

DIFFUSIVE LOGISTIC EQUATION WITH CONSTANT YIELD HARVESTING, I: STEADY STATES

SHOBHA ORUGANTI, JUNPING SHI, AND RATNASINGHAM SHIVAJI

ABSTRACT. We consider a reaction-diffusion equation which models the constant yield harvesting to a spatially heterogeneous population which satisfies a logistic growth. We prove the existence, uniqueness and stability of the maximal steady state solutions under certain conditions, and we also classify all steady state solutions under more restricted conditions. Exact global bifurcation diagrams are obtained in the latter case. Our method is a combination of comparison arguments and bifurcation theory.

1. INTRODUCTION

We study the nonlinear boundary value problem

$$(1.1) \quad \begin{cases} \Delta u + au - bu^2 - ch(x) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where a, b, c are positive constants, Ω is a smooth bounded region with $\partial\Omega$ in class C^2 in \mathbf{R}^n for $n \geq 1$, and $h : \overline{\Omega} \rightarrow \mathbf{R}$ satisfies $h(x) > 0$ for $x \in \Omega$, $\max_{x \in \overline{\Omega}} h(x) = 1$ and $h(x) = 0$ for $x \in \partial\Omega$. We assume $h \in C^\alpha(\overline{\Omega})$ throughout the paper. A direct consequence is that any solution of (1.1) belongs to the class $C^{2,\alpha}(\overline{\Omega})$.

(1.1) arises from the population biology of one species. Let $u(t, x)$ be the concentration of the species or the population density. We assume that (a) the species disperses randomly in the bounded environment Ω ; (b) the reproduction of the species follows the logistic growth; (c) the boundary $\partial\Omega$ of the environment is hostile to the species; and (d) the environment Ω is homogeneous (*i.e.*, the diffusion does not depend on x). Then it is well known that $u(t, x)$ satisfies the reaction-diffusion equation

$$(1.2) \quad \frac{\partial u}{\partial t} = D\Delta u + au \left(1 - \frac{u}{N}\right), \quad (t, x) \in (0, T) \times \Omega,$$

with the initial and boundary conditions

$$(1.3) \quad \begin{aligned} u(t, x) &= 0, & (t, x) &\in (0, T) \times \partial\Omega, \\ u(0, x) &= u_0(x) \geq 0, & x &\in \Omega, \end{aligned}$$

where $D > 0$ is the diffusion coefficient, $a > 0$ is the linear reproduction rate and $N > 0$ is the carrying capacity of the environment. (See Murray [M] for details.)

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Equation (1.2) is often called Fisher's equation after Fisher [F], and it was also studied by Kolmogoroff, Petrovsky and Piscounoff [KPP].

In many ecological systems, harvesting or predation of the species occurs. For example, fishing or hunting of the species u could happen. Hence it is natural to add a harvesting term to the right-hand side of (1.2), and the equation would be

$$(1.4) \quad \frac{\partial u}{\partial t} = D\Delta u + au \left(1 - \frac{u}{N}\right) - p(t, x, u),$$

where $p(t, x, u) \geq 0$ for all possible (t, x, u) values. In this paper, we consider the case of constant yield harvesting (not dependent on the density u or on t). In particular, we consider the case

$$(1.5) \quad p(t, x, u) \equiv ch(x),$$

where $c > 0$ is a parameter which represents the level of harvesting, $h(x) > 0$ for $x \in \Omega$, $h(x) = 0$ for $x \in \partial\Omega$ and $\|h\|_\infty = 1$. So $ch(\cdot)$ can be understood as the rate of the harvesting distribution, and the harvesting only occurs in the interior of the environment. Such a harvesting pattern arises naturally from fishery management problems, where $ch(x)$ is related to the fishing quota imposed by regulating authorities. The equation (1.4) with (1.5) is a generalization of the well-known ordinary differential equation logistic model with constant yield harvesting (see [Cl], [BC]). With a standard non-dimensionalization process, we can reduce (1.4) to

$$(1.6) \quad \frac{\partial u}{\partial t} = \Delta u + au - bu^2 - ch(x),$$

and the steady state solutions of (1.6) and (1.3) satisfy (1.1).

Mathematically, (1.6) and (1.3) generate a semiflow in the Sobolev space $W_0^{1,2}(\Omega)$. When $p(t, x, u) \equiv 0$ (*i.e.* the logistic case), the dynamics of (1.2) and (1.3) has been completely studied (see Henry [He]). Here we briefly describe the results for the logistic case. We denote by λ_k the k -th eigenvalue of

$$(1.7) \quad \begin{cases} \Delta\phi + \lambda\phi = 0, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

In particular, $\lambda_1 > 0$ is the principal eigenvalue with a positive eigenfunction ϕ_1 satisfying $\|\phi_1\|_\infty = 1$.

For (1.2) and (1.3), the following facts have been proved:

1. $v_0 \equiv 0$ is a steady state solution for any $a, N > 0$; when $0 < a \leq \lambda_1$, v_0 is the unique nonnegative steady state solution; and when $a > \lambda_1$, there is a unique positive steady state solution v_a .
2. The set

$$C_0 = \{\varphi \in W_0^{1,2}(\Omega) : \varphi(x) \geq 0 \text{ on } \Omega\}$$

is positively invariant; for any initial value $u_0(\cdot) \in C_0$, the solution $u(t, \cdot)$ exists for all $t \in (0, \infty)$ and is uniformly bounded in $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$.

3. Let $u_0(\cdot) \in C_0$. Then when $0 < a \leq \lambda_1$, $\|u(t, \cdot)\|_{W^{1,2}(\Omega)} \rightarrow 0$ as $t \rightarrow \infty$; when $a > \lambda_1$, $\|u(t, \cdot) - v_a(\cdot)\|_{W^{1,2}(\Omega)} \rightarrow 0$ (unless $u_0 \equiv 0$) as $t \rightarrow \infty$.

The results on the steady state solutions are well-known, see for example [BK], [SY], and for the sake of completeness, a proof based on bifurcation theory is given in Section 2.3, Theorem 2.5. The results on the dynamical systems for $\Omega = (0, 1)$

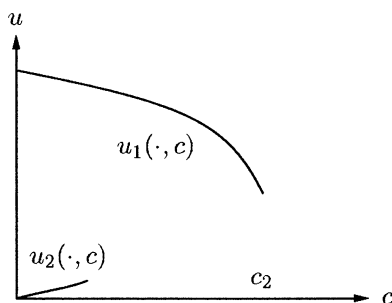


FIGURE 1.

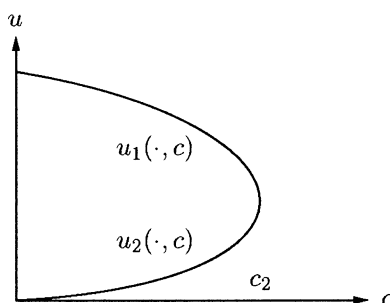


FIGURE 2.

can be found in [He], but the higher-dimensional version can be easily carried over using the same proof.

For the equation with harvesting term $ch(x)$, there is no such perfect structure as in the logistic equation. First, the set C_0 is *not* positively invariant, since the nonlinearity $f(x, u) = au - bu^2 - ch(x)$ does not satisfy $f(x, 0) \geq 0$, and so the maximum principle does not hold here. Second, the equation has possibly more than one positive steady state solution; one of them is stable, but the attraction basin of the stable steady state solution is not clear.

In this paper and a forthcoming paper we overcome some of these difficulties, and partially describe some important dynamical behavior of the system. In this paper we concentrate on the set of positive steady state solutions, and we shall study the dynamic behavior of the system in the forthcoming paper. When $a \leq \lambda_1$, $c > 0$, it is easy to show that (1.1) has no nonnegative solutions. When $a > \lambda_1$, it becomes more delicate, and we prove the following results:

1. When $0 < c \leq c_1$, (1.1) has a positive steady state solution u_1 , which is unique in the set

$$C_c = \{\varphi \in W_0^{1,2}(\Omega) : \lambda_1 \varphi(x) \geq ch(x) \text{ on } \Omega\}.$$

2. When $0 < c < c_2$ ($> c_1$), (1.1) has a positive stable steady state solution u_1 , and u_1 is the maximal steady state solution.
3. When $c > c_2$, there is no nonnegative steady state solution.

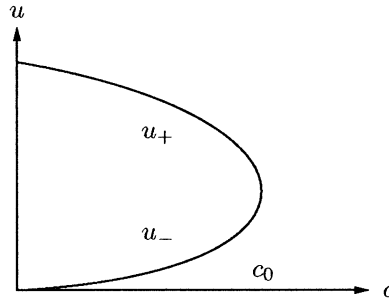


FIGURE 3.

In fact, when $c \in (0, c_1)$, we will show that the unique positive steady state u_1 also solves the “obstacle” problem $au_1 - bu_1^2 - ch(x) > 0$ for $x \in \Omega$. Thus u_1 is a subharmonic solution as v_a (solution of the logistic equation) and represents a biologically meaningful steady state solution for this harvesting case. (See Fig. 1 for illustration. (1.1) may have a second solution for small $c > 0$ —see Theorem 3.3 and the Remarks at the end of Section 3.)

The results above hold for any $a > \lambda_1$. When $a > \lambda_1$ is sufficiently close to λ_1 , we obtain a complete bifurcation diagram of (1.1), which is very interesting in a PDE context. We prove that when $0 < a - \lambda_1 < \delta$ for some $\delta > 0$, (1.1) has *exactly* two positive steady state solutions u_1 and u_2 when $c \in (0, c_2)$, exactly one when $c = c_2$, and no nonnegative steady state solution when $c > c_2$ (see Fig. 2).

It is also interesting to compare the PDE model (1.6) and (1.3) to the ODE model with constant yield harvesting (see [Cl], [BC]):

$$(1.8) \quad u' = au - bu^2 - c, \quad u(0) = u_0.$$

For (1.8), a complete bifurcation diagram in (c, u) space can be drawn (see Fig. 3). Fix $a, b > 0$; then there exists a critical number $c_0 = a^2/4b$ such that when $0 < c < c_0$, there are two equilibrium points

$$u_{\pm} = \frac{a \pm \sqrt{a^2 - 4bc}}{2b}$$

such that $u(t) \rightarrow u_+$ as $t \rightarrow \infty$ for any $u_0 > u_-$ and $u(t) < 0$ when $t > T$ for any $u_0 < u_-$; when $c > c_0$, for any $u_0 > 0$ we have $u(t) < 0$ when $t > T$.

The main mathematical tools in the paper include comparison methods for semi-linear elliptic equations and bifurcation theory in Banach spaces. The nonlinearity $f(x, u) = au - bu^2 - ch(x)$ satisfies $f(x, 0) < 0$ for $x \in \Omega$, which is often referred to as semi-positone nonlinearity, as the maximum principle does not hold in general. See [CMS] for a general survey for semi-positone problems.

When the harvesting term is homogeneous on x , there are more results available previously. When $p(t, x, u) = c$, a constant, see for example [ACS], [CS], [OS] and [S]. We should mention that $h(x) = 0$ on the boundary is not needed in the bifurcation type of results; it is only needed when we establish the existence and uniqueness of a solution such that $au - bu^2 > ch(x)$, and so many of the results in this paper can also be proved for the case $h(x) > 0$ for $x \in \bar{\Omega}$, in particular the case $p(t, x, u) = c$. Other types of predation terms have also been studied in the literature. Korman and Shi [KS] studied the bifurcation diagram of steady state solutions of (1.4) and (1.3) with $p(t, x, u) = cu/(1 + u)$ and Ω being the unit ball of

dimension $1 \leq n \leq 4$, and a complete classification of precise bifurcation diagrams was achieved for all $a, c > 0$. This type of nonlinearity is called Holling type II functional response of predator (see [Ho]). Other studies on the diffusive logistic equation can be found in [AB], [CC1], [CC2].

For a nonlinear operator F , we use F_u as the partial derivative of F with respect to the argument u . For a linear operator L , we use $N(L)$ as the null space of L and $R(L)$ as the range of L . We introduce the anti-maximum principle and bifurcation theory in Sections 2.1 and 2.2. In Section 2.3, we recall results on the logistic equation, and we prove some *a priori* estimates in Section 2.4. In Section 3, we prove the existence of solutions by comparison and bifurcation methods for all $a > \lambda_1$. The global bifurcation diagram for $0 < a - \lambda_1 < \delta$ is shown in Section 4.

2. PRELIMINARIES

2.1. Maximum principle and anti-maximum principle. This section is a recollection of some preliminaries and results from previous works. First we recall the anti-maximum principle of Clément and Peletier [CP]. Let $\Omega \subset \mathbf{R}^n$ be a bounded smooth domain ($\partial\Omega$ is of class C^2). Let L denote the differential operator

$$(2.1) \quad Lu = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial u}{\partial x_i} + au,$$

where $a_{ij} \in C(\bar{\Omega})$, $a_{ij} = a_{ji}$, and $\sum_{i,j=1}^n a_{ij}(x)\xi^i \xi^j > 0$ for $x \in \bar{\Omega}$ and $\xi = (\xi^i) \in \mathbf{R}^n \setminus \{0\}$, and $a_i, a \in L^\infty(\Omega)$.

Let $p > n$, let $X = \{u \in W^{2,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}$, and let $Y = L^p(\Omega)$. Let the operator $A : X \rightarrow Y$ be defined by $Au = Lu$. Then from [CP], pages 220-221, we know that A has a unique principal eigenvalue $\lambda_1(A)$, which is simple, and $Au = \lambda_1(A)u$ has a strict positive eigenfunction φ_1 such that

$$(2.2) \quad \varphi_1(x) > 0, \quad x \in \Omega, \quad \frac{\partial \varphi_1}{\partial n}(x) < 0, \quad x \in \partial\Omega.$$

Theorem 2.1 ([CP]). *Let A be the elliptic operator defined above and let $\lambda_1(A)$ be its principal eigenvalue. Suppose that $f \in L^p(\Omega)$, $p > n$, is such that $f > 0$, and suppose u satisfies the equation*

$$(2.3) \quad Au - \lambda u = f \quad \text{in } L^p(\Omega).$$

Then there exists $\delta_f > 0$, which depends on f , such that if $\lambda_1(A) < \lambda < \lambda_1(A) + \delta_f$, then

$$(2.4) \quad u(x) < 0, \quad x \in \Omega, \quad \frac{\partial u}{\partial n}(x) > 0, \quad x \in \partial\Omega,$$

and if $\lambda < \lambda_1(A)$, then

$$(2.5) \quad u(x) > 0, \quad x \in \Omega, \quad \frac{\partial u}{\partial n}(x) < 0, \quad x \in \partial\Omega.$$

Here the result for $\lambda_1(A) < \lambda < \lambda_1(A) + \delta_f$ is called an anti-maximum principle, and the result for $\lambda < \lambda_1(A)$ is an extended maximum principle.

2.2. Bifurcation theory. We use bifurcation theory to study the solution set, and our main tools are the implicit function theorem (see for example [CR1]) and two bifurcation theorems by Crandall and Rabinowitz [CR1], [CR2], which we recall below. In all three theorems, X and Y are Banach spaces.

Theorem 2.2 (Implicit function theorem, [CR1]). *Let $(\lambda_0, u_0) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y . Let $F(\lambda_0, u_0) = 0$. Suppose that $F_u(\lambda_0, u_0)$ is a linear homeomorphism of X onto Y . Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $(\lambda, u_0 + \lambda w_0 + z(\lambda))$, $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for some $\varepsilon > 0$, where $w_0 = -[F_u(\lambda_0, u_0)]^{-1}(F_\lambda(\lambda_0, u_0))$ and $\lambda \mapsto z(\lambda) \in X$ is a continuously differentiable function near $\lambda = \lambda_0$ with $z(\lambda_0) = z'(\lambda_0) = 0$.*

Theorem 2.3 (Bifurcation from a simple eigenvalue, [CR1]). *Let $\lambda_0 \in \mathbf{R}$ and let F be a continuously differentiable mapping of an open neighborhood $V \subset \mathbf{R} \times X$ of $(\lambda_0, 0)$ into Y . Suppose that*

1. $F(\lambda, 0) = 0$ for $\lambda \in \mathbf{R}$,
2. the partial derivative $F_{\lambda u}$ exists and is continuous,
3. $\dim N(F_u(\lambda_0, 0)) = \text{codim } R(F_u(\lambda_0, 0)) = 1$, and
4. $F_{\lambda u}(\lambda_0, 0)w_0 \notin R(F_u(\lambda_0, 0))$, where $w_0 \in X$ spans $N(F_u(\lambda_0, 0))$.

Let Z be any complement of $\text{span}\{w_0\}$ in X . Then there exist an open interval $I = (-\varepsilon, \varepsilon)$ and C^1 functions $\lambda : I \rightarrow \mathbf{R}$, $\psi : I \rightarrow Z$, such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$, and, if $u(s) = sw_0 + \psi(s)$ for $s \in I$, then $F(\lambda(s), u(s)) = 0$. Moreover, $F^{-1}(\{0\})$ near $(\lambda_0, 0)$ consists precisely of the curves $u = 0$ and $(\lambda(s), u(s))$, $s \in I$.

We recall from [S] that in Theorem 2.3, if F is C^2 in u , then we have

$$(2.6) \quad \lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, 0)[w_0, w_0] \rangle}{2\langle l, F_{\lambda u}(\lambda_0, 0) \rangle},$$

where $\langle \cdot, \cdot \rangle$ is the duality between Y and Y^* , the dual space of Y , and $l \in Y^*$ satisfies $N(l) = R(F_u(\lambda_0, 0))$.

Theorem 2.4 (Saddle-node bifurcation at a turning point, [CR2]). *Let $(\lambda_0, u_0) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y . Suppose that*

1. $\dim N(F_u(\lambda_0, u_0)) = \text{codim } R(F_u(\lambda_0, u_0)) = 1$, $N(F_u(\lambda_0, u_0)) = \text{span}\{w_0\}$, and
2. $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

If Z is a complement of $\text{span}\{w_0\}$ in X , then the solutions of $F(\lambda, u) = F(\lambda_0, u_0)$ near (λ_0, u_0) form a curve $(\lambda(s), u(s)) = (\lambda_0 + \tau(s), u_0 + sw_0 + z(s))$, where $s \rightarrow (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. Moreover, if F is k times continuously differentiable, so are $\tau(s)$ and $z(s)$.

We recall from [S] that in Theorem 2.4, if F is C^2 in u , then we have

$$(2.7) \quad \tau''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{\langle l, F_\lambda(\lambda_0, u_0) \rangle},$$

where $l \in Y^*$ satisfies $N(l) = R(F_u(\lambda_0, u_0))$.

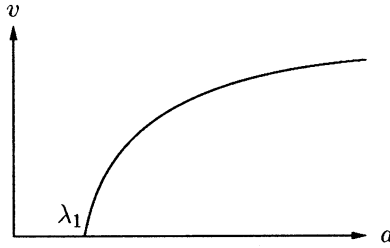


FIGURE 4.

2.3. Logistic equation. Here we recall the bifurcation diagram of the diffusive logistic equation. Indeed, we prove a more general result. Consider

$$(2.8) \quad \begin{cases} \Delta u + au - f(u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Theorem 2.5. *Assume that $f(u)$ satisfies*

$$(2.9) \quad \frac{d}{du} \left(\frac{f(u)}{u} \right) > 0, \quad \text{for } u > 0, \quad \lim_{u \rightarrow \infty} \frac{f(u)}{u} = \infty,$$

and $f(0) = f'(0) = 0$. Then (2.8) has no positive solution if $a \leq \lambda_1$, and has exactly one positive solution v_a if $a > \lambda_1$. Moreover, all v_a 's lie on a smooth curve, v_a is stable, and v_a is increasing with respect to a .

Theorem 2.5 is more or less known to the experts in the field of semilinear elliptic equations, but we have not been able to find an exact reference; so here we provide a proof based on the implicit function theorem (Theorem 2.2) and bifurcation from a simple eigenvalue (Theorem 2.3). The key to proving Theorem 2.5 is the following lemma, which we will also use in this paper (this lemma was first proved in [ABC], and the form here was first proved in Shi and Yao [SY]).

Lemma 2.6. *Suppose that $f : \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}$ is a continuous function such that $f(x, s)/s$ is strictly decreasing for $s > 0$ at each $x \in \Omega$. Let $w, v \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy*

- (a) $\Delta w + f(x, w) \leq 0 \leq \Delta v + f(x, v)$ on Ω ,
- (b) $w, v > 0$ on Ω and $w \geq v$ on $\partial\Omega$,
- (c) $\Delta v \in L^1(\Omega)$.

Then $w \geq v$ in $\bar{\Omega}$.

The stability of the solution is also an important subject in our study. We call a solution u of

$$(2.10) \quad \begin{cases} \Delta u + g(x, u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

a *stable solution* if all eigenvalues of

$$(2.11) \quad \begin{cases} \Delta \psi + g_u(x, u)\psi = -\mu\psi, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases}$$

are strictly positive, which can be inferred if the principal eigenvalue $\mu_1(u) > 0$. Otherwise u is *unstable*. When u is unstable, the number of negative eigenvalues

μ_i of (2.11) is the Morse index $M(u)$ of u . If 0 is an eigenvalue of (2.11), then u is a degenerate solution, otherwise nondegenerate.

Proof of Theorem 2.5. From (2.9) and $f(0) = f'(0) = 0$, we know that $f(u) > 0$ for all $u > 0$. Thus from (2.8) and (1.7), we have

$$\int_{\Omega} (a - \lambda_1)u\phi_1 dx = \int_{\Omega} f(u)\phi_1 dx,$$

if u is a positive solution of (2.8). So (2.8) has no positive solution if $a \leq \lambda_1$.

Next we apply Theorem 2.3 at $(a, u) = (\lambda_1, 0)$. Let $F(a, u) = \Delta u + au - bu^2$, where $a > 0$ and $u \in X \equiv C^{2,\alpha}(\bar{\Omega})$, and let $Y = C^\alpha(\bar{\Omega})$. $(a, u) = (a, 0)$ is a line of trivial solutions of (2.8); at $(\lambda_1, 0)$, $N(F_u(\lambda_1, 0)) = \text{span}\{\phi_1\}$, and $R(F_u(\lambda_1, 0)) = \{\psi \in Y : \int_{\Omega} \psi \phi_1 dx = 0\}$, which is codimension 1; $F_{au}(\lambda_1, 0)\phi_1 = \lambda_1\phi_1 \notin R(F_u(\lambda_1, 0))$ since $\lambda_1 \int_{\Omega} \phi_1^2 dx \neq 0$. Thus by Theorem 2.3, near $(\lambda_1, 0)$, the solutions of (2.8) are on two branches $\Sigma_0 = \{(a, 0)\}$ and $\Sigma_1 = \{(a(s), v(s)) : |s| \leq \delta\}$, where $a(0) = \lambda_1$, $v(s) = s\phi_1 + o(s^2)$; moreover $a(s) > \lambda_1$ for $s \in (0, \delta)$ from the last paragraph. Therefore there exists $\varepsilon > 0$ such that for $a \in (\lambda_1, \lambda_1 + \varepsilon)$, (2.8) has a positive solution v_a . We prove that any positive solution (a, v) of (2.8) is stable. Let $(\mu_1(v), \psi_1)$ be the principal eigenpair of (2.11) for $g(x, v) = av - f(v)$. Then from (2.11) and (2.8), we obtain

$$(2.12) \quad -\mu_1(v) \int_{\Omega} \psi_1 v dx = - \int_{\Omega} [f'(v)v - f(v)]\psi_1 dx.$$

Because of (2.9), $f'(v)v - f(v) > 0$ for $v > 0$. Thus $\mu_1(v) > 0$. In particular, any positive solution (a, v) is nondegenerate. Therefore, at any positive solution (a^*, v^*) , we can apply Theorem 2.2 to $F(a, v) = 0$, and all the solutions of $F(a, v) = 0$ near (a^*, v^*) are on a curve $(a, v(a))$ with $|a - a^*| \leq \varepsilon$ for some small $\varepsilon > 0$. Hence the portion of Σ_1 with $s > 0$ can be extended to a maximal set

$$(2.13) \quad \Sigma_1 = \{(a, v_a) : a \in (\lambda_1, a_M)\},$$

where a_M is the supremum of all $a > a_0$ such that v_a exists. We claim that $\lambda_M = \infty$. Suppose not. Then $\lambda_M < \infty$, and there are two possibilities: (a) $\lim_{a \rightarrow a_M^-} \|v_a\|_X = \infty$, or (b) $\lim_{a \rightarrow a_M^-} v_a = 0$; otherwise we can extend Σ_1 further beyond a_M . The case (a)

is impossible since $\lim_{u \rightarrow \infty} \frac{f(u)}{u} = \infty$; then, by the maximum principle,

$$\|u\|_{\infty} \leq K, \quad \text{where } K = \max \left\{ u > 0 : a > \frac{f(u)}{u} \right\}.$$

The case (b) is not possible either, since if so, $a = a_M$ must be a point where a bifurcation from the trivial solutions $v = 0$ occurs, a_M must be an eigenvalue λ_i of (1.7) with $i \geq 2$, and the eigenfunction ϕ_i is not of one sign, but the positive solution v_a satisfies $v_a/\|v_a\|_{\infty} \rightarrow \phi_i$ as $a \rightarrow a_M^-$, which is a contradiction. Thus $a_M = \infty$.

We prove v_a is increasing with respect to a . Since v_a is differentiable with respect to a (as a consequence of the implicit function theorem), then $\frac{dv_a}{da}$ satisfies

$$\Delta \frac{dv_a}{da} - a \frac{dv_a}{da} + f'(v_a) \frac{dv_a}{da} = -v_a \leq 0,$$

and v_a is stable; so $\mu_1(v_a) > 0$. Then, by Theorem 2.1, $\frac{dv_a}{da} \geq 0$. Finally, by Lemma 2.6, (2.8) has at most one positive solution for any possible $\lambda > 0$, which completes the proof. \square

2.4. Some a priori estimates. We close this section with some results on the dependence of solutions on the parameter $a > 0$. First we prove some nonexistence results:

Proposition 2.7. 1. If $a \leq \lambda_1$ and $c \geq 0$, (1.1) has no nonnegative solution except $u = 0$ when $c = 0$.
 2. If $a > \lambda_1$ and

$$(2.14) \quad c > \frac{a(a - \lambda_1) \int_{\Omega} \phi_1 dx}{b \int_{\Omega} h \phi_1 dx},$$

then (1.1) has no nonnegative solution.

Proof. (1) Multiplying (1.1) by ϕ_1 , and integrating over Ω , we obtain

$$(2.15) \quad (a - \lambda_1) \int_{\Omega} u \phi_1 dx = b \int_{\Omega} u^2 \phi_1 dx + c \int_{\Omega} h(x) \phi_1 dx.$$

Since $u \geq 0$, $\phi_1 > 0$, $b, c \geq 0$ and $a - \lambda_1 \leq 0$, then the equality can only be achieved when $u \equiv 0$ and $c = 0$.

(2) From the maximum principle, we have $\|u\|_{\infty} \leq a/b$ for any nonnegative solution u . Hence from (2.15), we obtain

$$(2.16) \quad c \int_{\Omega} h(x) \phi_1 dx \leq (a - \lambda_1) \int_{\Omega} u \phi_1 dx \leq \frac{a(a - \lambda_1)}{b} \int_{\Omega} \phi_1 dx,$$

a contradiction when (2.14) holds. \square

So $a > \lambda_1$ is a necessary condition for the existence of nonnegative solutions. When $a > \lambda_1$, we have the following estimate:

Proposition 2.8. If $a > \lambda_1$, $c \geq 0$, and u is a nonnegative solution to (1.1), then

$$(2.17) \quad \|u\|_{W^{2,2}(\Omega)}^2 \leq C_1(a - \lambda_1)^2,$$

where C_1 is a positive constant depending only on Ω , a , b and h .

Proof. Multiplying (1.1) by u , and integrating over Ω , we obtain

$$(2.18) \quad - \int_{\Omega} |\nabla u|^2 dx + a \int_{\Omega} u^2 dx = b \int_{\Omega} u^3 dx + c \int_{\Omega} h(x) u dx > 0.$$

Thus

$$(2.19) \quad \|u\|_{W^{1,2}(\Omega)}^2 \leq (a + 1) \int_{\Omega} u^2 dx.$$

On the other hand, from (2.18), we obtain

$$(2.20) \quad (a - \lambda_1) \int_{\Omega} u^2 dx \geq - \int_{\Omega} |\nabla u|^2 dx + a \int_{\Omega} u^2 dx > b \int_{\Omega} u^3 dx;$$

here in the first inequality, we use the fact that $\int_{\Omega} |\nabla u|^2 dx \geq \lambda_1 \int_{\Omega} u^2 dx$. Then, using Schwarz inequality inductively and (2.20), we can obtain

$$(2.21) \quad \int_{\Omega} u^n dx \leq \left(\frac{a - \lambda_1}{b} \right)^n |\Omega|,$$

for $n = 1, 2$ and 3 , which, together with (2.19), implies

$$(2.22) \quad \|u\|_{W^{1,2}(\Omega)}^2 \leq (a + 1) \left(\frac{a - \lambda_1}{b}\right)^2 |\Omega|.$$

Finally, from Lemma 2.6, we have $b/a \geq u(x)$ and $au(x) - bu^2(x) \geq 0$ for all $x \in \Omega$; thus $|\Delta u(x)| \leq |au(x) - bu^2(x)| + |ch(x)| \leq |au(x)| + |ch(x)|$ for any $x \in \Omega$. Then by $\|h\|_\infty = 1$ and (2.14), we have

$$(2.23) \quad \begin{aligned} \|\Delta u\|_{L^2(\Omega)}^2 &\leq a^2 \|u\|_{L^2(\Omega)}^2 + c^2 |\Omega| \\ &\leq a^2(a + 1) \left(\frac{a - \lambda_1}{b}\right)^2 |\Omega| + \left[\frac{a(a - \lambda_1) \int_\Omega \phi_1 dx}{b \int_\Omega h \phi_1 dx}\right]^2 |\Omega| \\ &\leq C_2(a - \lambda_1)^2, \end{aligned}$$

where $C_2 > 0$ depends on a, b, h and Ω , and, from standard elliptic estimates,

$$(2.24) \quad \|u\|_{W^{2,2}(\Omega)}^2 \leq C_3(\|u\|_{W^{1,2}(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2) \leq C_1(a - \lambda_1)^2,$$

where $C_3 > 0$ depends only on Ω . □

3. EXISTENCE OF LARGE AND SMALL SOLUTIONS

To consider the problem in an abstract setting, we define

$$(3.1) \quad F(c, u) = \Delta u + au - bu^2 - ch(x),$$

where $c \in \mathbf{R}$, $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Clearly, (1.1) has a solution (c, u) if and only if $F(c, u) = 0$. We remark that, though we consider the equation in Sobolev space $W^{2,2}(\Omega)$, all solutions to the equation (1.1) are classical solutions belonging to $C^{2,\alpha}(\overline{\Omega})$, since $g(x, u) = au - bu^2 - ch(x)$ belongs to $C^\alpha(\overline{\Omega})$ by our assumption. In particular, all solutions and related functions involved in the proofs also belong to $W^{2,p}(\Omega)$ for $p > n$, which is the necessary condition for applying the anti-maximum principle.

Our main results in this section are the following:

Theorem 3.1. *Suppose that $a > \lambda_1$ and $b > 0$. Then there exists $c_1 = c_1(a, b)$ such that for $0 < c < c_1$, (1.1) has a positive solution $u_1(x, c)$ such that*

$$(3.2) \quad au_1(x, c) - bu_1^2(x, c) > ch(x) > 0.$$

Theorem 3.2. *Suppose that $a > \lambda_1$ and $b > 0$. Then there exists $c_2(a, b) > c_1$ such that*

1. for $0 < c < c_2$, (1.1) has a maximal positive solution $u_1(x, c)$ such that for any solution $v(x, c)$ of (1.1), $u_1 \geq v$;
2. for $c > c_2$, (1.1) has no positive solution;
3. for $0 < c < c_2$, $u_1(\cdot, c)$ is stable with $\mu_1(u_1(\cdot, c)) > 0$; and
4. $u_1(\cdot, c)$ is decreasing with respect to the parameter c for $c \in (0, c_2)$.

Theorem 3.3. *Suppose that $a > \lambda_1$, $a \neq \lambda_i$, and $b > 0$. Then there exists $c_3 \in (0, c_2)$ such that*

1. for $c \in (0, c_3)$, (1.1) has a second solution $u_2(x, c) \neq u_1(x, c)$ such that $\lim_{c \rightarrow 0^+} \|u_2(\cdot, c)\|_{W^{2,2}(\Omega)} = 0$, and
2. if in addition $a \in (\lambda_1, \lambda_1 + \delta_h)$ for some $\delta_h > 0$, then $u_2(x, c) > 0$ for $x \in \Omega$.

Theorem 3.4. *Suppose that $a > 2\lambda_1$ and $b > 0$. Then there exists $0 < c_4 < c_1$ such that for $c \in (0, c_4)$, (1.1) has a unique positive solution (which must be $u_1(x, c)$) satisfying*

$$(3.3) \quad \lambda_1 u_1(x, c) \geq ch(x), \quad x \in \Omega.$$

Proof of Theorem 3.1. We use the method of sup-sub solutions. Let z_λ be the unique solution of

$$(3.4) \quad \begin{cases} \Delta z_\lambda + \lambda z_\lambda = 1, & x \in \Omega, \\ z_\lambda = 0, & x \in \partial\Omega, \end{cases}$$

where $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$, and $\delta_1 = \delta_1(\Omega) > 0$ is the constant in Theorem 2.1 for the validity of the anti-maximum principle. Then from Theorem 2.1, we have

$$(3.5) \quad z_\lambda(x) > 0, \quad x \in \Omega, \quad \frac{\partial z_\lambda}{\partial n}(x) < 0, \quad x \in \partial\Omega.$$

We construct a subsolution $\Psi(x)$ of (1.1) using z_λ such that

$$(3.6) \quad \lambda_1 \Psi(x) \geq ch(x).$$

Fix $\lambda_* \in (\lambda_1, \min\{a, \lambda_1 + \delta_1\})$. Let

$$(3.7) \quad \begin{aligned} \alpha &= \|z_{\lambda_*}\|_\infty, \quad K_0 = \inf\{K : \lambda_1 K z_{\lambda_*}(x) \geq h(x)\}, \\ K_1 &= \max\{1, K_0\}. \end{aligned}$$

Note that $K_0 > 0$ exists from (3.5). Define $\Psi(x) = Kcz_{\lambda_*}(x)$, where $K > 0$ is to be determined later. We will choose $K > 0$ and $c > 0$ properly so that Ψ is a subsolution. First we require that $K \geq K_1$; then $\lambda_1 \Psi(x) \geq ch(x)$. We have

$$(3.8) \quad \begin{aligned} \Delta \Psi + a\Psi - b\Psi^2 - ch(x) &= -Kc(\lambda_* z_{\lambda_*} - 1) + aKcz_{\lambda_*} - b(Kcz_{\lambda_*})^2 - ch(x) \\ &\geq -Kc(\lambda_* z_{\lambda_*} - 1) + aKcz_{\lambda_*} - b(Kcz_{\lambda_*})^2 - c \\ &= c[-bc(Kz_{\lambda_*})^2 + (a - \lambda_*)(Kz_{\lambda_*}) + (K - 1)]. \end{aligned}$$

Define $H(x) = -bcx^2 + (a - \lambda_*)x + (K - 1)$. Thus $\Psi(x)$ is a subsolution if $H(x) \geq 0$ for all $x \in [0, K\alpha]$. Notice that $H(0) = K - 1 > 0$, $H'(0) = a - \lambda_* \geq 0$, and $H''(0) = -2bc < 0$. Hence $H(x) \geq 0$ for all $x \in [0, K\alpha]$ if $H(K\alpha) \geq 0$, which is equivalent to

$$(3.9) \quad (a - \lambda_*)K\alpha + (K - 1) \geq bc(K\alpha)^2,$$

or

$$(3.10) \quad c \leq \frac{(a - \lambda_*)K\alpha + (K - 1)}{b(K\alpha)^2}.$$

We define

$$(3.11) \quad c_1 \equiv c_1(a, b) = \sup_{y \geq K_1} \frac{(a - \lambda_*)y\alpha + (y - 1)}{b\alpha^2 y^2} > 0.$$

Then when $c \in (0, c_1)$, there exists $\tilde{K} \geq K_1$ such that

$$(3.12) \quad c \leq \frac{(a - \lambda_*)\tilde{K}\alpha + (\tilde{K} - 1)}{b\alpha^2 \tilde{K}^2},$$

and hence $\Psi(x) = \tilde{K}cz_{\lambda_*}$ turns out to be a subsolution. On the other hand, it is easy to see that any large positive constant C is a supersolution to (1.1) for fixed

$a, b, c > 0$. Therefore, from the standard result of the sub-sup solution method (see for example [Sa]), when $c \in (0, c_1)$, there exists a solution $u_1(\cdot, c)$ of (1.1) satisfying $C \geq u_1(x, c) \geq \Psi(x) \geq (c/\lambda_1)h(x)$. Since $a > \lambda_1$, thus $au_1(x, c) > \lambda_1 u_1(x, c) \geq \lambda_1 \Psi(x) \geq ch(x)$.

Finally we prove that if we choose c_1 smaller, then

$$(3.13) \quad au_1(x, c) - bu_1^2(x, c) > ch(x).$$

Indeed, from a simple calculation, we can see that (3.13) will be satisfied if

$$(3.14) \quad u_1(x, c) > \frac{a - \sqrt{a^2 - 4bch(x)}}{2b}$$

and

$$(3.15) \quad u_1(x, c) < \frac{a + \sqrt{a^2 - 4bch(x)}}{2b}.$$

To prove (3.14), we notice that from our construction of u_1 , $u_1(x, c) \geq ch(x)/\lambda_1$. Hence (3.14) will be satisfied if

$$\frac{ch(x)}{\lambda_1} > \frac{a - \sqrt{a^2 - 4bch(x)}}{2b},$$

which is true if

$$\lambda_1 a - \lambda_1^2 > bch(x).$$

Therefore if we require

$$(3.16) \quad c < \frac{\lambda_1 a - \lambda_1^2}{b\|h\|_\infty},$$

then (3.14) holds. To prove (3.15), we consider the equation (1.1) with $c = 0$:

$$(3.17) \quad \begin{cases} \Delta u + au - bu^2 = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

From Theorem 2.5, we know that (3.17) has a unique positive solution v_a when $a > \lambda_1$. Let u be any nonnegative solution of (1.1). Then

$$\Delta v_a + av_a - bv_a^2 = 0 < ch(x) = \Delta u_1 + au_1 - bu_1^2$$

and $u_1 = v_a = 0$ on the boundary. By Lemma 2.6, $v_a(x) \geq u_1(x)$ for $x \in \Omega$, since $g(u) = au - bu^2$ satisfies $(g(u)/u)' < 0$ for $u \geq 0$. So (3.15) can be achieved if

$$(3.18) \quad v_a(x) < \frac{a + \sqrt{a^2 - 4bch(x)}}{2b}.$$

From a simple calculation, we can see that (3.18) is true if

$$(3.19) \quad c \leq \frac{a^2 - (2b\|v_a\|_\infty - a)^2}{4b\|h\|_\infty}.$$

Therefore, we choose c_1 such that both (3.16) and (3.19) are satisfied. Then (3.13) holds. \square

Proof of Theorem 3.2. We follow a similar proof in Shi and Shivaji [SS], as well as the earlier work by Shi and Yao [SY].

From the last part of the proof of Theorem 3.1, whenever (1.1) has a nonnegative solution u , then for the same parameters (a, c) , (1.1) also has a maximal solution $u_1(\cdot, c)$, which can be constructed as follows. We take v_a as a supersolution, any

solution u as a subsolution, and make the iteration sequences as in the sub-sup solution method. Then we obtain a solution u_1 in between v_a and u ; in particular, $u_1 \geq u$. Since u can be any solution, then the limit of the iterated sequence starting from v_a is the maximal solution. Clearly such u_1 is uniquely determined. (See details in [SY] or [SS].)

Thus we obtain a maximal positive solution $u_1(x, c)$ for $c \in (0, c_1)$, where c_1 is defined in Theorem 3.1, since we have proved (1.1) has a solution when $c \in (0, c_1)$ in Theorem 3.1. Moreover, it is clear that if $a > \lambda_1$ is fixed, then

$$(3.20) \quad \lim_{c \rightarrow 0^+} \|u_1(x, c) - v_a(x)\|_{C^{2,\alpha}(\bar{\Omega})} = 0.$$

Thus $(c, u_1(\cdot, c))$ is coincident with the branch of solutions of (1.1) perturbed from v_a by the implicit function theorem (Theorem 2.2). We define

$$c_2 = \sup\{c > 0 : (1.1) \text{ has a nonnegative solution with this } c\}.$$

Then $c_2 < \infty$ from Proposition 2.7. Then for $c \in (0, c_2)$, (1.1) has a maximal positive solution $u_1(x, c)$, and $u_1(\cdot, c)$ is continuous with respect to c ; from the construction of $u_1(\cdot, c)$.

We prove that

$$(3.21) \quad \mu_1(u_1(\cdot, c)) > 0, \quad \frac{\partial u_1(x, c)}{\partial c} < 0, \quad x \in \Omega.$$

First, (3.21) holds for $c = 0$. From Theorem 2.5, we have $\mu_1(u_1(\cdot, 0)) = \mu_1(v_a(\cdot)) > 0$. From Theorem 2.2, $\partial u_1(x, 0)/\partial c = -[F_u(0, v_a)]^{-1}(F_c(0, v_a))$ is the solution of

$$\Delta w + aw - 2bv_a w = h(x), \quad x \in \Omega, \quad w = 0, \quad x \in \partial\Omega.$$

Since $\mu_1(v_a(\cdot)) > 0$, then from Theorem 2.1,

$$\frac{\partial u_1(x, 0)}{\partial c} < 0, \quad x \in \Omega, \quad \frac{\partial}{\partial n} \frac{\partial u_1(x, 0)}{\partial c} > 0, \quad x \in \partial\Omega.$$

Since $u_1(\cdot, c)$ is continuous with respect to c , then (3.21) holds when $c \in (0, c^*)$ for some $c^* \in (0, c_2)$. We claim that (3.21) holds for all $c \in (0, c_2)$. Suppose this is not true; then at some $c_* \in (0, c_2)$, one of the statements in (3.21) is not true. If we have $\mu_1(u_1(\cdot, c_*)) > 0$, then using the same proof as above, we can show that $\partial u_1(x, c_*)/\partial c < 0$. Thus $\mu_1(u_1(\cdot, c_*)) = 0$. We apply Theorem 2.4 at (c_*, u_*) , where $u_* = u_1(\cdot, c_*)$. Since 0 is the principal eigenvalue of $F_u(c_*, u_*)$, then $\dim N(F_u(c_*, u_*)) = \text{codim} R(F_u(c_*, u_*))$, and $N(F_u(c_*, u_*)) = \text{span}\{w_0\}$, where w_0 is a solution of

$$\Delta w + aw - 2bu_* w = 0, \quad x \in \Omega, \quad w = 0, \quad x \in \partial\Omega.$$

Also $F_c(c_*, u_*) \notin R(F_u(c_*, u_*))$, since $F_c(c_*, u_*) = -h(x)$ and $-\int_{\Omega} h(x)w_0(x) dx \neq 0$, while $h > 0$ and $w_0 > 0$. Therefore near (c_*, u_*) , the solutions of (1.1) form a curve $(c(s), u(s)) = (c_* + o(|s|), u_* + sw_0 + o(|s|))$ with $|s| < \delta$. Moreover, by (2.6),

$$c''(0) = -\frac{2b \int_{\Omega} w_0^3(x) dx}{\int_{\Omega} h(x)w_0(x) dx} < 0.$$

Thus (1.1) has no solution near (c_*, u_*) when $c \in (c_*, c_* + \delta_1)$ for some $\delta_1 > 0$. However, $c_* < c_2$, and $u_1(\cdot, c)$ is continuous with respect to c ; so (1.1) has at least one solution $u_1(\cdot, c)$ for $c \in (c_*, c_* + \delta_1)$ which is also near u_* . That is a contradiction. Hence (3.21) holds for all $c \in (0, c_2)$. \square

Proof of Theorem 3.3. We apply the implicit function theorem (Theorem 2.2). Let $F(c, u)$ be defined as in (3.1). At $(c, u) = (0, 0)$, we have $F_u(0, 0)w = \Delta w + aw$. For $a \neq \lambda_i$, $F_u(0, 0)$ is an isomorphism from X to Y . Fix $a \neq \lambda_i$; then the solution set of (1.1) near $(0, 0)$ is of form $(c, u_2(\cdot, c))$ for $c \in (-\delta_1, \delta_1)$, $u_2(\cdot, 0) = 0$, and $u_2(\cdot, c) = cw_0 + o(|c|)$, where $w_0 = -[F_u(0, 0)]^{-1}(F_c(0, 0))$ is the solution of

$$(3.22) \quad \Delta w + aw = h(x), \quad x \in \Omega, \quad w = 0, \quad x \in \partial\Omega.$$

Since $h(\cdot) \in C^\alpha(\overline{\Omega})$, then $h \in L^p(\Omega)$ for any $p > n$. Suppose that $\delta_h > 0$ is the constant such that the anti-maximum principle holds for $A = -\Delta$, $f = -h < 0$; then, from Theorem 2.1, $w_0(x) > 0$ for $x \in \Omega$ and $\partial_n w_0(x) < 0$ for $x \in \partial\Omega$. In particular, $u_2(\cdot, c) > 0$ for $c \in (0, c_3)$. □

Proof of Theorem 3.4. Suppose that u is a nonnegative solution of (1.1) which satisfies (3.3). Then from (1.1) and $\Delta\phi_1 + \lambda_1\phi_1 = 0$, we obtain

$$(3.23) \quad \int_{\Omega} [a - \lambda_1 - bu]u\phi_1 dx - c \int_{\Omega} h\phi_1 dx = 0.$$

Let $a = 2\lambda_1 + \delta$ for some $\delta > 0$. Then, using (3.3) and (3.23), we obtain

$$\int_{\Omega} (\delta - bu)u\phi_1 dx < 0.$$

In particular,

$$\|u\|_{\infty} > \frac{\delta}{b}.$$

Since nonnegative solutions of (1.1) are bounded by Proposition 2.8, and when $c = 0$ the only nonnegative solutions of (1.1) are 0 and v_a , then for $c > 0$ sufficiently close to 0, the only possible nonnegative solutions are perturbations of 0 or v_a . In that case, nonnegative solutions of (1.1) can only be $u_1(x, c)$ or $u_2(x, c)$. From the proof of Theorem 3.3, $u_2(x, c) = cw_0 + o(|c|)$; thus if we choose $c > 0$ also satisfying

$$c < \frac{\delta}{2b\|w_0\|_{\infty}},$$

then $\|u_2(\cdot, c)\|_{\infty} < \delta/b$. In particular, u_2 does not satisfy (3.3), which implies the uniqueness of $u_1(x, c)$. □

Remarks. 1. In Theorem 3.2, there is no information on the solution(s) when $c = c_2$. It is easy to show that

$$u_1(x, c_2) = \lim_{c \rightarrow c_2^-} u_1(x, c)$$

is a classical nonnegative solution of (1.1) for $c = c_2$, and $(c_2, u_1(x, c_2))$ is a degenerate solution such that $\mu_1(u_1(\cdot, c_2)) = 0$. However it is not clear if

$$(3.24) \quad u_1(x, c_2) > 0, \quad x \in \Omega, \quad \frac{\partial u_1(x, c_2)}{\partial n} < 0, \quad x \in \partial\Omega,$$

is true. Note that Theorem 2.4 can be applied at $(c_2, u_1(x, c_2))$ in a way similar to the argument in the proof of Theorem 3.2 even when (3.24) is not true. But then we do not know whether the solutions on the lower branch are positive or not. In Section 4, we show that (3.24) is true if a is close enough to λ_1 , and further study on this problem will be reported in [SS].

2. In Theorem 3.3, $u_2(\cdot, c)$ can still be positive even when a is far away from λ_1 . From the proof of Theorem 3.3, it is sufficient to show that $w_0 > 0$ for the solution w_0 of (3.22). Consider the following example: $n = 1$, $\Omega = (0, \pi)$ and $h(x) = \sin x$. Then w_0 is the solution of

$$w'' + aw = \sin x, \quad x \in (0, \pi), \quad w(0) = w(\pi) = 0.$$

It is easy to verify that $w_0(x) = \sin x / (a - 1)$ for any $a > \lambda_1 = 1$. In that case, for any $a \neq \lambda_i = i^2$, $u_2(\cdot, c)$ is a positive solution for small $c > 0$.

4. GLOBAL BIFURCATION

In this section, we show that when a is slightly greater than λ_1 , a more precise global bifurcation diagram of positive solutions to (1.1) can be obtained by using some ideas from Shi [S]. In particular, we show that the branch of large solutions connects to the branch of small solutions (bifurcating from 0 as in Theorem 3.3), and the shape of the bifurcation diagram is exactly \supset -shaped as in the scalar ODE case. (See Fig. 2.)

Theorem 4.1. *If $b > 0$, then there exists $\delta_2 > 0$ such that for $a \in (\lambda_1, \lambda_1 + \delta_2)$,*

1. (1.1) has exactly two positive solutions $u_1(\cdot, c)$ and $u_2(\cdot, c)$ for $c \in [0, c_2)$, exactly one positive solution $u_1(\cdot, c)$ for $c = c_2$, and no positive solution for $c > c_2$;
2. the Morse index $M(u)$ is 1 for $u = u_2(\cdot, c)$, $c \in [0, c_2)$, and $u_1(\cdot, c_2)$ is degenerate with $\mu_1(u_1(\cdot, c_2)) = 0$;
3. all solutions lie on a smooth curve Σ that, on (c, u) space, starts from $(0, 0)$, continues to the right, reaches the unique turning point at $c = c_2$ where it turns back, then continues to the left without any turnings until it reaches $(0, v_a)$, where v_a is the unique positive solution of (1.1) with $c = 0$.

To prove Theorem 4.1, we first prove the following lemmas:

Lemma 4.2. *For $a \in (\lambda_1, \lambda_1 + \delta_3)$, (1.1) has a unique degenerate solution, which is positive.*

Proof. We apply the implicit function theorem in a different way here. Define

$$(4.1) \quad F(a, c, u) = \Delta u + au - bu^2 - ch(x),$$

and

$$(4.2) \quad H(a, c, u, w) = \begin{pmatrix} F(a, c, u) \\ F_u(a, c, u)[w] \end{pmatrix} = \begin{pmatrix} \Delta u + au - bu^2 - ch(x) \\ \Delta w + aw - 2buw \end{pmatrix},$$

where $a, c \in \mathbf{R}$, $b > 0$ is fixed, $u \in X \equiv W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $w \in X_1 = \{u \in X : \int_{\Omega} u^2(x)dx = 1\}$, $Y = L^2(\Omega)$. Then (1.1) has a degenerate solution (a, c, u) if and only if $H(a, c, u, w) = 0$ has a nontrivial solution (a, c, u, w) . We consider the operator H in a neighborhood M of $(\lambda_1, 0, 0, \phi_1)$:

$$(4.3) \quad M = \{(a, c, u, w) \in \mathbf{R}^2 \times X \times X_1 : |a - \lambda_1| < \delta_4, |c| \leq \delta_4, \|u\| \leq \delta_4, \|w - \phi_1\| \leq \delta_4\},$$

where δ_4 is a positive constant and $\|\cdot\|$ is the norm of $W^{2,2}(\Omega)$. We prove that there exists $\delta_5 > 0$ such that $H(a, c, u, w) = 0$ has a unique solution in M for each

$a \in (\lambda_1 - \delta_5, \lambda_1 + \delta_5)$. Let

$$(4.4) \quad \begin{aligned} K[\tau, v, \psi] &= H_{(c,u,w)}(\lambda_1, 0, 0, \phi_1)[\tau, v, \psi] \\ &= \begin{pmatrix} \tau F_c(\lambda_1, 0, 0) + F_u(\lambda_1, 0, 0)[v] \\ \tau F_{cu}(\lambda_1, 0, 0)[w_0] + F_{uu}(\lambda_1, 0, 0)[v, \phi_1] + F_u(\lambda_1, 0, 0)[\psi] \end{pmatrix}, \end{aligned}$$

where

$$(4.5) \quad F_u(a, c, u)[v] = \Delta v + av - 2buv,$$

$$(4.6) \quad F_c(a, c, u) = -h(x), \quad F_{cu}(a, c, u)[v] = 0,$$

$$(4.7) \quad F_{uu}(a, c, u)[v, \psi] = -2bv\psi,$$

$\tau \in \mathbf{R}$, $v \in X$ and $\psi \in X_2 \equiv T_{\phi_1}(X_1) = \{u \in X : \int_{\Omega} u \phi_1 dx = 0\}$, the tangent space of X_1 at ϕ_1 . We prove that K is a homeomorphism.

First we prove that K is injective. Suppose there exists (τ, v, ψ) such that $K(\tau, v, \psi) = (0, 0)$. Then (τ, v, ψ) satisfies

$$(4.8) \quad \Delta v + \lambda_1 v - \tau h(x) = 0,$$

$$(4.9) \quad \Delta \psi + \lambda_1 \psi - 2bv\phi_1 = 0.$$

We multiply (4.8) by ϕ_1 , and integrate over Ω , to get

$$(4.10) \quad \tau \int_{\Omega} h(x) \phi_1 dx = 0.$$

Since $h > 0$ and $\phi_1 > 0$, then $\tau = 0$. Thus $v = k\phi_1$ for some $k \in \mathbf{R}$. We multiply (4.9) by ϕ_1 , and integrate over Ω , to get

$$(4.11) \quad 2bk \int_{\Omega} \phi_1^3 dx = 0.$$

Thus $k = 0$ and $\Delta \psi + \lambda_1 \psi = 0$. But $\psi \in X_2$, so $\psi = 0$. So $(\tau, v, \psi) = 0$, and K is injective.

Next we prove K is surjective. Let $(f, g) \in Y \times Y$; then we need to find $(\tau, v, \psi) \in \mathbf{R} \times X \times X_2$ such that

$$(4.12) \quad \Delta v + \lambda_1 v - \tau h(x) = f,$$

$$(4.13) \quad \Delta \psi + \lambda_1 \psi - 2bv\phi_1 = g.$$

Again we multiply (4.12) by ϕ_1 , and integrate over Ω ; then

$$(4.14) \quad \tau = -\frac{\int_{\Omega} f \phi_1 dx}{\int_{\Omega} h \phi_1 dx}.$$

By the Fredholm alternative, (4.12) has a unique solution v_1 with τ given by (4.14) such that $\int_{\Omega} v_1 \phi_1 dx = 0$. We substitute $v = v_1 + k\phi_1$ in (4.13), multiply (4.13) by ϕ_1 , and integrate over Ω ; then k is determined by

$$(4.15) \quad -2b \int_{\Omega} v_1 \phi_1^2 dx - 2bk \int_{\Omega} \phi_1^3 dx = \int_{\Omega} g \phi_1 dx.$$

Finally, $\psi \in X_2$ can be uniquely solved for in (4.13) once k is determined as in (4.15). Therefore, $(f, g) \in R(K)$, and K is a bijection.

On the other hand, since F is twice differentiable, then K is continuous, and K^{-1} is also continuous by the open mapping theorem of Banach ([Y], pg.75). Thus K is a linear homeomorphism, and by the implicit function theorem (Theorem 2.2),

the solutions of $H(a, c, u, w) = 0$ near $(\lambda_1, 0, 0, \phi_1)$ in M can be written as the form $(a, c(a), u(a), w(a))$ such that

$$(4.16) \quad \frac{d}{da}(c(a), u(a), w(a)) \Big|_{a=\lambda_1} = -K^{-1}(H_a(\lambda_1, 0, 0, \phi_1)) = (0, k_1\phi_1, \psi_1),$$

where $k_1 = \int_{\Omega} \phi_1^2 dx / (2b \int_{\Omega} \phi_1^3 dx) > 0$, and $\psi_1 \in X_2$ satisfies $\Delta\psi + \lambda_1\psi = 2bk_1\phi_1^2 - \phi_1$. This is calculated using the proof of surjectivity. In particular, there exists $\delta_5 > 0$ such that for each $a \in (\lambda_1, \lambda_1 + \delta_5)$, $H = 0$ has a unique solution in M with the form $(a, o(|a - \lambda_1|), (a - \lambda_1)k_1\phi_1 + o(|a - \lambda_1|), \phi_1 + (a - \lambda_1)\psi_1 + o(|a - \lambda_1|))$. Notice that $u(a) > 0$ and $w(a) > 0$. \square

Lemma 4.3. *Define*

$$(4.17) \quad O_\delta = \{(c, u) \in \mathbf{R} \times X : 0 \leq c \leq \delta, \|u\|_{W^{2,2}(\Omega)} \leq \delta\}.$$

Then for any small $\delta > 0$, there exists $\eta = \eta(\delta) > 0$ such that, when $a \in (\lambda_1, \lambda_1 + \eta)$,

1. if (c, u) is a solution of (1.1) satisfying $c \geq 0$ and $u \geq 0$, then $(c, u) \in O_\delta$;
2. if $(c, u) \in O_\delta$ is a solution of (1.1), then $u \geq 0$.

Proof. The first statement can be obtained from Propositions 2.7 and 2.8. Since $c \geq 0$ and $u \geq 0$, then for some $C > 0$

$$(4.18) \quad |c| + \|u\|_{W^{2,2}(\Omega)} \leq C(a - \lambda_1).$$

For the second statement, we prove that when $a \in (\lambda_1, \lambda_1 + \eta)$ for small enough $\eta > 0$, any solution $(c, u) \in O_\delta$ satisfies

$$(4.19) \quad \begin{aligned} u &= \alpha\phi_1 + v, \alpha = \int_{\Omega} u\phi_1 dx > 0, \|v\|_{W^{2,2}(\Omega)} = o(\alpha), \\ c &= o(\alpha), \text{ as } \eta \rightarrow 0. \end{aligned}$$

First, from (2.15), we have

$$(4.20) \quad \alpha(a - \lambda_1) = (a - \lambda_1) \int_{\Omega} u\phi_1 dx > c \int_{\Omega} h\phi_1 dx.$$

Thus if $a > \lambda_1$, then any solution $(c, u) \in O_\delta$ of (1.1) satisfies $\alpha = \int_{\Omega} u\phi_1 dx > 0$ and $c < C\alpha(a - \lambda_1)$. In particular, $c = o(\alpha)$ as $\eta \rightarrow 0$. The smallness of v can be proved modifying an argument by Crandall and Rabinowitz [CR1]. In fact, when $c = 0$, we can directly apply Lemma 1.12 on pages 326-327 of [CR1], where the following (in our context) is proved:

$$(4.21) \quad \|v\|_{W^{2,2}(\Omega)} + |\alpha| \cdot |a - \lambda_1| \leq |\alpha|g(\alpha),$$

where $(a, u) = (a, \alpha\phi_1 + v) \in V$, a neighborhood of $(\lambda_1, 0)$, and $g(\cdot)$ is a continuous function on \mathbf{R} such that $g(0) = 0$. When $c \neq 0$, we can follow the proof in pages 326-327 of [CR1] to get

$$(4.22) \quad \begin{aligned} \|v\|_{W^{2,2}(\Omega)} + |\alpha| \cdot |a - \lambda_1| &\leq |\alpha|g(\alpha) + |c| \cdot \|h\|_{L^2(\Omega)} \\ &\leq |\alpha|g(\alpha) + C|\alpha| \cdot |a - \lambda_1|, \end{aligned}$$

which implies the estimate for v in (4.19).

Since ϕ_1 satisfies $\phi_1 > 0$ on Ω and $\partial_n\phi_1 < 0$ on $\partial\Omega$, then $u \geq 0$ when $\eta > 0$ is small enough. \square

Proof of Theorem 4.1. Fix a small $\delta > 0$. Then from Lemmas 4.2 and 4.3, there exists $\eta > 0$ such that for $a \in (\lambda_1, \lambda_1 + \eta)$, (1.1) has a unique degenerate solution $(c(a), u(a))$ in O_δ , all solutions in O_δ are nonnegative, and all nonnegative solutions are in O_δ .

By Theorem 2.5, (1.1) has exactly two nonnegative solutions, $(0, 0)$ and $(0, v_a)$, in O_δ when $c = 0$. We denote the degenerate solution by (c_2, u_*) , and $w(a) = w_*$. At (c_2, u_*) , we verify that Theorem 2.4 can be applied here. In fact, 0 is a simple eigenvalue of (2.11) from the uniqueness of a solution to $H = 0$, and $F_c(c_2, u_*) = -h(x) \notin R(F_u(c_2, u_*))$ since $\int_\Omega (-h(x))w_* dx \neq 0$. Therefore, by Theorem 2.4, the solution set of (1.1) near (c_2, u_*) can be written as a form $(c(s), u(s))$ for $s \in (-\delta_7, \delta_7)$ for some $\delta_7 > 0$, such that $c(0) = c_2$, $u(s) = u_* + sw_* + o(|s|)$, $c'(0) = 0$ and

$$(4.23) \quad c''(0) = -\frac{2b \int_\Omega w_*^3 dx}{\int_\Omega hw_* dx} < 0$$

from (2.7). Thus the branch of solutions turns to the left at (c_2, u_*) . We call the sub-branch containing $(c(s), u(s))$ with $s > 0$ the upper branch, and the one containing $(c(s), u(s))$ with $s < 0$ the lower branch. Both branches continue to the left up to $c = 0$ without any more turnings, since (c_2, u_*) is the unique degenerate solution, and the solutions on both branches are nonnegative by Lemma 4.3. The upper branch must be coincident to the branch of maximal solutions which we obtain in Theorem 3.2. In fact, the branch of maximal solutions emanates from $(c, u) = (0, v_a)$, continues as c increases until it reaches a degenerate solution (c^*, u^*) , which is nonnegative as a limit of (positive) maximal solutions, thus $(c^*, u^*) \in O_\delta$, and it must be coincident to (c_2, u_*) . The lower branch must meet $(0, 0)$ when $c = 0$.

All solutions in O_δ are positive except $(0, 0)$, since they have the form $u = \alpha\phi_1 + v$ with v small (see the proof of Lemma 4.3). There is no other nonnegative solution of (1.1), since all nonnegative solutions must lie in O_δ . The solutions on the upper branch are stable from Theorem 3.2. The solutions on the lower branch are nondegenerate, and they have Morse index 1 near the turning point or $(0, 0)$; hence each solution on the lower branch has Morse index 1 by the continuity. \square

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DEPARTMENT OF MATHEMATICS, MISSISSIPPI STATE UNIVERSITY, MISSISSIPPI STATE, MISSISSIPPI 39762

E-mail address: so1@ra.msstate.edu

DEPARTMENT OF MATHEMATICS, COLLEGE OF WILLIAM AND MARY, WILLIAMSBURG, VIRGINIA 23187 AND DEPARTMENT OF MATHEMATICS, HARBIN NORMAL UNIVERSITY, HARBIN, HEILONGJIANG, P. R. CHINA 150080

E-mail address: shij@math.wm.edu

DEPARTMENT OF MATHEMATICS, MISSISSIPPI STATE UNIVERSITY, MISSISSIPPI STATE, MISSISSIPPI 39762

E-mail address: shivaji@ra.msstate.edu