Diffusion limit of a semiconductor Boltzmann-Poisson system

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Abstract. The paper deals with the diffusion limit of the initial-boundary value problem for the multi-dimensional semiconductor Boltzmann-Poisson system. Here, we generalize the one dimensional results obtained in [6] to the case of several dimensions using global renormalized solutions. The method of moments and a velocity averaging lemma are used to prove the convergence of the renormalized solutions to the semiconductor Boltzmann-Poisson system towards a global weak solution of the Drift-Diffusion-Poisson model.

Key words. Kinetic transport equations, semiconductor Boltzmann-Poisson system, Drift-Diffusion model, Entropy dissipation, moment method, velocity averaging lemma, renormalized solution,. . .

1 Introduction and Main results

In this paper, we study the diffusion limit of the initial-boundary value problem for the semiconductor Boltzmann-Poisson system (see [27, 31]). The model we consider here is associated with a linear low density approximation of the electron-phonon collisions. In other words it is a low density approximation of the physically correct Fermi-Dirac system. When the potential is given and is smooth enough, Poupaud [31] has proved the convergence of the rescaled Boltzmann equation towards a linear Drift-Diffusion model. Let us recall that the Drift-Diffusion equation is a standard model for semiconductors physics, and suited for numerical computations since it does not involve the kinetic variable v. We refer to [10, 16, 17, 27] for a discussion about Drift-Diffusion models.

In the one dimensional case, the convergence results of [31] are extended in [6] to the semiconductor Boltzmann system with a Poisson coupling. In [6] the solutions considered are defined in a weak sense [1, 5, 6]. The entropy inequality and a hybrid-Hilbert expansion are used to approximate the entropy production term due to the boundary and allow to prove the convergence of the rescaled Boltzmann equation towards the Drift-Diffusion for self-consistent potential. The method is based essentially on the fact that solutions to the limit system are smooth, which gives useful uniform bounds on all terms of the Hilbert expansion and then allows to obtain a strong convergence and also to exhibit a convergence rate. The multi-dimensional case is different. Indeed, if we want to work with global solutions, we can only deal with solutions to the semiconductor Boltzmann-Poisson which are defined in the renormalized sense (see [12, 29]). Indeed, due to the presence of the Poisson term and the Boltzmann collision term in the equation for the density, we can not prove global uniform bounds in any L^p space for $p > 1$. On one hand, we can see that if we remove the collision term than we can easily get a priori estimates for f in any $L^\infty((0,T); L^p(dxdv))$, $1 \leq p \leq \infty$. On the other hand, if we remove the Poisson term, then we can get a priori estimates for f in any $L^\infty((0,T); L^p(dxM^{1-p}dv))$, $1 \leq p < \infty$. Hence, we can see that, mathematically, these two terms can not be treated in the same spaces. This is one of the major mathematical difficulties of this model.

Before recalling the Boltzmann-Poisson system, let us mention that Drift-Diffusion models can also be derived from other singular limits. We refer for instance to [32] where the Drift-Diffusion model is derived from a Vlasov-Fokker-Planck system.

1.1 Formulation of the problem

In this paper we study the parabolic limit of the rescaled Boltzmann-Poisson system. Hence, the rescaled system, defined on the phase space $\Omega = \omega \times \mathbb{R}^d$ where $d \geq 1$, reads as follows

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\Big(v.\nabla_x f^\varepsilon - \nabla_x(\phi^\varepsilon + \tilde{\phi}_b).\nabla_v f^\varepsilon\Big) - \frac{Q(f^\varepsilon)}{\varepsilon^2} = 0, \quad (x,v) \in \Omega, \tag{1}$$

where ε is a small parameter related to the mean free path and $f^\varepsilon(t,x,v)$ denotes the electron distribution function. The time variable t is nonnegative. The position x

belongs to an open set ω of $\mathbb{R}^d$, assumed to be smooth and bounded and the velocity v belongs to $\mathbb{R}^d$. This equation has to be complemented with initial and boundary conditions which take into account how particles are injected in the semiconductor device. We assume that the boundary $\partial\omega$ is sufficiently smooth. We denote by $n(x)$ the outward unit normal vector at the position $x \in \partial\omega$ and $d\sigma_x$ the Lebesgue measure on $\partial\omega$. The outgoing and incoming parts are defined as

$$\Gamma^{\pm} = \{(x,v) \in \partial\Omega;\ \pm v.n(x) > 0\}.$$

The initial data is assumed to be known and depend on the mean free path ε:

$$f^\varepsilon(0,x,v) = f_0^\varepsilon(x,v), \qquad (x,v) \in \Omega. \tag{2}$$

The incoming boundary data is assumed to be a well prepared function [5, 6, 31], in the sense that

$$f^\varepsilon(t,x,v) = f_b(t,x,v) := \rho_b(t,x)M(v), \qquad (x,v) \in \Gamma^- \tag{3}$$

where M is the normalized Maxwellian

$$M(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}$$

and $\rho_b(t,x)$ is a boundary data. The precise assumptions we choose on the initial and boundary conditions will be detailed later on. The linear operator Q describes physical conservation properties during collisions. Here, we only assume that the charge is conserved during the collision [2, 27]. A typical model for such situation is the linear approximation of the electron-phonon interaction, given by

$$Q(f)(v) = \int_{\mathbb{R}^d} \sigma(v,v')(M(v)f(v') - M(v')f(v))dv'. \tag{4}$$

The cross section σ is assumed to be symmetric (micro-reversibility principle) and bounded from above and below:

$$\begin{cases} \sigma(v,v') = \sigma(v',v), \quad (v,v') \in \mathbb{R}^{2d}, \\ \\ \exists\ \sigma_1,\, \sigma_2 > 0 \quad / \quad 0 < \sigma_1 \le \sigma \le \sigma_2. \end{cases} \tag{5}$$

Here, we are making an other approximation, by assuming that σ is bounded from below and above instead of taking delta measures concentrated on balls of constant kinetic energies (see [4, 7]).

The mean free path is defined to be an average of the collision frequency $\lambda(v)$ given by

$$\lambda(v) = \int_{\mathbb{R}^d} \sigma(v,v')M(v')dv'.$$

Here, for all v, we have $\sigma_1 \le \lambda(v) \le \sigma_2$. Hence, the mean free path in (1) is of order $1/\varepsilon$. We refer to [5, 6, 31] for the detailed properties of these kind of collision kernels. We assume that the potential ϕ^ε is self consistent:

$$\begin{cases} -\Delta_x \phi^\varepsilon = \displaystyle\int_{\mathbb{R}^d} f^\varepsilon dv, \\ \\ {\phi^\varepsilon}_{|\partial\omega} = 0. \end{cases} \tag{6}$$

The potential $\tilde{\phi}_b$ is given in $\overline{\omega}$. It takes for instance into account the distribution of positive background charges.

We define the charge density ρ^ε and the current density j^ε associated to the distribution f^ε by

$$\rho^\varepsilon(t,x) = \int_{\mathbb{R}^d} f^\varepsilon(t,x,v)dv, \qquad j^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} v f^\varepsilon(t,x,v)dv.$$

1.2 Assumptions and preliminaries

Throughout the paper we shall make the following assumptions and notations

A1: $f_0^\varepsilon \geq 0$, $\displaystyle\int_\Omega f_0^\varepsilon(1+|v|^2+|\log f_0^\varepsilon|) \leq C$ and $\phi^\varepsilon(t=0)$ is bounded in $H^1(\omega)$.

A2: $(\sqrt{\rho_b}, \partial_t\rho_b) \in L^\infty_{loc}(\mathbb{R}^+;\ H^{1/2}(\partial\omega)\times L^\infty(\partial\omega))$ and the density is bounded from above and below : there exist $\underline{c}$ and $\overline{c}$ such that $0 < \underline{c} \leq \rho_b(.,x) \leq \overline{c}$, for $x \in \partial\omega$.

A3: $\tilde{\phi}_b \geq 0$ and $(\tilde{\phi}_b, \partial_t\tilde{\phi}_b) \in L^\infty_{loc}(\mathbb{R}^+;\ W^{1,\infty}(\overline{\omega})\times L^\infty(\overline{\omega}))$.

We define the total charge (or mass), the kinetic energy and two distances to the local equilibrium by

$$\mathcal{M}^\varepsilon(t) = \int_\Omega f^\varepsilon(t,x,v)dxdv, \qquad \mathcal{K}^\varepsilon(t) = \int_\Omega \frac{|v|^2}{2} f^\varepsilon(t,x,v)dxdv,$$

$$\mathcal{R}^\varepsilon(t) = \frac{1}{2}\int_0^t\int_\Omega \left(\sqrt{f^\varepsilon} - \sqrt{\rho^\varepsilon M}\,\right)^2 dxdvds \tag{7}$$

$$\mathcal{R}_1{}^\varepsilon(t) = \frac{1}{2}\int_0^t\int_\Omega (f^\varepsilon - \rho^\varepsilon M\ )(\log f^\varepsilon - \log(\rho^\varepsilon M)\ )\,dxdvds.$$

The entropy and entropy fluxes through the inflow and outflow boundaries are defined by

$$\begin{aligned} \mathcal{E}^\varepsilon(t) &= \frac{1}{2}\|\nabla_x\phi^\varepsilon(t)\|^2_{L^2(\omega)} + \int_\Omega f^\varepsilon\left(\log f^\varepsilon + \frac{|v|^2}{2} + \tilde{\phi}_b\right)(t), \\ I^\pm_\varepsilon(t) &= \int_0^t\int_{\Gamma^\pm} f^\varepsilon\left(\log f^\varepsilon + \frac{|v|^2}{2} + \phi_b\right)|v.n(x)|\,d\sigma_x dvds. \end{aligned} \tag{8}$$

We also define the quasi-Fermi level (defined on the boundary $\partial\omega$)

$$\mathcal{E}_F(t,x) = \log\left(\frac{\rho_b(t,x)}{(2\pi)^{d/2}}\right) + \tilde{\phi}_b(t,x). \tag{9}$$

Let us recall two lemmas about the collision kernel (see [31])

Lemma 1.1 *(H-Theorem). Assume that* (5) *holds, then the operator Q is bounded in $L^1(dv)$ and satisfies for all $f \in L^1(dv)$, $f \geq 0$ and $f(|\log f| + |v|^2) \in L^1(dv)$*

$$\int_{\mathbb{R}^d} Q(f) = 0 \quad \textit{and} \quad \mathcal{H}(f) = \int_{\mathbb{R}^d} Q(f)\log\left(\frac{f}{M}\right) \leq -\frac{\sigma_1}{2}\int_{\mathbb{R}^d}\left(\sqrt{f} - \sqrt{\rho\,M}\right)^2$$

where $\rho = \int f(v)dv$. *Moreover,*

$$\mathcal{H}(f) = 0 \Leftrightarrow Q(f) = 0 \Leftrightarrow f(v) = \rho\, M(v).$$

Lemma 1.2 . *Assume that* (5) *holds, then*

1. $-Q$ *is a bounded, symmetric, nonnegative operator on* $L^2(\mathbb{R}^d;\ M^{-1}dv)$,
2. $\mathcal{K}erQ = \mathbb{R}\, M$,
3. $-Q$ *is coercive on* $\mathcal{R}(Q) = \mathcal{K}erQ^{\perp}$.

1.3 Statement of the result

Our motivation in this work is to prove the convergence of renormalized solutions $(f^{\varepsilon}, \phi^{\varepsilon})$ to (1–6) towards $(\rho M, \phi)$ where (ρ, ϕ) satisfies the following Drift-Diffusion-Poisson system [16, 21, 31]:

$$\textbf{(DD-P)} \quad \begin{cases} \partial_t \rho + \nabla_x . J(\rho, \phi) = 0, \\[2mm] J(\rho, \phi) = -\mathbf{D}[\nabla_x \rho + \rho \nabla_x(\phi + \tilde{\phi_b})], \\[2mm] \mathbf{D} = -\displaystyle\int_{\mathbb{R}^d} v \otimes\, Q^{-1}(vM)dv > 0, \\[2mm] -\Delta_x \phi = \rho, \\[2mm] \rho(t=0) = \rho_0, \quad (\rho, \phi)_{\,|\partial\omega} = (\rho_b, 0). \end{cases}$$

Definition 1.3 *We say that* (ρ, ϕ) *is a weak solution to the Drift-Diffusion-Poisson system* ***(DD-P)*** *if*

$$\begin{aligned} &\rho \in L^{\infty}(0, T;\, \mathrm{LlogL}(\omega)) \cap L^2(0, T;\ L^2(\omega)), \\ &\sqrt{\rho} \in L^2(0, T;\ H^1(\omega)), \\ &\partial_t \rho \in L^1(0, T;\ W^{-1,1}(\omega)), \\ &\phi \in L^2(0, T;\ H^1_0(\omega)). \end{aligned}$$

and (ρ, ϕ) *satisfies* ***(DD-P)*** *in the weak sense.* □

We recall here the definition of the space $\mathrm{LlogL}(\omega)$,

$$\mathrm{LlogL}(\omega) = \{f | f \geq 0 \,\text{and} \int_{\omega} [\, f\,(1 + |\mathrm{log} f|)] \text{ is finite}\} \tag{10}$$

and that $\rho \in L^{\infty}(0, T;\, \mathrm{LlogL}(\omega))$ if and only if $\int_{\omega} \rho(t)\,(1 + |\log \rho(t)|)\, dx \leq C$ where C is independent of $t \in (0, T)$.

We also point out that due to the fact that $\partial_t \rho \in L^1(0, T;\ W^{-1,1}(\omega))$, we deduce that ρ is continuous in time with values in $W^{-1,1}(\omega)$ and hence the initial data for ρ makes sense. The main result of this paper is the following theorem

Theorem 1.4 *Assume that Assumptions (A1), (A2) and (A3) hold. Let* $(f^\varepsilon, \phi^\varepsilon)$ *be a renormalized solution of (1–6), (in the sense of Theorem 2.2). Then,*

$$\begin{array}{lll} f^\varepsilon \to \rho M & \text{in } L^1((0,T)\times\Omega), & \\ \phi^\varepsilon \to \phi & \text{in } L^2((0,T); W^{1,p}(\omega)), & \forall\, p<2. \end{array} \tag{11}$$

where (ρ, ϕ) *is a weak solution of the Drift-Diffusion-Poisson system* ***(DD-P)****. Moreover,*

$$\phi \in L^\infty(0,T;\; H_0^1(\omega)) \cap L^2(0,T;\; H^2(\omega)).$$

□

The proof of this Theorem is as follows. In Section 2 we prove the existence of renormalized solutions to the semiconductor Boltzmann-Poisson system. In Section 3 we establish some a priori uniform estimates. These estimates generalize the estimates obtained in [6]. To get the convergence we argue in a different manner as in [6]. Indeed, in the one dimensional case the energy estimate of section 3 and the convergence are deduced from an hybrid-Hilbert expansion which is based on the regularity of the limiting system. In the present case, the solution to the Drift-Diffusion-Poisson is not regular enough and the solutions of the initial system are only renormalized. Instead, the method of moment and velocity averaging are used to pass to the limit ($\varepsilon \to 0$). In section 4, we use a velocity averaging lemma to prove the compactness of the charge density ρ. In section 5, we pass to the limit weakly in the equation. In section 6, we recover the boundary condition for ρ. Finally, section 7 is devoted to the proof of the regularity estimates on (ρ, ϕ) and that the limit solution (ρ, ϕ) is a weak solution of (**DD-P**).

2 Existence of renormalized solutions

For the existence of renormalized solutions to the full Boltzmann-Poisson system we refer to [12, 29]. It is noteworthy that even though the Boltzmann kernel we are considering here is linear, the combination of the Boltzmann term and the Poisson term makes the existence of weak solutions to (1–6) with uniform bounds a difficult problem and we were not able to construct such kind of solutions. This is coming from the fact that the Poisson term can be well treated in $L^p(dv)$ type of spaces whereas the linear Boltzmann term can be well treated in $L^p(M^{1-p}dv)$. This incompatibility is responsible for the lack of estimates. We notice then, that the entropy bound given in (15) is not enough to give a sense to $\nabla_x \phi^\varepsilon f_\varepsilon$.

Before stating an existence theorem for (1–6) let us give a definition for renormalized solution or more precisely the definition we are going to use.

Definition 2.1 *We say that* $(f^\varepsilon, \phi^\varepsilon)$ *is a renormalized solution to the semiconductor Boltzmann-Poisson system if it satisfies*

1. $\forall\, \beta \in \mathcal{C}^1(\mathbb{R}^+)$, $|\beta(t)| \le C(\sqrt{t}+1)$, *and* $|\beta'(t)| \le C$, $\beta(f^\varepsilon)$ *is a weak solution*

of

$$\begin{cases} \varepsilon\partial_t\beta(f^\varepsilon) + v.\nabla_x\beta(f^\varepsilon) - \nabla_v.(\nabla_x(\phi^\varepsilon + \tilde{\phi}_b)\beta(f^\varepsilon)) = \beta'(f^\varepsilon)\frac{Q(f^\varepsilon)}{\varepsilon} \\ \beta(f^\varepsilon)_{|\Gamma^-} = \beta(f_b^\varepsilon), \\ \beta(f^\varepsilon)(t=0) = \beta(f_0^\varepsilon), \end{cases} \tag{12}$$

2. $\forall\ \lambda > 0$, $\theta_{\varepsilon,\lambda} = \sqrt{f^\varepsilon + \lambda M}$ *satisfies*

$$\varepsilon\partial_t\theta_{\varepsilon,\lambda} + v.\nabla_x\theta_{\varepsilon,\lambda} - \nabla_v.[\nabla_x(\phi^\varepsilon + \tilde{\phi}_b)\theta_{\varepsilon,\lambda}] = \frac{Q(f^\varepsilon)}{2\varepsilon\theta_{\varepsilon,\lambda}} + \frac{\lambda M}{2\theta_{\varepsilon,\lambda}}v.\nabla_x(\phi^\varepsilon + \tilde{\phi}_b). \tag{13}$$

Theorem 2.2 *The semiconductor Boltzmann-Poisson system (1–6) has a renormalized solution in the sense of definition 2.1 which satisfies in addition*

1. *the continuity equation*
$$\partial_t\rho^\varepsilon + \nabla_x.j^\varepsilon = 0, \tag{14}$$
2. *the entropy inequality*
$$\begin{aligned}&\left[\int_\Omega f^\varepsilon\left(\tilde{\phi}_b + \frac{|v|^2}{2} + \log f^\varepsilon\right) + \frac{1}{2}\|\nabla\phi^\varepsilon\|_{L^2}^2\right]_0^t - \frac{1}{\varepsilon^2}\int_0^t\int_\Omega Q(f^\varepsilon)\log\left(\frac{f^\varepsilon}{M}\right) \\ &\leq \int_0^t\int_\Omega \partial_t\tilde{\phi}_b f^\varepsilon - \frac{1}{\varepsilon}\int_0^t\int_{\Gamma^+\cup\Gamma^-} f^\varepsilon\left(\phi_b + \frac{|v|^2}{2} + \log f^\varepsilon\right)(v.n(x)).\end{aligned} \tag{15}$$

Proof. For the convenience of the reader, we give an idea of the proof in the Appendix. We also refer to [29] for more details.

3 Uniform energy estimates

Lemma 3.1 *Assume that assumptions* **A1**, **A2** *and* **A3** *are satisfied. Then, any renormalized solution* $(f^\varepsilon, \phi^\varepsilon)$ *of the semiconductor Boltzmann-Poisson system (1-6) satisfies*

$$\mathcal{M}^\varepsilon(t) + \mathcal{K}^\varepsilon(t) + \|\nabla\phi^\varepsilon(t)\|_{L^2}^2 + \frac{\mathcal{R}_1^\varepsilon(t)}{\varepsilon^2} + \int_0^t \|j^\varepsilon(s)\|_{L^1}ds\ \leq C_T \tag{16}$$

uniformly in ε, *where* $\mathcal{M}^\varepsilon$, $\mathcal{K}^\varepsilon$ *and* $\mathcal{R}_1^\varepsilon$ *are defined in (7).*

Proof. Starting from the entropy inequality (15), one can write the entropy dissipation in the following form

$$\int_{\mathbb{R}^d} Q(f^\varepsilon)\log\left(\frac{f^\varepsilon}{M}\right)dv = -\frac{1}{2}\int_{\mathbb{R}^{2d}}\sigma MM'\left(\log\frac{f^\varepsilon(v')}{M(v')} - \log\frac{f^\varepsilon(v)}{M(v)}\right)\left(\frac{f^\varepsilon(v')}{M(v')} - \frac{f^\varepsilon(v)}{M(v)}\right)dvdv'.$$

Using the Jensen inequality

$$\int_{\mathbb{R}^d} Q(f^\varepsilon)\log\left(\frac{f^\varepsilon}{M}\right) \le -\frac{\sigma_1}{2}\int_{\mathbb{R}^d} M\left(\log\rho^\varepsilon - \log\frac{f^\varepsilon(v)}{M(v)}\right)\left(\rho^\varepsilon - \frac{f^\varepsilon(v)}{M(v)}\right).$$

Applying the relation

$$(a-b)\log(a/b) \ge \left(\sqrt{a}-\sqrt{b}\right)^2 \tag{17}$$

we obtain

$$-\int_0^t\int_\Omega Q(f^\varepsilon)\left(\log f^\varepsilon + \frac{|v|^2}{2}\right) \ge \sigma_1\mathcal{R}_1^\varepsilon(t) \ge \sigma_1\mathcal{R}^\varepsilon(t). \tag{18}$$

Moreover, we have also to approximate the entropy production term by the boundary, $(I_\varepsilon^+ - I_\varepsilon^-)(t)$, defined in (7). We write this quantity as follows

$$(I_\varepsilon^+ - I_\varepsilon^-) = \int_0^t\int_{\Gamma^+}|v.n|\left[f^\varepsilon(v)\log\left(\frac{f^\varepsilon(v)}{f^\varepsilon(-v)}\right) + (f^\varepsilon(v)-f^\varepsilon(-v))\,\mathcal{E}_F(s,x)\right].$$

Using the inequality $a\log(a/b) \ge a-b$ for $a,b>0$, we obtain

$$(I_\varepsilon^+ - I_\varepsilon^-)(t) \ge \int_0^t\int_{\Gamma^+}|v.n(x)|[f^\varepsilon(v)-f^\varepsilon(-v)]\,(1+\mathcal{E}_F(t,x))d\sigma_x\,dv\,ds. \tag{19}$$

Then, we can replace (15) according to (18) and (19) and obtain

$$[\mathcal{E}^\varepsilon(t)]_0^t + \frac{\sigma_1}{\varepsilon^2}\mathcal{R}_1^\varepsilon(t) \le \int_0^t\int_\Omega \partial_t\tilde{\phi}_b f^\varepsilon - \frac{1}{\varepsilon}\int_0^t\int_{\Gamma^+}|v.n(x)|[f^\varepsilon(v)-f^\varepsilon(-v)](1+\mathcal{E}_F) \tag{20}$$

where

$$\mathcal{E}^\varepsilon(t) = \frac{1}{2}\left\|\nabla_x\phi^\varepsilon(t)\right\|^2_{L^2(\omega)} + \int_\Omega f^\varepsilon\left(\log f^\varepsilon + \frac{|v|^2}{2} + \tilde{\phi}_b\right)(t)$$

and

$$\mathcal{E}_F(t,x) = \log\left(\frac{\rho_b(t,x)}{(2\pi)^{d/2}}\right) + \tilde{\phi}_b(t,x).$$

We extend the quasi-Fermi level on $\bar{\omega}$ (denoted by $\tilde{\mathcal{E}_F}$) and replace ρ_b by its harmonic extension, (in $\bar{\omega}$), $\tilde{\rho}_b$. According to assumptions **A2** and **A3**, $\nabla_x\tilde{\mathcal{E}_F}$ and $\partial_t\tilde{\mathcal{E}_F}$, defined on ω by

$$\begin{aligned}\partial_t\tilde{\mathcal{E}_F} &= (\partial_t\tilde{\rho}_b + \tilde{\rho}_b\,\partial_t\tilde{\phi}_b)/\tilde{\rho}_b,\\ \nabla_x\tilde{\mathcal{E}_F} &= (\nabla_x\tilde{\rho}_b + \tilde{\rho}_b\,\nabla_x\tilde{\phi}_b)/\tilde{\rho}_b\end{aligned}$$

are bounded. By multiplying (14) by $(1+\tilde{\mathcal{E}_F}(t,x))$ and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t\int_{\Gamma^+}(1+\mathcal{E}_F)(f^\varepsilon(v)-f^\varepsilon(-v))|v.n(x)| &= \int_0^t\int_\omega \partial_t\tilde{\mathcal{E}_F}\,\rho^\varepsilon + \int_0^t\int_\omega \nabla_x\tilde{\mathcal{E}_F}.j^\varepsilon\\ &- \left[\int_\omega(1+\tilde{\mathcal{E}_F})\rho^\varepsilon\right]_0^t\end{aligned} \tag{21}$$

and then (20) is equivalent to

$$[\mathcal{E}^\varepsilon(t)]_0^t + \frac{\sigma_1}{\varepsilon^2}\mathcal{R}_1^\varepsilon(t) \le \left[\int_\omega (1+\tilde{\mathcal{E}_F})\rho^\varepsilon\right]_0^t - \int_0^t\int_\omega \nabla_x \tilde{\mathcal{E}_F}.j^\varepsilon - \int_0^t\int_\omega \frac{\partial_t\tilde{\rho}_b}{\tilde{\rho}_b}\,\rho^\varepsilon$$

which implies, according to (**A1**), (**A2**) and (**A3**) that

$$\mathcal{E}^\varepsilon(t) + \frac{\mathcal{R}_1^\varepsilon(t)}{\varepsilon^2} \le C_T\left(1 + \int_0^t \mathcal{M}^\varepsilon(s)ds + \int_0^t \|j^\varepsilon(s)\|_{L^1}ds\right)$$

where C_T depends only on T. Let us estimate the current density in the following way

$$\begin{aligned}\int_0^t \|j^\varepsilon(s)\|_{L^1}ds &= \frac{1}{\varepsilon}\int_0^t\int_\omega\left|\int_{\mathbb{R}^d} v\left(\sqrt{f^\varepsilon}-\sqrt{\rho^\varepsilon M}\right)\left(\sqrt{f^\varepsilon}+\sqrt{\rho^\varepsilon M}\right)\right| \\ &\le \frac{1}{\varepsilon}\sqrt{\mathcal{R}^\varepsilon(t)}\left(\int_0^t\int_\Omega |v|^2\left(\sqrt{f^\varepsilon}+\sqrt{\rho^\varepsilon M}\right)^2\right)^{1/2}.\end{aligned}$$

The Young's inequality $(\alpha\, a^2 + \frac{1}{4\alpha}b^2 \ge ab, \quad \forall\, \alpha > 0)$, gives

$$\int_0^t \|j^\varepsilon(s)\|_{L^1}\,ds \le \frac{\alpha}{\varepsilon^2}\mathcal{R}^\varepsilon(t) + \frac{C_T}{4\alpha}\int_0^t (\mathcal{M}^\varepsilon(s) + \mathcal{K}^\varepsilon(s))ds \tag{22}$$

where α does not depend on ε (for example $\alpha = 1/2$). Then, one can deduce

$$\mathcal{E}^\varepsilon(t) + \frac{\mathcal{R}_1^\varepsilon(t)}{2\varepsilon^2} \le C_T\left(1 + \int_0^t \mathcal{M}^\varepsilon(s)ds + \int_0^t \mathcal{K}^\varepsilon(s)ds\right) \tag{23}$$

and bound $\mathcal{M}^\varepsilon$ and $\mathcal{K}^\varepsilon$ in terms of $\mathcal{E}^\varepsilon$ using that

$$\int_\Omega f^\varepsilon \log\left(\frac{f^\varepsilon}{e^{-|v|^2/4}}\right) \ge \mathcal{M}^\varepsilon - |\omega| e\int_{\mathbb{R}^d} e^{-|v|^2/4}. \tag{24}$$

Hence, we deduce

$$\mathcal{M}^\varepsilon(t) + \mathcal{K}^\varepsilon(t) + \|\nabla_x\phi^\varepsilon(t)\|_{L^2}^2 + \frac{\mathcal{R}_1^\varepsilon(t)}{\varepsilon^2} \le C_T\left(1 + \int_0^t \mathcal{M}^\varepsilon(s)ds + \int_0^t \mathcal{K}^\varepsilon(s)ds\right).$$

The Gronwall inequality leads to a uniform bound of $\mathcal{M}^\varepsilon$, $\mathcal{K}^\varepsilon$, $\varepsilon^{-2}\,\mathcal{R}_1^\varepsilon$ and $\|\nabla_x\phi^\varepsilon\|_{L^2}$. Then we get the L^1-bound on j^ε using (22). □

Corollary 3.2 *The renormalized solution satisfies*

$$\int_\Omega f^\varepsilon(1+|v|^2+|\log f^\varepsilon|) + \int_0^t\int_{\Gamma^+} f^\varepsilon(1+|v|^2+|\log f^\varepsilon|)|v.n(x)| \le C_T.$$

Moreover, f^ε and its trace $f^\varepsilon_{|_{\Gamma^+}}$ are weakly relatively compact in $L^1((0,T)\times\Omega))$ and $L^1((0,T)\times\Gamma^+,\,|v.n(x)|dtd\sigma_x dv))$ respectively. □

Proof. Let us remark that

$$\int f^\varepsilon |\log f^\varepsilon| = \int_{f^\varepsilon \geq 1} f^\varepsilon \log f^\varepsilon - \int_{f^\varepsilon \leq 1} f^\varepsilon \log f^\varepsilon. \tag{25}$$

Estimates (16), (23) and (24) imply

$$|\mathcal{E}^\varepsilon(t)| \leq C_T,$$

$$\begin{aligned} \int_{f^\varepsilon \leq 1} f^\varepsilon |\log f^\varepsilon| &= -\int_{f^\varepsilon \leq 1} f^\varepsilon \log(f^\varepsilon / e^{-|v|^2}) + \int |v|^2 f^\varepsilon \\ &\leq \int |v|^2 f^\varepsilon dx dv + |\omega| \int e^{-|v|^2} dv \leq C_T \end{aligned}$$

and

$$\int_{f^\varepsilon \geq 1} f^\varepsilon \log f^\varepsilon \leq |\mathcal{E}^\varepsilon(t)| + \int_{f^\varepsilon \leq 1} f^\varepsilon |\log f^\varepsilon| \leq C_T.$$

The Dunford-Pettis Theorem [15] implies the weak compactness of f^ε in the mentioned space. We obtain the bound and the weak compactness of $f^\varepsilon_{|_{\Gamma+}}$ by a similar argument. Indeed, from the entropy bound we can deduce that

$$\int_0^t \int_{\Gamma+} |v.n(x)| \left[\frac{f^\varepsilon(v)}{\rho_b M} \log \left(\frac{f^\varepsilon(v)}{\rho_b M} \right) \right] \rho_b M(v) dv d\sigma dt \leq C\varepsilon \tag{26}$$

and then, we can argue as above. □

The above Corollary will be used to approximate uniformly f^ε by bounded function. Indeed, let β_δ be an approximation of the identity, namely $\beta_\delta(s) = \frac{1}{\delta}\beta(\delta s)$ where β is a $\mathcal{C}_0^\infty$ function satisfying $\beta(s) = s$ for $s \leq 1$, $0 \leq \beta'(s) \leq 1$ for all s and $\beta(s) = 2$ for $s \geq 3$. As a consequence of the equi-integrability of f^ε, we deduce that $(\beta_\delta(f^\varepsilon))_{\delta,\varepsilon}$ is weakly relatively compact in $L^1((0,T) \times \Omega))$. Indeed, we have

$$\int_0^T \int_\Omega |\beta_\delta(f^\varepsilon) - f^\varepsilon| \leq C \int_{f^\varepsilon \geq 1/\delta} f^\varepsilon \to 0, \quad \text{as } \delta \to 0$$

and

$$\int_0^T \int_{\Gamma+} |\beta_\delta(f^\varepsilon) - f^\varepsilon| (v.n(x)) \leq \int_0^T \int_{\Gamma+ \cap \{f^\varepsilon \geq 1/\delta\}} |f^\varepsilon| (v.n(x)) \to 0, \quad \text{as } \delta \to 0$$

uniformly in ε. Up to extraction of a subsequence, we also have

$$\begin{aligned} \beta_\delta(f^\varepsilon) - f^\varepsilon &\to 0, \qquad \text{a.e.} \quad \text{as } \delta \to 0, \\ \beta'_\delta(f^\varepsilon) &\to 1, \qquad \text{a.e.} \quad \text{as } \delta \to 0. \end{aligned}$$

More precisely

$$\sup_{\varepsilon < 1} \|\beta_\delta(f^\varepsilon) - f^\varepsilon\|_{L^1_{t,x,v}} \to 0 \qquad \text{as } \delta \to 0. \tag{27}$$

Proposition 3.3 *The renormalized solution* $(f^\varepsilon, \phi^\varepsilon)$ *satisfies*

1. ρ^ε *is weakly relatively compact in* $L^1((0,T)\times\omega)$.
2. $\frac{Q(f^\varepsilon)}{\varepsilon}$ *is weakly relatively compact in* $L^1((0,T)\times\Omega))$.
3. $\nabla\phi^\varepsilon$ *is relatively compact in* $L^2(0,T;\ L^p(\omega))$ *for all* $1\le p<2$.

Proof. Let $\log^+ s = \max(0, \log s)$. Applying the Jensen inequality we get

$$\rho^\varepsilon \log^+\rho^\varepsilon = \int\left(\frac{f^\varepsilon}{M}\,M dv\right)\left(\log^+\int\frac{f^\varepsilon}{M}\,M dv\right) \le \int f^\varepsilon \log^+\frac{f^\varepsilon}{M} dv.$$

The uniform energy bound (16) and Corollary 3.2 lead to

$$\int_0^t\int_\omega \rho^\varepsilon(1+\log^+\rho^\varepsilon)\le C_T$$

which implies the $L^1((0,T)\times\omega)$ weak compactness of the sequence ρ^ε.
Let us define

$$r_\varepsilon := \frac{\sqrt{f^\varepsilon}-\sqrt{\rho^\varepsilon\,M}}{\varepsilon\,\sqrt{M}}.$$

Hence, from the energy bound (16), we deduce that r_ε is bounded in $L^2((0,T)\times\Omega,\ M dt dx dv)$. Extracting a subsequence if necessary, we denote by r its weak limit. Using r_ε, we can rewrite

$$f^\varepsilon = \rho^\varepsilon\,M + 2\varepsilon\,M\sqrt{\rho^\varepsilon}\,r_\varepsilon + \varepsilon^2 r_\varepsilon^2 M$$

and

$$\frac{Q(f^\varepsilon)}{\varepsilon} = 2\sqrt{\rho^\varepsilon}\,Q(r_\varepsilon\,M) + \varepsilon\,Q(r_\varepsilon^2 M)$$

where $r_\varepsilon^2 M$ and $r_\varepsilon\,M$ are respectively bounded in L^1 and $L^2(M^{-1}dv)$. The operator Q is bounded in L^1 and $L^2(M^{-1}dv)$. This implies that $\frac{Q(f^\varepsilon)}{\varepsilon}$ is bounded in $L^1_{t,x,v}$ and

$$\frac{Q(f^\varepsilon)}{\varepsilon} = 2\sqrt{\rho^\varepsilon}\,Q(r_\varepsilon\,M) + \mathbf{O}(\varepsilon)_{L^1(0,T)\times\Omega}.$$

Moreover, let $\alpha>0$, then

$$\int_A |\sqrt{\rho^\varepsilon}\,Q(r_\varepsilon\,M)| \le C\int_A \rho^\varepsilon\,M + \frac{1}{4C}\int_A\frac{Q^2(r_\varepsilon\,M)}{M}$$

and choose C such that $\frac{1}{4C}\|\frac{Q^2(r_\varepsilon\,M)}{M}\|_{L^1_{t,x,v}} \le \alpha/2$. For such fixed α and C the equi-integrability of $\rho^\varepsilon M$ implies

$$\exists\,\delta>0, \forall\,A\subset(0,T)\times\Omega,\quad |A|<\delta \Rightarrow C\int_A \rho^\varepsilon M \le \alpha/2.$$

So, $\sqrt{\rho^\varepsilon}\, Q(r_\varepsilon M)$ is equi-integrable. Besides, if $\Omega_R := \omega \times B(0,R)$, then

$$\int_0^T\int_{\Omega_R^c} |\sqrt{\rho^\varepsilon}\, Q(r_\varepsilon M)| \leq \|\rho^\varepsilon\|_{L^1_{t,x}}^{1/2} \left(\int_{B(O,R)^c} M dv\right)^{1/2} \left(\int_0^T\int_{\Omega_R^c} Q(r_\varepsilon M)^2 M^{-1}\right)^{1/2} \to 0$$

as R goes to infinity uniformly in ε. This proves the weak compactness of $\frac{Q(f^\varepsilon)}{\varepsilon}$ in $L^1((0,T)\times\Omega)$.

The third assertion of the proposition is a consequence of the Lions-Aubin theorem (see [25] lemma 5.1). Indeed, using that $\nabla_x\phi^\varepsilon$ is bounded in $L^2(0,T;\ L^2(\omega))$ and $\Delta_x\phi^\varepsilon$ is bounded in $L^2(0,T,\ L^1(\omega))$ we deduce that $\nabla_x\phi^\varepsilon$ is bounded in $L^2(0,T;\ L^2(\omega)\cap W^{r,q}(\omega))$ for some $q>1$ and $0<r<1$ such that $\frac{1-r}{d} > 1-\frac{1}{q}$. And, using that $\partial_t\nabla_x\phi^\varepsilon = \nabla_x(\Delta_x)^{-1}\nabla_x.(j^\varepsilon)$ we see that $\partial_t\nabla_x\phi^\varepsilon$ is bounded in $L^1(W^{-s,p}_{loc}(\omega))$ for some $p>1$ and $s>d-\frac{d}{p}$. Hence, $\nabla_x\phi^\varepsilon$ is compact in $L^2(0,T;\ L^q_{loc}(\omega))$ for some $1<q<2$. And using that $\nabla_x\phi^\varepsilon$ is bounded in $L^2(0,T;\ L^2(\omega))$ we deduce the compactness in $L^2(0,T;\ L^p(\omega))$ for all $p<2$. □

Proposition 3.4 *r_ε is such that $\varepsilon|r_\varepsilon|^2|v|^2M$ is bounded in $L^1((0,T)\times\Omega)$ and $\sqrt{\varepsilon}|r_\varepsilon|^2|v|M$ is bounded in $L^1((0,T)\times\Omega)$.*

Proof. The proof uses Young inequality (see [8] and [26] where a similar argument is used to control the distance to the Maxwellian). Let us denote $r(z) = z\log(1+z)$ and

$$r^*(p) = \sup_{z>-1}(pz - r(z))$$

its Legendre transform. Hence $r^*(p)$ behaves like e^p when p goes to $+\infty$. Moreover, $r^*(p)$ has a superquadratic homogeneity, namely for $0<\alpha<1$ and $p>0$, we have $r^*(\alpha p)\leq \alpha^2 r^*(p)$. We also denote $z_\varepsilon = \frac{f^\varepsilon}{\rho_\varepsilon M} - 1$ and $z_\varepsilon = 0$ if $\rho_\varepsilon = 0$. Hence

$$\varepsilon|r_\varepsilon|^2|v|^2 \leq \frac{1}{\varepsilon}\rho_\varepsilon|z_\varepsilon||v|^2. \tag{28}$$

By Young inequality, we have

$$\begin{aligned}\frac{1}{\varepsilon}\rho_\varepsilon|v|^2|z_\varepsilon| &\leq \frac{4\rho_\varepsilon}{\varepsilon^2}\left[r^*(\frac{\varepsilon}{4}|v|^2) + r(|z_\varepsilon|)\right]\\ &\leq \frac{4\rho_\varepsilon}{\varepsilon^2}\left[\varepsilon^2 r^*(\frac{|v|^2}{4}) + r(z_\varepsilon)\right]\end{aligned}$$

which is clearly bounded in $L^1((0,T)\times\Omega, dtdxMdv)$ by using the growth of r^* and the entropy dissipation bound $\mathcal{R}_1^\varepsilon(t)\leq C\varepsilon^2$. This proves the first assertion. Interpolating with the fact that r_ε is bounded in $L^2((0,T)\times\Omega, dtdxMdv)$, we deduce the second bound. This ends the proof of the proposition. □

4 Compactness by velocity averaging

Proposition 4.1 *The density ρ^ε is relatively compact in $L^1((0,T)\times\omega)$: there exists $\rho\in L^1((0,T)\times\omega)$ such that, up to extraction of a subsequence if necessary,*

$$\rho^\varepsilon \to \rho \quad \textit{in } L^1 \textit{and a. e.}$$ □

Using (27), it suffices to show the compactness of $(\beta_\delta(f^\varepsilon))_\varepsilon$ for all (fixed) $\delta > 0$. This is a consequence of the averaging lemma (see [20, 14]) and the continuity equation.

Let us recall the following averaging lemma

Lemma 4.2 *Assume that h^ε is bounded in $L^2((0,T)\times\Omega)$, that h_0^ε and h_1^ε are bounded in $L^1((0,T)\times\Omega)$, and that*

$$\varepsilon\, \partial_t h^\varepsilon + v.\, \nabla_x h^\varepsilon = h_0^\varepsilon + \nabla_v \,.\, h_1^\varepsilon. \tag{29}$$

Then, for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\| \int_{\mathbb{R}^d} (h^\varepsilon(t,x+y,v) - h^\varepsilon(t,x,v)\,)\,\psi(v)dv\|_{L^1_{t,x}} \;\rightarrow\; 0 \,when\, y \;\rightarrow\; 0 \,uniformly\ in\, \varepsilon \tag{30}$$

where $h^\varepsilon(t,x,v)$ has been prolonged by 0 for $x \notin \omega$.

Remark 4.3 *The above lemma only gives the compactness in the x variable of the averages in v of $h^\varepsilon(t,x,v)$. This is due to the presence of an ε in front of the time derivative in (29). We also refer to [21] and [8] for similar averaging lemmas where there is only gain of regularity in the x variable.*

This lemma can be deduced from the proof of theorem 1.8 of [9] or from the proof of theorem 6 of [14] (see also the proof of theorem 3 of [14]). The only difference here is the presence of the time derivative which comes with the factor ε in front and hence does not imply regularity in time as in theorem 6 of[14]. Actually, following the proof of theorem 3 of [14] with $q = m = 1$, $p = 2$, $\tau = 0$, and writing the problem in the whole space $\mathbb{R}_t \times \mathbb{R}^d_x$, we can prove that $\int_{\mathbb{R}^d} \psi(v)h^\varepsilon(t,x,v)dv$ is in the Besov space $L^{r,\infty}((0,T); B^{s,r}_{\infty,\infty})$ where $r = \frac{5}{3}$ and $s = \frac{1}{5}$. For the definition of the Besov space $B^{s,r}_{\infty,\infty}$, we refer to [14]. A sketch of the proof will be given in the appendix. This, of course, yields the compactness stated in (30).

Proof of Proposition 4.1. Let δ be a (fixed) nonnegative parameter. Let us verify that the rescaled Boltzmann equation (in the renormalized sense) satisfies the assumptions of Lemma 4.2. Indeed, $\beta_\delta(f^\varepsilon)$ is a weak solution of

$$\varepsilon\, \partial_t \beta_\delta(f^\varepsilon) + v.\nabla_x \beta_\delta(f^\varepsilon) = h_0^\varepsilon + \nabla_v .\, h_1^\varepsilon$$

where

$$h_0^\varepsilon = \frac{Q(f^\varepsilon)}{\varepsilon}\beta'_\delta(f^\varepsilon) \qquad \text{and} \qquad h_1^\varepsilon = \nabla(\phi^\varepsilon + \tilde{\phi}_b)\,\beta_\delta(f^\varepsilon).$$

The sequences $(\beta_\delta(f^\varepsilon))_\varepsilon$ is bounded in $L^\infty \cap L^1((0,T)\times\Omega)$ and hence in $L^2((0,T)\times\Omega)$. Moreover, h_0^ε is weakly relatively compact in $L^1_{t,x,v}$ and by applying Holder's inequality and using the uniform bound of $\beta_\delta(f^\varepsilon)$ in L^2 (for fixed δ), we obtain

$$\|h_1^\varepsilon\|_{L^1((0,T)\times\omega;L^2(\mathbb{R}^d))} \leq \frac{C}{\sqrt{\delta}} \sup_{t\leq T} \|\nabla(\phi^\varepsilon + \tilde{\phi}_b)(t)\|_{L^2_x}.$$

Since we are using a compactly supported function to localize in space, the L^2 bound in v also yields and L^1 bound.

Assumptions of Lemma 4.2 are satisfied and hence we get the L^1-compactness in x of $\int_{\mathbb{R}^d} \psi(v)\beta_\delta(f^\varepsilon)dv$, namely (30) holds with h^ε replaced by $\beta_\delta(f^\varepsilon)$.

Next, using that $(\beta_\delta(f^\varepsilon))_\varepsilon$ is bounded in $L^\infty(0,T;\, L^1((1+|v|^2)dxdv))$, we see that we can take $\psi(v)$ to be constant equal to 1 in (30) and hence we deduce, after also sending δ to 0 and using the equi-integrability of f^ε, that

$$\|\rho^\varepsilon(t,x+y)-\rho^\varepsilon(t,x)\|_{L^1_{t,x}} \to 0 \,\text{when}\, y \to 0 \,\text{uniformly in}\, \varepsilon$$

Finally, using that $\partial_t\rho^\varepsilon = -\nabla_x .j^\varepsilon$ is bounded in $L^1((0,T);W^{-1,1}(w))$, we deduce that ρ^ε is relatively compact in $L^1((0,T)\times\omega)$ and Proposition 4.1 is proved. □

5 Passing to the limit

Using the previous section, there exists $\rho \in L^1((0,T)\times\omega)$ such that

$$\rho^\varepsilon \to \rho \quad \text{in } L^1_{t,x} \quad \text{and a. e.}$$

The inequality $\left(\sqrt{a}-\sqrt{b}\right)^2 \le |a-b|$ leads to

$$\sqrt{\rho^\varepsilon} \to \sqrt{\rho} \quad \text{in } L^2_{t,x} \quad \text{and a. e.}$$

The entropy dissipation given by (15) leads to

$$f^\varepsilon \to \rho\, M \quad \text{in } L^1_{t,x,v} \quad \text{and a. e.} \tag{31}$$

Moreover, we have

$$\frac{Q(f^\varepsilon)}{\varepsilon} = \left(2\sqrt{\rho^\varepsilon}\, Q(r_\varepsilon\, M) + \varepsilon\, Q(r_\varepsilon^2 M)\right) \quad \rightharpoonup \quad 2\sqrt{\rho}\, Q(r\, M), \quad \text{in } L^1$$

where r is the weak limit of r^ε in $L^2((0,T)\times\Omega,\, M(v)dtdxdv)$. So, one can pass to the limit in (13) for $\lambda > 0$, up to extraction of a subsequence, and gets

$$\begin{aligned} v.\nabla_x\sqrt{(\rho+\lambda)M} - \nabla_v.(\nabla_x(\phi+\tilde{\phi_b})\sqrt{(\rho+\lambda)M}) \;&= \frac{\sqrt{\rho}\, Q(r\, M)}{\sqrt{(\rho+\lambda)M}} \\ &+ \frac{\lambda M\, v.\nabla_x(\phi+\tilde{\phi_b})}{2\sqrt{(\rho+\lambda)M}} \end{aligned} \tag{32}$$

where $\nabla_x\phi$ is the $L^2_{t,x}$−weak limit of $\nabla_x\phi^\varepsilon$. Sending λ to 0, we infer that

$$\left(\nabla_x\sqrt{\rho} + \frac{1}{2}\sqrt{\rho}\,\nabla_x(\phi+\tilde{\phi_b})\right).vM = Q(r\, M).$$

Using that $Q(r\, M)$ is bounded in $L^2((0,T)\times\Omega,\, M^{-1}(v)dtdxdv)$, we deduce that $\nabla_x\sqrt{\rho}+\frac{1}{2}\sqrt{\rho}\,\nabla_x(\phi+\tilde{\phi_b})$ is bounded in $L^2((0,T)\times\omega)$. Besides, the current density is given by

$$j^\varepsilon = 2\sqrt{\rho}^\varepsilon \int_{\mathbb{R}^d} r^\varepsilon\, vMdv + \varepsilon\int r_\varepsilon^2 vMdv.$$

Using Proposition 3.4, we deduce that

$$j^\varepsilon = 2\sqrt{\rho}^\varepsilon \int_{\mathbb{R}^d} r^\varepsilon\, vM dv + \mathbf{O}(\sqrt{\varepsilon})_{L^1((0,T)\times\Omega)} \rightharpoonup 2\sqrt{\rho}\int_{\mathbb{R}^d} r\, vM dv, \quad \text{in } L^1_{t,x}.$$

The function $vM \in \mathcal{R}(Q) := \mathcal{K}er Q^\perp$, therefore

$$\int r\, vM\, dv = \int Q(r\, M) Q^{-1}(vM)\frac{dv}{M}$$

and

$$j^\varepsilon \rightharpoonup J(\rho,\phi) := 2\sqrt{\rho}\left[\int_{\mathbb{R}^d}(v\otimes Q^{-1}(vM))dv\right]\left(\nabla_x\sqrt{\rho} + \frac{1}{2}\sqrt{\rho}\,\nabla_x(\phi+\tilde{\phi_b})\right).$$

Passing to the limit ($\varepsilon \to 0$) in (14) and the Poisson equation $-\Delta\phi^\varepsilon = \rho^\varepsilon$, we obtain

$$\begin{cases} \partial_t\rho + \nabla_x J(\rho,\phi) = 0, \\ J(\rho,\phi) = 2\sqrt{\rho}\left[-\mathbf{D}\left(\nabla_x\sqrt{\rho} + \frac{1}{2}\sqrt{\rho}\,\nabla_x(\phi+\tilde{\phi_b})\right)\right], \\ \mathbf{D} = -\int_{\mathbb{R}^d}(v\otimes Q^{-1}(vM))dv \quad \text{and} \quad -\Delta_x\phi = \rho. \end{cases}$$

6 The limit boundary condition

In this section, we want to pass to the limit in the boundary condition and prove that $\rho = \rho_b$ on $\partial\omega$. First notice that from the fact that $\nabla_x\sqrt{\rho} + \frac{1}{2}\sqrt{\rho}\,\nabla_x(\phi+\tilde{\phi_b})$ is bounded in $L^2((0,T)\times\omega)$, we deduce that $\nabla_x\sqrt{\rho}$ is bounded in $L^1((0,T)\times\omega)$ and hence the trace of $\sqrt{\rho}$ makes sense on $\partial\omega$.

For each sequence $(g_\varepsilon)_\varepsilon$, $\overline{g_\varepsilon}$ will denote the weak limit of $(g_\varepsilon)_\varepsilon$ when ε goes to zero, extracting a subsequence if necessary. In particular $\overline{f^\varepsilon_{|\Gamma}}$ denotes the weak limit of $f^\varepsilon_{|\Gamma}$ in $L^1((0,T)\times\Gamma^+, |v.n(x)|dtd\sigma_x dv))$. We recall that $\beta_\delta(f^\varepsilon)$ is a weak solution of the renormalized semiconductor Boltzmann equation:

$$\varepsilon\partial_t\beta_\delta(f^\varepsilon) + v\,.\nabla_x\beta_\delta(f^\varepsilon) - \nabla_x(\phi^\varepsilon+\tilde{\phi_b}).\nabla_v\beta_\delta(f^\varepsilon) = \frac{Q(f^\varepsilon)}{\varepsilon}\beta_\delta'(f^\varepsilon) \tag{33}$$

with the following boundary condition and initial data

$$\beta_\delta(f^\varepsilon)_{|\Gamma^-} = \beta_\delta(\rho_b M) \quad \text{and} \quad \beta_\delta(f^\varepsilon)_{|t=0} = \beta_\delta(f_0^\varepsilon).$$

Passing to the limit in (33), we infer

$$v\,.\nabla_x\beta_\delta(\rho M) - \nabla_x(\phi+\tilde{\phi_b}).\nabla_v\beta_\delta(\rho M) = 2\sqrt{\rho}\,Q(r\,M)\beta_\delta'(\rho M). \tag{34}$$

On one hand, using $\xi(t,x,v) \in \mathcal{C}^\infty([0,T]\times\bar{\Omega})$, as a test function in (34) we get

$$\begin{aligned} &-\int_0^T\int_\Omega \beta_\delta(\rho M)v.\nabla_x\xi + \int_0^T\int_\Omega \beta_\delta(\rho M)\nabla_x(\phi+\tilde{\phi_b}).\nabla_v\xi \\ &-\int_0^T\int_\Omega 2\sqrt{\rho}\,Q(r\,M)\beta_\delta'(\rho M)\xi + \int_0^T\int_{\partial\Omega}\xi\beta_\delta(\rho M)(v.n(x)) = 0. \end{aligned}$$

On the other hand, using $\xi(t,x,v)$ as a test function in (33) and passing to the limit, we deduce that

$$\beta_\delta(\rho M)_{|\partial\Omega} = \overline{\beta_\delta(f^\varepsilon)_{|\partial\Omega}}. \tag{35}$$

From Corollary 3.2, we deduce that $f^\varepsilon_{|\partial\Omega} \in L^\infty(0,T;\ \mathrm{LlogL}(|v.n(x)|d\sigma_x dv))$ and hence $\beta_\delta(f^\varepsilon)_{|\partial\Omega} = \beta_\delta(f^\varepsilon_{|\partial\Omega}) \in L^\infty(0,T;\ \mathrm{LlogL}(|v.n(x)|d\sigma_x dv))$ uniformly in ε,δ. Hence, $\beta_\delta(\rho M)_{|\partial\Omega}$ is uniformly bounded in $L^\infty(0,T;\ \mathrm{LlogL}(|v.n(x)|d\sigma_x dv))$ and converges to $(\rho M)_{|\partial\Omega}$ when δ goes to 0.

Using, $\psi(t,x) \in \mathcal{C}^\infty([0,T]\times\bar\omega)$ as a test function in (33) and passing to the limit, we get

$$\int_0^T\int_\Omega \beta_\delta(\rho M)v.\nabla_x\psi + \int_0^T\int_\Omega 2\sqrt{\rho}\, Q(r\,M)\beta'_\delta(\rho M)\psi$$
$$= \int_0^T\int_{\Gamma^+}\psi\left(\overline{\beta_\delta(f^\varepsilon)_{|\Gamma^+}} - \beta_\delta(\rho_b M)\right)(v.n(x))$$

Sending δ to 0 and using that

$$\int_0^T\int_\Omega \rho M v.\nabla_x\psi + \int_0^T\int_\Omega 2\sqrt{\rho}\, Q(r\,M)\psi = 0,$$

we deduce that

$$\lim_{\delta\to 0}\int_0^T\int_{\Gamma^+}\left[\overline{\beta_\delta(f^\varepsilon)_{|\Gamma^+}} - \beta_\delta(\rho_b M)\right](v.n(x))\psi = 0 \tag{36}$$

Using (35) and the fact that

$$\int_{\mathbb{R}^d\cap\{v.n(x)\geq 0\}} M(v.n(x))dv = \frac{1}{2\pi}$$

we infer

$$\int_0^T\int_{\partial\omega}[\rho_{|\partial\omega} - \rho_b]\psi = 0 \tag{37}$$

and hence $\rho = \rho_b$ on $\partial\omega$.

7 Regularity of the density

In this section we shall prove that the limit $\rho \in L^\infty(0,T;\ L^2(\omega))$ and that $\sqrt{\rho} \in L^2(0,T;\ H^1(\omega))$.

We assume that $\rho_b(t,x)$ is defined in the whole domain ω. We can for instance extend $\sqrt{\rho_b(t,x)}$ in ω as a harmonic function. Hence, $\rho_b(t,x)$ satisfies

$$\rho_b \in L^\infty((0,T)\times\omega) \quad \textit{and} \quad \sqrt{\rho_b} \in L^2(0,T;\ H^1(\omega)).$$

Lemma 7.1 *Let ω be a regular and bounded open subset of $\mathbb{R}^d$ and ρ a positive function of $L^\infty(0,T;\ L^1(\omega))$ satisfying*

$$\begin{cases} \nabla_x\sqrt{\rho}+\frac{1}{2}\nabla_x\phi\sqrt{\rho}=G\in L^2(0,T;\ L^2(\omega)), \\ -\Delta_x\phi=\rho, \\ \nabla_x\phi\in L^\infty(0,T;\ L^2) \\ \rho=\rho_b \quad \text{on } \partial\omega. \end{cases} \tag{38}$$

Then

$$\rho\in L^2(0,T;\ L^2(\omega)), \qquad \sqrt{\rho}\in L^2(0,T;\ H^1(\omega))$$

and $$\nabla\phi\sqrt{\rho}\in L^2(0,T;\ L^2(\omega)).$$ □

Proof. The first and third equations of (38) imply that $\nabla_x\sqrt{\rho}\in L^1$ and hence the boundary condition $\rho=\rho_b$ on $\partial\omega$ makes sense. Let us take β_δ as in the proof of Corollary 3.2. But since we will deal with possibly negative values, we take $\beta_\delta(s)=\frac{1}{\delta}\beta(\delta s)$ where β is a C^∞ function satisfying $\beta(s)=s$ for $-1\le s\le 1$, $0\le\beta'(s)\le 1$ for all s and $\beta(s)=2$ for $|s|\ge 3$.
We denote $\psi=\sqrt{\rho}-\sqrt{\rho_b}$. Hence

$$\nabla_x\beta_\delta(\psi)=\nabla_x\psi\,\beta_\delta'(\psi).$$

Hence, after subtracting $\sqrt{\rho_b}$ from $\sqrt{\rho}$, we can renormalize the first equation appearing in (38) , it gives

$$\nabla_x\beta_\delta(\psi)+\frac{1}{2}\nabla_x\phi\beta_\delta'(\psi)\psi=\tilde{G}\beta_\delta'(\psi) \tag{39}$$

where $\tilde{G}=G-\nabla_x\sqrt{\rho_b}-\frac{1}{2}\nabla_x\phi\sqrt{\rho_b}$ is also in $L^2(0,T;\ L^2(\omega))$. Then using that for fixed $\delta>0$,

$$|\nabla_x\phi\beta_\delta'(\psi)\psi|\le\frac{1}{2\delta}|\nabla\phi|\in L^2$$

we deduce that $\nabla_x\beta_\delta(\psi)\in L^2$ for fixed δ. Taking the L^2 norm of (39), we get

$$\|\nabla_x\beta_\delta(\psi)\|_{L^2}^2+\frac{1}{4}\|\nabla_x\phi\beta_\delta'(\psi)\psi\|_{L^2}^2+\int_0^T\int_\omega\nabla_x\phi\nabla_x\beta_\delta(\psi)\beta_\delta'(\psi)\psi\le\|\tilde{G}\|_{L^2}^2. \tag{40}$$

Let $\tilde{\beta}$ be given by $\tilde{\beta}(s)=\int_0^s\tau\beta'(\tau)^2d\tau$ and $\tilde{\beta}_\delta(s)=\frac{1}{\delta^2}\tilde{\beta}(\delta s)$. Hence, $\tilde{\beta}_\delta(s)$ goes to $\frac{s^2}{2}$ when δ goes to 0.
Computing the third term in (40), we get

$$\int_\omega\nabla_x\phi.\nabla_x\beta_\delta(\psi)\beta_\delta'(\psi)\psi \;=\; \int_\omega\nabla_x\phi.\nabla_x\tilde{\beta}_\delta(\psi)=\int_\omega\rho\tilde{\beta}_\delta(\psi).$$

Hence, we deduce that for all $\delta>0$,

$$\|\nabla_x\beta_\delta(\psi)\|_{L^2}^2+\frac{1}{4}\|\nabla_x\phi\beta_\delta'(\psi)\psi\|_{L^2}^2+\int_0^T\int_\omega\rho\tilde{\beta}_\delta(\psi)\le\|\tilde{G}\|_{L^2}^2.$$

Letting δ go to zero, we get that

$$\|\nabla_x(\sqrt{\rho}-\sqrt{\rho_b})\|_{L^2}^2+\frac{1}{4}\|\nabla_x\phi(\sqrt{\rho}-\sqrt{\rho_b})\|_{L^2}^2+\frac{1}{2}\int_0^T\int_\omega \rho(\sqrt{\rho}-\sqrt{\rho_b})^2\le\|\tilde{G}\|_{L^2}^2.$$

Using that $\sqrt{\rho_b}$ is bounded, we conclude the proof of the lemma. □
Now, using the lemma, we can see easily that we can rewrite the current

$$J(\rho,\phi)=2\sqrt{\rho}\left[-\mathbf{D}\left(\nabla_x\sqrt{\rho}+\frac{1}{2}\sqrt{\rho}\,\nabla_x(\phi+\tilde{\phi_b})\right)\right]=-\mathbf{D}[\nabla_x\rho+\rho\nabla_x(\phi+\tilde{\phi_b})].$$

Finally, the regularity of ϕ can be easily deduced from that of ρ and this ends the proof of the main Theorem 1.4.

Appendix 1 : Existence of renormalized solution

We present here a proof of the existence of renormalized solution to (1–6) satisfying the conditions of Theorem 2.2. We refer to [29] for the existence of renormalized solution to the Vlasov-Poisson-Boltzmann system with a nonlinear Boltzmann kernel.

To simplify the notations we take $\varepsilon=1$. We begin by regularizing the collision operator, and both Boltzmann and Poisson equations. Let us define

$$Q_R(f)=\int_{\mathbb{R}^d}\sigma_R(v,v')(Mf'-M'f)dv' \tag{41}$$

where

$$\sigma_R(v,v')=\sigma(v,v')1_{|v|\le R}(v)1_{|v'|\le R}(v'). \tag{42}$$

the regularized semiconductor Boltzmann-Poisson system reads

$$(VBP)_{\alpha,R}\qquad\begin{cases}\partial_t f_{\alpha,R}+\Big(v.\nabla_x f_{\alpha,R}-\nabla_x(\phi_{\alpha,R}+\tilde{\phi_b}).\nabla_v f_{\alpha,R}\Big)=Q_R(f_{\alpha,R}),\\ -(1-\alpha\,\Delta_x)^{2m}\Delta_x\phi_{\alpha,R}=\rho_{\alpha,R}=\displaystyle\int_{|v|\le R}f_{\alpha,R}dv,\\ f_{\alpha,R}(0,x,v)=f_0(x,v),\qquad (x,v)\in\Omega\\ f_{\alpha,R}(t,x,v)=f_b(t,x,v),\qquad (x,v)\in\Gamma^-\\ \phi_{\alpha,R}=\Delta_x\phi_{\alpha,R}=\ldots=\Delta_x^{2m}\phi_{\alpha,R}=0,\quad x\in\partial\omega\end{cases}$$

where α is a nonnegative parameter and $m\in\mathbb{N}^*$. We refer to [1, 6, 33] for further details about this approximation.

A simple computation gives the following H-Theorem

Lemma 7.2 *The collision operator Q_R is bounded in L^1 and L^∞ and satisfies*

$$\int_{|v|<R} Q_R(f)dv = \int_{\mathbb{R}^d} Q_R(f)dv = 0$$

and

$$\int_{|v|<R} Q_R(f)\log\frac{f}{M} = \int_{\mathbb{R}^d} Q_R(f)\log\frac{f}{M} \leq -\frac{\sigma_1}{2}\int\left(\sqrt{f} - \sqrt{M\int fdv}\right)^2.$$

As a consequence of these conservation properties, one can prove, by a fixed point procedure and by using the characteristic method, that the modified system $(VBP)_{\alpha,R}$ has a weak solution $(f_{\alpha,R}, \phi_{\alpha,R})$. More precisely, multiplying the semiconductor Boltzmann equation by $(1+|v|^2/2+\log f_{\alpha,R})$, and integrating with respect to $dtdxdv$, we get

$$\left[\frac{1}{2}\int_\omega |(1-\alpha\Delta)^m\nabla_x\phi_{\alpha,R}|^2(s) + \int_\Omega f_{\alpha,R}\left(\log f_{\alpha,R} + \frac{|v|^2}{2} + \tilde{\phi}_b\right)(s)\right]_0^t$$
$$\int_0^t\int_{\Gamma^+} f_{\alpha,R}\left(\log f_{\alpha,R} + \frac{|v|^2}{2} + \tilde{\phi}_b\right)|v.n(x)| + \frac{\sigma_1}{2}\int_0^t\int_\Omega\left(\sqrt{f_{\alpha,R}} - \sqrt{\rho_{\alpha,R}M}\right)^2$$
$$\leq \int_0^t\int_{\Gamma^-} f_b\left(\log f_b + \frac{|v|^2}{2} + \phi_b\right)|v.n(x)|$$

For fixed α, the solution is weak. one can pass to the limit $R\to\infty$ and shows that there exists a weak solution $(f_\alpha,\ \rho_\alpha,\ \phi_\alpha)$ of

$$(VBP)_\alpha \qquad \begin{cases} \partial_t f_\alpha + v.\nabla_x f_\alpha - \nabla_x(\phi_\alpha + \tilde{\phi}_b).\nabla_v f_\alpha = Q(f_\alpha), \\ -(1-\alpha\Delta_x)^{2m}\Delta_x\phi_\alpha = \rho_\alpha = \displaystyle\int_{\mathbb{R}^d} f_\alpha dv, \\ f_\alpha(0,x,v) = f_0(x,v), \qquad (x,v)\in\Omega, \\ f_\alpha(t,x,v) = f_b(t,x,v) \qquad (x,v)\in\Gamma^- \\ \phi_\alpha = \Delta_x\phi_\alpha = \ldots = \Delta_x^{2m}\phi_\alpha = 0, \qquad x\in\partial\omega \end{cases}$$

Moreover, this weak solution satisfies equations (12), (13) and (14) and

$$\left[\frac{1}{2}\int_\omega |(1-\alpha\Delta)^m\nabla_x\phi_\alpha|^2(t) + \int_\Omega f_\alpha\left(\log f_\alpha + \frac{|v|^2}{2} + \tilde{\phi}_b\right)(s)\right]_0^t$$
$$\int_0^t\int_{\Gamma^+} f_\alpha\left(\log f_\alpha + \frac{|v|^2}{2} + \phi_b\right)|v.n(x)| + \frac{\sigma_1}{2}\int_0^t\int_\Omega\left(\sqrt{f_\alpha} - \sqrt{\rho_\alpha M}\right)^2$$
$$\leq \int_0^t\int_{\Gamma^-} f_b\left(\log f_b + \frac{|v|^2}{2} + \phi_b\right)|v.n(x)| \leq C_T.$$

As a consequence of this identity we get the following proposition

Proposition 7.3

1. *f_α and $f_{\alpha|_{\Gamma^+}}$ are respectively weakly relatively compact in $L^1((0,T)\times\Omega)$ and $L^1((0,T)\times\Gamma^+, |v.n(x)|dtd\sigma_x dv)$.*

2. *$\|\nabla_x\phi_\alpha\|_{L^2} \le \|(1-\Delta_x)^m\nabla_x\phi_\alpha\|_{L^2} \le C_T$ and $\nabla_x\phi_\alpha$ is relatively compact in $L^p((0,T)\times\omega)$, for all $p<2$.*

3. *ρ_α is relatively compact in $L^1((0,T)\times\omega)$.*

The proof of this proposition follows the same lines as the proofs of compactness given in the paper (see the proofs of Corollary 3.2, Proposition 3.3 and Proposition 4.1). Notice however, that we do not get immediately the compactness of f_α as in (31). □

Using this Proposition, we can end the proof of Theorem 2.2.

Let f, ρ, ϕ and j be the weak limits of subsequences of f_α, ρ_α, ϕ_α and $j^\alpha = \int vf_\alpha dv$. To prove that (f,ϕ) satisfies (12) and (13), we argue as in [24] (see also [29]). The method is based on a double renormalization. First, we write the equation satisfied by $\beta_\delta(f_\alpha)$ where β_δ was defined in section 3 and then weakly pass to the limit when α goes to zero. Then, we renormalize the resulting limit equation using the function β or the function $\sqrt{s+\lambda M}$. Finally, we let α go to zero and recover (12) and (13). The continuity equation (14) can be easily deduced from the continuity equation for ρ_α and the entropy inequality (15) can be deduced from a classical convexity argument (see also [29]). This ends the proof of Theorem 2.2.

Appendix 2 : Sketch of the proof of lemma 4.2

Here we would like to prove lemma 4.2.

Step 1 : First, we rewrite the problem in the whole space in t and x. This step only uses the equiintegrability of h_ε. Indeed, for α small enough, we can find C^∞ cut-off functions $\chi_1(t)$ and $\chi_2(x)$ such that $\chi_1 = 1$ on $(\alpha, T-\alpha)$, has compact support in $(0,T)$ and $\chi_2 = 1$ on $\{x\in\omega, |, \mathrm{dist}(x,\partial\omega) > \alpha$ and has compact support in ω. Denoting $\chi(t,x) = \chi_1(t)\chi_2(x)$ and $\tilde{h}^\varepsilon = \chi(t,x)h^\varepsilon$, we get

$$\varepsilon\,\partial_t\tilde{h}^\varepsilon + v.\nabla_x\tilde{h}^\varepsilon = \chi h_0^\varepsilon + \nabla_v\,.\,(\chi h_1^\varepsilon) + (\varepsilon\partial_t + v.\nabla_x)\chi h^\varepsilon. \tag{43}$$

Moreover, due to the uniform bound of h^ε in L^2, we have

$$\|\int_{\mathbb{R}^d}\Big(h^\varepsilon(t,x,v) - \tilde{h}^\varepsilon(t,x,v)\Big)\psi(v)dv\|_{L^1((0,T)\times\omega)} \to 0$$

when α goes to zero, uniformly in ε.

Step 2: From step 1, we see that it is enough to prove the lemma when, we replace $(0,T)\times\omega$ by $\mathbb{R}_t\times\mathbb{R}^d_x$ since the extra term on the right hand side of (43) is in L^2. Notice also that since we localize in the v variable by integrating against $\psi(v)$, the L^2 norm controls the L^1 norm.

More precisely, we can prove that under the conditions of the lemma, $\int_{\mathbb{R}^d}\psi(v)h^\varepsilon(t,x,v)dv \in L^{r,\infty}((0,T);B^{s,r}_{\infty,\infty})$ where $r=\frac{5}{3}$ and $s=\frac{1}{5}$ (see [14] for the definition of the Besov space built on the Lorentz space $L^{r,\infty}$). One way of proving the bound in the Besov space is to use the Littlewood-Paley decomposition and follow the proof of theorem 3 of [14]. The only difference is that $\xi.v$ should be replaced by $\varepsilon\tau+\xi.v$ and that we take the Fourier transform in t and x (see also theorem 1.8 of [9]).

Here, we would like to sketch a proof which follows the idea used in [22]. Adding λh^ε to both sides of (29), we get

$$\lambda h^\varepsilon + \varepsilon\,\partial_t h^\varepsilon + v.\nabla_x h^\varepsilon = h_0^\varepsilon + \nabla_v\,.\,h_1^\varepsilon + \lambda h^\varepsilon.$$

Hence,

$$\int_{\mathbb{R}^d} h^\varepsilon(t,x,v)\psi(v)dv = T_\lambda(h_0^\varepsilon + \nabla_v\,.\,h_1^\varepsilon + \lambda h^\varepsilon) \tag{44}$$

where

$$T_\lambda g(t,x) = \int_0^\infty \int_{\mathbb{R}^d} g(t-\varepsilon s, x-sv, v)e^{-\lambda s}\phi(v)dv\,ds. \tag{45}$$

Applying Proposition 3.1 of [22], we deduce that

$$\|T_\lambda(g)\|_{L^2_tH^{1/2}_x} \le C\lambda^{-1/2}\|g\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \tag{46}$$

$$\|T_\lambda(g)\|_{\lambda^{-2}L^1_tW^{-1,1}_x+\lambda^{-1}L^1_tL^1_x} \le C\|g\|_{L^1(\mathbb{R}_t\times\mathbb{R}^d_x;W^{-1,1}(\mathbb{R}^d))} \tag{47}$$

Hence,

$$\int_{\mathbb{R}^d} h^\varepsilon(t,x,v)\psi(v)dv = \rho = \rho^1+\rho^2 \tag{48}$$

where

$$\begin{aligned}\|\rho^1\|_{L^2_tH^{1/2}_x} &\le C\lambda^{1/2}\|h^\varepsilon\|_{L^2} &(49)\\ \|\rho^2\|_{\lambda^{-2}L^1_tW^{-1,1}_x+\lambda^{-1}L^1_tL^1_x} &\le C(\|h_0^\varepsilon\|_{L^1}+\|h_1^\varepsilon\|_{L^1}) &(50)\end{aligned}$$

This can also be written as $\rho^2=\rho^2_1+\rho^2_2$, where

$$\|\rho^2_1\|_{L^1_tW^{-1,1}_x} \le C\lambda^{-2}(\|h_0^\varepsilon\|_{L^1}+\|h_1^\varepsilon\|_{L^1}) \tag{51}$$

$$\|\rho^2_2\|_{L^1_tL^1_x} \le C\lambda^{-1}(\|h_0^\varepsilon\|_{L^1}+\|h_1^\varepsilon\|_{L^1}). \tag{52}$$

We would like to deduce that $\rho\in[L^2H^{1/2},L^1W^{-1,1}]_{(1/5,\infty)}$, the real interpolation of order $(1/5,\infty)$ of the couple $(L^2H^{1/2},L^1W^{-1,1})$. For all $t\in\mathbb{R}_+$, we define the function

$$K(t) = \inf_{a_1+a_2=\rho}\|a_1\|_{L^2H^{1/2}} + t\|a_2\|_{L^1W^{-1,1}} \tag{53}$$

To conclude it is enough to prove that $K(t)\le Ct^{1/5}$ For $t>0$, we take λ such that $t=\lambda^{5/2}$.

If $0<t<1$, then ρ^2_2 also satisfies $\|\rho^2_2\|_{L^1_tW^{-1,1}_x}\le C\lambda^{-2}$ and hence taking $a_1=\rho^1$ and $a_2=\rho^2$, we deduce that $K(t)\le Ct^{1/5}$.

If $t > 1$, then we write ρ_2^2 as the sum of two terms $\rho_2^2 = \rho_3^2 + \rho_4^2$ such that $\rho_3^2 \in \lambda^{-2} L^1 W^{-1,1}$ and $\rho_4^2 \in \lambda^{1/2} L^1 W^{3/2,1}$. Hence, if we define

$$K_1(t) = \inf_{a_1+a_2=\rho} \|a_1\|_{L^2 H^{1/2} + L^1 W^{3/2,1}} + t\|a_2\|_{L^1 W^{-1,1}} \tag{54}$$

we get that $K_1(t) \leq C t^{1/5}$ by taking $a_1 = \rho^1 + \rho_4^2$ and $a_2 = \rho_1^2 + \rho_3^2$.

This proves that $\rho \in [L^2 H^{1/2} + L^1 W^{3/2,1}, L^1 W^{-1,1}]_{(1/5,\infty)}$. This is of course enough to get the compactness stated in the lemma.

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