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Diffusion approximation in heterogeneous media

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Abstract. We investigate the behaviour of the solutions of kinetic equations as a parameter related to both the mean free path and a characteristic length of heterogeneities goes to 0. We obtain in the limit a diffusion equation for the macroscopic density which combines the homogenization effects to the diffusion approximation. In particular the limit equation contains drift terms related to the behaviour of the kernel of the collision operator.

1. Introduction

This paper is a contribution to the study of the behaviour of solutions of kinetic equations with respect to certain small parameters. Such kind of questions naturally arises in various fields of statistical physics like, for instance, in nuclear engineering, in radiative tranfer or in semiconductors physics. Generally speaking, we consider a cloud of particles (which are neutrons, electrons, ions, photons, gas molecules and so on . . .) described through a distribution function $f(t,x,v)$. While $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ stand for time and space variables, respectively, v represents the degrees of freedom of the particles. Usually, v is nothing but the translational velocity but it can also be related to internal degrees of freedom. This variable lies in a certain measured space $(V, \mathrm{d}\mu(v))$ so that

$$\int_{\mathcal{V}} f(t,x,v)\,\mathrm{d}\mu(v)$$

is the density of particles having, at time t and position x, the parameter v in the (measurable) domain $\mathcal{V} \subset V$. Modelization of the evolution of the cloud of particles leads to the following kinetic equation

$$\partial_t f + a(v)\cdot\nabla_x f = Q(f).$$

The left-hand side describes the transport of particles whereas the right-hand side is related to various interaction phenomena. In absence of interactions, we have $Q = 0$ and the particles involve freely on the characteristic lines $x + ta(v)$; $Q(f) \neq 0$ describes interparticles collisions and interaction with the media in which the particles involve. The function $a : V \to \mathbb{R}^N$ is the kinetic velocity of the particles, which is

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assumed to be given. The classical framework is given by $V = \mathbb{R}^N$ or $V = S^{N-1}$ and $a(v) = v$, but we can also consider relativistic particles $a(v) = v/\sqrt{1+v^2}$, or cases, in neutronic applications, where V is a reunion of spheres. Another framework of interest is the case of discrete velocities where V is a set of indices and $\mathrm{d}\mu(v)$ is the associated discrete measure. It is worth pointing out that our study also covers this case.

Some characteristic lengths have to be considered in the model. Comparison of these length scales to a macroscopic length of reference L (for instance the size of the domain in which the phenomenom under consideration occurs), leads to asymptotic questions. Here, we deal with two characteristic length scales: λ is the mean free path of the particles and ℓ is related to inhomogeneities in the media. Variations of the physical properties of the media translates naturally into oscillations in the coefficients of the interaction operator Q. We set $\varepsilon = \lambda/L$, $\delta = \ell/L$ and we consider the evolution of the system on time scale of order $1/\varepsilon$. We are thus interested in the following penalized problem

$$\varepsilon\partial_t f + a(v)\cdot\nabla_x f = \frac{1}{\varepsilon}Q_\delta(f)$$

and we address the question of the behaviour of the solutions with respect to both ε and δ. For example, a nuclear reactor can be viewed as a very complex and highly heterogeneous assembly of different materials in which neutrons involve with a relatively small mean free path. Therefore, determination of asymptotic models is a relevant question in this field.

We do not enter further in the details, instead we refer to the review of Golse [20] and to the classical treatises of Cercignani [13] and Dautray and Lions [14]. Study of the problem $\varepsilon \to 0$ with δ fixed is now classical: we obtain a diffusion equation satisfied by the macroscopic density $\rho(t,x)$, limit of $\rho_\varepsilon(t,x) = \int_V f_\varepsilon(t,x,v)\,\mathrm{d}\mu(v)$. This fact is known as the Rosseland approximation: see Bardos et al. [7–9], Degond et al. [15], Lions and Toscani [26]. For application specialized to the field of semiconductors physics and the Pauli equation, we quote Poupaud [31], Golse and Poupaud [22], Mellet [28]. On the other hand, as δ becomes small with ε fixed, we keep a kinetic description involving an homogeneized interaction operator as shown in Dumas and [16], Gérard and Golse [17], Golse [18,19]. Here, we consider the critical case where ε and δ have the same order. This combines the effects of the diffusion approximation to homogenization aspects. This situation is presented in the seminal papers by Wigner [36], Bensoussan et al. [10], Larsen and Keller [24], Larsen and Williams [25]. Recently, Allaire and Bal [2,5] have studied the related spectral problem, having in mind applications in nuclear enginneering (for which we also refer to Malvagi et al. [27]), while the homogenization of the diffusive approximation is performed by Allaire and Capdeboscq [3,11,12]. The approach of these works is very close to our, in particular with the use of double scale limit arguments, following Allaire [1], Nguetseng [29].

According to the scaling introduced above, we shall investigate the behaviour of sequence f_ε of solutions of

$$\begin{cases} \partial_t f_\varepsilon + \dfrac{1}{\varepsilon}a(v)\cdot\nabla_x f_\varepsilon = \dfrac{1}{\varepsilon^2}Q_\varepsilon(f_\varepsilon) \quad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x^N \times V_v, \\ f_\varepsilon|_{t=0} = f_{0,\varepsilon} \quad \text{in } \mathbb{R}_x^N \times V_v. \end{cases} \tag{1}$$

We particularly focus on linear situation where the interaction term reads

$$\begin{cases} Q_\varepsilon(f) = K_\varepsilon(f) - \Sigma_\varepsilon(x,v)f, \\ K_\varepsilon(f) = \int_V \sigma_\varepsilon(x,v,w)f(w)\,\mathrm{d}\mu(w). \end{cases}$$

The functions σ_ε and Σ_ε are naturally nonnegative; since they depend on the media, they are varying on the scale of the inhomogeneities which leads to oscillations in the space variable. We will describe more precisely the assumptions on these coefficients later, with in particular the introduction of suitable periodicity assumptions. Furthermore, we restrict to conservative operators which means that, up to integrability questions, one has

$$\int_V Q_\varepsilon(f)\,\mathrm{d}\mu(v) = 0 \tag{2}$$

for all distribution function f. We are thus led to assume that the following relation holds

$$\Sigma_\varepsilon(x,v) = \int_V \sigma_\varepsilon(x,w,v)\,\mathrm{d}\mu(w).$$

Of course, further assumptions, more or less technical, will be necessary; they are precised in Section 3 below. Hence, this work is a continuation of the first attempts presented in [23,15]. Clearly, the kernel of the interaction operator is of crucial importance in this kind of problem: it determines the first term in a formal expansion of the solution. Hence, the hypothesis we will need are mainly concerned with Ker(Q_ε). In [23] the situation was quite simple since Ker(Q_ε) $= \mathbb{R}$ does not depend on ε (but nonlinearities are also dealt with). The usual assumption concerning the cross section (see [13,31] for instance) is the so-called detailed balance principle (also known as the microreversibility condition):

$$\sigma(v,w)M(v) = \sigma(w,v)M(w),$$

where $M(v)$ is the Maxwellian function. In such a case, the steady states are obviously Maxwellian functions: the collision integral $Q(M)$ vanishes since the integrand vanishes. In [15], homogenization aspects are neglected (Q does not depend on ε) but the detailed balance hypothesis is removed; it is only assumed the existence of a space dependent function $F(x,v) > 0$ in the kernel of the collision operator. This space dependence produces effects through an additional drift term in the limit equation. Here, our aim is to point out, in some quite simple situation, the combination of these effects to homogenization, considering cases where Ker(Q_ε) $=$ Span$\{F_\varepsilon\}$.

One could also improve the model by adding in the Boltzmann equation a force term $\frac{1}{\varepsilon}\nabla_x V_\varepsilon(x)\cdot\nabla_v f_\varepsilon$, with an electric potential V_ε depending on ε. Such a situation has been investigated by Tayeb in [35], when the electric potential reads $V_\varepsilon(x) = V_1(x) + V_2(x/\varepsilon)$, and $V_2(y)$ has a periodic behaviour.

This paper is organized as follows: first, we guess the result by inserting a formal double scale expansion of the solution in (1). Then, in Section 3, we will precise the assumptions on the collision operator and then, we will be able to give the statement of our main result. Section 4 is devoted to a priori estimates on the sequence f_ε. Finally, in Section 5, we achieve the proof of the convergence result. Auxiliary results are postponed in the appendix.

2. Formal asymptotics

We can assume as a first approximation that oscillations of the physical properties of the media enjoy some periodicity property. Let us expand, at least formally, the oscillating kernel σ_ε as a double scale power series; namely we set

$$\sigma_\varepsilon(x,v,w) = \sum_{k=0}^{\infty} \varepsilon^k \sigma^{(k)}\left(x, \frac{x}{\varepsilon}, v, w\right),$$

where, for each k, $\sigma^{(k)}(x,y,v,w)$ is nonnegative and Y-periodic with respect to the variable y. Here and below Y stands for $[0,1]^N$. Therefore, we naturally introduce the sequence of operators $\mathcal{L}^{(k)}$ associated to these kernels and acting on functions of y and v as follows

$$\mathcal{L}^{(k)}(\Phi)(y,v) = \int_{\mathbb{R}^N} \sigma^{(k)}(x,y,v,w)\,\Phi(y,w)\,\mathrm{d}\mu(w) - \int_{\mathbb{R}^N} \sigma^{(k)}(x,y,w,v)\,\mathrm{d}\mu(w)\,\Phi(y,v).$$

Then, we also expand the solution as a formal series

$$f_\varepsilon(t,x,v) = \sum_{k=0}^{\infty} \varepsilon^k \mathcal{F}^{(k)}\Big(t,x,\frac{x}{\varepsilon},v\Big).$$

Inserting this ansatz into (1) and identifying the terms arising with the same power of ε yield

$$\begin{aligned} &\varepsilon^{-2} \text{ term:} \qquad a(v)\cdot\nabla_y \mathcal{F}^{(0)} - \mathcal{L}^{(0)}\big(\mathcal{F}^{(0)}\big) = 0,\\ &\varepsilon^{-1} \text{ term:} \qquad a(v)\cdot\nabla_y \mathcal{F}^{(1)} - \mathcal{L}^{(0)}\big(\mathcal{F}^{(1)}\big) = \mathcal{L}^{(1)}\big(\mathcal{F}^{(0)}\big) - a(v)\cdot\nabla_x \mathcal{F}^{(0)}. \end{aligned}$$

Note that the $\mathcal{L}^{(1)}$ term is unusual in this kind of problem; it will induce an additional drift term in the limit equation. Next, we have

$$\varepsilon^{0} \text{ term:} \qquad a(v)\cdot\nabla_y \mathcal{F}^{(2)} - \mathcal{L}^{(0)}\big(\mathcal{F}^{(2)}\big) = \mathcal{L}^{(1)}\big(\mathcal{F}^{(1)}\big) - a(v)\cdot\nabla_x \mathcal{F}^{(1)} + \mathcal{L}^{(2)}\big(\mathcal{F}^{(0)}\big) - \partial_t \mathcal{F}^{(0)}.$$

Integrating this equation with respect to both y and v provides the following relation between $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$

$$\partial_t\bigg(\int_Y\int_V \mathcal{F}^{(0)}\,\mathrm{d}y\,\mathrm{d}\mu(v)\bigg) + \mathrm{div}_x\bigg(\int_Y\int_V a(v)\mathcal{F}^{(1)}\,\mathrm{d}y\,\mathrm{d}\mu(v)\bigg) = 0,$$

which will actually produce the limit drift diffusion equation. Equations for ε^{-2} and ε^{-1} have the general form

$$a(v)\cdot\nabla_y \mathcal{F} - \mathcal{L}^{(0)}(\mathcal{F}) = H$$

with periodic boundary condition. This is a cell problem where the relevant variables are y and v while t and x appear as parameters (in the definition of the right-hand side H). It can be easily seen that, if a solution $\mathcal{F}$ exists, then, H should be of null $y,\ v$-average (integrate the equation to see this fact). We immediately remark that the right-hand side for the ε^{-2} equation fulfills this condition as well as the first part of the ε^{-1} equation. Let $\overline{\mathcal{F}^{(0)}}$ be a normalized solution of the ε^{-2} equation and let $\phi,\ \chi_i,\ \psi$ be solution of the following cell problems

$$\begin{cases} a(v)\cdot\nabla_y \chi_i - \mathcal{L}^{(0)}(\chi_i) = -a_i(v)\,\overline{\mathcal{F}^{(0)}} \quad \text{for } i\in\{1,\ldots,N\},\\ a(v)\cdot\nabla_y \phi - \mathcal{L}^{(0)}(\phi) = -a(v)\cdot\nabla_x \overline{\mathcal{F}^{(0)}},\\ a(v)\cdot\nabla_y \psi - \mathcal{L}^{(0)}(\psi) = \mathcal{L}^{(1)}(\overline{\mathcal{F}^{(0)}}). \end{cases}$$

In view of the previous remark, such equations make sense provided, at least, the compatibility condition

$$\int_Y \int_V a_i(v)\,\overline{\mathcal{F}^{(0)}}\,\mathrm{d}y\,\mathrm{d}\mu(v) = 0,$$

holds; this crucially motivates assumption (B3) below. Hence, denoting by $\rho(t,x)$ the average (with respect to v) of $\mathcal{F}^{(0)}$, which is expected to be the limit of $\rho_\varepsilon(t,x) = \int_V f_\varepsilon(t,x,v)\,\mathrm{d}\mu(v)$,

$$\mathcal{F}^{(1)} = \chi \cdot \nabla_x \rho + \phi\rho + \psi\rho$$

provides a solution of the ε^{-1} equation. Coming back to the integrated ε^0 equation, we are led to the following drift diffusion equation for the macroscopic density ρ

$$\begin{cases} \partial_t \rho - \mathrm{div}_x\big(D(x)\nabla_x\rho - (c+b)\rho\big) = 0, \\ D(x) = -\displaystyle\int_Y \int_V \chi \otimes a(v)\,\mathrm{d}y\,\mathrm{d}\mu(v), \\ c(x) = \displaystyle\int_Y \int_V a(v)\phi\,\mathrm{d}y\,\mathrm{d}\mu(v), \\ b(x) = \displaystyle\int_Y \int_V a(v)\psi\,\mathrm{d}y\,\mathrm{d}\mu(v). \end{cases}$$

We observe two drift terms: the former is related to space dependence of the equilibrium state $\overline{\mathcal{F}_0}$, as in [15], the latter is related to the fluctuation of the collision operator. Of course, this development is purely formal. In particular, it assumes the solvability of the cell problems, which is far from being garanteed. We refer for instance to Bardos et al. [6] for some counter-examples. Actually, solvability relies on the application of the Fredholm alternative; when it does not apply, we are led to different asymptotic behaviour as investigated in [2,11]. Before we give the statement of this expected convergence, let us now specify the assumptions on the interaction operator.

3. Interaction operator

3.1. Notation

As mentioned in the previous sections, we assume that the fluctuations of the collision kernel are mainly periodic; precisely, we set

$$\sigma_\varepsilon(x,v,w) = \sigma^{(0)}\left(x, \frac{x}{\varepsilon}, v, w\right) + \varepsilon\widehat{\sigma}_\varepsilon(x,v,w), \tag{3}$$

where we assume that $y \mapsto \sigma^{(0)}(x,y,v,w)$ is Y-periodic. We recall that $Y = [0,1]^N$. Hence, $\sigma^{(0)}$ defines a conservative linear operator $\mathcal{L}^{(0)}$ as follows

$$\begin{cases} \mathcal{L}^{(0)}(\Phi) = \displaystyle\int_V \sigma^{(0)}(x,y,v,w)\Phi(w)\,\mathrm{d}\mu(w) - \Sigma^{(0)}(x,y,v)\Phi(v) = \mathcal{K}^{(0)}(\Phi) - \Sigma^{(0)}\Phi, \\ \Sigma^{(0)}(x,y,v) = \displaystyle\int_V \sigma^{(0)}(x,y,w,v)\,\mathrm{d}\mu(w). \end{cases}$$

We shall denote by $Q_\varepsilon^{(0)}$ the associated operator evaluated with $y = x/\varepsilon$. Furthermore, $\widehat{Q_\varepsilon}$ stands for the conservative operator associated to $\widehat{\sigma}_\varepsilon(x, v, w)$,

$$\begin{cases} \widehat{Q_\varepsilon}(\phi) = \displaystyle\int_V \widehat{\sigma}_\varepsilon(x, v, w)\phi(w)\,\mathrm{d}\mu(w) - \widehat{\Sigma}_\varepsilon(x, v, w)\phi(v), \\ \widehat{\Sigma}_\varepsilon(x, v, w) = \displaystyle\int_V \widehat{\sigma}_\varepsilon(x, w, v)\,\mathrm{d}\mu(w). \end{cases}$$

We have also mentionned in the introduction that a crucial role is played by eigenfunctions F_ε of Q_ε. We suppose that the collision operator Q_ε has a kernel with dimension one, spanned by a positive function $F_\varepsilon(x, v)$. Such a property can be obtained as a consequence of the Krein–Rutman operator, see [32]. According to (3), main contribution in F_ε comes from eigenfunction of $\mathcal{L}^{(0)}$. Assuming that

$$\mathrm{Ker}\big(\mathcal{L}^{(0)}\big) = \mathrm{Span}\big\{F^{(0)}\big\}, \quad F^{(0)} > 0, \quad \int_V F^{(0)}\,\mathrm{d}\mu(v) = 1$$

(with $F^{(0)}$ a priori depending on $x,\ y,\ v$), we naturally expand

$$F_\varepsilon = F_\varepsilon^{(0)} + \varepsilon\widehat{F_\varepsilon},$$

with $F_\varepsilon^{(0)}(x, v) = F^{(0)}(x, x/\varepsilon, v)$. Therefore, $Q_\varepsilon(F_\varepsilon) = 0$ leads to the following relation between $F_\varepsilon^{(0)}$ and $\widehat{F_\varepsilon}$

$$-\widehat{Q_\varepsilon}\big(F_\varepsilon^{(0)}\big) = \big(Q_\varepsilon^{(0)} + \varepsilon\widehat{Q_\varepsilon}\big)\widehat{F_\varepsilon}. \tag{4}$$

In the next subsection, we will assume that $F^{(0)}$ exists (see (A2) below). The existence of $\widehat{F_\varepsilon}$ (and, as a consequence, those of F_ε) relies on the solvability of (4), that requires some connection between the collision kernels $\sigma^{(0)}$ and $\widehat{\sigma}_\varepsilon$ which will be investigated in Section 3.4. Hence, let us first detail the assumptions on the zeroth order operator and some consequences for the cell problem in Section 3.3.

3.2. Zeroth order operator

We introduce now the hypothesis on the differential cross section:

$$\begin{cases} \sigma^{(0)}(x, y, v, w) \text{ is measurable, positive, } Y\text{-periodic and} \\ \text{there exist a positive function } \Sigma(x, v) \text{ and a constant } \overline{\Sigma} > 0 \\ \text{such that } \Sigma^{(0)}(x, y, v) = \int_V \sigma^{(0)}(x, y, w, v)\,\mathrm{d}\mu(w) \text{ satisfies} \\ 0 < \Sigma(x, v) \leqslant \Sigma^{(0)}(x, y, v) \leqslant \overline{\Sigma}, \quad \mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}\mu(v)\text{-a.e.,} \end{cases} \tag{A1}$$

where "a.e." means "almost everywhere". Note that $\mathcal{L}^{(0)}$ satisfies the conservation property (2). The assumption on $\Sigma^{(0)}$ are certainly not optimal, but we prefer to avoid too much technicalities and keep as clear as possible our development. Notice also that $\Sigma^{(0)}$ can depend on the "macroscopic" variable x. In [2], such a situation is excluded (up to ε^2 perturbations; but we will introduce other restrictions ...) due to the use of factorization techniques, see some comments in Appendix D.

Our second assumption concerns the existence of an equilibrium:

$$\begin{cases} \text{There exists an a.e. positive measurable function } F^{(0)} \\ \text{which depends only on } x, v \text{ such that } \mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}\mu(v)\text{-a.e.} \\ \Sigma^{(0)}(x,y,v)F^{(0)}(x,v) = \int_V \sigma^{(0)}(x,y,v,w)F^{(0)}(x,w)\,\mathrm{d}\mu(w). \\ \text{Furthermore, there exists a positive constant } M \text{ such that } \mathrm{d}x\text{-a.e.} \\ \int_V \left(\frac{1}{\Sigma(x,v)} + \Sigma(x,v)\right)F^{(0)}(x,v)\,\mathrm{d}\mu(v) \leqslant M, \quad \int_V F^{(0)}(x,v)\,\mathrm{d}\mu(v) = 1. \end{cases} \tag{A2}$$

The last condition is only a normalization condition, while the existence of $F^{(0)}$ is for instance fulfilled if we can apply the Krein–Rutmann theorem, see [32]. Note that the condition $\int_V F^{(0)}(x,v)\Sigma(x,v)\,\mathrm{d}\mu(v) \leqslant M$ is already a consequence of (A1) and the normalization condition. Main restriction comes from the fact that, despite the dependence of $\sigma^{(0)}$ on the variable y, we assume that the generalized Maxwellian $F^{(0)}$ does not oscillate; this assumption will be crucial in the sequel. In particular, assumption (A2) immediately provides a solution for the ε^{-2} equation derived in the formal asymptotics: $\rho(t,x)F^{(0)}(x,v)$ satisfies it. We will also use (A2) to obtain some compactness properties.

Before we give more technical details, let us give the following simple example of a ε and space dependent linear Boltzmann operator

$$\begin{cases} \sigma_\varepsilon(x,v,w) = b\left(x, \frac{x}{\varepsilon}, v, w\right)M_\varepsilon(x,v), \\ b(x,y,v,w) = b(x,y,w,v), \quad y \mapsto b(x,y,v,w) \text{ is } Y\text{-periodic}, \\ M_\varepsilon(x,v) = \left(2\pi\left(\mathcal{T}^{(0)}(x) + \varepsilon\mathcal{T}\left(x, \frac{x}{\varepsilon}\right)\right)\right)^{-N/2} \exp\left(-\frac{|v - \varepsilon u(x, x/\varepsilon)|^2}{2(\mathcal{T}^{(0)}(x) + \varepsilon\mathcal{T}(x, x/\varepsilon))}\right), \end{cases}$$

which yields

$$\begin{cases} \sigma^{(0)}(x,y,v,w) = b(x,y,v,w)M^{(0)}(x,v), \\ M^{(0)}(x,v) = \left(2\pi\mathcal{T}^{(0)}(x)\right)^{-N/2} \exp\left(-\frac{v^2}{2\mathcal{T}^{(0)}(x)}\right), \\ \mathrm{Ker}\left(\mathcal{L}^{(0)}\right) = \mathrm{Span}\left\{M^{(0)}\right\}. \end{cases}$$

Let us temporarily drop x, y-dependence (but keeping in mind that the following properties hold uniformly with respect to these arguments). We introduce the functional spaces

$$H = \left\{f\colon V \to \mathbb{R},\ \mathrm{d}\mu \text{ measurable such that } f^2\frac{\Sigma^{(0)}}{F^{(0)}} \in L^1(\mathrm{d}\mu)\right\},$$

$$G = \left\{f\colon V \to \mathbb{R},\ \mathrm{d}\mu \text{ measurable such that } f^2\frac{1}{\Sigma^{(0)}F^{(0)}} \in L^1(\mathrm{d}\mu)\right\}.$$

The spaces H and G are Hilbert spaces endowed with the scalar products

$$(f,g)_H := \int_V f(v)g(v)\frac{\Sigma^{(0)}(v)}{F^{(0)}(v)}\,\mathrm{d}\mu(v), \qquad (h,k)_G := \int_V h(v)k(v)\frac{1}{\Sigma^{(0)}(v)F^{(0)}(v)}\,\mathrm{d}\mu(v),$$

respectively. We remark that for $g \in G$, and $h \in H$, the product $gh/F^{(0)}$ is integrable, as a matter of fact we have:

$$\left(g, \frac{h}{F^{(0)}}\right)_{L^2} = \left(g, \Sigma^{(0)} h\right)_G,$$

with $g,\ \Sigma^{(0)} h$ lying in G.

Remark 1. Thanks to assumption (A2), H and G both embed in $L^1(\mathrm{d}\mu)$, with

$$\|h\|_{L^1} \leqslant \sqrt{M}\|h\|_H, \qquad \|g\|_{L^1} \leqslant \sqrt{M}\|g\|_G.$$

Therefore it makes sense to introduce the following closed subspace

$$H_0 := \left\{ f \in H \text{ such that } \int_V f(v)\,\mathrm{d}\mu(v) = 0 \right\},$$

and G_0 can be defined similarly. Assumption (A2) also yields that $F^{(0)} \in H \cap G$. Finally, we remark that $G \subset H$ (nevertheless, the G-norm will appear as the adapted norm in the sequel, see Proposition 1).

Finally, one also needs the following technical hypothesis:

$$\begin{cases} \text{There exists a positive constant } \kappa \text{ such that } \mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}\mu(v) \otimes \mathrm{d}\mu(w)\text{-a.e.} \\ F^{(0)}(x,v) \leqslant \kappa\left(\Sigma^{(0)}(x,y,w) + \dfrac{1}{\Sigma^{(0)}(x,y,w)}\right) \dfrac{1}{\Sigma^{(0)}(x,y,v)}\,\sigma^{(0)}(x,y,v,w). \end{cases} \tag{A3}$$

Now, we recall from [15] the following claim:

Proposition 1 ([15]).

(i) *Assume* (A1) *and* (A2). *Then, the operators* $\mathcal{K}^{(0)}$ *and* $\mathcal{L}^{(0)}$ *belong to* $\mathcal{L}(H,G)$, $\|\mathcal{K}^{(0)}(f)\|_G \leqslant \|f\|_H$, *and* $\mathrm{Ker}(\mathcal{L}^{(0)}) = \mathrm{Span}(F^{(0)})$.

(ii) *The bilinear form*

$$B^{(0)}(f,g) := -\int_V \mathcal{L}^{(0)}(f) g \frac{1}{F^{(0)}}\,\mathrm{d}\mu = -\left(\mathcal{L}^{(0)}(f), \Sigma^{(0)} g\right)_G$$

is well defined, nonnegative and continuous on $H \times H$. *We have the following dissipative entropy inequality*

$$\begin{aligned} B^{(0)}(f,f) &= \frac{1}{2}\int_V\int_V \left(\frac{f(v)}{F^{(0)}(v)} - \frac{f(w)}{F^{(0)}(w)}\right)^2 \sigma^{(0)}(v,w)\,F^{(0)}(w)\,\mathrm{d}\mu(v)\,\mathrm{d}\mu(w) \\ &\geqslant \frac{1}{2}\left\|\left(\frac{1}{\Sigma^{(0)}}\right)\mathcal{L}^{(0)}(f)\right\|_H^2 \geqslant 0. \end{aligned}$$

(iii) *Assume* (A1), (A2) *and* (A3). *For* $f \in H$ *set* $\rho_f := \int_V f(v)\,\mathrm{d}\mu(v)$. *Then, the coercivity inequality*

$$B^{(0)}(f,f) \geqslant \frac{1}{2M\kappa}\|f - \rho_f F^{(0)}\|_H^2$$

holds.

(iv) *For any* $h \in G$ *there exists* $f \in H$ *such that* $L(f) = h$ *if and only if* $\int_V h(v)\,\mathrm{d}\mu(v) = 0$ (*or* $h \in G_0$). *The solution is unique in* H_0 *and satisfies*

$$\|f\|_H \leqslant 2M\kappa\left(\int_V h^2(v)\frac{1}{\Sigma^{(0)}(v)F^{(0)}(v)}\,\mathrm{d}\mu(v)\right)^{1/2} = 2M\kappa\|h\|_G.$$

Let us also describe the assumptions which will be used in the discussion of the diffusion limit. The equilibrium function $F^{(0)}$ should have a vanishing mean velocity and in some sense the collision term has to control the drift term. We are thus led to assume

$$\text{there exists } C_1 > 0 \quad \text{such that} \quad |a(v)|\,\big|\nabla_x F^{(0)}(x,v)\big| \leqslant C_1 \Sigma(x,v)F^{(0)}(x,v), \quad \text{a.e.;} \tag{B1}$$

$$\text{there exists } C_2 > 0 \quad \text{such that} \quad \int_V \big|a(v)\big|^2 \frac{F^{(0)}(x,v)}{\Sigma(x,v)}\,\mathrm{d}\mu(v) \leqslant C_2, \quad \mathrm{d}x\text{-a.e.;} \tag{B2}$$

$$\int_V a(v)F^{(0)}(x,v)\,\mathrm{d}\mu(v) = 0, \quad \mathrm{d}x\text{-a.e.} \tag{B3}$$

Remark that $a(v)F^{(0)}(x,v)$ is integrable with respect to v for a.a. x because of (A2) and (B2), so that (B3) makes sense.

Remark 2. Assumption (B3) is crucial: it appears as the compatibility condition necessary to apply the Fredholm alternative and solve the cell problems which determines the coefficients of the limit equation (see the formal ansatz and also Eq. (6) below). The conservation property (2) also means that $F_*^{(0)}(x,y,v) = 1$ is a solution of the adjoint cell problem $a(v)\cdot\nabla_y F_*^{(0)} + \mathcal{L}^{(0)*}F_*^{(0)} = 0$. Therefore, we can rewrite (B3) as

$$\int_V\int_Y a(v)F^{(0)}F_*^{(0)}\,\mathrm{d}\mu(v)\,\mathrm{d}y = 0.$$

We recover the form of the solvability condition which appears in previous works as [2].

Finally, we need the following geometrical assumption on the velocities to obtain some useful compactness properties.

$$\text{For any } \xi \in \mathbb{R}^N \setminus \{0\}, \quad \mu\big(\{v \in V,\ \text{such that } a(v)\cdot\xi \neq 0\}\big) > 0. \tag{C}$$

This assumption appears first in Lions and Toscani [26] and has been used successfully in Goudon and Poupaud [23] and Degond et al. [15]. We recall that (C) is weaker than the following property:

$$\text{For any } \xi \in \mathbb{R}^N \setminus \{0\}, \quad \mu\big(\{v \in V,\ \text{such that } a(v)\cdot\xi = 0\}\big) = 0, \tag{5}$$

which arises with averaging lemma methods of Golse et al. [21]. Since $\mu(V) \neq 0$ the assumption (5) clearly implies (C). But (5) fails for discrete velocities models while (C) means nothing but the fact that the subspace spanned by $\{a(v),\ v \in V\}$ coincides with the whole space $\mathbb{R}^N$. Assumption (C) will essentially be used through Lemma 2.

3.3. Consequences: Solvability of the cell problem

Assumptions introduced above can be used when interested in the cell problem

$$a(v) \cdot \nabla_y f - \mathcal{L}^{(0)} f = h$$

which appears in the formal asymptotic. Since in these problems, the variables are $y \in Y$ and $v \in V$, while $t,\ x$ are only parameters, we only keep the dependence with respect to $y,\ v$. In view of the coerciveness property detailed in Proposition 1, we set

$$\mathbb{H} = \left\{ f\colon \mathbb{R}^N \times V \to \mathbb{R},\ \text{such that } f \text{ is } Y\text{-periodic and } \int_Y \int_V f(y,v)^2 \frac{\Sigma^{(0)}(y,v)}{F^{(0)}(v)} \,\mathrm{d}\mu(v)\,\mathrm{d}y < \infty \right\}$$

endowed with the natural scalar product. We define in a similar way the space $\mathbb{G} \subset \mathbb{H}$ (with the weight $1/(\Sigma^{(0)}(y,v)F^{(0)}(v))$). As we did for H and G, we remark that the product $g\,f/F^{(0)}$ is integrable, for $f \in \mathbb{H},\ g \in \mathbb{G}$. From Proposition 1, $\mathcal{L}^{(0)}$ is a bounded operator from $\mathbb{H}$ to $\mathbb{G}$. Then, for $h \in \mathbb{G}$, we search for

$$f \in \{g \in \mathbb{H} \mid a(v) \cdot \nabla_y g \in \mathbb{G}\}, \quad \text{verifying} \quad a(v) \cdot \nabla_y f - \mathcal{L}^{(0)} f = h \qquad (\mathcal{P})$$

with periodic boundary condition. We recall that $F^{(0)}(v)$ does not depend on y, hence it belongs to the kernel of $a(v) \cdot \nabla_y - \mathcal{L}^{(0)}$, which provides an obvious solution of $(\mathcal{P})$ when $h = 0$. We also notice that, if $(\mathcal{P})$ has a solution f, then, integrating the equation with respect to both variables proves that the average of h should vanish. Then, we have the following statement whose (quite classical) proof is postponed in the appendix.

Proposition 2. *Assume* (A)–(C). *The problem* $(\mathcal{P})$ *has a unique solution* (*up to a constant*) *iff the right-hand side* h *satisfies* $\int_V \int_Y h \,\mathrm{d}y\,\mathrm{d}\mu(v) = 0$. *In particular, there exists a unique solution with null* y, v*-average*.

3.4. The remainder operator, double-scale limit and main result

Let us now discuss some properties of the remainder operator $\widehat{Q_\varepsilon}$. We are going to introduce further assumptions, referred to by the letter (D).

• First, let us go back to the equilibrium expansion $F_\varepsilon(x,v) = F^{(0)}(x,v) + \varepsilon \widehat{F_\varepsilon}(x,v)$. We can rewrite (4) as follows

$$(I + \varepsilon (Q_\varepsilon^{(0)})^{-1} \widehat{Q_\varepsilon}) \widehat{F}_\varepsilon = -(Q_\varepsilon^{(0)})^{-1} \widehat{Q}_\varepsilon F^{(0)},$$

where we define $(Q_\varepsilon^{(0)})^{-1}$ by using Proposition 1. This equation makes sense as soon as $\widehat{Q_\varepsilon}$ is a bounded operator from H to G, since the null average property is garanteed by the definition of $\widehat{Q_\varepsilon}$. Furthermore, we assume that

$$\text{there exists a constant } C_3 > 0 \text{ such that } \|\widehat{Q_\varepsilon}\|_{\mathcal{L}(H,G)} \leqslant C_3. \tag{D1}$$

This holds uniformly with respect to both x and ε. Hence, the left-hand side of (4) lies in H_0 and the operator $(Q_\varepsilon^{(0)})^{-1}\widehat{Q_\varepsilon}$ belongs to $\mathcal{L}(H_0)$. Furthermore, for ε small enough, its norm is $< 1/\varepsilon$ so that $(I + \varepsilon(Q_\varepsilon^{(0)})^{-1}\widehat{Q_\varepsilon})$ is invertible and we can write

$$\widehat{F_\varepsilon} = \left(I + \varepsilon\left(Q_\varepsilon^{(0)}\right)^{-1}\widehat{Q_\varepsilon}\right)^{-1}\left(-\left(Q_\varepsilon^{(0)}\right)^{-1}\widehat{Q_\varepsilon}F^{(0)}\right) = -\sum_{k=0}^{\infty}(-1)^k\varepsilon^k\left(\left(Q_\varepsilon^{(0)}\right)^{-1}\widehat{Q_\varepsilon}\right)^k\left(Q_\varepsilon^{(0)}\right)^{-1}\widehat{Q_\varepsilon}F^{(0)}.$$

One deduces that

$$\|\widehat{F_\varepsilon}\|_H \leqslant \frac{2M\kappa C_3\overline{\Sigma}}{1 - \varepsilon_0 2M\kappa C_3}$$

holds for $0 < \varepsilon < \varepsilon_0 \ll 1/(2M\kappa C_3)$, by using Proposition 1 and (D1).

By its definition $\widehat{F_\varepsilon}$ has null v-average. This implies that F_ε produces a flux with size of order ε, namely

$$\int_V a(v)F_\varepsilon(x,v)\,\mathrm{d}\mu(v) = \varepsilon u_\varepsilon(x)$$

with

$$u_\varepsilon(x) = \int_V a(v)\widehat{F_\varepsilon}(x,v)\,\mathrm{d}\mu(v).$$

• Next, we have seen in the formal asymptotic that the "remainder operator" $\widehat{Q_\varepsilon}$ plays a role in the limit procedure through its action on $F^{(0)}$. Hence, we should precise a little bit its behaviour. We introduce the following double scale limit:

$$\begin{cases} \text{There exists a nonnegative bounded function } \widehat{\sigma}(x,y,v,w) \\ Y\text{-periodic with respect to the variable } y \text{ such that } \forall\phi \in \mathcal{D}_\# \\ \lim_{\varepsilon\to 0}\displaystyle\int_{\mathbb{R}^N}\int_V \widehat{Q}_\varepsilon(F^{(0)})\phi\left(x, \frac{x}{\varepsilon}, v\right)\mathrm{d}\mu(v)\,\mathrm{d}x \\ \quad = \displaystyle\int_{\mathbb{R}^N}\int_V F^{(0)}(x,v)\left(\int_V\int_Y \widehat{\sigma}(x,y,w,v)\big(\phi(x,y,w) - \phi(x,y,v)\big)\,\mathrm{d}\mu(w)\,\mathrm{d}y\right)\mathrm{d}\mu(v)\,\mathrm{d}x, \end{cases} \tag{D2}$$

where the space $\mathcal{D}_\#$ of admissible test functions is defined by

$$\mathcal{D}_\# = \big\{\phi \in \mathcal{C}^\infty\big((0,T)\times\mathbb{R}_x^N\times\mathbb{R}_y^N; L^\infty(V)\big),\ \phi \text{ is } Y\text{-periodic with respect to } y,$$
$$\text{there exists a compact set } K \subset (0,T)\times\mathbb{R}^N, \text{ such that } \operatorname{supp}(\phi) \in K\times\mathbb{R}_y^N\times V\big\}.$$

When $\widehat{\sigma}_\varepsilon(x,w,v)$ is bounded in $L^\infty(\mathbb{R}^N_x \times V_v; L^1 \cap L^\infty(V_w))$ the existence of $\widehat{\sigma}$ is more or less a consequence of the existence of a double-scale limit for a bounded sequence in L^∞ (since for all $\phi \in \mathcal{D}_\#$, the functions $F^{(0)}(x,v)\phi(x,y,v)$ and $F^{(0)}(x,v)\phi(x,y,w)$ are in L^1 spaces with value in $\mathcal{C}_\#(\mathbb{R}^N_y)$), see [1,29]. We denote by $\widehat{\mathcal{L}}(F^{(0)})$ the (conservative) operator arising in the right-hand side

$$\begin{cases} \widehat{\mathcal{L}}(F^{(0)})(x,y,v) = \displaystyle\int_V \widehat{\sigma}(x,y,v,w)F^{(0)}(x,w)\,\mathrm{d}\mu(w) - \widehat{\Sigma}(x,y,v)F^{(0)}(x,v), \\ \widehat{\Sigma}(x,y,v) = \displaystyle\int_V \widehat{\sigma}(x,y,w,v)\,\mathrm{d}\mu(w). \end{cases}$$

By using Proposition 2, we can consider $\phi,\ \chi,\ \psi$ solution of the following auxiliary problems

$$\begin{cases} a(v)\cdot\nabla_y\chi_i - \mathcal{L}^{(0)}(\chi_i) = -a_i(v)\,F^{(0)} \quad \text{for } i \in \{1,\dots,N\}, \\ a(v)\cdot\nabla_y\phi - \mathcal{L}^{(0)}(\phi) = -a(v)\cdot\nabla_x F^{(0)}, \\ a(v)\cdot\nabla_y\psi - \mathcal{L}^{(0)}(\psi) = \widehat{\mathcal{L}}\big(F^{(0)}\big). \end{cases} \tag{6}$$

• Finally, we need another assumption related to the use of double-scale technics as introduced in [1,29] in order to pass to the limit in a suitable weak formulation of (1), see Eq. (12) below. This is mainly a (technical) regularity assumption on the kernel $\sigma^{(0)}$ which garantees that a certain function is "admissible" (see in particular the discussion in Section 5 of [1])

$$\begin{cases} \text{either } \sigma^{(0)}(x,y,v,w) \text{ belongs to } L^2_{\mathrm{loc}}(\mathbb{R}^N_x; C^0(Y; L^\infty(V_w; L^1(V_v)))) \\ \text{or } \sigma^{(0)}(x,y,v,w) \text{ belongs to } L^2(Y; C^0(\mathbb{R}^N_x; L^\infty(V_w; L^1(V_v)))) \\ \text{and } F^{(0)}/\Sigma(x,v) \text{ is in } C^0(\mathbb{R}^N_x; L^1(V_v)). \end{cases} \tag{D3}$$

The first assumption is maybe a bit simpler, however the second one is certainly more acceptable on a physical viewpoint since it applies for instance to a composite device containing distinct materials with different physical properties.

Then, our main result states as follows.

Theorem 1. *Assume* (A)–(D). *Let the initial data $f_{\varepsilon,0} \geqslant 0$ for* (1) *satisfies*

$$\int_{\mathbb{R}^N}\int_V \frac{f^2_{\varepsilon,0}(x,v)}{F^{(0)}(x,v)}\,\mathrm{d}\mu(v)\,\mathrm{d}x < \infty,$$

and let f_ε be a corresponding solution of (1) (*such a solution is assumed to exists, see Remark* 3). *Then, there exists a subsequence from $\rho_\varepsilon(t,x) = \int_V f_\varepsilon(t,x,v)\,\mathrm{d}\mu(v)$ which converges strongly in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^N)$ to $\rho(t,x)$, solution of the following drift diffusion equation*

$$\begin{cases} \partial_t\rho - \operatorname{div}\big(D(x)\nabla_x\rho - (c+b)\rho\big) = 0, \\ D(x) = -\displaystyle\int_Y\int_V \chi\otimes v\,\mathrm{d}y\,\mathrm{d}\mu(v), \\ c(x) = \displaystyle\int_Y\int_V a(v)\phi\,\mathrm{d}y\,\mathrm{d}\mu(v), \\ b(x) = \displaystyle\int_Y\int_V a(v)\psi\,\mathrm{d}y\,\mathrm{d}\mu(v). \end{cases} \tag{7}$$

The limit is actually a drift diffusion since the coefficients enjoy the properties listed in the following lemma whose proof is postponed in the appendix.

Lemma 1. *Let* (A)–(D) *hold. Then, the symmetric part of the matrix $D(x)$ is positive definite. Precisely, there exist $C, c > 0$ such that for all $\xi \in \mathbb{R}^N$*

$$0 < \frac{(\widetilde{\theta}(x)\xi \cdot \xi)^2}{c|\xi|^2} \leqslant D(x)\xi \cdot \xi \leqslant C|\xi|^2,$$

where

$$\widetilde{\theta}(x) = \int_V a(v) \otimes a(v) F^{(0)}(x, v)\,\mathrm{d}\mu(v).$$

In particular, $D(x) + {}^tD(x)$ is uniformly coercive on compact sets of $\mathbb{R}^N$. Furthermore $c(x)$ and $b(x)$ belong to $L^\infty(\mathbb{R}^N)$.

Remark 3. We shall not discuss in the sequel on the existence theory for problem (1). It may be not obvious at all in the full generality introduced here, both on the cross section σ and on the set of velocities V. Instead, we refer for instance to Petterson's works [30] or to results in [13]. Furthermore, we shall also prove that ρ_ε is compact in $C^0([0, T])$ with value in some negative Sobolev space, so that the initial data is also recovered in (7)

$$\rho_{|t=0} = \rho_0 = \lim_{\varepsilon \to 0} \int_V f_{\varepsilon,0}\,\mathrm{d}\mu(v).$$

Finally, it is also clear that if we are able to obtain the uniqueness for the limit problem (7), for instance when (the symmetric part of) the matrix D is shown to be uniformly coercive, then the convergence stated in Theorem 1 applies to the whole sequence.

Remark 4. Here the "drift" refers to the influence of the extra-convective terms c, b in the limit equation. In neutron transport theory this term is also used when dealing with some concentration effects due to the violation of the condition (B3) and it is worth pointing that these situations are radically different. We refer for this interesting aspect to the work of Larsen and Williams [25] or, for recent study on the diffusion approximation, Capdeboscq [11,12].

4. Estimates

This section is devoted to various estimates on the solutions f_ε of (1). As mentioned in Remark 3, we assume the existence, for $\varepsilon > 0$, of $f_\varepsilon \in C^0(\mathbb{R}^+; L^1(\mathbb{R}^N \times V, \mathrm{d}\mu(v)\,\mathrm{d}x))$ solution of (1) with $Q_\varepsilon(f_\varepsilon) \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^N; L^1(V, \mathrm{d}\mu(v)))$. Furthermore, $f_\varepsilon \geqslant 0$ when the initial data $f_{0,\varepsilon}$ is nonnegative and f_ε belongs to the weighted spaces $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^N \times V; 1/F^{(0)}\,\mathrm{d}\mu(v)\,\mathrm{d}x))$ and $L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^N \times V; \Sigma/F^{(0)}\,\mathrm{d}\mu(v)\,\mathrm{d}x))$. There is no topology on the set of velocities V and, as in [15], Eq. (1) is understood in the sense that

$$\partial_t\left(\int_V f_\varepsilon \phi\,\mathrm{d}\mu(v)\right) + \mathrm{div}_x\left(\frac{1}{\varepsilon}\int_V a(v) f_\varepsilon \phi\,\mathrm{d}\mu(v)\right) = \int_V \frac{1}{\varepsilon^2} Q_\varepsilon(f_\varepsilon)\phi\,\mathrm{d}\mu(v)$$

holds in $\mathcal{D}'((0,T)\times\mathbb{R}^N)$ for all $\phi \in L^\infty(V)$. Let us now formally derive the uniform estimates necessary for the asymptotics.

Clearly, conservation property gives a bound in $L^\infty((0,T);L^1(\mathbb{R}^N \times V))$. We shall establish other uniform estimates in weighted spaces, involving the equilibrium state $F^{(0)}$. Indeed, intuitively, we guess that f_ε behaves as F_ε for small values of ε, while F_ε essentially looks like $F^{(0)}$. This suggests to write

$$f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)F^{(0)}(x,v) + \varepsilon\gamma_\varepsilon, \tag{8}$$

where

$$\rho_\varepsilon(t,x) = \int_V f_\varepsilon \,\mathrm{d}\mu(v).$$

We note that

$$\int_V \gamma_\varepsilon \,\mathrm{d}\mu(v) = 0,$$

and we introduce the current

$$j_\varepsilon = \frac{1}{\varepsilon}\int_V a(v)\, f_\varepsilon \,\mathrm{d}\mu(v) = \int_V a(v)\gamma_\varepsilon \,\mathrm{d}\mu(v).$$

With (8) we realize that the collision term is actually of order $1/\varepsilon$ since the right-hand side of (1) reads $1/\varepsilon(\widehat{Q}_\varepsilon(\rho_\varepsilon F^{(0)}) + Q_\varepsilon(\gamma_\varepsilon))$.

Multiplying (1) by $f_\varepsilon/F^{(0)}$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\int_V \frac{f_\varepsilon^2}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x + \mathit{Transp}_\varepsilon + \mathit{Coll}_\varepsilon = 0, \tag{9}$$

where the transport term is

$$\mathit{Transp}_\varepsilon = \frac{1}{2\varepsilon}\int_{\mathbb{R}^N}\int_V f_\varepsilon^2 \frac{a(v)\cdot\nabla_x F^{(0)}}{(F^{(0)})^2}\,\mathrm{d}\mu(v)\,\mathrm{d}x$$

(after integration by part) while the collision term reads

$$\mathit{Coll}_\varepsilon = -\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_V \frac{f_\varepsilon}{F^{(0)}}\big(\widehat{Q}_\varepsilon\big(\rho_\varepsilon F^{(0)}\big) + Q_\varepsilon^{(0)}(\gamma_\varepsilon) + \varepsilon\widehat{Q}_\varepsilon(\gamma_\varepsilon)\big)\,\mathrm{d}\mu(v)\,\mathrm{d}x.$$

Let us estimate these terms. First, we have

$$\begin{aligned}
\mathit{Transp}_\varepsilon &= \frac{1}{2\varepsilon}\int_{\mathbb{R}^N}\int_V \big(\rho_\varepsilon^2\big(F^{(0)}\big)^2 + 2\varepsilon\gamma_\varepsilon\rho_\varepsilon F^{(0)} + \varepsilon^2\gamma_\varepsilon^2\big)\frac{a(v)\cdot\nabla_x F^{(0)}}{(F^{(0)})^2}\,\mathrm{d}\mu(v)\,\mathrm{d}x \\
&= \frac{1}{2\varepsilon}\int_{\mathbb{R}^N}\rho_\varepsilon^2 \mathrm{div}_x\bigg(\int_V a(v)F^{(0)}\,\mathrm{d}\mu(v)\bigg)\,\mathrm{d}x + \int_{\mathbb{R}^N}\int_V \gamma_\varepsilon\rho_\varepsilon\frac{a(v)\cdot\nabla_x F^{(0)}}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x \\
&\quad + \frac{\varepsilon}{2}\int_{\mathbb{R}^N}\int_V \gamma_\varepsilon^2\frac{a(v)\cdot\nabla_x F^{(0)}}{(F^{(0)})^2}\,\mathrm{d}\mu(v)\,\mathrm{d}x.
\end{aligned}$$

By (B3), the first term vanishes. Then, let $\nu > 0$ to be precised later. By using (B1), the second term is dominated by

$$\int_{\mathbb{R}^N}\int_V \left(\frac{\nu\gamma_\varepsilon^2\Sigma}{F^{(0)}} + C_\nu\rho_\varepsilon^2\Sigma\, F^{(0)}\right) \mathrm{d}\mu(v)\,\mathrm{d}x.$$

Then, we get

$$\mathit{Transp}_\varepsilon \leqslant \left(\nu + \frac{\varepsilon}{2}\right)\int_{\mathbb{R}^N}\int_V \gamma_\varepsilon^2\frac{\Sigma}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x + C_\nu\int_{\mathbb{R}^N}\int_V \rho_\varepsilon^2\Sigma F^{(0)}\,\mathrm{d}\mu(v)\,\mathrm{d}x.$$

However, we remark that, since $F^{(0)}$ is normalized,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_V \rho_\varepsilon^2\Sigma F^{(0)}\,\mathrm{d}\mu(v)\,\mathrm{d}x &= \int_{\mathbb{R}^N}\rho_\varepsilon^2\left(\int_V \Sigma F^{(0)}\,\mathrm{d}\mu(v)\right)\mathrm{d}x \leqslant \overline{\Sigma}\int_{\mathbb{R}^N}\rho_\varepsilon^2\,\mathrm{d}x\\ &\leqslant \overline{\Sigma}\int_{\mathbb{R}^N}\int_V \frac{f_\varepsilon^2}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x.\end{aligned} \tag{10}$$

It follows that

$$\mathit{Transp}_\varepsilon \leqslant \left(\nu + \frac{\varepsilon}{2}\right)\int_{\mathbb{R}^N}\int_V \gamma_\varepsilon^2\frac{\Sigma}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x + \overline{\Sigma}C_\nu\int_{\mathbb{R}^N}\int_V \frac{f_\varepsilon^2}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x.$$

On the other hand, the collision term reads

$$\begin{aligned}\mathit{Coll}_\varepsilon &= -\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_V \left(\rho_\varepsilon + \frac{\varepsilon\gamma_\varepsilon}{F^{(0)}}\right)\left(Q_\varepsilon^{(0)}(\gamma_\varepsilon) + \widehat{Q}_\varepsilon(f_\varepsilon)\right)\mathrm{d}\mu(v)\,\mathrm{d}x\\ &= -\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\rho_\varepsilon\left(\int_V \left(Q_\varepsilon^{(0)}(\gamma_\varepsilon) + \widehat{Q}_\varepsilon(f_\varepsilon)\right)\mathrm{d}\mu(v)\right)\mathrm{d}x\\ &\quad - \int_{\mathbb{R}^N}\int_V \frac{\gamma_\varepsilon}{F^{(0)}}Q_\varepsilon^{(0)}(\gamma_\varepsilon)\,\mathrm{d}\mu(v)\,\mathrm{d}x - \int_{\mathbb{R}^N}\int_V \frac{\gamma_\varepsilon}{F^{(0)}}\widehat{Q}_\varepsilon(f_\varepsilon)\,\mathrm{d}\mu(v)\,\mathrm{d}x.\end{aligned}$$

By conservation property the first integral vanishes. The second one is bounded from below

$$-\int_{\mathbb{R}^N}\int_V \frac{\gamma_\varepsilon}{F^{(0)}}Q_\varepsilon^{(0)}(\gamma_\varepsilon)\,\mathrm{d}\mu(v)\,\mathrm{d}x \geqslant \frac{1}{2M\kappa}\int_{\mathbb{R}^N}\int_V \frac{\gamma_\varepsilon^2\Sigma_\varepsilon^{(0)}}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x$$

by using Proposition 1(iii) (we denote $\Sigma_\varepsilon^{(0)} = \Sigma^{(0)}(x, x/\varepsilon, v)$). The last term is estimated by using (D1). We have

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\int_V \frac{\gamma_\varepsilon}{F^{(0)}}\widehat{Q}_\varepsilon(f_\varepsilon)\,\mathrm{d}\mu(v)\,\mathrm{d}x\right| &\leqslant \int_{\mathbb{R}^N}\|\gamma_\varepsilon\|_H\big\|\widehat{Q}_\varepsilon(f_\varepsilon)\big\|_G\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N}\|\gamma_\varepsilon\|_H C_3\|f_\varepsilon\|_H\,\mathrm{d}x\\ &\leqslant \nu\int_{\mathbb{R}^N}\int_V \frac{\gamma_\varepsilon^2\Sigma_\varepsilon^{(0)}}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x + C_\nu\int_{\mathbb{R}^N}\int_V \frac{f_\varepsilon^2\Sigma_\varepsilon^{(0)}}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x.\end{aligned}$$

Combining these informations to (9) and (10) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\int_V \frac{f_\varepsilon^2}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x + \left(\frac{1}{2M\kappa} - 2\nu - \frac{\varepsilon}{2}\right)\int_{\mathbb{R}^N}\int_V \frac{\gamma_\varepsilon^2 \Sigma_\varepsilon^{(0)}}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x$$
$$\leqslant C_\nu \int_{\mathbb{R}^N}\int_V \frac{f_\varepsilon^2}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x,$$

where we used also (A1). Hence we choose ν small enough to ensure, say $1/(2M\kappa) - 2\nu > 1/(4M\kappa)$; thus for $0 < \varepsilon < \varepsilon_0$, the quantity $1/(2M\kappa) - 2\nu - \varepsilon/2$ remains $>1/(8M\kappa)$. Then the following statement follows by applying Gronwall's lemma on a finite time interval $(0, T)$.

Proposition 3. *Assume* (A), (B), (D1). *Let the initial data $f_{\varepsilon,0}$ satisfy*

$$\int_{\mathbb{R}^N}\int_V \frac{f_{\varepsilon,0}^2(x,v)}{F^{(0)}}\,\mathrm{d}x\,\mathrm{d}\mu(v) \leqslant C_0.$$

Then, there exists $\varepsilon_0 > 0$ such that, the associated sequence $(f_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ of solutions of (1) *satisfies*

$$\begin{cases} f_\varepsilon \textit{ bounded in } L^2((0,T)\times\mathbb{R}^N\times V, \Sigma/F^{(0)}\,\mathrm{d}x\,\mathrm{d}\mu(v)\,\mathrm{d}t) \\ \quad \textit{and in } L^\infty((0,T); L^2(\mathbb{R}^N\times V, 1/F^{(0)}\,\mathrm{d}x\,\mathrm{d}\mu(v))) \cap L^\infty((0,T); L^1(\mathbb{R}^N\times V)), \\ \rho_\varepsilon(t,x) = \int_V f_\varepsilon\,\mathrm{d}\mu(v) \textit{ is bounded in } L^2((0,T)\times\mathbb{R}^N), \\ \gamma_\varepsilon = (1/\varepsilon)(f_\varepsilon - \rho_\varepsilon F^{(0)}) \textit{ is bounded in } L^2((0,T)\times\mathbb{R}^N\times V, \Sigma/F^{(0)}\,\mathrm{d}x\,\mathrm{d}\mu(v)\,\mathrm{d}t), \\ j_\varepsilon = (1/\varepsilon)\int_V a(v) f_\varepsilon\,\mathrm{d}\mu(v) \textit{ is bounded in } L^2((0,T)\times\mathbb{R}^N). \end{cases}$$

Proof. Estimate on f_ε in $L^\infty((0,T); L^1(\mathbb{R}^N\times V))$ follows from mass conservation; estimates on f_ε on $L^\infty((0,T); L^2(\mathbb{R}^N\times V, 1/F^{(0)}\,\mathrm{d}x\,\mathrm{d}\mu(v)))$ as well as estimate on

$$\int_0^T\int_{\mathbb{R}^N}\int_V |\gamma_\varepsilon|^2 \frac{\Sigma_\varepsilon^{(0)}}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t$$

have been obtained in the discussion above. The bound on ρ_ε is a consequence of (10) combined to the estimate on $f_\varepsilon^2/F^{(0)}$. Note that, by (A1), γ_ε is bounded in $L^2((0,T)\times\mathbb{R}^N\times V, \Sigma/F^{(0)}\,\mathrm{d}x\,\mathrm{d}\mu(v)\,\mathrm{d}t)$; main advantage here is that the weight does not depend on ε. Consequently, $f_\varepsilon = \rho_\varepsilon F^{(0)} + \varepsilon\gamma_\varepsilon$ is also bounded in $L^2((0,T)\times\mathbb{R}^N\times V, \Sigma/F^{(0)}\,\mathrm{d}x\,\mathrm{d}\mu(v)\,\mathrm{d}t)$. For the current, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N} |j_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^N}\left|\int_V a(v)\gamma_\varepsilon\,\mathrm{d}\mu(v)\right|^2\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_0^T\int_{\mathbb{R}^N}\left(\int_V a(v)^2\frac{F^{(0)}}{\Sigma}\,\mathrm{d}\mu(v)\right)\int_0^T\int_{\mathbb{R}^N}\left(\int_V \frac{\gamma_\varepsilon^2\Sigma}{F^{(0)}}\,\mathrm{d}\mu(v)\right)\mathrm{d}x\,\mathrm{d}t \\ &\leqslant C_2\int_0^T\int_{\mathbb{R}^N}\int_V \frac{\gamma_\varepsilon^2\Sigma}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \leqslant C(T),\end{aligned}$$

which achieves the proof. □

5. Passing to the limit

Having obtained uniform estimates, we wish to pass to the limit in (1) at least for a suitable subsequence. The proof falls into five steps.

Step 1: Preliminaries

First, we can assume that there exists $\gamma \in L^2((0,T) \times \mathbb{R}^N \times V, \Sigma/F^{(0)}\,\mathrm{d}x\,\mathrm{d}\mu(v)\,\mathrm{d}t)$ such that

$$\begin{cases} \displaystyle\lim_{\varepsilon\to 0} \int_0^T\!\!\int_{\mathbb{R}^N}\int_V \gamma_\varepsilon \phi\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_{\mathbb{R}^N}\int_V \gamma\phi\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \\ \text{holds for all for test functions } \phi \text{ verifying} \\ \displaystyle\int_0^T\!\!\int_{\mathbb{R}^N}\int_V \phi^2(t,x,v)\frac{F^{(0)}(x,v)}{\Sigma(x,v)}\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t < +\infty. \end{cases} \tag{11}$$

A similar conclusion applies for the convergence of f_ε to a certain f. In particular, we notice that $\phi(t,x,v) = \zeta(t,x)\psi(v)$ is an admissible test function with $\zeta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ and $\psi(v) \in L^\infty(V)$. By (B2), we can also use $\phi(t,x,v) = \zeta(t,x)a(v)\psi(v)$, with $\zeta,\ \psi$ as before. Moreover, we can suppose that

$$\rho_\varepsilon \rightharpoonup \rho, \quad j_\varepsilon \rightharpoonup j \quad \text{in } L^2\big((0,T) \times \mathbb{R}^N\big).$$

Then, it can be easily checked that the limits are connected as follows

$$f = \rho F^{(0)}, \qquad \rho = \int_V f\,\mathrm{d}\mu(v), \qquad j = \int_V a(v)\gamma\,\mathrm{d}\mu(v).$$

Next, let us end this first step by rewritting Eq. (1) in a suitable weak formulation. In order to exploit the periodicity properties, we use test functions $\varepsilon\phi(t,x,x/\varepsilon,v)$. We get

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^T\!\!\int_{\mathbb{R}^N}\int_V Q_\varepsilon(f_\varepsilon)\phi\Big(t,x,\frac{x}{\varepsilon},v\Big)\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_0^T\!\!\int_{\mathbb{R}^N}\int_V \big(\widehat{Q}_\varepsilon\big(\rho_\varepsilon F^{(0)}\big) + Q_\varepsilon(\gamma_\varepsilon)\big)\phi\Big(t,x,\frac{x}{\varepsilon},v\Big)\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -\varepsilon\int_0^T\!\!\int_{\mathbb{R}^N}\int_V f_\varepsilon(t,x,v)\partial_t\phi\Big(t,x,\frac{x}{\varepsilon},v\Big)\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \frac{1}{\varepsilon}\int_0^T\!\!\int_{\mathbb{R}^N}\int_V f_\varepsilon(t,x,v)a(v)\cdot\nabla_y\phi\Big(t,x,\frac{x}{\varepsilon},v\Big)\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_0^T\!\!\int_{\mathbb{R}^N}\int_V f_\varepsilon(t,x,v)a(v)\cdot\nabla_x\phi\Big(t,x,\frac{x}{\varepsilon},v\Big)\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -\varepsilon\int_0^T\!\!\int_{\mathbb{R}^N}\int_V \Big(f_\varepsilon(t,x,v)\partial_t\phi\Big(t,x,\frac{x}{\varepsilon},v\Big) + \gamma_\varepsilon a(v)\cdot\nabla_x\phi\Big(t,x,\frac{x}{\varepsilon},v\Big)\Big)\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_0^T\!\!\int_{\mathbb{R}^N}\int_V \Big(\gamma_\varepsilon a(v)\cdot\nabla_y\phi\Big(t,x,\frac{x}{\varepsilon},v\Big) + \rho_\varepsilon F^{(0)}a(v)\cdot\nabla_x\phi\Big(t,x,\frac{x}{\varepsilon},v\Big)\Big)\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \frac{1}{\varepsilon}\int_0^T\!\!\int_{\mathbb{R}^N}\int_V \rho_\varepsilon F^{(0)}a(v)\cdot\nabla_y\phi\Big(t,x,\frac{x}{\varepsilon},v\Big)\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{12}$$

Step 2: Strong convergence of the macroscopic density

Let us use (12) with the test function $\phi = \eta(v)\zeta(t,x)$, with $\zeta \in C_0^\infty((0,T)\times\mathbb{R}^N)$, and $\eta(v)$ defined by

$$\eta(v) = \begin{cases} \dfrac{a(v)}{|a(v)|} & \text{if } a(v) \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

We obtain the following relation in $\mathcal{D}'((0,T)\times\mathbb{R}^N)$

$$\begin{aligned} &\varepsilon\Big[\partial_t\Big(\int_V \eta(v)\, f_\varepsilon \,\mathrm{d}\mu(v)\Big) + \mathrm{Div}_x\Big(\int_V \frac{a(v)\otimes a(v)}{|a(v)|}\gamma_\varepsilon \,\mathrm{d}\mu(v)\Big)\Big] \\ &\qquad + \mathrm{Div}_x\Big(\int_V \frac{a(v)\otimes a(v)}{|a(v)|} F^{(0)} \,\mathrm{d}\mu(v)\,\rho_\varepsilon\Big) \\ &\quad = \rho_\varepsilon \int_V \eta(v)\,\widehat{Q}_\varepsilon(F^{(0)})\,\mathrm{d}\mu(v) + \int_V \eta(v)\, Q_\varepsilon(\gamma_\varepsilon)\,\mathrm{d}\mu(v). \end{aligned} \tag{13}$$

By using assumptions (A)–(D) and Proposition 3, we check that the right-hand side is bounded in $L^2((0,T)\times\mathbb{R}^N)$. On the other hand, the first term in the left-hand side is ε times first derivatives (with respect to time or space) of bounded functions in $L^2((0,T)\times\mathbb{R}^N)$, therefore it converges strongly to 0 in $H^{-1}((0,T)\times\mathbb{R}^N)$. Finally, the last term in the left-hand side is $\mathrm{Div}(\theta(x)\rho_\varepsilon)$ where θ is the matrix

$$\theta(x) = \int_V \frac{a(v)\otimes a(v)}{|a(v)|} F^{(0)}(x,v)\,\mathrm{d}\mu(v) = \int_V a(v)\otimes \eta(v) F^{(0)}(x,v)\,\mathrm{d}\mu(v).$$

As a consequence of assumption (C), (B1) and (B2), we have

Lemma 2. *The matrix $\theta(x)$ is positive definite, and its coefficients belong to $W^{1,\infty}(\mathbb{R}^N)$. Therefore, for each x, $\theta(x)$ is invertible and we have*

$$\theta(x) \geqslant \alpha_K I > 0 \quad \textit{for all } x \in K,\ K \subset \mathbb{R}^N \textit{ compact}.$$

Therefore, we realize that (13) implies

$$\nabla_x \rho_\varepsilon \in \text{Compact Set in } H^{-1}_{\mathrm{loc}}\big((0,T)\times\mathbb{R}^N\big).$$

We combine this information to the mass conservation relation

$$\partial_t \rho_\varepsilon + \mathrm{div}_x\, j_\varepsilon = 0 \tag{14}$$

obtained by choosing $\phi = \zeta(t,x)1(v)$ in (12). Indeed, let us introduce the vector fields (in $\mathbb{R}^{N+1}$) $U_\varepsilon = (\rho_\varepsilon, j_\varepsilon)$, $V_\varepsilon = (\rho_\varepsilon, 0, \ldots, 0)$. We have $\mathrm{div}_{t,x}(U_\varepsilon) = 0 \in$ Compact Set in $H^{-1}((0,T)\times\mathbb{R}^N)$ by (14) and

$$\mathrm{curl}_{t,x}(V_\varepsilon) = \begin{pmatrix} 0 & -{}^t\nabla_x\rho_\varepsilon \\ \nabla_x\rho_\varepsilon & 0 \end{pmatrix}$$

also lies in a Compact Set in $(H^{-1}((0,T)\times\mathbb{R}^N))^{(N+1)\times(N+1)}$. By applying the celebrated div–curl lemma of Tartar [33,34], we conclude that $U_\varepsilon\cdot V_\varepsilon=\rho_\varepsilon^2\rightharpoonup\rho^2$ which in turn implies that ρ_ε converges strongly to ρ in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^N)$.

Step 3: Regularity of ρ

Furthermore, the properties of the matrix θ also lead to an improvement of the regularity for the limit ρ. Indeed, since the right-hand side in (13) is bounded in $L^2((0,T)\times\mathbb{R}^N)$, we can suppose it converges weakly to some ν. Thus, passing to the limit $\varepsilon\to 0$ in (13) yields

$$0+\mathrm{Div}(\theta\rho)=\nu\in L^2\big((0,T)\times\mathbb{R}^N\big).$$

Therefore we write $\nabla_x\rho=\theta^{-1}(\mathrm{Div}_x(\theta\rho)-\mathrm{Div}_x(\theta)\rho)$ which shows that

$$\rho\in L^2\big(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^N)\big).$$

Our next aim will be to obtain more explicit relations between ρ, ν and j and to rely this to the behaviour of γ_ε.

Step 4: Double scale limit

In order to investigate more precisely (12), we shall use the notion of double scale convergence. We have

$$\gamma_\varepsilon(t,x,v)\left(\frac{\Sigma}{F^{(0)}}(x,v)\right)^{1/2}\quad\text{bounded in } L^2\big((0,T)\times\mathbb{R}^N\times V\big).$$

Thus, adapting the seminal arguments of Allaire [1], Nguetseng [29] (see also [23] for some slight variations), possibly at the cost of extracting a subsequence, we can say that $\gamma_\varepsilon(t,x,v)$ double scale converges to $\Gamma(t,x,y,v)$ in the following way:

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^N}\int_V\gamma_\varepsilon(t,x,v)\phi\left(t,x,\frac{x}{\varepsilon},v\right)\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t\\&\quad=\int_0^T\int_{\mathbb{R}^N}\int_V\int_Y\Gamma(t,x,y,v)\phi(t,x,y,v)\,\mathrm{d}y\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

for all test function $\phi\in\mathcal{D}_\#$ (using the fact that $(\frac{F^{(0)}}{\Sigma}(x,v))^{1/2}\phi(t,x,y,v)$ lies in $L^2(B(0,R);\ \mathcal{C}_\#(\mathbb{R}^N_y;\mathcal{H}))$ as soon as $\phi\in\mathcal{D}_\#$ where $\mathrm{supp}_x(\phi)\in B(0,R)$ and $\mathcal{H}=L^2((0,T)\times V)$). We recall that, if a sequence converges strongly in $L^2((0,T)\times\mathbb{R}^N\times V;\Sigma/F^{(0)}\,\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t)$ then its double scale limit does not depend on the variable y and coincides with the strong limit. On the other hand, weak limit, in the sense of (11), equals the y-average of the double-scale limit.

Having disposed of these preliminaries, we remark that it remains a singular term in (12), namely

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^N}\int_V\rho_\varepsilon F^{(0)}a(v)\cdot\nabla_y\phi\left(t,x,\frac{x}{\varepsilon},v\right)\mathrm{d}\mu(v)\,\mathrm{d}x\,\mathrm{d}t\\&\quad=\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^N}\rho_\varepsilon(t,x)\left(\int_V a(v)F^{(0)}\cdot\nabla_y\phi\left(t,x,\frac{x}{\varepsilon},v\right)\mathrm{d}\mu(v)\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Then, interesting things can be obtained only if this term disappears. To this end, we restrict ourselves to test functions in the set

$$E_\# = \left\{\phi \in \mathcal{D}_\#,\ \mathrm{div}_y\left(\int_V a(v)F^{(0)}\,\phi\,\mathrm{d}\mu(v)\right) = \int_V a(v)F^{(0)}\cdot\nabla_y\phi\,\mathrm{d}\mu(v) = 0\right\}.$$

For such a test function, by using estimates in Proposition 3 and the strong convergence of ρ_ε, the right-hand side in (12) tends to

$$\begin{aligned}\lim_{\varepsilon\to 0}\text{r.h.s}_\varepsilon = &-\int_0^T\!\!\int_{\mathbb{R}^N}\int_V\int_Y \rho(t,x)F^{(0)}(x,v)a(v)\cdot\nabla_x\phi(t,x,y,v)\,\mathrm{d}\mu(v)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\!\!\int_{\mathbb{R}^N}\int_V\int_Y \varGamma a(v)\cdot\nabla_y\phi(t,x,y,v)\,\mathrm{d}\mu(v)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{15}$$

The convergence of the second term holds since assumption (B2) gives $(F^{(0)}/\varSigma)^{1/2}a(v)\cdot\nabla_y\phi \in L^2(\mathbb{R}^N, \mathcal{C}_\#(\mathbb{R}^N_y;\mathcal{H}))$. On the other hand, by using (D2) and (D3), the left-hand side in (12) has the following limit

$$\lim_{\varepsilon\to 0}\text{l.h.s}_\varepsilon = +\int_0^T\!\!\int_{\mathbb{R}^N}\int_V\int_Y \big(\widehat{\mathcal{L}}(\rho F^{(0)})\phi(t,x,y,v) + \mathcal{L}^{(0)}(\varGamma)\phi(t,x,y,v)\big)\,\mathrm{d}\mu(v)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \tag{16}$$

where the operator $\widehat{\mathcal{L}}$ is defined by (D2).

Combining (15) and (16), we obtain that $a(v)\cdot\nabla_y\varGamma - \mathcal{L}^{(0)}(\varGamma) + \mathrm{div}_x(\rho\, a(v)F^{(0)}) - \widehat{\mathcal{L}}(\rho F^{(0)})$ lies in the orthogonal set of $E_\#$. Then, the following claim will be useful.

Lemma 3. *Let T be in the dual $\mathcal{D}'_\#$ of $\mathcal{D}_\#$. Then, T belongs to the orthogonal set of $E_\#$ iff $T = a(v)F^{(0)}\cdot\nabla_y Q$, where $Q\in\mathcal{D}'_\#$ does not depend on v.*

Step 5: Conclusion

Therefore, there exists $Q\in\mathcal{D}'_\#$ such that $\mathcal{F} = \varGamma - F^{(0)}Q$ satisfies the following cell problem

$$a(v)\cdot\nabla_y\mathcal{F} - \mathcal{L}^{(0)}\mathcal{F} = -a(v)\cdot\nabla_x\big(\rho F^{(0)}\big) + \widehat{\mathcal{L}}\big(\rho F^{(0)}\big). \tag{17}$$

Furthermore, Q is solution of

$$a(v)F^{(0)}\cdot\nabla_y Q = a(v)F^{(0)}\cdot\nabla_y\varGamma - a(v)\cdot\nabla_x\big(\rho F^{(0)}\big) + \widehat{\mathcal{L}}\big(\rho F^{(0)}\big) + \mathcal{L}^{(0)}\varGamma.$$

Following [23], multiplying by $\eta(v)$ and integrating with respect to v, we get

$$\nabla_y Q = \theta(x)^{-1}\left[\mathrm{Div}_y\int_V \frac{a(v)\otimes a(v)}{|a(v)|}F^{(0)}\varGamma\,\mathrm{d}\mu(v) + \int_V \eta(v)\big(\widehat{\mathcal{L}}\big(\rho F^{(0)}\big) + \mathcal{L}^{(0)}\varGamma\big)\,\mathrm{d}\mu(v)\right] + \nabla_x\rho.$$

Since we can assume, without loss of generality, that $\int_V Q\,\mathrm{d}\mu(v) = 0$, this determines Q as a function of $L^2((0,T)\times\mathbb{R}^N_x\times Y)$ since the right-hand side of the previous relation involves only y-first order derivatives of $L^2((0,T)\times\mathbb{R}^N_x\times Y)$ functions. Thus $\mathcal{F}\in L^2((0,T)\times\mathbb{R}^N_x;\mathbb{H})$, and (17) gives $a(v)\cdot\nabla_y\mathcal{F}\in$

$L^2((0,T) \times \mathbb{R}^N_x; \mathbb{G})$. Since the right-hand side in (17) fulfills the requirements of Proposition 2, we can consider $\phi,\ \chi_i,\ \psi$ solution of the auxiliary problem (6). Then, we get

$$\Gamma(t,x,y,v) = F^{(0)}(x,v)Q(t,x,y) + r(t,x)F^{(0)} + \chi \cdot \nabla_x \rho + \phi\rho + \psi\rho.$$

This seems to be a very partial information, however it suffices to describe precisely the limit current as follows

$$j(t,x) = \lim_{\varepsilon\to 0} \int_V a(v)\gamma_\varepsilon \, \mathrm{d}\mu(v) = \int_V \int_Y a(v)\Gamma \, \mathrm{d}y \, \mathrm{d}\mu(v).$$

Inserting this expression into (14) gives the expected drift diffusion Eq. (7) and ends the proof of Theorem 1. It only remains to say that (14) combined to Proposition 3 and Ascoli's theorem ensures that ρ_ε belongs to a compact set in $C^0([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R}^N))$. Hence, the initial data ρ_0 for the limit problem (7) corresponds to the weak limit of the $\rho_{0,\varepsilon}$'s. $\square$

Appendix A. Proof of Proposition 2

The scheme of the proof follows the strategy adopted in [23]. It is worth pointing out that, in view of (C), Proposition 2 and the proof below apply to discrete velocities models. Again, the keypoint relies on the properties of the matrix θ.

Step 1: Uniqueness

Uniqueness follows from coercivity of the operator $\mathcal{L}^{(0)}$. Indeed, let f satisfy ($\mathcal{P}$) with $h = 0$. Multipying par $f/F^{(0)}$ and integrating yield

$$0 = \int_V \int_Y a(v) \cdot \nabla_y \left(\frac{f^2}{2}\right) \frac{1}{F^{(0)}} \, \mathrm{d}y \, \mathrm{d}\mu(v) - \int_V \int_Y L^{(0)}(f) \frac{f}{F^{(0)}} \, \mathrm{d}y \, \mathrm{d}\mu(v) = 0 + \mathcal{B}^{(0)}(f,f),$$

where we used the fact that F_0 does not depend on y and we denoted by $\mathcal{B}^{(0)}$ the following bilinear mapping

$$\mathcal{B}^{(0)}(f,g) = -\int_V \int_Y \mathcal{L}^{(0)}(f) \frac{g}{F^{(0)}} \, \mathrm{d}y \, \mathrm{d}\mu(v).$$

By Proposition 1, $\mathcal{B}^{(0)}$ is continuous and satisfies

$$\mathcal{B}^{(0)}(f,f) \geqslant \kappa \int_Y \int_V \left|f - \rho_f F^{(0)}\right|^2 \frac{\Sigma^{(0)}}{F^{(0)}} \, \mathrm{d}y \, \mathrm{d}\mu(v).$$

Hence, we deduce that $f(y,v) = \rho_f(y)F^{(0)}(v)$. In turn, the equation reads now $a(v) \cdot \nabla_y(\rho F^{(0)}) = a(v) \cdot \nabla_y(\rho)F^{(0)} = 0$. Since $F^{(0)} > 0$, this means that $\nabla_y \rho(y)$ is orthogonal to V; therefore, by (C), ρ is constant (with respect to y).

Step 2: Existence

Let $\lambda > 0$. By integrating along characteristic curves, one gets a periodic solution of

$$\lambda f + a(v) \cdot \nabla_y f + \Sigma^{(0)}(y,v) f = h.$$

Then, we set

$$\mathcal{T}_\lambda : f_1 \mapsto f,$$

where f is solution of the problem above with right-hand side $g = h + \mathcal{K}^{(0)}(f_1)$. We now show that $\mathcal{T}_\lambda$ is a contraction for the norm

$$\|f\| = \int_V \frac{(\lambda + \Sigma)}{F^{(0)}(v)} \left|f(v)\right|^2 \mathrm{d}\mu(v).$$

For two function $f_1,\ f_1'$, denoting by $\delta_1 = f_1 - f_1'$ and $\delta = \mathcal{T}_\lambda(f_1) - \mathcal{T}_\lambda(f_1')$, we get

$$\begin{aligned}
&\int_V \mathcal{K}^{(0)}(\delta_1)(v) \frac{\delta(v)}{F^{(0)}(v)} \,\mathrm{d}\mu(v) = \int_V \int_V \sigma^{(0)}(v,w) \delta_1(w) \frac{\delta(v)}{F^{(0)}(v)} \,\mathrm{d}\mu(v) \\
&\quad \leqslant \left(\int_V \int_V \sigma^{(0)}(v,w) \frac{\delta_1(w)^2}{F^{(0)}(w)} \,\mathrm{d}\mu(w) \,\mathrm{d}\mu(v) \right)^{1/2} \left(\int_V \int_V \sigma^{(0)}(v,w) F^{(0)}(w) \frac{\delta(v)^2}{F^{(0)}(v)^2} \,\mathrm{d}\mu(w) \,\mathrm{d}\mu(v) \right)^{1/2} \\
&\quad \leqslant \left(\int_V \Sigma^{(0)}(w) \frac{\delta_1(w)^2}{F^{(0)}(w)} \,\mathrm{d}\mu(w) \right)^{1/2} \left(\int_V \Sigma^{(0)}(v) \frac{\delta(v)^2}{F^{(0)}(v)} \,\mathrm{d}\mu(v) \right)^{1/2} \\
&\quad \leqslant \frac{\overline{\Sigma}}{\lambda + \overline{\Sigma}} \left(\int_V \left(\lambda + \Sigma^{(0)}(w)\right) \frac{\delta_1(w)^2}{F^{(0)}(w)} \,\mathrm{d}\mu(w) \right)^{1/2} \left(\int_V \left(\lambda + \Sigma^{(0)}(v)\right) \frac{\delta(v)^2}{F^{(0)}(v)} \,\mathrm{d}\mu(v) \right)^{1/2},
\end{aligned} \tag{18}$$

since $(\Sigma^{(0)}(w))/(\lambda + \Sigma^{(0)}(w)) \leqslant \overline{\Sigma}/(\lambda + \overline{\Sigma})$ for all $w \in V$. Futhermore, δ, δ_1 solve

$$\lambda\delta + a(v) \cdot \nabla_y \delta + \Sigma^{(0)}(y,v)\delta = \mathcal{K}^{(0)}(\delta_1).$$

Therefore, multiplying by $\delta/F^{(0)}$, an integration by part yields

$$\int_V \left(\lambda + \Sigma^{(0)}\right) \frac{\delta^2}{F^{(0)}} \,\mathrm{d}\mu(v) \leqslant \int_V \mathcal{K}^{(0)}(\delta_1)(v) \frac{\delta(v)}{F^{(0)}(v)} \,\mathrm{d}\mu(v),$$

and the results falls from (18), since $\overline{\Sigma}/(\lambda + \overline{\Sigma}) < 1$.

Finally, thanks to the Banach fixed point theorem, we deal with a sequence f_λ satisfying

$$\lambda f_\lambda + a(v) \cdot \nabla_y f_\lambda + \mathcal{L}^{(0)}(f_\lambda) = h,$$

and we wish to pass to the limit $\lambda \to 0$ in this equation. To this end, let us assume temporarily that f_λ remains in a bounded set in $\mathbb{H}$. Therefore, up to a subsequence, we have

$$\begin{cases} \lim\limits_{\lambda\to 0} \int_Y \int_V f_\lambda \phi \, \mathrm{d}y \, \mathrm{d}\mu(v) = \int_Y \int_V f\phi \, \mathrm{d}y \, \mathrm{d}\mu(v) \\ \text{for test functions } \phi \text{ verifying} \\ \int_Y \int_V \phi^2(y,v) \dfrac{F^{(0)}(v)}{\Sigma^{(0)}(y,v)} \, \mathrm{d}y \, \mathrm{d}\mu(v). \end{cases} \tag{19}$$

Let $\phi(y,v) = \zeta(y)\psi(v)$, with $\zeta \in C_0^\infty(Y)$ and $\psi \in L^\infty(V)$. Then, ϕ fulfills this condition as well as $a(v)\psi(v) \cdot \nabla_y \zeta(y)$ (by using (B2)) while $\phi F^{(0)} \in \mathbb{H}$. Thus, we can write

$$\begin{aligned} \int_Y \int_V h\phi(y,v) \, \mathrm{d}y \, \mathrm{d}\mu(v) = \lambda \int_Y \int_V f_\lambda \phi(y,v) \, \mathrm{d}y \, \mathrm{d}\mu(v) \\ + \int_Y \int_V a(v)\psi(v) \cdot \nabla_y \zeta(y) \, f_\lambda(y,v) + \mathcal{B}^{(0)}\big(f_\lambda, \phi F^{(0)}\big). \end{aligned}$$

With (19), letting λ goes to 0 shows that the limit f is a solution of $(\mathcal{P})$.

It remains to show that f_λ is bounded in $\mathbb{H}$. Assume that, for some subsequence, $N_\lambda = \|f_\lambda\|_{\mathbb{H}} \to \infty$, and set $\mathcal{F}_\lambda = f_\lambda / N_\lambda$, $\rho_\lambda = \int_V \mathcal{F}_\lambda \, \mathrm{d}\mu(v)$. We can assume that $\mathcal{F}_\lambda$ has a weak limit $\mathcal{F}$ in the sense of (19). Furthermore ρ_λ is bounded in $L^2(Y)$ (by using (A3)) and converges weakly to a limit denoted by ρ. We have

$$\lambda \mathcal{F}_\lambda + a(v) \cdot \nabla_y \mathcal{F}_\lambda + \mathcal{L}^{(0)}(F_\lambda) = \frac{h}{N_\lambda} = h_\lambda \to 0 \quad \text{in } \mathbb{G}. \tag{20}$$

First, multiply by $\mathcal{F}_\lambda / F^{(0)}$ and integrate. This proves the strong convergence of $\mathcal{F}_\lambda - \rho_\lambda F^{(0)}$ towards 0 in $\mathbb{H}$ since one has

$$\begin{aligned} \kappa \big\| \mathcal{F}_\lambda - \rho_\lambda F^{(0)} \big\|_{\mathbb{H}}^2 &\leqslant \lambda \int_V \int_Y \mathcal{F}_\lambda^2 / F^{(0)} \, \mathrm{d}y \, \mathrm{d}\mu(v) + \mathcal{B}^{(0)}(\mathcal{F}_\lambda, \mathcal{F}_\lambda) \\ &\leqslant \int_V \int_Y \frac{h_\lambda \, \mathcal{F}_\lambda}{F^{(0)}} \, \mathrm{d}y \, \mathrm{d}\mu(v) \leqslant \|h_\lambda\|_{\mathbb{G}} \, \|\mathcal{F}_\lambda\|_{\mathbb{H}}. \end{aligned}$$

Therefore, $\mathcal{F}_\lambda = \rho_\lambda F^{(0)} + (\mathcal{F}_\lambda - \rho_\lambda F^{(0)})$ converges to $\mathcal{F} = \rho F^{(0)} + 0$ in the sense of (19).

Next multiplying (20) by $\eta(v)$ and integrating with respect to v give

$$\lambda \int_V \mathcal{F}_\lambda \eta(v) \, \mathrm{d}\mu(v) + \int_V a(v) \cdot \nabla_y \mathcal{F}_\lambda \, \eta(v) \, \mathrm{d}\mu(v) - \int_V \mathcal{L}^{(0)}(\mathcal{F}_\lambda) \eta(v) \, \mathrm{d}\mu(v) = \int_V h_\lambda \eta(v) \, \mathrm{d}\mu(v).$$

Then we rewrite the second integral of the left-hand side as

$$\theta \nabla_y \rho_\lambda + \mathrm{Div}_y \bigg(\int_V \frac{a(v) \otimes a(v)}{|a(v)|} \big(\mathcal{F}_\lambda - \rho_\lambda F^{(0)}\big) \, \mathrm{d}\mu(v) \bigg).$$

This leads to a relation looking like

$$\theta\nabla_y\rho_\lambda = I_\lambda + \mathrm{div}_y(II_\lambda),$$

where I_λ and II_λ tend strongly to 0 in $L^2(Y)$. Therefore, since $\theta(x)$ is invertible and does not depend on y (see Lemma 2), $\nabla_y\rho_\lambda$ is compact in $H^{-1}(Y)$, and, in turn, ρ_λ is compact in $L^2(Y)$ (since the y-average of ρ_λ vanishes).

This proves that the convergence $\mathcal{F}_\lambda \to \mathcal{F} = \rho F^{(0)}$ holds strongly in $\mathbb{H}$ and leads to a contradiction. Indeed, repeating previous arguments, we can pass to the limit in (20). We obtain

$$a(v)\cdot\nabla_y\mathcal{F} - \mathcal{L}^{(0)}(\mathcal{F}) = 0 = a(v)F^{(0)}\cdot\nabla_y\rho$$

which implies that ρ is constant. On the other hand, integrating (20) we get

$$\lambda\int_Y\int_V \mathcal{F}_\lambda\,\mathrm{d}y\,\mathrm{d}\mu(v) + 0 = \int_Y\int_V h_\lambda\,\mathrm{d}y\,\mathrm{d}\mu(v) = 0 = \lambda\int_Y \rho_\lambda\,\mathrm{d}y$$

by using the null average condition on h. It follows that

$$\int_Y \rho\,\mathrm{d}y = \rho = \lim_{\lambda\to 0}\int_Y \rho_\lambda\,\mathrm{d}y = 0.$$

Hence $\mathcal{F}_\lambda$ would converge strongly to 0 while this sequence has norm 1. This ends the proof of Proposition 2. □

Appendix B. Proof of Lemma 3

In this annex we are concerned with the determination of the orthogonal set $E_\#^\perp$ of

$$E_\# = \left\{\phi\in\mathcal{D}_\#,\ \mathrm{div}_y\left(\int_V a(v)F^{(0)}\phi\,\mathrm{d}\mu(v)\right) = 0\right\}.$$

Clearly, if T reads $vF^{(0)}\cdot\nabla_y Q$ where Q does not depend on the v variable, then T belongs to $E_\#^\perp$. Let us prove this actually characterizes $E_\#^\perp$. Let $T\in\mathcal{D}'_\#$ be a distribution such that

$$\forall\phi\in E_\#,\quad \int_0^T\int_Y\int_{\mathbb{R}^N}\int_V T\phi(t,x,y,v)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\mu(v) = 0.$$

In the above formula and in the sequel the integral has to be taken as a duality between distribution and test functions. Our characterization of T relies on Fourier series. For $n\in\mathbb{Z}^N$ the formula

$$\widehat{T}(t,x,n,v) = \int_Y T(t,x,y,v)\,\mathrm{e}^{-2\mathrm{i}\pi y\cdot n}\,\mathrm{d}y$$

defines a distribution on $t,\ x,\ v$ and we have

$$\lim_{M\to\infty}\sum_{|n|\leqslant M}\widehat{T}(t,x,n,v)\,\mathrm{e}^{2i\pi y\cdot n} = T\quad \text{in } \mathcal{D}'_\#.$$

By taking first ϕ constant with respect to y we obtain $\widehat{T}(t,x,0,v) = 0$. Second, for a fixed $n \neq 0$ we choose $\phi = \beta(x,v)\zeta(t,x)\,\mathrm{e}^{-2\mathrm{i}\pi y\cdot n}$ where $\beta(x,v)$ is any function in the orthogonal set of *Vect*$(n\cdot a(v)F^{(0)})$ in $L^2(\mathrm{d}\mu(v))$, and $\zeta \in \mathcal{D}((0,T)\times\mathbb{R}^N)$. (Note that that $v \mapsto n\cdot a(v)F^{(0)}$ cannot be the null function in view of the condition (C).) For any ζ we have

$$\int_0^T\int_{\mathbb{R}^N}\int_V \widehat{T}(t,x,n,v)\beta(x,v)\zeta(t,x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mu(v) = 0,$$

which means

$$\int_V \widehat{T}(t,x,n,v)\beta(x,v)\,\mathrm{d}\mu(v) = 0 \in \mathcal{D}'\big((0,T)\times\mathbb{R}^N\big).$$

It implies that

$$\widehat{T}(t,x,n,v) = n\cdot a(v)F^{(0)}\widehat{Q}(t,x,n)$$

with $\widehat{Q}(t,x,n) \in \mathcal{D}'((0,T)\times\mathbb{R}^N)$. Multiplying by $\eta(v)$ and integrating give

$$\widehat{Q}(t,x,n)\theta(x)n = \int_V \widehat{T}(t,x,n,v)\eta(v)\,\mathrm{d}\mu(v)$$

and thus, it follows that

$$\widehat{Q}(t,x,n) = \frac{n}{|n|^2}\cdot\theta^{-1}(x)\bigg(\int_V \widehat{T}(t,x,n,v)\eta(v)\,\mathrm{d}\mu(v)\bigg)$$

holds for all $n \neq 0$. Let us set $Q = \sum_{n\neq 0}\widehat{Q}(t,x,n)\frac{1}{2\mathrm{i}\pi}\,\mathrm{e}^{2\mathrm{i}\pi y\cdot n}$ which converges in $\mathcal{D}'_{\#}$. Finally, we have

$$a(v)F^{(0)}\cdot\nabla_y Q = \sum_{n\neq 0} n\cdot a(v)F^{(0)}\widehat{Q}(t,x,n)\,\mathrm{e}^{2\mathrm{i}\pi y\cdot n} = T - \widehat{T}(t,x,0,v) = T.$$

This achieves the proof. □

Appendix C. Proof of Lemma 1

Let $\xi \in \mathbb{R}^N$. On the one hand, by using the definition of χ and Proposition 1(ii), we have,

$$\begin{aligned}
D\xi\cdot\xi &= \int_V\int_Y a(v)\cdot\xi\chi\cdot\xi\,\mathrm{d}\mu(v)\,\mathrm{d}y = \int_V\int_Y \big(a(v)\cdot\nabla_y(\chi\cdot\xi) - \mathcal{L}^{(0)}(\chi\cdot\xi)\big)\frac{\chi\cdot\xi}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}y\\
&= -\int_V\int_Y \mathcal{L}^{(0)}(\chi\cdot\xi)\frac{\chi\cdot\xi}{F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}y = \mathcal{B}^{(0)}(\chi\cdot\xi,\chi\cdot\xi)\\
&\geqslant \int_V\int_Y \Big(\mathcal{L}^{(0)}(\chi\cdot\xi)\Big)^2\,\frac{1}{\Sigma^{(0)}F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}y.
\end{aligned}$$

On the other hand, Cauchy–Schwarz's inequality yields

$$\widetilde{\theta}\xi\cdot\xi = \int_V\int_Y \big(a(v)\cdot\xi\big)^2 F^{(0)}\,\mathrm{d}\mu(v)\,\mathrm{d}y = \int_V\int_Y a(v)\cdot\xi\,\mathcal{L}^{(0)}(\chi\cdot\xi)\,\mathrm{d}\mu(v)\,\mathrm{d}y$$
$$\leqslant \left(\int_V\int_Y \big(\mathcal{L}^{(0)}(\chi\cdot\xi)\big)^2\frac{1}{\Sigma^{(0)}F^{(0)}}\,\mathrm{d}\mu(v)\,\mathrm{d}y\right)^{1/2}\left(\int_V\int_Y \big(a(v)\cdot\xi\big)^2\,\Sigma^{(0)}F^{(0)}\,\mathrm{d}\mu(v)\,\mathrm{d}y\right)^{1/2}.$$

By combining these estimates and using (B2), and (A1), we get, for $\xi\neq 0$

$$\frac{(\widetilde{\theta}\xi\cdot\xi)^2}{c|\xi|^2}\leqslant D\xi\cdot\xi.$$

We notice that $\widetilde{\theta}(x)$ is positive definite by (C) and (A2) and the coefficients belong to $W^{1,\infty}(\mathbb{R}^N)$, by (B1). Hence one deduces that $D(x)\xi\cdot\xi\geqslant\alpha_K|\xi|^2$, for $x\in K$, $K\subset\mathbb{R}^N$ compact.

We now check the L^∞ estimates on $c(x)$ and $b(x)$. Thanks to Proposition 1, and equality (6), we have

$$\|\phi\|_H\leqslant 2M\kappa\big\|a(v)\cdot\nabla_x F^{(0)}\big\|_G = 2M\kappa\int_V |a(v)|^2\big|\nabla_x F^{(0)}\big|^2\frac{1}{\Sigma^{(0)}F^{(0)}}\,\mathrm{d}\mu(v).$$

Therefore, using (B1) and (A2), we get

$$\int_V \phi^2\frac{\Sigma^{(0)}}{F^{(0)}}\,\mathrm{d}\mu(v)\leqslant 2M^2\kappa,$$

which implies

$$\big|c(x)\big|\leqslant\int_Y\int_V\big|a(v)\big|\,|\phi|\,\mathrm{d}y\,\mathrm{d}\mu(v)$$
$$\leqslant\left(\int_Y\int_V\frac{|a(v)|^2F^{(0)}}{\Sigma^{(0)}}\,\mathrm{d}y\,\mathrm{d}\mu(v)\right)^{1/2}\left(\int_Y\int_V\phi^2\frac{\Sigma^{(0)}}{F^{(0)}}\,\mathrm{d}y\,\mathrm{d}\mu(v)\right)^{1/2}\leqslant 2C_2M^2\kappa,$$

thanks to assumption (B2). Same estimates hold for $b(x)$. □

Appendix D. Notes on factorization techniques

In this section, we briefly summarize how can be treated the case of a cross section $\sigma_\varepsilon(x,v,w) = \sigma(x/\varepsilon,v,w)$ that does not depend on the macroscopic scale x by using a factorization strategy. This idea is a well known engineering procedure; we refer for mathematical presentation and rigourous justification of this principle to [2,5,11]. For the sake of simplicity we assume here $a(v)=v$. Let $\Theta(y,v)$ be the (normalized) solution of the cell problem

$$v\cdot\nabla_y\Theta - Q(\Theta) = 0,$$

which has been proved to exists (at least when we can apply average lemma and V is bounded) by Bal in [5]. Note that $\Theta(y, v)$ now depends on the fast variable y, but does not depend on x anymore: this case is therefore like the opposite case from the one we have studied in this paper.

Then, we consider the factorized function $g_\varepsilon = f_\varepsilon/\Theta_\varepsilon$, with $\Theta_\varepsilon(x, v) = \Theta(x/\varepsilon, v)$. This new unknown satisfies

$$\partial_t g_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x g_\varepsilon = \frac{1}{\varepsilon^2} \widetilde{Q}_\varepsilon(g_\varepsilon)$$

with

$$\widetilde{Q}_\varepsilon(g) = \int_V \frac{\sigma(x/\varepsilon, v, w)}{\Theta(x/\varepsilon, v)} \big(g(x, w) - g(x, v)\big)\, \mathrm{d}\mu(w).$$

Now, the kernel of the (conservative) collision operator $\widetilde{Q}$ reduces to constants. Therefore, we can follow the strategy of [2,22] to describe the limit problem, namely $g_\varepsilon \to \rho(t, x)$ satisfying a diffusion equation

$$\partial_t \rho - \mathrm{div}_x(A \nabla_x \rho) = 0$$

whose coefficient are defined through a cell problem. In other words the behaviour of f_ε is mainly given by $\Theta(x/\varepsilon, v)\rho(t, x)$.

This factorization strategy obviously breaks down as soon as the steady state Θ depends on x. It is remarkable that we can treat separately the cases where the collision operator does not depend on the macroscopic variable or the case where its kernel does not depend on the fast variable. The mixed case where Θ depends of both the micro- and macroscopic variables seems to be much more difficult.

References

[1] G. Allaire, Homogenization and two scale convergence, *SIAM J. Math. Anal.* **23** (1992), 1482–1518.

[2] G. Allaire and G. Bal, Homogenization of the criticality spectral equation in neutron transport, *Model. Math. Anal. Numer.* **33** (1999), 721–746. Announced in: Homogénéisation d'une équation spectrale du transport neutronique, *C. R. Acad. Sci.* **325** (1997), 1043–1048.

[3] G. Allaire and Y. Capdeboscq, Homogenization of a spectral problem for a multigroup neutronic diffusion model, *Comput. Methods Appl. Mech. Engrg.* **187** (2000), 91–117.

[4] G. Allaire and F. Malige, Analyse asymptotique spectrale d'un problème de diffusion neutronique, *C. R. Acad. Sci* **324** (1997), 939–944.

[5] G. Bal, Couplage d'équations et homogénéisation en transport neutronique, Thèse de doctorat de l'Université Paris 6, 1997.

[6] C. Bardos, L. Dumas, P. Gérard and F. Golse, Systèmes dynamiques et équations cinétiques: résultats et perspectives, in: *Les Grands Systèmes des Sciences et de la Technologie* (en hommage à R. Dautray), J. Horowitz and J.L. Lions, eds, RMA Res. Notes Appl. Math., Vol. 28, Masson, 1994, pp. 49–62.

[7] C. Bardos, F. Golse and B. Perthame, The Rosseland approximation for the radiative transfer equations, *Comm. Pure Appl. Math.* **40** (1987), 691–721 and *Comm. Pure Appl. Math.* **42** (1989), 891–894.

[8] C. Bardos, F. Golse, B. Perthame and R. Sentis, The nonaccretive radiative transfer equations: existence of solutions and Rosseland approximations, *J. Funct. Anal.* **77** (1988), 434–460.

[9] C. Bardos, R. Santos and R. Sentis, Diffusion approximation and computation of the critical size, *Trans. Amer. Math. Soc.* **284** (1984), 617–649.

[10] A. Bensoussan, J.-L. Lions and G. Papanicolaou, Boundary layers and homogenization of transport processes, *Publ. RIMS Kyoto Univ.* **15** (1979), 53–157.

[11] Y. Capdeboscq, Homogénéisation des modèles de diffusion en neutronique, Thèse Université Paris 6, 2000.

[12] Y. Capdeboscq, Homogenization of a spectral problem with drift, Preprint 2000. Homogenization of a diffusion equation with drift, *C. R. Acad. Sci.* **327** (2000), 807–812.
[13] C. Cercignani, *The Boltzmann Equation and its Applications*, Appl. Math. Sci., Vol. 67, Springer, 1988.
[14] R. Dautray and J.-L. Lions, *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, Vol. 3, Masson, 1985.
[15] P. Degond, T. Goudon and F. Poupaud, Diffusion limit for non-homogeneous and non-microreversible processes, *Indiana Univ. Math. J.* **49**(3) (2000), 1175–1198.
[16] L. Dumas and F. Golse, Homogenization of transport equations, *SIAM J. Appl. Math.* **60** (2000), 1447–1470.
[17] P. Gérard and F. Golse, Averaging regularity results for pdes under transversality assumptions, *Comm. Pure Appl. Math.* **45** (1992), 1–26.
[18] F. Golse, Remarques sur l'homogénéisation des équations de transport, *C. R. Acad. Sci.* **305** (1987), 801–804.
[19] F. Golse, Particle transport in nonhomogeneous media, in: *Mathematical Aspects of Fluids and Plasma Dynamics*, Proc. Salice Terme 1988, Boffi, Rionero and Toscani, eds, Lecture Notes in Math., Vol. 1460, Springer, 1991, pp. 152–170.
[20] F. Golse, From kinetic to macroscopic models, Session "Etat de la Recherche" SMF, Orléans, 4–6 juin 1998.
[21] F. Golse, P.-L. Lions, B. Perthame and R. Sentis, Regularity of the moments of the solution of a transport equation, *J. Funct. Anal.* **76** (1988), 110–125.
[22] F. Golse and F. Poupaud, Limite fluide des équations de Boltzmann des semi-conducteurs pour une statistique de Fermi–Dirac, *Asymptotic Anal.* **6** (1992), 135–160.
[23] T. Goudon and F. Poupaud, Approximation by homogenization and diffusion of kinetic equations, *Comm. Partial Differential Equations* **26**(3–4) (2001), 537–569.
[24] E. Larsen and J. Keller, Asymptotic solution of neutron transport processes for small free paths, *J. Math. Phys.* **15** (1974), 75–81.
[25] E. Larsen and M. Williams, Neutron drift in heterogeneous media, *Nuclear Sci. Engrg.* **65** (1978), 290–302.
[26] P.-L. Lions and G. Toscani, Diffuse limit for finite velocity Boltzmann kinetic models, *Rev. Mat. Iberoamericana* **13**(3) (1997), 473–513.
[27] F. Malvagi, C. Levermore and G. Pomraning, Asymptotics limits of a statistical transport description, in: *Modern Math. Methods in Transport Theory*, W. Greenberg and J. Polewczak, eds, Oper. Theory Adv. Appl., Vol. 51, Birkhäuser, 1991, pp. 200–207.
[28] A. Mellet, Derivation of a non-linear regime without detailed balance principle, *Monatsh. Math.*, to appear.
[29] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20** (1989), 608–623.
[30] R. Petterson, Existence theorems for the linear, space-inhomogeneous transport equation, *IMA J. Appl. Math.* **30** (1983), 81–105.
[31] F. Poupaud, Diffusion approximation of the linear semiconductor Boltzmann equation: Analysis of boundary layers, *Asymptotic Anal.* **4** (1991), 293–317.
[32] H. Shaefer, *Banach Lattices and Positive Operators*, Springer, 1974.
[33] L. Tartar, Compensated compactness and applications to pde, in: *Herriot–Watt Symp.*, Vol. IV, R. Knopps, ed., Res. Notes Math., Vol. 39, Pitman, 1979, pp. 136–212.
[34] L. Tartar, Compacité par compensation: résultats et perspectives, in: *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar*, Vol. IV, H. Brezis, J.L. Lions and D. Cioranesu, eds, Res. Notes Math., Vol. 84, Pitman, 1983, pp. 350–369.
[35] M.L. Tayeb, Analyse mathématique de quelques équations de Boltzmann des semi-conducteurs, Thése École Nationale d'Ingénieurs de Tunis, 2001.
[36] E. Wigner, *Nuclear Reactor Theory*, Amer. Math. Soc., 1961.