs. 6 and 8 in addition to the methods and results described in the present paper and in ref. 14. It might be that such a result would in fact be slightly easier to establish than the one in refs. 7 and 9, although one cannot be completely sure as of now.

REFERENCES

1. C. Bardos, L. Dumas, and F. Golse, *C. R. Acad. Sci. Paris II* **315**:1433–1437 (1992).
2. C. Bardos, F. Golse, and D. Levermore, *J. Stat. Phys.* **63**:323–344 (1991).
3. C. Bardos, F. Golse, and D Levermore, *Commun. Pure Appl. Math.* **46**:667–753 (1993).
4. C. Bardos and S. Ukai, *Math. Meth. Models Appl. Sci.* **1**:235–257 (1991).
5. A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Study of Periodic Structures* (North Holland, Amsterdam, 1978).
6. L. Bunimovich and Ya. Sinai, *Commun. Math. Phys.* **78**:247–280 (1980).
7. L. Bunimovich and Ya. Sinai, *Commun. Math. Phys.* **78**:479–497 (1981).
8. L. Bunimovich, Ya. Sinai, and N. Chernov, *Russ. Math. Surv.* **45**:105–152 (1990).
9. L. Bunimovich, Ya. Sinai, and N. Chernov, *Russ. Math. Surv.* **46**(4):47–106 (1991).
10. R. Caflisch, *Commun. Pure Appl. Math.* **33**:651–666 (1980).
11. C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases* (Springer-Verlag, New York, 1994).
12. M. Cessenat, *C. R. Acad. Sci. Paris I* **299**:831–834 (1984).
13. A. De Masi, R. Esposito, and J. Lebowitz, *Commun. Pure Appl. Math.* **42**:1189–1214 (1990).
14. L. Dumas, In preparation.
15. A. Garrett, In *From Statistical Physics to Statistical Inference and Back*, P. Grassberger and J.-P. Nadal, eds. (1994), pp. 45–75.
16. J. Lebowitz, *Physica A* **194**:1–27 (1993).
17. T. Nishida, *Commun. Math. Phys.* **61**:119–148 (1978).