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DIGITAL FILTERS FOR NON-REAL-TIME DATA PROCESSING

by James T. Taylo

Prepared by

NORTHEAST LOUISIANA STATE COLLEGE

Monroe, La.

for George C. Marshall Space Flight Center

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ABSTRACT

Digital filtering techniques have become significant methods for data processing. This report presents the general theory through the definition of a digital filter and also presents a class of digital filters, called Martin-Graham filters, which are particularly well-suited to the operation of data smoothing. Included in this class are filters for non-real-time smoothing; smoothing and differentiation; smoothing and interpolation; smoothing, differentiation, and interpolation; and smoothing and integration. Application of these filters requires that the data be band-limited. In most cases, error bounds are given. Sample programs and sample results are also included.

PREFACE

On November 6, 1964, a project sponsored by the Computation Laboratory of the Marshall Space Flight Center, Huntsville, Alabama, was initiated with Northeast Louisiana State College to perform a research study of numerical smoothing methods and numerical aspects of finite difference methods. The research was supported in its entirety by the National Aeronautics and Space Administration, Huntsville, Alabama, under Contract No. NAS 8-11492 and was performed by members of the Mathematics Department of Northeast Louisiana State College. The Contract Technical Representatives were Mr. Ronald J. Graham and Mr. David G. Aichele of the Computation Laboratory.

Mathematics Department members involved in the research during the term of the contract were Dr. Edward B. Anders, Principal Investigator from November 6, 1964 to September 1, 1966, Mr. James T. Taylo, Investigator, November 6, 1964 to September 1, 1966, and Principal Investigator, September 1, 1966 to March 1, 1967; and for various periods, Dr. Daniel E. Durpee, Dr. Lonnie T. Bennett, Mr. James O'Neil, Dr. Dale R. Bedgood, Mr. Stephen Hamm, and Mr. Kenneth R. Russell. Typing of the final report was done by Mrs. Betty Stone and the proofreading was done by Mr. Russell Rainbolt.

Two of the investigators on this contract were also involved in the research performed under Contract No. NAS 8-11492 at Auburn University, Auburn, Alabama. The final report on that contract, CR-136, was well-received, and one project undertaken under NAS 8-11492 was revision and rewriting of that final report. The report presented here completes that project, and also incorporates significant results obtained under the present contract.

In writing this report, it was assumed that the reader is familiar with Fourier series. A very readable presentation of the Fourier theory can be found in [1] .

The methods employed here in the applications assumes that the transfer function of a filter is given analytically, and that it is such that its inverse Fourier transform can be found. Cases do arise where only values of the transfer function of a filter are known at equally spaced points on one-half the period of the filter. A method for computing the corresponding filters weights is given in Appendix B.

In Appendix C, a method is given for determining coefficients in the Fourier series representation of a function. Application requires that the series either be finite or the coefficients a_n and b_n be negligible for large n, and that the samples of the function can be obtained at the required points.

A reader interested only in the weight expressions and the applications may go directly to Chapter IV.

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CHAPTER I

CLASSICAL FOURIER ANALYSIS

1.0 INTRODUCTION

We shall give here some definitions and results from the classical Fourier analysis. We shall not attempt to establish the Fourier integral theorem and we refer the reader to [1] for a proof with integration in the sense used here. The reader familiar with Lebesgue integration will find a proof in [2].

There are several different forms of the Fourier integral theorem. The so-called complex form of the theorem states that if h(t) is a function of the real variable t, then

$$h(t) = \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} h(x)\, e^{-2\pi i f(x-t)} dx \qquad (1.0)$$

provided h(t) satisfies one of the variety of sufficient conditions (see Section 1.2 for two such conditions).

The results in this chapter are obtained again in the second chapter in a more general setting. Many of the restrictions placed on the functions in the classical theory are removed there. The duplication is intentional and serves two purposes. First, for the reader not familiar with the Fourier transform, this chapter will serve as an introduction. Secondly, if the reader is willing to accept a few results from the second chapter, he can read this chapter and go directly to the third chapter and the applications.

We shall use integration in the sense of Riemann and integration will be over the entire real line. Furthermore, our functions can have a finite number of points of discontinuity at which they may be bounded or unbounded. Thus the integrals we encounter shall be improper Riemann integrals of the so-called "third kind".

1.1 IMPROPER INTEGRALS AND ABSOLUTELY INTEGRABLE FUNCTIONS

Let $h(t)$ be a function defined for all real t. We shall say that $h(t)$ is <u>integrable</u> if $h(t)$ has at most a finite number of points of discontinuity on the real line and the improper Riemann integral

$$\int_{-\infty}^{\infty} h(t)\, dt \tag{1.1}$$

exists (finite). Thus if $t_1, t_2, \ldots\ldots, t_n$ are points of discontinuity at which $h(t)$ is unbounded, choosing $a_1 < t_1 < a_2 < t_2 < a_3 < t_3 < \ldots\ldots < a_n < t_n < a_{n+1}$, then (1.1) is the limit

$$\lim_{\substack{b \to \infty \\ \epsilon \to 0 \\ a \to -\infty}} \left\{ \int_a^{a_1} h(t)dt + \sum_{i=1}^{n} \left[\int_{a_i}^{t_i - \epsilon} h(t)dt + \int_{t_i+\epsilon}^{a_{i+1}} h(t)dt \right] + \int_{a_{n+1}}^{b} h(t)dt \right\}$$

if this limit exists, and we say that $h(t)$ is integrable. The integral (1.1) is usually said to be <u>convergent</u> or <u>divergent</u> according to whether the above limit does or does not exist. Thus when we say that $h(t)$ is integrable, we simply mean that $h(t)$ has at most a finite number of points of discontinuity and the integral (1.1) is convergent.

Suppose that $h(t)$ is continuous at t_o. Then from the inequality

$$\Big| |h(t)| - |h(t_o)| \Big| \leq |h(t) - h(t_o)|$$

it follows that the function $|h(t)|$ is continuous at t_o. The converse is not always true. A simple example is the function

$$h(t) = \begin{cases} 1 & t \leq 0 \\ -1 & t > 0 \end{cases}$$

The function $|h(t)|$ is continuous at t=0, but h(t) is not. Thus if a function h(t) has at most a finite number of points of discontinuity, then so does the function $|h(t)|$, but the converse is not true in general.

We shall say that a function h(t) which has at most a finite number of points of discontinuity is <u>absolutely integrable</u> if the function $|h(t)|$ is integrable in the above sense, that is, the improper Riemann integral

$$\int_{-\infty}^{\infty} |h(t)| \, dt \tag{1.2}$$

exists. The continuity of $|h(t)|$ except at a finite number of points follows from that of h(t). This is sometimes expressed by saying that the integral (1.1) is <u>absolutely convergent</u>. Noting that

$$-|h(t)| \leq h(t) \leq |h(t)|$$

and adding $|h(t)|$ to each member, we have

$$0 \leq h(t) + |h(t)| \leq 2|h(t)| \tag{1.3}$$

By (1.3) and the comparison test for integrals, the existence of (1.2) implies that the integral

$$\int_{-\infty}^{\infty} [h(t) + |h(t)|] \, dt$$

exists. But then we have that

$$\int_{-\infty}^{\infty} [h(t) + |h(t)|] \, dt - \int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} h(t) dt$$

and the integral (1.1) exists. This proves the following theorem.

Theorem 1.10 If $h(t)$ is absolutely integrable, then $h(t)$ is integrable.

The converse is not always true, for example, the function

$$h(t) = \frac{\sin t}{t}$$

is integrable, but it is not absolutely integrable.

Special forms of other theorems on improper integrals apply here and we shall use them when needed. These theorems are found in most advanced calculus texts. Some other results for improper integrals containing a parameter shall be needed and we list these for easy reference. Proofs of these are usually found in advanced calculus texts also.

Let $h(t,\beta)$ be a function of t involving the parameter β and suppose that $h(t,\beta)$ is integrable with respect to t for $\beta_1 \leq \beta \leq \beta_2$, that is, $h(t,\beta)$ has at most a finite number of points of discontinuity as a function of t and the improper integral

$$\Phi(\beta) = \int_{-\infty}^{\infty} h(t,\beta)dt \qquad (1.4)$$

exists for all β in $[\beta_1,\beta_2]$. The integral (1.4) is said to be uniformly convergent in $[\beta_1,\beta_2]$ if for each $\epsilon > 0$ there exists a number $N(\epsilon) > 0$ such that

$$\left| \Phi(\beta) - \int_{-a}^{b} h(t,\beta)dt \right| < \epsilon$$

for all $a,b > N(\epsilon)$ and all β in $[\beta_1,\beta_2]$.

Theorem 1.11 Weierstrass M test. If there exists a function $M(t) \geq 0$ such that

(a) $|h(t,\beta)| \leq M(t)$ for all t and all β in $[\beta_1,\beta_2]$

(b) $\int_{-\infty}^{\infty} M(t)dt$ converges,

then $h(t,\beta)$ is absolutely integrable with respect to t and the integral (1.4) is uniformly convergent in $[\beta_1,\beta_2]$.

Theorem 1.12 If $h(t,\beta)$ is integrable with respect to t and continuous as a function of β for $\beta_1 \leq \beta \leq \beta_2$ and if (1.4) is uniformly convergent in $[\beta_1,\beta_2]$, then

$$\Phi(\beta) = \int_{-\infty}^{\infty} h(t,\beta)dt$$

is a continuous function of β on $[\beta_1,\beta_2]$. In particular,

$$\lim_{\beta \to \beta_0} \Phi(\beta) = \lim_{\beta \to \beta_0} \int_{-\infty}^{\infty} h(t,\beta)dt = \int_{-\infty}^{\infty} \lim_{\beta \to \beta_0} h(t,\beta)dt.$$

Theorem 1.13 Under the conditions of Theorem 1.12, the function $\Phi(\beta)$ is integrable (in the proper sense) on $[\beta_1,\beta_2]$ and

$$\int_{\beta_1}^{\beta_2} \Phi(\beta)d\beta = \int_{\beta_1}^{\beta_2} d\beta \int_{-\infty}^{\infty} h(t,\beta)dt = \int_{-\infty}^{\infty} dt \int_{\beta_1}^{\beta_2} h(t,\beta)d\beta,$$

that is, the order of integration may be interchanged.

Theorem 1.14 If $h(t,\beta)$ is continuous as a function of the two variables t and β, $\beta_1 \leq \beta \leq \beta_2$, and is integrable with respect to t, and if

(a) $\frac{\partial h(t,\beta)}{\partial \beta}$ exists and is continuous with respect to β,

(b) $\int_{-\infty}^{\infty} \frac{\partial h(t,\beta)}{\partial \beta} dt$ exists and is uniformly convergent in $[\beta_1,\beta_2]$ (and hence is continuous there), then the function

$$\Phi(\beta) = \int_{-\infty}^{\infty} h(t,\beta)dt \text{ is differentiable in } [\beta_1,\beta_2] \text{ and}$$

$$\Phi'(\beta) = \frac{d}{d\beta} \int_{-\infty}^{\infty} h(t,\beta)dt = \int_{-\infty}^{\infty} \frac{\partial h(t,\beta)}{\partial\beta}dt$$

1.2 THE FOURIER TRANSFORM

If the integral (1.0) exists, it can be written as

$$h(t) = \int_{-\infty}^{\infty} df \; [\, e^{2\pi ift} \int_{-\infty}^{\infty} h(x)\, e^{-2\pi ifx}dx \,]$$

and letting

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi ift}dt \tag{1.5}$$

we have

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{2\pi ift}df \tag{1.6}$$

The function H(f) is called the Fourier transform of h(t). A sufficient but not necessary condition for the existence of (1.5) is that h(t) be absolutely integrable. To see this, we note that

$$|e^{-2\pi ift}| = 1, \qquad |h(t)\, e^{-2\pi ift}| = |h(t)|$$

and h(t) absolutely integrable implies that $h(t)e^{-2\pi ift}$ is absolutely integrable. Hence $h(t)e^{-2\pi ift}$ is integrable for each f and (1.5) exists. By Theorem 1.12, H(f) is continuous for all f. Also it can be shown that H(f) converges to zero as $|f| \longrightarrow \infty$ (see [1]). This condition for the existence of (1.5) is sufficient but not necessary.

The validity of (1.0), and hence of (1.6), is a different matter. These are valid if $h(t)$ is absolutely integrable and also satisfies one of the following conditions:

(a) $h(t)$ is of bounded variation on every finite interval.

(b) $h(t)$ is piecewise smooth on every finite interval.

These conditions are sufficient but not necessary.

If (1.6) holds, then $h(t)$ is called the <u>inverse Fourier transform</u> of $H(f)$. To denote that two functions are related by (1.5) and (1.6) we write

$$h(t) \longleftrightarrow H(f)$$

The Fourier transform is the only type transform we shall use and no confusion should arise if we drop the word "Fourier" and speak of the "transform of h(t)" and the "inverse transform of H(f)".

If we interpret the variable t as time, then the variable f is interpreted as frequency (cycles per second). Letting $w = 2\pi f$ in (1.5) and (1.6) yields the following form of the transform pair:

$$\overline{H}(w) = \int_{-\infty}^{\infty} h(t)\, e^{-iwt} dt$$

$$h(t) = (1/2\pi) \int_{-\infty}^{\infty} \overline{H}(w)\, e^{iwt} dw$$

where $\overline{H}(w) = H(f)$ and $w = 2\pi f$ is <u>angular frequency</u>. This form of the transform pair does not possess the symmetry of (1.5) and (1.6) due to the constant $(1/2\pi)$ appearing in the second expression. Symmetry can be obtained by multiplying the first expression by $(2\pi)^{-\frac{1}{2}}$ and taking a factor of $(2\pi)^{-\frac{1}{2}}$ under the integral sign in the second, and then replacing $(2\pi)^{-\frac{1}{2}}\, \overline{H}(w)$ by $H(w)$ in both expressions. The forms (1.5) and (1.6) suit our purposes best and shall be used. The exponents $\pm 2\pi i f t$ are cumbersome and we shall use the notation

$$\exp(x) = e^x \tag{1.7}$$

which will avoid some notation problems and is somewhat more tractable.

1.3 SPECIAL FORMS OF THE FOURIER TRANSFORMS

In general, $h(t)$ and $H(f)$ may be complex. If $h(t)$ is complex, letting $h_1(t)$ and $h_2(t)$ denote its real and imaginary parts, we have

$$h(t) = h_1(t) + ih_2(t)$$

Using $\exp(-2\pi ift) = \cos 2\pi ft - i\sin 2\pi ft$, from (1.5) we obtain

$$H(f) = \int_{-\infty}^{\infty} [\, h_1(t) \cos 2\pi ft + h_2(t) \sin 2\pi ft \,]\, dt$$

$$-i \int_{-\infty}^{\infty} [\, h_1(t) \sin 2\pi ft - h_2(t) \cos 2\pi ft \,]\, dt$$

Thus $H(f) = H_1(f) + iH_2(f)$ where

$$H_1(f) = \int_{-\infty}^{\infty} [h_1(t) \cos 2\pi ft + h_2(t) \sin 2\pi ft]\, dt$$

$$H_2(f) = - \int_{-\infty}^{\infty} [\, h_1(t) \sin 2\pi ft - h_2(t) \cos 2\pi ft \,]\, dt \tag{1.8}$$

In a similar manner, we obtain

$$h_1(t) = \int_{-\infty}^{\infty} [\, H_1(f) \cos 2\pi ft - H_2(f) \sin 2\pi ft \,]\, df$$

$$h_2(t) = \int_{-\infty}^{\infty} [\, H_1(f) \sin 2\pi ft + H_2(f) \cos 2\pi ft \,]\, df \tag{1.9}$$

If $h(t)$ is real, then $h_2(t) = 0$ and $h_1(t) = h(t)$. Then the expressions (1.8) reduce to

$$H_1(f) = \int_{-\infty}^{\infty} h(t) \cos 2\pi ft \, dt \tag{1.10a}$$

$$H_2(f) = - \int_{-\infty}^{\infty} h(t) \sin 2\pi ft \, dt \qquad (1.10b)$$

Replacing f by -f in (1.10a) and (1.10b) we see that

$$H_1(-f) = H_1(f) \text{ and } H_2(-f) = -H_2(f) \qquad (1.11)$$

Therefore $H_1(f)$ is an even function of f and $H_2(f)$ is an odd function of f. Then

$$H(-f) = H_1(-f) + iH_2(-f) = H_1(f) - iH_2(f)$$

and hence

$$H(-f) = H^*(f) \qquad (1.12)$$

Conversely, if $H(-f) = H^*(f)$, then

$$H_1(f) - iH_2(f) = H_1(-f) + iH_2(-f)$$

and equating the real and imaginary parts we see that $H_1(f)$ is even and $H_2(f)$ is odd. Then the integrand in the first integral of (1.9) is even and the integrand in the second is odd. Hence $h_2(t) = 0$ and h(t) is real. Furthermore,

$$h(t) = 2 \int_0^{\infty} [H_1(f) \cos 2\pi ft - H_2(f) \sin 2\pi ft] \, df \qquad (1.13)$$

A special case which we shall encounter later is when H(f) is real and even. Then (1.12) holds and putting $H_2(f) = 0$ in (1.13) we obtain

$$h(t) = 2 \int_0^{\infty} H(f) \cos 2\pi ft \, df \qquad (1.14)$$

Another special case is when H(f) is purely imaginary and odd.

Then $H(f) = iH_2(f)$, $H(-f) = -H(f) = -iH_2(f) = H^*(f)$ and (1.12) holds. Putting $H_1(f) = 0$ and $iH(f) = i^2H_2(f) = -H_2(f)$ in (1.13) we obtain

$$h(t) = 2i \int_0^\infty H(f) \sin 2\pi ft \, df \tag{1.15}$$

If $h(t)$ is purely imaginary, then $h(t) = ih_2(t)$ and

$$\begin{aligned} H_1(f) &= \int_{-\infty}^\infty h_2(t) \sin 2\pi ft \, dt \\ H_2(f) &= \int_{-\infty}^\infty h_2(t) \cos 2\pi ft \, dt \end{aligned} \tag{1.16}$$

Thus $H_1(f)$ is odd and $H_2(f)$ is even and

$$H(-f) = H_1(-f) + iH_2(-f) = -H_1(f) + iH_2(f) = -H^*(f) \tag{1.17}$$

It is easy to show that the converse is true, that is, if $H(f)$ is such that $H(-f) = -H^*(f)$, then $h(t)$ is purely imaginary.

1.4 SOME SIMPLE THEOREMS

We present here some simple theorems from the classical theory. These theorems will be restated in the second chapter in a more general setting and proved with less restrictive conditions.

The following theorem is an immediate consequence of the linearity of integration.

Linearity Theorem. If $h(t) \longleftrightarrow H(f)$, $g(t) \longleftrightarrow G(f)$ and if a,b are arbitrary constants, then

$$ah(t) + bg(t) \longleftrightarrow aH(f) + bG(f) \tag{1.18}$$

Symmetry Theorem. If $h(t) \longleftrightarrow H(f)$, then

$$H(t) \longleftrightarrow h(-f) \tag{1.19}$$

Proof: We have

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-2\pi ift)dt$$

Replacing f by t and t by -f gives

$$H(t) = \int_{\infty}^{-\infty} h(-f) \exp(-2\pi it(-f))(-df)$$

$$= \int_{-\infty}^{\infty} h(-f) \exp(2\pi ift)df$$

Scaling Theorem. If $h(t) \longleftrightarrow H(f)$ and a is any non-zero real constant, then

$$h(at) \longleftrightarrow \frac{H(f/a)}{|a|} \tag{1.20}$$

Proof: We have

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-2\pi ift)dt$$

and replacing f by (f/a) gives

$$H(f/a) = \int_{-\infty}^{\infty} h(t) \exp(-2\pi ift/a)dt$$

Now let t = ax. Then dt = adx and if $a > 0$,

$$H(f/a) = a \int_{-\infty}^{\infty} h(ax) \exp(-2\pi ifx)dx$$

If $a < 0$, then the order of the integration is reversed and

$$H(f/a) = a \int_{\infty}^{-\infty} h(ax) \exp(-2\pi ifx)dx$$

$$= -a \int_{-\infty}^{\infty} h(ax) \exp(-2\pi ifx)dx$$

Hence, for any $a \neq 0$,

$$H(f/a) = |a| \int_{-\infty}^{\infty} h(ax) \exp(-2\pi i f x) dx$$

Replacing x by t and dividing both sides by $|a|$ completes the proof.

First Shifting Theorem. If $h(t) \longleftrightarrow H(f)$ and t_o is a real constant, then

$$h(t - t_o) \longleftrightarrow H(f) \exp(-2\pi i t_o f) \tag{1.21}$$

Proof: We have

$$h(t) = \int_{-\infty}^{\infty} H(f) \exp(2\pi i f t) df$$

and replacing t by $t - t_o$ gives

$$h(t - t_o) = \int_{-\infty}^{\infty} H(f) \exp(2\pi i f(t - t_o)) df$$

$$= \int_{-\infty}^{\infty} [H(f) \exp(-2\pi i t_o f)] \exp(2\pi i f t) df$$

which proves the theorem.

The following theorem is proved in a similar manner.

Second Shifting Theorem. If $h(t) \longleftrightarrow H(f)$ and f_o is a real constant, then

$$h(t) \exp(2\pi i f_o t) \longleftrightarrow H(f - f_o) \tag{1.22}$$

From (1.20), (1.21), and (1.22), we obtain

$$h(at) \exp(2\pi i f_o t) \longleftrightarrow \frac{1}{|a|} H\left(\frac{f - f_o}{a}\right) \tag{1.23}$$

$$h(at - t_o) \longleftrightarrow \frac{1}{|a|} H\left(\frac{f}{a}\right) \exp(-2\pi i t_o f/a) \tag{1.24}$$

Also, letting a = -1 in (1.20) gives

$$h(-t) \longleftrightarrow H(-f) \tag{1.25}$$

<u>First Differentiation Theorem</u>. If h(t) is continuous and $t^n h(t)$ is absolutely integrable, then

$$(2\pi it)^k h(t) \longleftrightarrow H^{(k)}(f) \tag{1.26}$$

for k = 0, 1, 2, . . . , n. [$H^{(0)}(f) \equiv H(f)$]

Proof: Let

$$A = \max_{|t| \leq 1} |(2\pi)^n h(t)|$$

and let

$$M(t) = \begin{cases} A & |t| \leq 1 \\ |(2\pi t)^n h(t)| & |t| > 1 \end{cases}$$

By the continuity of h(t) on [1,-1], A is finite and $t^n h(t)$ is absolutely integrable by hypothesis. Thus the integral

$$\int_{-\infty}^{\infty} M(t)dt$$

converges. Furthermore, for k = 0, 1, 2,...., n,

$$|(-2\pi it)^k h(t)\exp(-2\pi ift)| = |(-2\pi it)^k h(t)| \leq M(t)$$

for all t and all f. By the Weierstrass M test, the integral

$$H_k(f) = \int_{-\infty}^{\infty} (2\pi it)^k h(t)\exp(-2\pi ift)dt \tag{1.27}$$

exists and is uniformly convergent in f, k = 0, 1, 2,...., n.

The integrand in (1.27) satisfies the conditions of Theorem 1.14

for k = 0, 1, 2,...., n-1, and hence

$$H_{k+1}(f) = H_k^{(1)}(f) = \frac{\partial}{\partial f} \int_{-\infty}^{\infty} (-2\pi it)^k h(t)\exp(-2\pi ift)dt$$

$$= \int_{-\infty}^{\infty} (-2\pi it)^{k+1} h(t)\exp(-2\pi ift)dt$$

For k = 0, $H_0(f) = H(f) = H^{(0)}(f)$, and hence $H_1(f) = H^{(1)}(f)$, $H_2(f) = H_1^{(1)}(f) = H^{(2)}(f)$,......, $H_k(f) = H^{(k)}(f)$,......, $H_n(f) = H^{(n)}(f)$.

Finally, by the continuity and absolute integrability of $(2\pi it)^k h(t)$ and the Fourier integral theorem, the inversion formula holds for k = 0, 1, 2,....., n.

<u>Second Differentiation Theorem</u>. If $h(t) \longleftrightarrow H(f)$ and

(1) h(t) is continuous and converges to zero as $|t| \longrightarrow \infty$, and

(2) $h^{(1)}(t)$ is absolutely integrable, then

$$(2\pi if)H(f) = \int_{-\infty}^{\infty} h^{(1)}(t)\exp(-2\pi ift)dt \qquad (1.28)$$

Proof: We have

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-2\pi ift)dt$$

Integrating by parts with

$$u = h(t) \qquad dv = \exp(-2\pi ift)dt$$

$$du = h^{(1)}(t)dt \qquad v = -(2\pi if)^{-1}\exp(-2\pi ift)$$

we obtain

$$H(f) = (2\pi if)^{-1} \left[-h(t)\exp(-2\pi ift) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} h^{(1)}(t)\exp(-2\pi ift)dt \right]$$

and since $h(t) \longrightarrow 0$ as $|t| \longrightarrow \infty$, the first term in the brackets is zero. Multiplying both sides by $(2\pi if)$, we obtain (1.28).

If $h^{(1)}(t)$ and $h^{(2)}(t)$ satisfy the conditions of the theorem, then integration by parts again yields

$$(2\pi if)^2 H(f) = \int_{-\infty}^{\infty} h^{(2)}(t)\exp(-2\pi ift)dt$$

Continuing in this manner, if $h^{(n)}(t)$ and $h^{(n+1)}(t)$ satisfy the conditions of the theorem, we obtain

$$(2\pi if)^{n+1} H(f) = \int_{-\infty}^{\infty} h^{(n+1)}(t)\exp(-2\pi ift)dt$$

Then for $k \leq n$, $h^{(k)}(t)$ satisfies conditions sufficient for the inversion formula to hold, and we obtain

$$h^{(k)}(t) \longleftrightarrow (2\pi if)^k H(f) \tag{1.29}$$

<u>Conjugate Function Theorem</u>. If $h(t) \longleftrightarrow H(f)$, then

$$h^*(t) \longleftrightarrow H^*(-f) \tag{1.30}$$

Proof: With $h(t) = h_1(t) + ih_2(t)$, we have

$$H(f) = \int_{-\infty}^{\infty} [\, h_1(t) + ih_2(t) \,] \exp(-2\pi ift)dt$$

and with $H(f) = H_1(f) + iH_2(f)$ and equations (1.8)

$$H^*(f) = \int_{-\infty}^{\infty} [\, h_1(t)\cos 2\pi ft + h_2(t)\sin 2\pi ft \,]\ dt$$

$$+ i \int_{-\infty}^{\infty} [\, h_1(t)\sin 2\pi ft - h_2(t)\cos 2\pi ft \,]\ dt$$

$$= \int_{-\infty}^{\infty} [h_1(t) - ih_2(t)] \, [\cos 2\pi ft + i \sin 2\pi ft] \, dt$$

$$= \int_{-\infty}^{\infty} h^*(t) \exp(2\pi ift) dt$$

Replacing f by -f shows that $H^*(-f)$ is the transform of $h^*(t)$. The validity of the inversion formula can be verified similarly, starting with

$$h(t) = \int_{-\infty}^{\infty} H(f) \exp(2\pi ift) \, df$$

1.5 The Convolution Theorems.

Second only to the transform and inverse transform, the convolution theorems are the most powerful tools in Fourier analysis. These theorems in their generalized form play a central role in filter theory.

Let $g(t)$ and $h(t)$ be functions of a real variable t, and let

$$q(t) = \int_{-\infty}^{\infty} g(x)h(t - x)dx \tag{1.31}$$

If this integral exists, then $q(t)$ is called the convolution of $g(t)$ and $h(t)$. This is usually denoted by writing $q(t) = (g*h)(t)$. By letting $z = t - x$ in (1.31) it is easy to show that the convolution is commutative, that is,

$$(g*h)(t) = (h*g)(t) \tag{1.32}$$

Also, from the linearity property of integration, it follows that

$$(g* [h_1+h_2])(t) = (g*h_1 + g*h_2)(t) \tag{1.33}$$

The following theorem is valid when $g(t)$ and $h(t)$ are absolutely integrable. The proof is not difficult, but it is long and will not be given here.

<u>Time Domain Convolution Theorem</u>. If h(t) and g(t) are absolutely integrable and H(f) and G(f) are their Fourier transforms, then the convolution $q(t) = (g*h)(t)$ is also absolutely integrable. Furthermore, $Q(f) = G(f)H(f)$.

Under the conditions of the theorem, a change in the order of integration is justified in

$$Q(f) = \int_{-\infty}^{\infty} dt \left[\exp(-2\pi i f t) \int_{-\infty}^{\infty} g(x)h(t-x)dx \right]$$

Hence

$$Q(f) = \int_{-\infty}^{\infty} dx \left[g(x) \int_{-\infty}^{\infty} h(t-x)\exp(-2\pi i f t)dt \right]$$

Using (1.21) we obtain

$$Q(f) = \int_{-\infty}^{\infty} g(x)[H(f)\exp(-2\pi i f x)]dx$$

$$= H(f) \int_{-\infty}^{\infty} g(t)\exp(-2\pi i f t)dt$$

$$= H(f)G(f)$$

The conditions of the above theorem are sufficient but not necessary. If, in addition to the conditions of the theorem, q(t) is bounded on every finite interval, the inversion formula holds and we have

$$(g*h)(t) \longleftrightarrow G(f)H(f) \qquad (1.34)$$

If(1.34) holds, the following theorem follows from the symmetry property (1.19).

Frequency Domain Convolution Theorem. If $g(t) \longleftrightarrow G(f)$ and $h(t) \longleftrightarrow H(f)$, then

$$g(t)h(t) \longleftrightarrow (G*H)(f) \tag{1.35}$$

Parseval's Formula. If (1.35) holds, then

$$\int_{-\infty}^{\infty} g(t)h(t)dt = \int_{-\infty}^{\infty} G(f)H(-f)df \tag{1.36}$$

Proof. From (1.35) we have

$$(G*H)(f) = \int_{-\infty}^{\infty} G(x)H(f-x)dx = \int_{-\infty}^{\infty} g(t)h(t)\exp(-2\pi ift)dt$$

and (1.36) follows by letting $f = 0$ and replacing x by f in the first integral.

Note that if $h(t)$ is real, then by (1.12) we have $H(-f) = H^*(f)$ which gives

$$\int_{-\infty}^{\infty} g(t)h(t)dt = \int_{-\infty}^{\infty} G(f)H^*(f)df$$

Letting $g(t) = h^*(t)$, from (1.30), $G(f) = H^*(-f)$ and we have

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} H^*(-f)H(-f)df = \int_{-\infty}^{\infty} |H(f)|^2 df \tag{1.37}$$

If we write $H(f)$ in polar form,

$$H(f) = A(f)\exp(i\theta(f)) \tag{1.38}$$

then the real function $A(f)$ is called the <u>Fourier spectrum</u> of $h(t)$, $A^2(f)$ is called the <u>energy spectrum</u> of $h(t)$, and $\theta(f)$ its <u>phase angle</u>. From (1.38) we have $|H(f)|^2 = A^2(f)$, and thus (1.37) can be written as

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} A^2(f) df \tag{1.39}$$

CHAPTER II

GENERALIZED FUNCTIONS AND THEIR FOURIER TRANSFORMS

2.0 INTRODUCTION

Several approaches to the definition of a digital filter are possible. In the choice of approach, one is influenced by purpose and background. The approach we choose here requires the Dirac delta function and some of its properties. This is not proposed to be the shortest or easiest way of arriving at the definition of a digital filter, but it is proposed as one of the clearest and most meaningful approaches.

The Dirac delta function $\delta(t)$ is often defined by one of the following statements:

(A) If $g(t)$ is a continuous function at $t = t_o$, then $\delta(t)$ has the property that

$$\int_{-\infty}^{\infty} g(t)\ \delta(t-t_o)dt = g(t_o);$$

(B) $\delta(t) = 0$ if $t \neq 0$, and

$$\int_{-\infty}^{\infty} \delta(t)\ dt = 1;$$

(C) $\delta(t) = \lim_{n\to\infty} g_n(t)$ where $\{g_n(t)\}$ is a sequence of functions satisfying the conditions

(i) if $t \neq 0$, then $\lim_{n\to\infty} g_n(t) = 0$, and

(ii) $$\int_{-\infty}^{\infty} g_n(t)dt = 1.$$

These definitions are meaningless if we attempt to think of $\delta(t)$ as a function in the ordinary sense. By introducing the delta function as a new concept, a generalized function, (A) can be given a precise meaning, but definitions (B) and (C) do not uniquely describe $\delta(t)$.

There is no shortage of theories to justify (A). One particularly suited to our purposes is given by Lighthill [3]. The development is similar to Cantor's extension of the rational numbers to the real numbers, an analogy we shall return to after making a definition and some comments.

Definition 2.00 A function $g(t)$ of the real variable t is called a test function if

(i) $g(t)$ is everywhere differentiable any number of times, and

(ii) $g(t)$ and all of its derivatives are $O(|t|^{-N})$ as $|t| \longrightarrow \infty$ for all integers N.

As a reminder, the "big O" notation, $g(t) = O(h(t))$ as $t \longrightarrow a$, means that there exists a positive constant A such that

$$|g(t)| < A|h(t)| \text{ as } t \longrightarrow a.$$

We shall denote the set of all test functions by S. Each $g(t)$ in S is a function of the real variable t, but these functions may be complex-valued. Note that the function $g(t) = 0$ is in S, and S is non-empty. A non-trivial example of a function of S is $g(t) = e^{-t^2}$. We note that Lighthill calls the functions of S "good functions" but the terminology we have adopted is more commonly used. Some other minor changes in terminology will be made.

Cantor extended the rationals to the reals by using equivalence classes of Cauchy sequences of rational numbers. The set analogous to the rationals in Lighthill's development is the set S of test functions. Cantor's scheme was as follows: Let R denote the set of rational numbers and let

$$C = \{ \{r_n\} \mid \{r_n\} \text{ is a Cauchy sequence of rationals} \}.$$

Define a relation on C as follows: If $\{r_n\}$ and $\{s_n\}$ are elements of C, then $\{r_n\} \sim \{s_n\}$ if and only if $\lim_{n \to \infty} (r_n - s_n) = 0$. That is, the sequence of rational numbers $\{r_n - s_n\}$ must be <u>null</u>. It is easy to show that $\sim$ is an equivalence relation and hence partitions C into disjoint subclasses, called <u>equivalence classes</u>. Let

$$\overline{R} = \{ r \mid r \text{ is an equivalence class determined by } \sim \},$$

then the elements of $\overline{R}$ are called real numbers, and $\overline{R}$ is called the set of real numbers. If r and s are real numbers, their sum and product are defined as follows: Let $\{r_n\}$ be a sequence of r and $\{s_n\}$ be a sequence of s. Then r+s is the subclass of C containing $\{r_n+s_n\}$ and rs is the subclass of C containing $\{r_n s_n\}$. Of course, it is necessary to show that $\{r_n+s_n\}$ and $\{r_n s_n\}$ are Cauchy and that r+s and rs are uniquely determined, that is, if $\{r'_n\}$ and $\{s'_n\}$ are in r and s, then the subclasses determined by $\{r'_n+s'_n\}$ and $\{r'_n s'_n\}$ are the same as those determined by $\{r_n+s_n\}$ and $\{r_n s_n\}$. The various field axioms are verified next. Finally, the mapping T defined on R by writing

$$T(a) = \{a,a,a,\ldots,a,\ldots\} = \overline{a} \text{ in } \overline{R}, \ a \text{ in } R,$$

embeds R in $\overline{R}$.

An outline of Lighthill's construction of the set of generalized functions from the set S of test functions is as follows (new terms are defined later): With S the set of all test functions, let

$$C = \{ \quad \{g_n(t)\} \quad | \quad \{g_n(t)\} \text{ is a regular sequence of test functions}\}$$

Introduce an equivalence relation $\sim$ on C by writing $\{g_n(t)\} \sim \{h_n(t)\}$ if and only if

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(t)H(t)dt = \lim_{n \to \infty} \int_{-\infty}^{\infty} h_n(t)H(t)dt$$

for all H(t) in S. Let $\overline{S}$ be the set whose elements are the equivalence classes determined by the relation $\sim$. An element of $\overline{S}$ is called a generalized function. Let g and h be generalized functions and let $\{g_n(t)\}$ be a sequence of g and $\{h_n(t)\}$ be a sequence of h. Define the sum g+h to be the generalized function (subclass of C) determined by the sequence $\{(g_n+h_n)(t)\}$, where $(g_n+h_n)(t) = g_n(t) + h_n(t)$ for all t. For any complex number a, define ag to be the generalized function determined by $\{(ag_n)(t)\}$, where $(ag_n)(t) = ag_n(t)$ for all t. Show that these definitions are consistent, that is, show that each sequence above is regular and that the definition is independent of the choice of $\{g_n(t)\}$ in g and $\{h_n(t)\}$ in h. Next, show that $\overline{S}$ with this sum and product of a complex number and a generalized function is a linear (vector) space. Finally, embed S in $\overline{S}$.

An alternate approach is found in functional analysis. There, a generalized function F is a continuous linear functional on the linear space S of test functions, that is, F is a mapping of S into the complex numbers such that

$$F(ag+bh) = aF(g) + bF(h)$$

for all g,h in S and all complex numbers a and b. Of course, the use of the word continuous implies that either a topology is explicitly given on S or that convergence in some sense is defined there.

As indicated, we will not use the last approach. However, some notation and terminology from this approach will be helpful in interpreting some definitions and results.

2.1 THE TEST SPACE S

Let V be a set with an operation(+) called addition defined on it and let R be a field (usually the real or complex numbers). V is called a linear space over R if

(i) V is a commutative group with respect to +,

(ii) for each a in R and each x in V a product ax is defined such that ax is in V and for all a,b in R and all x,y in V,

a) $a(bx) = (ab)x$

b) $a(x+y) = ax + ay$

c) $(a+b)x = ax + bx$

d) $1x = x$ where 1 is the multiplicative identity of R.

A linear space over R is often called a vector space over R. The elements of R are called scalars and the operation ax is usually called scalar multiplication. The classical example of a linear space is n-dimensional Euclidean space.

The set S of all test functions is a linear space with respect to addition and scalar multiplication defined by

$$(g+h)(t) = g(t) + h(t)$$
$$(ag)(t) = ag(t)$$

and S is called a test space.

The functions of S are very "well-behaved" as is implied by Lighthill's terminology "good functions". Some of the "good" properties of these functions are

(i) they are everywhere continuous on the real line,

(ii) they are absolutely integrable on the real line,

(iii) they are of bounded variation on every finite interval,

(iv) they are square integrable.

In fact, each g in S satisfies conditions sufficient for the existence of its Fourier transform and for the inversion formula to hold. Every result and theorem of Chapter I applies to functions of S since in each case these functions satisfy sufficient conditions. Thus we can apply the results of Chapter I to test functions without restrictions of any sort.

It is convenient to adopt a notation to denote the operation of taking the transform and inverse transform of a function. For a test function, g, we shall see later that the transform and inverse transform of g are both in S, and hence the operations of taking transforms and inverse transforms can be thought of as mappings of S into S. We let F denote the operation of taking the transform,

$$F(g) = \int_{-\infty}^{\infty} g(t)\exp(-2\pi ift)dt$$

and let F^{-1} denote the operation of taking the inverse transform,

$$F^{-1}(g) = \int_{-\infty}^{\infty} g(f)\exp(2\pi ift)df.$$

We shall use the symbol "f" to denote a real variable called frequency, and this symbol shall not be used to denote a function. The functions of S may be thought of as functions of f, or of t, the symbol used for the real variable being immaterial.

If g(t) is a function such that g(t)h(t) is in S for all h(t) in S, then g(t) is called a multiplier on S. Clearly, every constant function is a multiplier on S.

Let M denote the set of all functions m(t) which are everywhere differentiable any number of times and such that m(t) and all of its derivatives are $O(|t|^{N_0})$ as $|t| \longrightarrow \infty$ **for some integer N_0.** We show that every function of M is a multiplier on S.

Theorem 2.10 If m(t) is in M and h(t) is in S, then m(t)h(t) is in S.

Proof: We have

$$\frac{d^p(m(t)h(t))}{dt^p} = \sum_{j=0}^{p} \binom{p}{j} m^{(j)}(t)h^{(p-j)}(t)$$

It suffices to show each term in the right side is in S. From the definition of M, we have that there exist numbers $A > 0$, $K > 0$, and an integer N such that

$$|\ m^{(j)}(t)\ | < A|t|^N \quad \text{for all } t \text{ such that } |t| > K.$$

From the definition of S, we have that if N' is any integer, then there exist numbers $A' > 0$, $K' > 0$ such that

$$|h^{(p-j)}(t)| < A'|t|^{-N'} \quad \text{for all } t \text{ such that } |t| > K'.$$

Then for all t such that $|t| > \max\{K,K'\}$,

$$|m^{(j)}(t)h^{(p-j)}(t)| < AA'|t|^{N-N'}$$

But N-N' is arbitrary because N' is arbitrary. Thus m(t)h(t) is in S.

Note that if a function is contained in one of the sets M or S, then every derivative of that function is contained in the same set. Thus, if m(t) is in M and h(t) is in S, then $m^{(j)}(t)h^{(k)}(t)$ is in S for all integers $j,k \geq 0$. A familiar class of functions contained in M is the set of all polynomials.

The following theorem lists some of the properties which the functions of S possess.

Theorem 2.11 If h(t) is in S, then

(i) F(h) = H(f) is in S

(ii) $F^{-1}(h) = g(t)$ is in S

(iii) h(-t) is in S

(iv) h*(t) is in S

(v) h(at+b) is in S, a,b, constants, $a \neq 0$.

Proof: For part (i), note that the conditions of Theorem 1.14 are satisfied by

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-2\pi ift)dt$$

and each of the derivatives $H^{(p)}(f)$. Also, we may integrate by parts repeatedly. Differentiating p times and integrating by parts n times, we have

$$|H^{(p)}(f)| = |(2\pi i f)^{-n} \int_{-\infty}^{\infty} \frac{d^n}{dt^n}\{(-2\pi i t)^p h(t)\} \exp(-2\pi i f t)dt|$$

$$\leq \frac{(2\pi)^{p-n}}{|f|^n} \int_{-\infty}^{\infty} \left| \frac{d^n}{dt^n}\{t^p h(t)\} \right| dt$$

Now $t^p h(t)$ is in S by Theorem 2.10, and hence the nth derivative of $t^p h(t)$ is in S. Thus the integral on the right side above exists and is finite, and we have

$$H^{(p)}(f) = O(|f|^{-n}).$$

Part (ii) is proved by replacing $\exp(-2\pi i f t)$ by $\exp(2\pi i f t)$ and interchanging the roles of f and t in the proof of (i).

For part (iii), if we let $h(t) \longleftrightarrow H(f)$, then by (i), both $H(f) = F(h)$ and the function

$$F(H) = \int_{-\infty}^{\infty} H(f)\exp(-2\pi i f t)df = h(-t)$$

are in S (see(1.19)).

For part (iv), we have that h(t) is everywhere differentiable any number of times, and it is obvious that $h^*(t)$, the complex conjugate of h(t), also has this property. To complete this part, all we need do is note that $|h^{(n)}(t)| = |h^{*(n)}(t)|$ for all $n \geq 0$.

To show the last part, let m be a non-negative integer and N be any positive integer. Then there exist numbers $A_m > 0$, $K_m > 0$ such that for all t such that $|t| > K_m$,

$$|h^{(m)}(t)| < A_m |t|^{-N} .$$

Letting $g(t) = h(at+b)$, we have $g(t)$ differentiable any number of times and for all t such that $|at+b| > K_m$,

$$|g^{(m)}(t)| = |a^m h^{(m)}(at+b)|$$
$$< |a|^m A_m |at+b|^{-N}$$
$$= |a|^{m-N} A_m |t + (b/a)|^{-N}$$

Note that it suffices to show that $|t + (b/a)|^{-1} = O(|t|^{-1})$. To do this, let $c = b/a$ and choose t such that $|t| > 2|c|$. Then $|c/t| < 1/2$, and $-1/2 < c/t < 1/2$. Adding 1 to each member of this inequality gives $1/2 < 1 + c/t < 3/2$, or taking reciprocals, $2/3 < 1/(1+c/t) < 2$. Thus

$$\left|\frac{1}{1 + c/t}\right| < 2 ,$$

and hence

$$\left|\frac{t}{t + c}\right| < 2 ,$$

$$\left|\frac{1}{t + c}\right| < \frac{2}{|t|} ,$$

or $|t + c|^{-1} < 2|t|^{-1}$, and hence $|t + c|^{-N} < 2^N|t|^{-N}$ for $|t| > 2|c|$. Finally, for all t such that $|t| > 2|c|$ and $|at+b| > K_m$, we have

$$|g^{(m)}(t)| < |a|^{m-N} A_m |t + c|^{-N} < 2^N |a|^{m-N} A_m |t|^{-N} ,$$

which completes the proof.

2.2 GENERALIZED FUNCTIONS

We now define the class of sequences of functions of S which play a role in the construction of generalized functions similar to that of the Cauchy sequences of rational numbers in the construction of the real numbers.

<u>Definition 2.20</u> A sequence $\{g_n(t)\}$ of test functions is called <u>regular</u> if the limit

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(t)G(t)dt \tag{2.00}$$

exists and is finite for all test functions $G(t)$ in S.

We denote by C the class of all regular sequences and note that C is not empty since

$$\begin{aligned} \lim_{n \to \infty} \int_{-\infty}^{\infty} \exp(-t^2/n^2)G(t)dt &= \int_{-\infty}^{\infty} \lim_{n \to \infty} \{\exp(-t^2/n^2)\}G(t)dt \\ &= \int_{-\infty}^{\infty} G(t)dt \end{aligned} \tag{2.01}$$

and hence the sequence $\{\exp(-t^2/n^2)\}$ is regular.

<u>Definition 2.21</u> A sequence $\{h_n(t)\}$ in C is said to be <u>equivalent</u> to the sequence $\{g_n(t)\}$ in C, denoted by writing $\{h_n(t)\} \sim \{g_n(t)\}$, if and only if

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} h_n(t)G(t)dt = \lim_{n \to \infty} \int_{-\infty}^{\infty} g_n(t)G(t)dt$$

for every $G(t)$ in S.

The limits and the integrals in the above definition exist, so we could rewrite the condition for equivalence as

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \{h_n(t) - g_n(t)\}G(t)dt = 0$$

for every G(t) in S. This resembles the null condition taken in the construction of the reals.

The relation $\sim$ is clearly an equivalence relation, that is, we have

(i) $\{h_n(t)\} \sim \{h_n(t)\}$ for all regular sequences,

(ii) if $\{h_n(t)\} \sim \{g_n(t)\}$, then $\{g_n(t)\} \sim \{h_n(t)\}$,

(iii) if $\{h_n(t)\} \sim \{g_n(t)\}$ and $\{g_n(t)\} \sim \{k_n(t)\}$, then $\{h_n(t)\} \sim \{k_n(t)\}$.

Thus $\sim$ partitions C into disjoint subclasses, the equivalence classes determined by $\sim$. We let $\overline{S}$ denote the collection of all the subclasses of C determined by $\sim$.

<u>Definition 2.22</u> An element s of $\overline{S}$ is called a <u>generalized</u> <u>function</u>.

Thus a generalized function is a class of equivalent regular sequences, that is, if $\{s_n(t)\}$ is regular, then the class s of all regular sequences equivalent to $\{s_n(t)\}$ is a generalized function. A sequence $\{s_n(t)\}$ in the class s is called a <u>representative</u> of the generalized function s.

Note that if s is a generalized function, then the limit,

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} s_n(t)G(t)dt,$$

is a complex number whose value is independent of the choice of representative $\{s_n(t)\}$ of s. However, the limit does vary with G(t) in S.

This leads us to make the following definition.

Definition 2.23 Let s be a generalized function and let $\{s_n(t)\}$ be a representative of s. Then for each G(t) in S, we define

$$s(G) = \lim_{n \to \infty} \int_{-\infty}^{\infty} s_n(t)G(t)dt \qquad (2.02)$$

We now see that a generalized function s can be thought of as a mapping of the set S into the complex numbers (see Figure 2.1). We shall use this interpretation and the mapping notation in preference to the "integral" notation used by Lighthill, the latter being somewhat confusing at times.

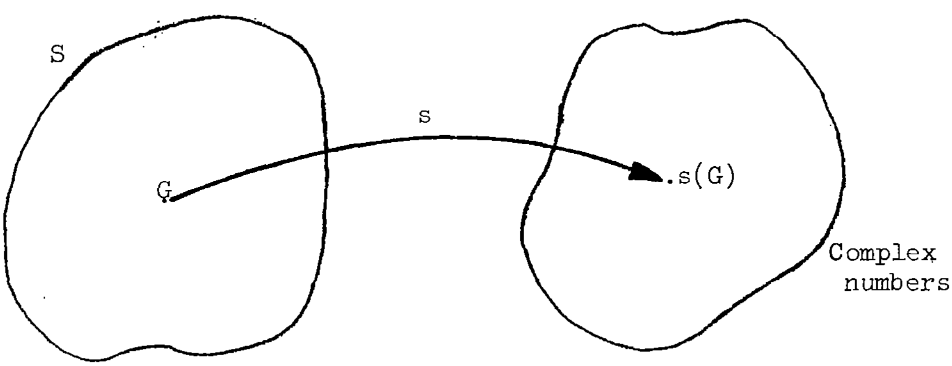


FIGURE 2.1

As an example, let I denote the generalized function with representative $\{\exp(-t^2/n^2)\}$, then from (2.01) and above we have that

$$I(G) = \int_{-\infty}^{\infty} G(t)dt,$$

and hence I is a mapping which maps each G(t) in S onto its integral over the interval $(-\infty,\infty)$.

Let r and s be generalized functions and let $\{r_n(t)\}$ and $\{s_n(t)\}$ be representatives of r and s, respectively. Thinking of r and s as sets, to say that r and s are <u>equal</u> means that r and s are the same subclass of C. Hence the representatives of these generalized functions must be equivalent because they belong to the same class, and we have that for every G(t) in S,

$$r(g) = \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(t)G(t)dt$$

$$= \lim_{n \to \infty} \int_{-\infty}^{\infty} s_n(t)G(t)dt$$

$$= s(G)$$

But this is the familiar requirement for writing r = s where r and s are interpreted as mappings of S into the complex numbers. Hence it is clear that if r and s are generalized functions, then r = s if and only if r(G) = s(G) for every G(t) in S.

<u>Definition 2.24</u> Let r and s be generalized functions and let $\{r_n(t)\}$ and $\{s_n(t)\}$ be representatives of r and s, respectively.

(i) The <u>sum</u> of the generalized functions r and s, denoted by r+s, is defined to be the generalized function with representative $\{r_n(t) + s_n(t)\}$;

(ii) The <u>derivative</u> of the generalized function r, denoted by r', is defined to be the generalized function with representative $\{r_n'(t)\}$;

(iii) $r_{a,b}$ is defined to be the generalized function with representative $\{r_n(at+b)\}$;

(iv) For each m(t) in M, the <u>product</u> mr is defined to be the generalized function with representative $\{m(t)r_n(t)\}$;

(v) The <u>Fourier transform</u> Fr of the generalized function r is defined to be the generalized function with representative $\{F(r_n)\}$. The <u>inverse Fourier transform</u> $F^{-1}r$ is defined to be the generalized function with representative $\{F^{-1}(r_n)\}$.

We must show that these definitions are consistent, that is, we must show that each one uniquely determines a generalized function. To do this, we show that

(a) each sequence named is a sequence of test functions,

(b) each sequence named is regular and hence defines a generalized function, and

(c) that the definitions are independent of the choice of representatives of r and s, that is, the generalized functions defined are unique.

Part (a) follows from previous remarks and Theorems 2.10 and 2.11. We now verify (b) and (c) for each part of the definition.

<u>Part (i)</u> Let G(t) be in S. Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \{r_n(t)+s_n(t)\}G(t)dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} \{r_n(t)G(t)+s_n(t)G(t)\}dt$$

$$= \lim_{n\to\infty} \left[\int_{-\infty}^{\infty} r_n(t)G(t)dt + \int_{-\infty}^{\infty} s_n(t)G(t)dt\right].$$

$$= \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(t)G(t)dt$$

$$+ \lim_{n \to \infty} \int_{-\infty}^{\infty} s_n(t)G(t)dt \tag{2.03}$$

Now each limit in the last line on the right exists and is independent of the choice of representative of r and s. Hence the limit on the left side exists and is independent of the choice of representatives $\{r_n(t)\}$ and $\{s_n(t)\}$. This verifies (b) and (c).

In terms of the notation (2.02), (2.03) yields

$$(r+s)(G) = r(G) + s(G)$$

for all G(t) in S. Thus r+s is just the sum of the mappings r and s.

Part (ii) With U = G(t) and $dV = r_n'(t)dt$, integrating by parts one time, we have

$$\int_{-\infty}^{\infty} r_n'(t)G(t)dt = r_n(t)G(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} r_n(t)G(t)dt \tag{2.04}$$

$$= - \int_{-\infty}^{\infty} r_n(t)G'(t)dt$$

Letting $n \longrightarrow \infty$ in both sides, since $\{r_n(t)\}$ is regular and G'(t) is in S, the limit in the right side exists and is independent of the representative $\{r_n(t)\}$ of r. Thus the left side has the same properties.

In our adopted notation, letting $n \longrightarrow \infty$ in (2.04) yields

$$r'(G) = -r(G') \tag{2.05}$$

for all G(t) in S.

Part (iii) By making a change of variable, we have for each $r_n(t)$,

$$\int_{-\infty}^{\infty} r_n(at+b)G(t)dt = |a|^{-1} \int_{-\infty}^{\infty} r_n(t)G((t-b)/a)dt \qquad (2.06)$$

By (v) of Theorem 2.11, $G((t-b)/a)$ is in S. Since $\{r_n(t)\}$ is regular, the limit as $n \longrightarrow \infty$ of the right side exists and is independent of the choice of the representative of r. Therefore, the left side also has these properties.

Letting $\overline{G}(t) = G((t-b)/a)$ and taking the limit in both sides of (2.06) yields

$$r_{a,b}(G) = r(\overline{G}) \qquad (2.07)$$

for all $G(t)$ in S.

Part (iv) This part follows easily from

$$\int_{-\infty}^{\infty} \{m(t)r_n(t)\}G(t)dt = \int_{-\infty}^{\infty} r_n(t)\{m(t)G(t)\}dt, \qquad (2.08)$$

noting that $\{r_n(t)\}$ is regular, $m(t)G(t)$ is in S, and letting $n \longrightarrow \infty$ in both sides. In the mapping notation, we have that for every $G(t)$ in S,

$$mr(G) = r(m \cdot G) \qquad (2.09)$$

where $m \cdot G$ is the ordinary function $(m \cdot G)(t) = m(t)G(t)$.

Part (v) Recall that for ordinary functions if $h(t) \longleftrightarrow H(f)$ and $g(t) \longleftrightarrow G(f)$, then by (1.19), $H(t) \longleftrightarrow h(-f)$; by (1.25), $h(-t) \longleftrightarrow H(-f)$; and by Parseval's formula (1.36),

$$\int_{-\infty}^{\infty} g(t)h(t)dt = \int_{-\infty}^{\infty} G(f)H(-f)df.$$

Using (1.19) and (1.25), several different forms of Parseval's formula are obtained. One form of interest to us here is

$$\int_{-\infty}^{\infty} g(t)h(-t)dt = \int_{-\infty}^{\infty} G(f)H(f)df. \qquad (2.10)$$

In what follows, we shall not assume a fixed role for the variables f and t as has been previously taken. We have to this point written the transform as a function of f and the inverse transform as a function of t. However, the roles of f and t are interchangeable in (1.5) and (1.6). That is, whether we have the transform or the inverse transform is determined by the sign of the exponent in the integrals of (1.5) and (1.6), not on the manner in which the variables are denoted.

Parseval's formula is valid for test functions, and hence if $H(f)$ is in S and $h(t) \longleftrightarrow H(f)$, then $h(t)$ is in S and from (2.10) we have

$$\int_{-\infty}^{\infty} F(r_n)H(f)df = \int_{-\infty}^{\infty} r_n(t)h(-t)dt \qquad (2.11)$$

By interchanging the roles of f and t in (1.19) and using (iii) of Theorem 2.11, we have $h(-t) = F(H)$ is in S. The sequence $\{r_n(t)\}$ is regular, so the limit as $n \longrightarrow \infty$ in the right side of (2.11) exists and is independent of the representative of r. Hence the limit of the left side exist and is independent of the representative of r. In the mapping notation, this yields

$$Fr(H) = \lim_{n \to \infty} \int_{-\infty}^{\infty} F(r_n)H(f)df$$

$$= \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(t)h(-t)dt$$

$$= \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(t)F(H)dt \qquad (2.12)$$

$$= r(F(H)),$$

that is, the image of H under Fr is the same as the image of F(H) under the mapping determined by r.

For the second part, we consider the functions of the representative sequence of r as functions of the variable f. Then $F^{-1}(r_n)$ is a function of t for each n, and by Parseval's formula,

$$\int_{-\infty}^{\infty} F^{-1}(r_n)h(t)dt = \int_{-\infty}^{\infty} r_n(f)H(-f)df \qquad (2.13)$$

For each h(t) in S, H(f) = F(h) is in S. Thus so is H(-f), and letting $n \longrightarrow \infty$ in both sides of (2.13) shows that $F^{-1}r$ is a uniquely determined generalized function.

Now, by (1.25), H(-f) = F(h(-t)), and by making a change in the variables in (1.5), we find that $F(h(-t)) = F^{-1}(h)$. In the mapping notation, this yields that for all h(t) in S,

$$F^{-1}r(h) = \lim_{n \to \infty} \int_{-\infty}^{\infty} F^{-1}(r_n)h(t)dt$$

$$= \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(f)H(-f)df$$

$$= \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(f)F^{-1}(h)df \qquad (2.14)$$

$$= r(F^{-1}(h)).$$

This gives an interpretation of $F^{-1}r$ similar to the one for Fr above.

Applying F^{-1} to (2.13), applying F to (2.14), and noting that $F(F^{-1}(H)) = H = F^{-1}(F(H))$ for all H in S, we have that for each r in $\overline{S}$,

$$F^{-1}Fr(H) = F^{-1}r(F(H)) = r(F^{-1}(F(H))) = r(H);$$

$$FF^{-1}r(H) = Fr(F^{-1}(H)) = r(F(F^{-1}(H))) = r(H).$$

Thus we see that if F is thought of as a mapping of $\overline{S}$ into itself, then F^{-1} is the inverse mapping of F, that is,

$$FF^{-1} = I_S = F^{-1}F$$

where I_S is defined by $I_S(r) = r$ for all r in $\overline{S}$.

We have already noted that every constant function $m(t) = a$ is in M and thus if r is a generalized function, by Definition 2.24, part (iv), $a \cdot r$ is a generalized function. In part (i) of the same definition, a sum is defined on $\overline{S}$. It is easy to verify the following theorem.

<u>Theorem 2.20</u> The set $\overline{S}$ of all generalized functions with addition as defined in (i) of Definition 2.24 and with scalar multiplication defined by letting $m(t) = a$ in (iv) of Definition 2.24 is a linear space over the complex numbers.

We have already noted that each generalized function in $\overline{S}$ determines a mapping of the space S of all test functions into the complex numbers. Such a mapping is usually called a functional. We now show that these functionals are linear.

Let r be in $\overline{S}$ and let $\{r_n(t)\}$ be a representative of r. Then if

$G(t)$ and $H(t)$ are elements of S, and a,b are complex numbers, we have

$$r(aG+bH) = \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(t)\{aG(t)+bH(t)\}dt$$

$$= a \cdot \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(t)G(t)dt + b \cdot \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(t)H(t)dt$$

$$= a \cdot r(G) + b \cdot r(H)$$

Hence the mapping determined by r is a linear functional on S.

Note that by reapplying part (ii) of Definition 2.24 to the derivative r' of r, we obtain $r'' = (r')'$; the <u>second derivative</u> of the generalized function r (note that the proof of consistency is valid with $r_n(t)$ and $r_n'(t)$ replaced by $r_n'(t)$ and $r_n''(t)$, respectively). In fact, since each function of the sequence $\{r_n(t)\}$ representing r is differentiable any number of times, we may reapply part (ii) and its proof of consistency any number of times. Thus, by induction, the <u>kth derivative</u> of a generalized function is defined, and we see that every generalized function has derivatives of all orders. We denote the kth derivative of r by the symbol $r^{(k)}$.

We have shown that every generalized function r has a Fourier transform $s = Fr$. Applying F^{-1} to both sides, we have $F^{-1}s = F^{-1}Fr = I_S(r) = r$, and the generalized functions r and s form a transform pair. In the notation of the first chapter, we have $r \longleftrightarrow s$.

<u>Theorem 2.21</u> Let r be a generalized function and let s be its Fourier transform, that is, $r \longleftrightarrow s$. Then

(i) $s \longleftrightarrow r_{-1,0}$ (2.15)

(ii) $r_{a,b} \longleftrightarrow |a|^{-1}\exp(2\pi ibf/a)s_{1/a,0}$ (2.16)

(iii) $r^{(k)} \longleftrightarrow (2\pi if)^k s, \ k = 1,2,3,\ldots\ldots$ (2.17)

(iv) $(2\pi it)^k r \longleftrightarrow s^{(k)}, \ k = 1,2,3,\ldots\ldots$ (2.18)

Proof: We shall prove part (iv). The proofs of the other parts are done in a similar manner.

Let $\{r_n(t)\}$ be a representative of r. Then by part (v) of Definition 2.24, $\{s_n(f)\}$ where

$$s_n = F(r_n) \ , \ n = 1,2,3,\ldots\ldots$$

is a representative of the generalized function s. Each $s_n(f)$ is a test function and (1.26) holds. Thus,

$$s_n^{(k)} = F((2\pi it)^k r_n), \ n = 1,2,3,\ldots\ldots \tag{2.19}$$

By part (iv) of Definition 2.24, the sequence $\{(2\pi it)^k r_n(t)\}$ represents the generalized function $(2\pi it)^k r$. From part (ii) of Definition 2.24 and the comments preceding this theorem, the sequence $\{s_n^{(k)}(f)\}$ represents the generalized function $s^{(k)}$. Therefore, by (2.19) and part (v) of Definition 2.24,

$$(2\pi it)^k r \longleftrightarrow s^{(k)} \ , \ k = 1,2,3,\ldots\ldots$$

Let h(t) be an ordinary function having the property that h(t)G(t) is integrable on $(-\infty, \infty)$ for every G(t) in S, and write

$$\bar{h}(G) = \int_{-\infty}^{\infty} h(t)G(t)dt \tag{2.20}$$

It is easy to see that this defines a linear functional on S, for if a and b are complex numbers and G(t) and H(t) are in S, then from the linearity of integration, we have that

$$\bar{h}(aG+bH) = a\bar{h}(G) + b\bar{h}(H) \ .$$

This naturally leads to the question of whether or not the ordinary function h(t) determines a generalized function in the above manner. To answer this, we must determine if there exists a generalized function $\bar{h}$ such that if $\{h_n(t)\}$ is a representative of $\bar{h}$, then

$$\bar{h}(G) = \lim_{n \to \infty} \int_{-\infty}^{\infty} h_n(t)G(t)dt = \int_{-\infty}^{\infty} h(t)G(t)dt.$$

Lighthill, [3], pp. 22-23, shows that if h(t) is an ordinary function such that $(1+t^2)^{-N}h(t)$ is absolutely integrable on $(-\infty,\infty)$ for some integer N, then there exists a regular sequence $\{h_n(t)\}$ such that for all G(t) in S,

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} h_n(t)G(t)dt = \int_{-\infty}^{\infty} h(t)G(t)dt,$$

where the integral on the right side exists in the ordinary sense because

$$\int_{-\infty}^{\infty} h(t)G(t)dt = \int_{-\infty}^{\infty} \{(1+t^2)^{-N}h(t)\}\{(1+t^2)^{N}G(t)\}\, dt,$$

with $(1+t^2)^{-N}h(t)$ absolutely integrable for some N and $(1+t^2)^{N}G(t)$ a test function. It is easy to see that the set of all such functions h(t) forms a linear space K and that K is embedded in $\bar{S}$ by the mapping Φ which maps each h(t) onto the class of all sequences equivalent to $\{h_n(t)\}$ (see Figure 2.2).

Definition 2.25 If $h(t)$ is an ordinary function such that $(1+t^2)^{-N}h(t)$ is absolutely integrable, then the image under Φ of $h(t)$ in $\overline{S}$ is called the generalized function defined by $h(t)$ and is denoted by the symbol $\overline{h}$.

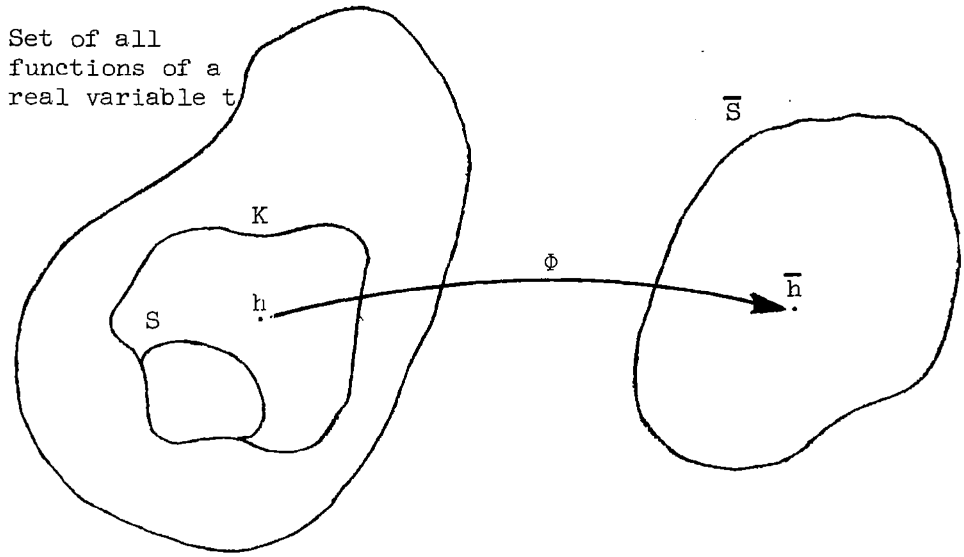


Figure 2.2

Let $h(t)$ be an ordinary function which defines a generalized function $\overline{h}$. We already know that the generalized function $\overline{h}$ has a generalized derivative $(\overline{h})'$. Suppose that $h(t)$ is differentiable and that $h'(t)$ defines a generalized function $\overline{(h')}$. Then we have the following theorem.

Theorem 2.22 Let $h(t)$ and $h'(t)$ be ordinary functions which define generalized functions $\overline{h}$ and $\overline{(h')}$, respectively. Then the generalized functions $(\overline{h})'$ and $\overline{(h')}$ are equal.

Proof: From (2.05), we have that for the generalized function $\overline{h}$ and $(\overline{h})'$,

$$(\overline{h})'(G) = -\overline{h}(G') \text{ for all } G(t) \text{ in } S.$$

Now $h'(t)$ defines $\overline{(h')}$ by

$$\overline{(h')}(G) = \int_{-\infty}^{\infty} h'(t)G(t)dt.$$

We note that it suffices to show that $\overline{(h')}(G) = -\overline{h}(G')$ for all $G(t)$ in S. Due to the conditions on the ordinary function $h(t)$ and $h'(t)$, each of the integrals

$$\int_{-\infty}^{\infty} h'(t)G(t)dt, \quad \int_{-\infty}^{\infty} h(t)G(t)dt, \text{ and } \int_{-\infty}^{\infty} h(t)G'(t)dt$$

exist (finite). Integrating the first by parts, we have

$$\int_{-\infty}^{\infty} h'(t)G(t)dt = \lim_{\substack{a \to \infty \\ b \to \infty}} h(t)G(t)\Big|_{-a}^{b} - \int_{-\infty}^{\infty} h(t)G'(t)dt.$$

Hence

$$\lim_{b \to \infty} h(b)G(b) \qquad \text{and} \qquad \lim_{a \to \infty} h(-a)G(-a)$$

must both be finite. But the existence of the integral $\int_{-\infty}^{\infty} h(t)G(t)dt$ implies that both limits are zero. Therefore

$$\overline{(h')}(G) = \int_{-\infty}^{\infty} h'(t)G(t)dt = -\int_{-\infty}^{\infty} h(t)G'(t)dt = -\overline{h}(G') = (\overline{h})'(G)$$

for all $G(t)$ in S, and hence $\overline{(h')} = (\overline{h})'$.

<u>Theorem 2.23</u> If $h(t)$ is an ordinary function which is absolutely integrable on $(-\infty,\infty)$--so that its Fourier transform $H(f)$ exists by the classical Fourier integral theorem--then the Fourier transform of the generalized function $\overline{h}$ defined by $h(t)$ is the generalized function $\overline{H}$ defined by $H(f)$.

<u>Proof</u>: We have that

$$\int_{-\infty}^{\infty} |(1+f^2)^{-1}H(f)|df = \int_{-\infty}^{\infty} |(1+f^2)^{-1} \int_{-\infty}^{\infty} h(t)\exp(-2\pi ift)dt|df$$

$$\leq \int_{-\infty}^{\infty} (1+f^2)^{-1}df \int_{-\infty}^{\infty} |h(t)|dt$$

$$< \infty$$

Hence $(1+f^2)H(f)$ is absolutely integrable on $(-\infty,\infty)$ and does define a generalized function. Let $g(t)$ be any test function and let $G(f)$ be its Fourier transform. Then we have

$$\bar{H}(G) = \int_{-\infty}^{\infty} H(f)G(f)df$$

$$= \int_{-\infty}^{\infty} h(t)g(-t)dt$$

$$= \int_{-\infty}^{\infty} h(t)F(G)dt$$

$$= \bar{h}(F(G))$$

$$= F\bar{h}(G)$$

where we have used Parseval's formula and (2.12). Since this holds for all $G(t)$ in S, we have $F\bar{h} = \bar{H}$ and the theorem is proved.

2.3 THE DIRAC DELTA FUNCTION AND ITS TRANSFORM

We now show that the sequence $\{(n/\pi^{\frac{1}{2}}) \exp(-nt^2)\}$ is regular and represents the important Dirac delta function δ. This generalized function has the property that for every $H(t)$ in S,

$$\delta(H) = H(0) \tag{2.21}$$

To prove this, we shall need the following definite integrals:

(a) $$\int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}} \exp(-nt^2)dt = 1 \ ; \ n = 1,2,3,\ldots\ldots$$

(b) $$\int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}}\, t \exp(-nt^2)dt = (n\pi)^{\frac{1}{2}}$$

To establish (2.21), we must show that if H(t) is a test function, then

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}} \exp(-nt^2)H(t)dt = H(0)$$

Multiplying both sides of (a) by H(0), we may write

$$\left| \int_{-\infty}^{\infty} (n/\pi) \exp(-nt^2)H(t)dt - H(0) \right| = \left| \int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}} \exp(-nt^2)\{H(t)-H(0)\}dt \right|$$

$$= \left| \int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}} t \exp(-nt^2) \frac{\{H(t)-H(0)\}}{t} dt \right|$$

$$\leq \int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}} |t| \exp(-nt^2) \left| \frac{H(t)-H(0)}{t} \right| dt$$

Now by the Mean Value Theorem for derivatives, on each interval $[0,t]$ (or $[t,0]$) there exists a β_t, $0 < \beta_t < t$ $(t < \beta_t < 0)$, such that

$$H'(\beta_t) = \frac{H(t) - H(0)}{t}$$

Now H'(t) is in S and hence is bounded on the real line, so we have for each t

$$A = \sup_t\{|H'(t)|\} \geq |H'(\beta_t)| = \left|\frac{H(t) - H(0)}{t}\right|$$

Putting into the above and then using (b) gives

$$\left|\int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}}\exp(-nt^2)H(t)dt - H(0)\right| \leq A \int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}}|t|\exp(-nt^2)dt$$

$$= A(n\pi)^{\frac{1}{2}}$$

and this last expression tends to zero as $n \longrightarrow \infty$. Hence we have

$$\delta(H) = \lim_{n \to \infty} \int_{-\infty}^{\infty} (n/\pi)^{\frac{1}{2}}\exp(-nt^2)H(t)dt = H(0)$$

It is well known (see [4]) that $(n/\pi)^{\frac{1}{2}}\exp(-nt^2)$ and $\exp(-\pi^2 f^2/n)$ form an ordinary Fourier transform pair, that is,

$$(n/\pi)^{\frac{1}{2}} e(-nt^2) \longleftrightarrow \exp(-\pi^2 f^2/n)$$

and hence the sequence $\{\exp(-\pi^2 f^2/n)\}$ is a representative of the Fourier transform of the generalized function δ. Now for any test function $H(f)$, we have

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \exp(-\pi^2 f^2/n)H(f)df = \int_{-\infty}^{\infty} 1\cdot H(f)df = I(H) \qquad (2.22)$$

where I is the generalized function of the example following Definition 2.23. But from (2.22), we see that I is the generalized function defined by the ordinary function $h(t) = 1$ for all t. Thus, by Definition 2.25, we have that $I = \bar{1}$, and hence $F(\delta) = \bar{1}$ and $F^{-1}(\bar{1}) = \delta$. More briefly,

$$\delta \longleftrightarrow \bar{1} \qquad (2.23)$$

From (2.23) and (2.16), we have

$$\delta_{1,-t_o} \longleftrightarrow \overline{\exp}(-2\pi i t_o f) \tag{2.24}$$

Note that $\exp(-2\pi i t_o f)$ does define a generalized function because $(1+t^2)^{-1}\exp(-2\pi i t_o f)$ is absolutely integrable.

From (2.24) and Theorem 2.21,

$$\overline{\exp}(2\pi i t f_o) \longleftrightarrow \delta_{1,-f_o} \tag{2.25}$$

Putting $f_o = 0$ in (2.25) yields

$$\overline{1} \longleftrightarrow \delta \tag{2.26}$$

First noting that cos at and sin at define generalized functions, and then writing $\cos 2\pi t f_o = \frac{1}{2}[\exp(2\pi i t f_o) + \exp(-2\pi i t f_o)]$, using (2.25) and the linearity of the Fourier transform, we obtain

$$\overline{\cos}\ 2\pi t f_o \longleftrightarrow \tfrac{1}{2}[\delta_{1,-f_o} + \delta_{1,f_o}] \tag{2.27}$$

Writing $\sin 2\pi t f_o = \frac{1}{2i}[\exp(2\pi i t f_o) - \exp(-2\pi i t f_o)]$, in a similar manner we find that

$$\overline{\sin}\ 2\pi t f_o \longleftrightarrow \frac{1}{2i}[\delta_{1,-f_o} - \delta_{1,f_o}] \tag{2.28}$$

Hence the generalized functions $\overline{\sin}\ 2\pi t f_o$ and $\overline{\cos}\ 2\pi t f_o$ defined by the ordinary functions $\sin 2\pi t f_o$ and $\cos 2\pi t f_o$ have Fourier transforms, a property which the ordinary functions do not have.

2.4 COMMENTS ON NOTATION

It has been mentioned that in [3] Lighthill uses an "integral" notation for s(G), s in $\overline{S}$, G in S. There, to denote the number s(G), the symbol

$$\int_{-\infty}^{\infty} s(t)G(t)dt, \tag{2.29}$$

is used, that is,

$$s(G) \equiv \int_{-\infty}^{\infty} s(t)G(t)dt \equiv \lim_{n \to \infty} \int_{-\infty}^{\infty} s_n(t)G(t)dt \tag{2.30}$$

In general, the expression (2.29) has no meaning as an integral, in fact, the notation s(t) has no meaning in general since s is not an ordinary function. However, for the space K of ordinary functions h(t) such that $(1+t^2)^{-N}h(t)$ is absolutely integrable for some N, each quantity in the notation

$$\overline{h}(G) = \int_{-\infty}^{\infty} h(t)G(t)dt = \lim_{n \to \infty} \int_{-\infty}^{\infty} h_n(t)G(t)dt \tag{2.31}$$

has a well-defined meaning. Furthermore, the integral notation is preferred here because it is more explicit than the notation $\overline{h}(G)$. This, along with some manipulative advantages of the integral notation, leads us to make the following changes in notation.

Definition 2.40 A generalized function s will be denoted by the symbol s(t) and for each G in S, s(G) will be denoted by (2.29). Furthermore, for each h(t) in K, the symbol h(t) will be used to denote both the ordinary function h(t) and the generalized function $\overline{h}$ defined by h(t).

We shall now point out some changes which this makes in previously encountered generalized functions. We now have that

(1) The generalized function $r_{a,b}$ is now denoted by the symbol $r(at+b)$ in order to be consistent with Definition 2.40, for we have

$$\int_{-\infty}^{\infty} r(t)G(t)dt = \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(t)G(t)dt$$

and the defining sequence for $r_{a,b}$ is obtained by replacing t by $at+b$ in each $r_n(t)$, hence

$$\int_{-\infty}^{\infty} r(at+b)G(t)dt = \lim_{n \to \infty} \int_{-\infty}^{\infty} r_n(at+b)G(t)dt \qquad (2.32)$$

(2) The transform pair of (2.23) are now written as

$$\delta(t) \longleftrightarrow 1 \qquad (2.33)$$

(3) The transform pairs (2.24) through (2.28) are now written as

$$\delta(t-t_o) \longleftrightarrow \exp(-2\pi i t_o f) \qquad (2.34)$$

$$\exp(2\pi i t f_o) \longleftrightarrow \delta(f-f_o) \qquad (2.35)$$

$$1 \longleftrightarrow \delta(f) \qquad (2.36)$$

$$\cos 2\pi t f_o \longleftrightarrow \tfrac{1}{2}[\delta(f-f_o) + \delta(f+f_o)] \qquad (2.37)$$

$$\sin 2\pi t f_o \longleftrightarrow \frac{1}{2i}[\delta(f-f_o) - \delta(f+f_o)] \qquad (2.38)$$

(4) For each $H(t)$ in S, in place of (2.21) we now have

$$\int_{-\infty}^{\infty} \delta(t)H(t)dt = H(0) \qquad (2.39)$$

2.5 EQUALITY OF ORDINARY AND GENERALIZED FUNCTIONS ON AN INTERVAL

In (B) of Section 2.1, it was stated that $\delta(t)$ is sometimes described as having the property that $\delta(t) = 0$ if $t \neq 0$. We can now give a more precise meaning to this part of (B).

Definition 2.50 Let $g(t)$ be an ordinary function such that, for any test function $G(t)$ which is zero outside of the interval (a,b), $g(t)G(t)$ is integrable on (a,b), $a < b$. If $s(t)$ is a generalized function such that

$$\int_{-\infty}^{\infty} s(t)G(t)dt = \int_{a}^{b} g(t)G(t)dt \tag{2.40}$$

then we define $s(t) = g(t)$ for $a < t < b$.

In the sense of this definition, we have $\delta(t) = 0$ for $0 < t < \infty$. For suppose $G(t) = 0$ for all $t \leq 0$, $G(t)$ in S. Then $G(0) = 0$ and we have

$$\int_{-\infty}^{\infty} \delta(t)G(t)dt = G(0) = 0 = \int_{0}^{\infty} 0 \cdot G(t)dt$$

where the first equality is obtained from (2.39). In a similar manner, we find that $\delta(t) = 0$ for $-\infty < t < 0$. Thus, in the sense of Definition 2.50, $\delta(t) = 0$ if $t \neq 0$.

2.6 CONVOLUTION OF GENERALIZED FUNCTIONS

We shall not attempt a complete discussion of the convolution of two generalized functions. A convolution of generalized functions cannot in general be defined without imposing some restrictions on one of the functions. A complete discussion may be found in [5] and [8].

One immediate problem we would encounter in such a discussion would be the lack of the concepts of convergence in S and continuity of generalized functions. For a proof of the continuity in a certain sense of every generalized function defined here, see [6].

Convolution of a generalized function and a test function

The convolution of a generalized function and a test function is derived from a previously defined generalized function. Putting $a = 1$ and $b = -t$ in the definition of the generalized function $s_{a,b}$, we obtain for each G in S,

$$\begin{aligned} s_{1,-t}(G) &= \lim_{n \to \infty} \int_{-\infty}^{\infty} s_n(x-t)G(x)dx \\ &= \int_{-\infty}^{\infty} s(x-t)G(x)dx \end{aligned} \tag{2.41}$$

Fixing G and letting t vary, we see that this defines an ordinary function of t. The convolution of s in $\overline{S}$ and G in S is defined to be the ordinary function

$$s(t)*G(t) = s_{1,-t}(G) \tag{2.42}$$

By making a change of variable in (2.41), we see that

$$\begin{aligned} s(t)*G(t) &= \lim_{n \to \infty} \int_{-\infty}^{\infty} s_n(x)G(t+x)dx \\ &= \int_{-\infty}^{\infty} s(x)G(t+x)dx \end{aligned} \tag{2.43}$$

In the last lines of (2.41) and (2.43), we have reverted to the integral notation.

Note that if G is any function of S, then

$$\delta(t)*G(t) = \int_{-\infty}^{\infty} \delta(x)G(t+x)dx$$
$$= G(t) \tag{2.44}$$

<u>Convolution of generalized functions</u>

Let r and s be generalized functions and suppose s(t)*G(t) is in S for all G in S. The <u>convolution</u> of r and s is defined by

$$(r*s)(G) = r(s*G) \tag{2.45}$$

When using the r(t), s(t) notation, the convolution will be denoted by writing r(t)*s(t). The corresponding integral notation is obtained as follows. We have

$$(r*s)(G) = \int_{-\infty}^{\infty} (r(t)*s(t))G(t)dt \tag{2.46}$$

and

$$(r*s)(G) = r(s*G)$$
$$= \int_{-\infty}^{\infty} r(x) \left[\int_{-\infty}^{\infty} s(t-x)G(t)dt\right] dx \tag{2.47}$$
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} r(x)s(t-x)dx\right] G(t)dt$$

Comparing (2.46) and (2.47), we have in the integral notation that

$$r(t)*s(t) = \int_{-\infty}^{\infty} r(x)s(t-x)dx \tag{2.48}$$

As noted above, $\delta(t)*G(t) = G(t)$ for all G in S, and hence for any $r(t)$ in $\overline{S}$, we have

$$r*\delta(G) = r(\delta*G) = r(G) \tag{2.49}$$

Therefore, $r(t)*\delta(t) = r(t)$ for every generalized function $r(t)$. In the integral notation, (2.49) yields

$$r(t) = \int_{-\infty}^{\infty} r(x)\delta(t-x)dx \tag{2.50}$$

We have already shown that $F(\delta(t)) = 1$, and hence for any $s(t)$ in $\overline{S}$ we have that

$$F(s(t)*\delta(t)) = F(s(t))\cdot 1 = F(s(t))*F(\delta(t)). \tag{2.51}$$

Clearly we have

$$s(t)*\delta(t-t_o) = s(t-t_o)$$

and by (2.16) and (2.34)

$$F(s(t-t_o)) = \exp(-2\pi i f t_o)F(s(t)),$$

$$F(\delta(t-t_o)) = \exp(-2\pi i f t_o).$$

Therefore, we have

$$F(s(t)*\delta(t-t_o)) = F(s(t))F(\delta(t-t_o)) \tag{2.52}$$

Suppose that $\Phi(t)$ is a finite linear combination of delta functions, that is,

$$\Phi(t) = \sum_{j=-M}^{N} a_j \delta(t-t_j) \tag{2.53}$$

where the a_j and x_j are constants. Then it is easy to show that convolution is linear, and hence if s(t) is in $\overline{S}$

$$s(t)*\Phi(t) = \sum_{j=-M}^{N} a_j s(t-t_j) \tag{2.54}$$

Applying F to both sides, using its linearity and (2.52), we obtain

$$F(s(t)*\Phi(t)) = F(s(t))F(\Phi(t)) \tag{2.55}$$

Letting $r(f) = F(s(t))$ and $q(f) = F(\Phi(t))$, applying F^{-1} to both sides of (2.55) and using (2.54), we obtain

$$F^{-1}(r(f)q(f)) = \sum_{j=-M}^{N} a_j s(t-t_j) \tag{2.56}$$

2.7 TRIGONOMETRIC SERIES

If $s_z(t)$ is a generalized function for each value of the parameter z and if s(t) is a generalized function such that

$$\lim_{z \to a} \int_{-\infty}^{\infty} s_z(t)G(t)dt = \int_{-\infty}^{\infty} s(t)G(t)dt \tag{2.57}$$

for all G(t) in S, then $s_z(t)$ is said to <u>converge</u> to s(t) and we write

$$\lim_{z \to a} s_z(t) = s(t).$$

With this definition of convergence in $\overline{S}$, we have the following theorem (see [3] for a proof).

Theorem 2.70 The trigonometric series

$$\sum_{n=-\infty}^{\infty} a_n \exp(in\pi t/p) \tag{2.58}$$

converges in the sense of (2.57) to a generalized function s(t) if and only if $a_n = O(|n|^N)$ for some N as $|n| \longrightarrow \infty$. If (2.58) converges, then its Fourier transform is

$$r(f) = \sum_{n=-\infty}^{\infty} a_n \delta(f-n/2p) \tag{2.59}$$

Also, s(t) = 0 only if $a_n = 0$ for all n.

The function r(f) is called a "row of deltas" of spacing 1/2p. This function is represented graphically by drawing vertical lines of amplitudes a_n at the points f = n/2p (see Figure 2.3). This repre-

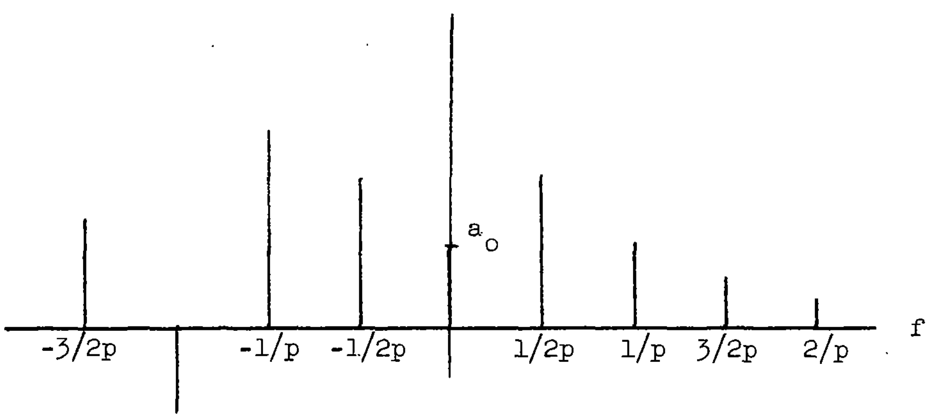


FIGURE 2.3

sentation arises from the equality in the sense of Definition 2.50 of r(f) and an ordinary function which is zero on (n/2p, (n+1)/2p). If g(t) is an ordinary periodic function which has a Fourier series representation, then the a_n are the Fourier coefficients (see [3])

$$a_n = (1/2p) \int_{-p}^{p} g(t)\exp(-in\pi t/p)dt.$$

This is equivalent to the statement that convergence of a trigonometric series in the ordinary sense implies convergence in the sense of (2.57) and that the limits are the same. The converse is not true, for by Theorem 2.70, the series

$$\sum_{n=-\infty}^{\infty} \cos(n\pi t/p)$$

converges in the sense of (2.57), but obviously not to an ordinary function.

CHAPTER III

FILTERS

3.0 LINEAR SYSTEMS

A linear system, for our purposes, is a linear operator (mapping) L of $\overline{S}$ into $\overline{S}$. That is, if g(t), h(t) are in $\overline{S}$ and a,b are scalars, then

$$L\{ag(t) + bh(t)\} = aL\{g(t)\} + bL\{h(t)\} \tag{3.0}$$

We have already encountered some linear operators on $\overline{S}$. The Fourier transform and inverse Fourier transform are both linear operators on $\overline{S}$. Another example is the operation of taking the generalized derivative of a generalized function.

Under certain conditions, a linear system L is completely characterized by the effect of applying L to the set of generalized functions of the form $\delta(t-x)$. That is, suppose that for every value of the parameter x, we have $L\{\delta(t-x)\} = h_x(t)$, and that the family of generalized functions $h_x(t)$ is known. Let g(t) be an arbitrary element of $\overline{S}$, and let $r(t) = L\{g(t)\}$. The function g(t) is usually called the <u>input</u> of the linear system L, and the function r(t) is called the <u>output</u> of L. From (2.50) we have

$$g(t) = \int_{-\infty}^{\infty} g(x)\delta(t-x)dx, \tag{3.1}$$

and applying L to both sides, we obtain

$$r(t) = L\{ \int_{-\infty}^{\infty} g(x)\delta(t-x)dx\}.$$

Assuming that (3.0) is sufficient to write

$$L\{\int_{-\infty}^{\infty} g(x)\delta(t-x)dx\} = \int_{-\infty}^{\infty} L\{g(x)\delta(t-x)\}dx$$

then

$$r(t) = \int_{-\infty}^{\infty} g(x)L\{\delta(t-x)\}dx$$

$$= \int_{-\infty}^{\infty} g(x)h_x(t)dx$$

Suppose that L satisfies the condition:

(A) If $L\{g(t)\} = r(t)$ and t_o is real constant, then $L\{g(t-t_o)\} = r(t-t_o)$, i.e., L is time-invariant.

Then if $L\{\delta(t)\} = h(t)$, $L\{\delta(t-t_o)\} = h(t-t_o)$ and

$$r(t) = \int_{-\infty}^{\infty} g(x)h(t-x)dx, \tag{3.2}$$

that is, the output of L is given in terms of the input and a unique function h(t). The function h(t) is called the impulse response or weight function of the linear system L, and its Fourier transform

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-2\pi ift)dt \tag{3.3}$$

is called the system or transfer function of L.

Note that (3.2) is the convolution $g(t)*h(t)$. If $r(t)\longleftrightarrow R(f)$ $g(t)\longleftrightarrow G(f)$, then using (3.2) and assuming that the convolution theorem holds for these generalized functions, we have

$$r(t) = g(t)*h(t) \longleftrightarrow G(f)H(f)$$

and

$$R(f) = G(f)H(f), \qquad (3.4)$$

$$r(t) = \int_{-\infty}^{\infty} G(f)H(f)\exp(2\pi ift)df$$

That is, the Fourier transform of the output of the linear system L is equal to the product of the transforms of the input and the weight function h(t). We also note that if G(f) is the transform of an input and R(f) is the transform of a desired output, then from (3.4) the transfer function of the linear system L giving the desired output is

$$H(f) = \frac{R(f)}{G(f)}. \qquad (3.5)$$

H(f) may in general be complex {see (1.38)}

$$H(f) = A(f)\exp(i\theta(f))$$

where A(f) and $\theta(f)$ have already been defined in the classical case as the Fourier spectrum and phase angle of h(t), respectively.

Definition 9. A linear system L which satisfies (A) is called a filter if A(f) is small in some sense on certain parts of the frequency axis. A low-pass filter is a filter for which A(f) is small for $|f| > f_c$ where f_c is called the cut-off frequency. A band-pass filter is a filter for which A(f) is small outside the intervals $[\bar{f}_j, f_j]$, $j = 1, 2, \ldots\ldots, n$. A frequency $\bar{f}$ is said to be passed by a filter if $A(\bar{f})$ is not small.

3.1 IDEAL LOW-PASS FILTERS

Ideal smoothing filter.

This, by defintion, is a low-pass filter which passes all frequencies f such that $|f| \leq f_c$ without change and deletes all frequencies greater than f_c. No phase shift is involved, and hence $\theta(f) = 0$. Thus

$$H(f) = A(f) = \begin{cases} 1 & |f| \leq f_c \\ 0 & |f| > f_c. \end{cases} \tag{3.6}$$

See figure 3.1.

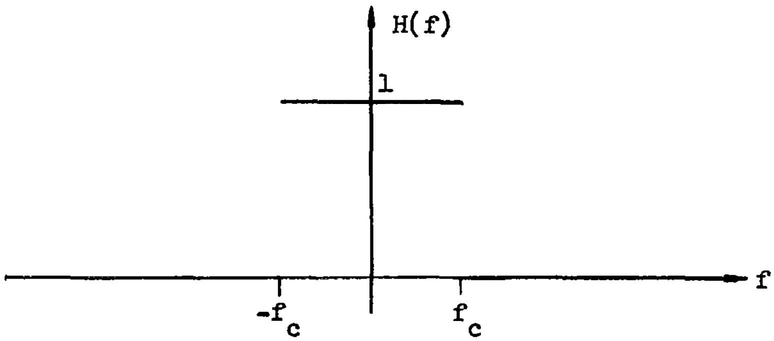


FIGURE 3.1

The corresponding weight function is

$$h(t) = \int_{-f_c}^{f_c} \exp(2\pi ift)df \tag{3.7}$$

$$= 2\int_0^{f_c} \cos 2\pi ftdf$$

$$= \frac{\sin 2\pi f_c t}{\pi t}$$

If g(t) is the input to this filter, then the output is

$$r(t) = \int_{-\infty}^{\infty} g(z)h(t-z)dz,$$

which has transform {see (3.4)}

$$R(f) = \begin{cases} G(f)H(f) & |f| \leq f_c \\ 0 & |f| > f_c \end{cases}$$

where $g(t) \longleftrightarrow G(f)$.

Ideal smoothing and differentiating filter.

By (2.17), if $g(t) \longleftrightarrow G(f)$, then for n = 1, 2,......, we have

$$g^{(n)}(t) \longleftrightarrow (2\pi if)^n G(f) \tag{3.8}$$

From (3.5) we see that to find the $n^{\underline{th}}$ derivative of an input g(t) the transfer function must be $(2\pi if)^n$. Then, in order to smooth using the ideal filter and find the $n^{\underline{th}}$ derivative, the transfer function is given by

$$H(f) = \begin{cases} (2\pi if)^n & |f| \leq f_c \\ 0 & |f| > f_c \end{cases} \tag{3.9}$$

and the weight function is

$$h_n(t) = \int_{-f_c}^{f_c} (2\pi if)^n \exp(2\pi ift)\, df \tag{3.10}$$

But differentiating (3.7) n times, we have

$$h^{(n)}(t) = \int_{-f_c}^{f_c} (2\pi if)^n \exp(2\pi ift)df \tag{3.11}$$

and so

$$h_n(t) = h^{(n)}(t) \tag{3.12}$$

Thus to find the weight function of the ideal smoothing and differentiating filter we simply differentiate the weight function of the smoothing filter the appropriate number of times. Then the output of the filter is given by

$$g^{(n)}(t) = \int_{-\infty}^{\infty} g(z)h^{(n)}(t-z)dz. \tag{3.13}$$

3.2 THE SAMPLING THEOREM

Ideal filters of the type discussed above are not physically realizable because of the jump discontinuities at $\pm f_c$. Furthermore, in digital filtering the input consists of a finite number of equally spaced values $g_m, M \leq m \leq N$, which we may assume are samples of some function $g(t)$ for $t = m\Delta t = \frac{m}{f_s}$. We may also assume that $g(t)$ defines a generalized function, for, recalling Definition 2.25 and Theorem 2.23, this does not place a serious restriction on $g(t)$. It is obvious that $g(t)$ is not uniquely determined by the values g_m, and hence the set of values g_m are associated with a subset G_{MN} of $\overline{S}$.

If we know that the samples g_m arise from a function $g(t)$ whose transform $G(f)$ is zero for $|f| > f_\beta$, then the subset G_{MN} of $\overline{S}$ is reduced to a subset $G^1_{MN} \subset G_{MN}$. In this case $g(t)$ is said to be band-limited.

Theorem 8. Shannon's sampling theorem (see [9]).

If $g(t)$ is band-limited, i.e., if $g(t) \longleftrightarrow G(f)$ where

$$G(f) = 0 \qquad |f| > f_\beta \tag{3.14}$$

then $g(t)$ can be uniquely determined from its values

$$g_n = g(\frac{n}{2f_\beta}) \tag{3.15}$$

at a sequence of equidistant points of distance $\frac{1}{2f_\beta}$ apart.

Furthermore

$$g(t) = \sum_{n=-\infty}^{\infty} g_n \frac{\sin \pi (2f_\beta t-n)}{\pi (2f_\beta t-n)} . \tag{3.16}$$

Proof: We first compute the g_n. We have, using (3.14),

$$g(t) = \int_{-f_\beta}^{f_\beta} G(f)\exp(2\pi ift)df$$

hence

$$g_n = g(\frac{n}{2f_\beta}) = \int_{-f_\beta}^{f_\beta} G(f)\exp(n\pi if/f_\beta)df. \tag{3.17}$$

Expanding $G(f)$ in a Fourier series on $(-f_\beta, f_\beta)$ we have

$$G(f) = \sum_{n=-\infty}^{\infty} G_n \exp(-n\pi if/f_\beta), \quad -f_\beta < f < f_\beta, \tag{3.18}$$

where

$$G_n = \frac{1}{2f_\beta} \int_{-f_\beta}^{f_\beta} G(f)\exp(n\pi if/f_\beta)df \tag{3.19}$$

comparing (3.17) and (3.19), we have

$$G_n = \frac{g_n}{2f_\beta}.$$

The function

$$\overline{G}(f) = \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\beta} \exp(-n\pi i f/f_\beta), \ -\infty < f < \infty,$$

is the periodic extension of G(f) and

$$\overline{G}(f) = G(f) \quad \text{for } -f_\beta < f < f_\beta.$$

Hence we may write

$$G(f) = H(f)\overline{G}(f)$$

where

$$H(f) = \begin{cases} 1 & |f| \leq f_\beta \\ 0 & |f| > f_\beta \ . \end{cases}$$

Now (see (3.6) and (3.8))

$$\frac{\sin 2\pi f_\beta t}{\pi t} \longleftrightarrow H(f) \qquad (3.20)$$

So we have

$$G(f) = H(f) \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\beta} \exp(-n\pi i f/f_\beta)$$

$$= \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\beta} H(f)\exp(-n\pi i f/f_\beta)$$

and

$$g(t) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\beta} H(f)\exp(-n\pi if/f_\beta) \right] \exp(2\pi ift)df$$

$$= \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\beta} \int_{-\infty}^{\infty} H(f)\exp(-n\pi if/f_\beta)\exp(2\pi ift)df.$$

Applying the First Shifting Theorem gives

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{g_n}{2f_\beta} \frac{\sin 2\pi f_\beta(t-n/2f_\beta)}{\pi(t-n/2f_\beta)}$$

$$= \sum_{n=-\infty}^{\infty} g_n \frac{\sin \pi(2f_\beta t-n)}{\pi(2f_\beta t-n)}$$

If f_s is any number such that $f_s \geq 2f_\beta$, then the theorem remains true if in the proof the periodic function $\overline{G}(f)$ is assumed to be of period f_s, and $H(f) = 1$ for $|f| \leq \frac{f_s}{2}$; $H(f) = 0, |f| > \frac{f_s}{2}$. Therefore

$$g(t) = \sum_{n=-\infty}^{\infty} g_n \frac{\sin \pi(f_s t-n)}{\pi(f_s t-n)} \tag{3.21}$$

where

$$g_n = g(\frac{n}{f_s}). \tag{3.22}$$

If the g_n are known, as assumed above, for $M \leq n \leq N$, then the function

$$g_{MN}(t) = \sum_{n=M}^{N} g_n \frac{\sin \pi(f_s t-n)}{\pi(f_s t-n)} \tag{3.23}$$

differs from each function $\bar{g}(t)$ of G^1_{MN} by

$$\epsilon_{\bar{g}}(t) = \sum_{n=-\infty}^{M-1} \bar{g}_n \frac{\sin \pi(f_s t-n)}{\pi(f_s t-n)} + \sum_{n=N+1}^{\infty} \bar{g}_n \frac{\sin \pi(f_s t-n)}{\pi(f_s t-n)} \tag{3.24}$$

where $\bar{g}_n = \bar{g}(\frac{n}{f_s})$. Hence, at least in the cases where the series in (3.21) converges uniformly to g(t), the maximum difference

$$\operatorname*{Max}_t \left|\epsilon_{\bar{g}}(t)\right| = \max_t \left|g(t)-g_{MN}(t)\right| \tag{3.25}$$

can be made as small as we please by taking a sufficient number of terms in $g_{MN}(t)$. Thus we can associate with the samples $\{g_n\}$ a unique function g(t) in the sense that (3.25) can be made arbitrarily small by taking a sufficient number of samples.

3.3 DEFINITION OF A DIGITAL FILTER

Suppose that the sampled function g(t) is band-limited. Then $G(f) = 0$ for $|f| > f_\beta$. If H(f) is a desired transfer function, then $H(f)G(f) = 0$ for $|f| > f_\beta$. Thus if $\bar{H}(f)$ is a periodic extension of H(f) with period $f_s \geq 2f_\beta$, we have the transform R(f) of the output r(t) given by

$$R(f) = H(f)G(f) = \bar{H}(f)G(f), \tag{3.26}$$

for all f.

If H(f) is such that $\overline{H}(f)$ can be written as a trigonometric series,

$$\overline{H}(f) = \sum_{n=-\infty}^{\infty} a_n \exp(2n\pi if/f_s) \tag{3.27}$$

with $a_n = O(|n|^N)$ for some N as $|n| \longrightarrow \infty$, then, by Theorem 2.70, $\overline{H}(f) \in \overline{S}$ and is the transform of

$$\overline{h}(t) = \sum_{n=-\infty}^{\infty} a_n \, \delta(t+n/f_s) \tag{3.28}$$

Now g(t) is time sampled. In order to obtain a time sampled version of the output r(t) we might define a convolution $g(t)*\overline{h}(t)$ and extend (2.54) and (2.55) to functions $\Phi(t) = \overline{h}(t)$. Assuming that we did this, we would have

$$r(t) = \sum_{n=-\infty}^{\infty} a_n g(t+n/f_s)$$

which would yield the sampled version of r(t) for $t = m/f_s$ as

$$r(m/f_s) = \sum_{n=-\infty}^{\infty} a_n g((m+n)/f_s)$$

which is impossible to use digitally since it requires infinitely many samples.

Alternately, let

$$H_{MN}(f) = \sum_{n=M}^{N} a_n \exp(2n\pi if/f_s) \tag{3.29}$$

be a trigonometric polynomial which approximates $H(f)$ in some sense. Then (3.29) is the transform of

$$h_{MN}(t) = \sum_{n=M}^{N} a_n \, \delta(t+n/f_s), \tag{3.30}$$

and by (2.54) the convolution $g(t)*h_{MN}(t)$ is defined for all $g(t) \in \overline{S}$. Also (2.55) holds. Thus

$$R(f) = G(f)H(f) \doteq G(f)H_{MN}(f) = \overline{R}(f) \tag{3.31}$$

and $\overline{r}(t) \longleftrightarrow \overline{R}(f)$ is given by

$$\begin{aligned}\overline{r}(t) &= g(t)*h_{MN}(t) \\ &= \sum_{n=M}^{N} a_n \int_{-\infty}^{\infty} g(z)\delta(t-z+n/f_s)dz \\ &= \sum_{n=M}^{N} a_n g(t+n/f_s).\end{aligned}$$

For $t = m/f_s$, $\overline{r}_m = \overline{r}(m/f_s)$, $g_m = (m/f_s)$, we have

$$\overline{r}_m = \sum_{n=M}^{N} a_n g_{m+n} \tag{3.32}$$

This is the <u>fundamental formula</u> of digital filtering.

Note that any pair (3.29) and (3.30) determine a linear operator L on $\overline{S}$ which satisfies condition (A) and which, on the subspace G_s of all band-limited functions $g(t)$ with $2f_\beta \leq f_s$, acts as a low-pass filter. Now any finite set of constants a_n determines a generalized function (3.30), which determines (3.29) and hence a linear operator L.

<u>Definition 3.</u> Let a_n, $M \leq n \leq N$, be any set of constants. Then the linear system $\overline{L}$ determined by the a_n is called a <u>digital</u> or <u>numerical filter</u>. The constants a_n are called the <u>weights</u> of the digital filter.

Application of $\overline{L}$ must be limited to the subspace G_s. Otherwise "frequency folding" occurs, i.e., frequencies in the intervals $(\frac{(2n-1)f_s}{2}, \frac{(2n+1)f_s}{2})$, $n = \pm 1, \pm 2, \ldots$, are folded back into the $(-\frac{f_s}{2}, \frac{f_s}{2})$. For example, suppose the input contains a frequency component $A\cos 2\pi(f_o+kf_s)t$ where $f_o < \frac{f_s}{2}$ and k is a positive integer. Then if we sample at $t = n/f_s$,

$$A\cos\ 2\pi(f_o+kf_s)n/f_s = A\cos\left[2\pi f_o n/f_s + 2nk\pi\right]$$

$$= A\cos\ (2\pi f_o n/f_s).$$

The sample values would be the same as those obtained from a component $A\cos 2\pi f_o t$ for $t = n/f_s$. Hence the filter treats the frequency $f_o + kf_s > f_s/2$ in the same manner as f_o.

<u>3.4 EVEN AND ODD TRANSFER FUNCTIONS</u>

In most cases of interest here, the transfer function H(f) is either even or odd. Hence the trigonometric polynomial $H_{MN}(f)$ which approximates H(f) can be written in terms of $\cos 2n\pi f/f_s$ and $\sin 2n\pi f/f_s$ respectively. If we take $M = -N$ some advantages are gained. Let

$$H_{MN}(f) = H_N(f) = \sum_{n=-N}^{N} a_n \exp(2\pi n i f/f_s). \tag{3.33}$$

For even functions,

$$H_N(f) = a_o + 2 \sum_{n=1}^{N} a_n \cos 2n\pi f/f_s . \tag{3.34}$$

For odd functions,

$$H_N(f) = 2i \sum_{n=1}^{N} a_n \sin 2n\pi f/f_s . \tag{3.35}$$

Two questions now arise:

(1) given H(f), how are the weights a_n to be chosen, and

(2) what is the error introduced by the approximation $\overline{R}(f) \doteq R(f)$?

3.5 METHODS OF FILTER APPROXIMATION

If H(f) is an ordinary function, there are several methods of approximating H(f) and obtaining the weights a_n. One of these methods--the Min-Max technique--is given by Martin [11]. Essentially, it assumes continuity of H(f) in which case, if $Q_n(f)$ is a set of N continuous and linearly independent functions on $[-\frac{f_s}{2}, \frac{f_s}{2}]$, then there exists a polynomial

$$P_N(f) = a_1Q_1(f) + . . . + a_NQ_N(f)$$

which deviates the least from H(f) on $(-\frac{f_s}{2}, \frac{f_s}{2})$, i.e.,

$$\max_{f \in (-\frac{f_s}{2}, \frac{f_s}{2})} |H(f)-P_N(f)| \leq \max_{f \in (-\frac{f_s}{2}, \frac{f_s}{2})} |H(f) - \sum_{n=1}^{N} x_n Q_n(f)|$$

for any numbers $x_1, x_2, \ldots, x_N$. The $Q_n(f)$ are obtained after putting a constraint (or constraints) on a trigonometric polynomial (3.33). $P_N(f)$ is then fitted at a finite number of points to H(f) in the above sense. A good approximation of the a_n is obtained by an iterative process, but the technique is long and complex, and not very versatile. That is, any change in H(f) necessitates a complete repetition of the process for finding the a_n.

The method we shall use assumes that H(f) can be approximated by a Fourier series,

$$\overline{H}(f) = \sum_{n=-\infty}^{\infty} h_n \exp(2n\pi i f/f_s), \tag{3.36}$$

where

$$h_n = 1/f_s \int_{-f_2/2}^{f_s/2} H(f)\exp(-2n\pi i f/f_s)df, \tag{3.37}$$

and $H_N(f)$ is taken to be the truncated series for H(f),

$$H_N(f) = \sum_{n=-N}^{N} h_n \exp(2n\pi i f/f_s). \tag{3.38}$$

This gives a function which is the best fit to H(f) in the least mean square sense. (See [1] for a discussion of Fourier series.)

Noting that, since H(f) = 0 for $|f| > \frac{f_s}{2}$, the inverse transform of H(f) is

$$h(t) = \int_{-f_s/2}^{f_s/2} H(f)\exp(-2\pi ift)df, \qquad (3.39)$$

and comparing (3.39) and (3.37), we see that

$$h_n = 1/f_s \; h(-n/f_s). \qquad (3.40)$$

This is the basic formula for computing the $h_n = a_n$ to use in (3.32). Therefore (3.32) can be written as

$$\bar{r}_m = \sum_{n=-N}^{N} h_n g_{m+n}. \qquad (3.41)$$

The Min-Max technique uses a finite number of values of the transfer function H(f), while the second approach assumes that H(f) is given for all f, and hence the h_n may be computed from (3.37), or from (3.40) if h(t) is computed first. In some applications, H(f) is known at only a finite number of points and this second method is not applicable. In particular, the case sometimes arises that

$$H(f) = A(f)\exp(i\theta(f))$$

and only values of A(f) and $\theta(f)$ are known at equally spaced points on the interval $(0, f_s/2)$. A method for computing the h_n for such a filter is discussed in Appendix B.

3.6 ERROR ANALYSIS

With an approximation $H_N(f)$ of H(f), (3.31) becomes

$$R(f) = G(f)H(f) \doteq G(f)H_N(f) = \overline{R}(f),$$

and so

$$R(f)-\overline{R}(f) = G(f)\,[H(f)-H_N(f)].$$

This gives the pointwise error between the spectrum of the desired output and the spectrum of the actual output.

For a complex frequency component $g_o(t) = A\exp(2\pi i f_o t)$ in the input we have

$$g_o(t) = A\exp(2\pi i f_o t) \longleftrightarrow A\cdot\delta(f-f_o) = G(f_o)$$

and

$$R(f_o) = A\cdot\delta(f-f_o)H(f_o),$$

also

$$\overline{R}(f_o) = A\cdot\delta(f-f_o)H_N(f_o).$$

Denoting the difference in the outputs by $\epsilon(f_o,t)$ we have

$$|\epsilon(f_o,t)| = \left| \int_{-\infty}^{\infty} A\cdot\delta(f-f_o) \quad H(f_o)-H_N(f_o) \quad \exp(2\pi i f t)df \right|$$

$$= \left| A\exp(2\pi i f_o t) \quad H(f_o) - H_N(f_o) \quad \right|$$

$$= \left| A\exp(2\pi i f_o t) \right| \cdot \left| \epsilon(f_o,N) \right|,$$

where $\epsilon(f_o,N) = H(f_o) - H_N(f_o)$.

In the time sampled version:

$$|\epsilon(f_o, n/f_s)| = |A \exp(2\pi i f_o n/f_s)| \cdot |\epsilon(f_o, N)|. \tag{3.42}$$

Thus the magnitude of the error in a component of the actual sampled output is given in terms of the magnitude of the corresponding component of the input function, and of the magnitude of the error in the approximation of H(f).

Approximations of ϵ,

$$\epsilon = \max_f |\epsilon(f,N)| = \max_f |H(f) - H_N(f)| \tag{3.43}$$

derived mathematically are usually found to be so large as to render them useless in applications. In applications of the smoothing filter discussed later, acceptable values of ϵ are in the range $.005 \leq \epsilon \leq .01$, or referred to unity, $\frac{1}{2}$% and 1%. When speaking of percent error we will always mean ϵ referred to unity. For a given H(f), an N is found empirically such that $H_N(f)$ approximates H(f) within the desired limits.

However, satisfying the requirement that $.005 \leq \epsilon \leq .01$ does not imply the output error is within these bounds (see Chapter VII).

3.7 THE GIBBS' PHENOMENON

When approximating an ideal or designed transfer function H(f) having one or more jump discontinuities with a truncated Fourier series, there exist oscillations in the approximating transfer function $H_N(f)$ near the discontinuities of H(f) due to the Gibbs' phenomenon (see [12]). No matter how large N is taken, ϵ cannot be brought within the acceptable range $.005 \leq \epsilon \leq .01$.

To avoid this difficulty H(f) is first approximated by a function which is continuous. In most cases, this imposes a restriction on the input g(t). The particular cases of interest here shall be dealt with in the next chapter.

CHAPTER IV

FILTER DESIGN

4.0 ASSUMPTIONS ABOUT THE INPUT

In order to apply a digital filter to a set of samples $\{g_n\}$, we have made two assumptions about the data:

I. It arises from a function $g(t)$ which defines a generalized function, and

II. $g(t)$ is band-limited.

In many cases of interest, the Fourier spectrum $G(f)$ of a signal $g(t)$ consists of a desired signal spectrum in an interval $[-f_c, f_c]$, an unwanted signal spectrum (noise spectrum) in intervals $[-f_\beta, -f_c)$ and $(f_c, f_\beta]$, and $G(f) = 0$ for $|f| > f_\beta$. When applying a low-pass filter, elimination of the unwanted spectrum is desired. Hence the ideal filter transfer function, $H_I(f)$, is such that $H_I(f) = 0$, $|f| > f_c$. Usually $H_I(\pm f_c) \neq 0$ and $H_I(f)$ has jump discontinuities at $f = \pm f_c$. If the truncated Fourier series of $H_I(f)$ is used to approximate $H_I(f)$, then, due to the Gibbs' phenomenon, large oscillations persist in a neighborhood of $\pm f_c$. Furthermore, the amplitude of these oscillations remains constant with increasing N. The truncated Fourier series is continuous everywhere because it is a finite sum of everywhere continuous functions. Since $H_I(f_c) \neq 0$, we expect that the truncated series, $H_N(f)$, is such that $H_N(f_c) \neq 0$. Then, by continuity, $H_N(f)$ is non-zero on some interval $(f_c, f_c + \Delta f)$ where $\Delta f > 0$ and depends on N. Any unwanted frequencies which appear in this interval are passed--though somewhat attenuated--by the approximating filter. Hence, in addition to the large oscillations which appear near $\pm f_c$, unwanted frequencies arbitrarily close to $\pm f_c$ cannot be eliminated by increasing N. This undesirable property must be tolerated because it is a property of any truncated Fourier series such that $H_N(f_c) \neq 0$. However, the large oscillations are caused by non-uniform convergence of the Fourier series of $H_I(f)$. This can be remedied by redefining $H_I(f)$ so that it is a continuous function. We choose to do this on the intervals $(-f_c-\Delta f, -f_c)$ and $(f_c, f_c+\Delta f)$ for some $\Delta f > 0$. Any unwanted frequencies in these intervals will be passed to some extent by the filter, but, as pointed out above, this cannot be avoided anyway. However, in many applications unwanted frequencies do not appear near $\pm f_c$. Therefore, we make the following third assumption about the data:

III. The desired signal spectrum and the unwanted spectrum of g(t) are disjoint.

Then there exists a $\Delta f > 0$ such that the signal spectrum $G(f) = 0$ on $(-f_c-\Delta f, -f_c)$ and $(f_c, f_c+\Delta f)$. Letting $f_T = f_c+\Delta f$, we may modify $H_I(f)$ on $[-f_T, -f_c)$ and $(f_c, f_T]$ to obtain a function H(f) continuous for all f and thereby eliminate the Gibbs' phenomenon. H(f), as defined on the intervals $[-f_T, -f_c)$ and $(f_c, f_T]$, is called the roll-off of the filter, and the frequency f_T is called the termination frequency.

4.1 FILTER DESIGN BY CONVOLUTION

The usual approach to the design of a filter is to select the ideal transfer function $H_I(f)$ on $[-f_c, f_c]$ and then to specify the roll-off. This gives the filter transfer function H(f) from which the weight function h(t) is found. The weights of the filter to be used in (3.41) are then computed from (3.40). In addition to not being very versatile, this approach usually involves some rather long and tedious integration in determining h(t).

We propose a different approach to the design which simplifies the integration and gives considerable freedom in varying the roll-off shape of the filter. We shall use the convolution theorem of Chapter I:

$$k(t)g(t) \longleftrightarrow \int_{-\infty}^{\infty} k(z)G(f-z)dz \tag{4.1}$$

where $g(t) \longleftrightarrow G(f)$ and $k(t) \longleftrightarrow K(f)$.

Before continuing, we note that filters for simultaneously performing smoothing and differentiation can be found from the weight-transfer functions, h(t) and H(f), of the smoothing filter by applying (2.17) in a manner analogous to that in the ideal case [see Section (3.1)]. That is, to smooth and find the n^{th} derivative, the transfer function is

$$Y^n(f) = (2\pi i f)^n H(f). \tag{4.2}$$

With

$$y^n(t) \longleftrightarrow Y^n(f),$$

we have

$$y^n(t) = h^{(n)}(t) \tag{4.3a}$$

where

$$h(t) \longleftrightarrow H(f).$$

In (4.3a), let $t = -x/f_s$. Then $t^n = (-x/f_s)^n$ and $dt^n = (-1/f_s)^n dx^n$.

Hence

$$y^n(-x/f_s) = \frac{d^n h(-x/f_s)}{(-1/f_s)^n dx^n}$$

$$= (-1)^n f_s{}^n \frac{d^n h(-x/f_s)}{dx^n}$$

Using (3.40) to compute the weights of the filter, we have

$$y_k^n = 1/f_s \; y^n(-k/f_s)$$

$$= 1/f_s y^n(-x/f_s)|_{x=k}$$

$$= (-1)^n f_s{}^n \frac{\{d^n \; 1/f_s \; h(-x/f_s)\}}{dx^n}\Big|_{x=k}$$

We now see that we may write

$$y_k^n = (-1)^n f_s^n \frac{d^n h_k}{dk^n} \tag{4.3b}$$

where $h_k = 1/f_s h(-k/f_s)$ and, for purposes of differentiating, k is treated as a variable in the right side of (4.3b)

Returning to the problem of designing the filter, we conclude from the above that we may restrict ourselves to the design of smoothing filters. Hence suppose that

$$H(f) = \int_{-\infty}^{\infty} K(z)G(f-z)dz. \tag{4.4}$$

Ideally, for smoothing we want H(f) to be continuous, and

$$H(f) = \begin{cases} 1, \; 0 \leq f \leq f_c, \\ \text{monotonic decreasing}, \; f_c < f < f_T, \\ 0, \; f \geq f_c, \\ H(-f), \; f < 0. \end{cases} \qquad (4.5)$$

We attempt to find functions G(f) and K(f) such that H(f) given by (4.4) has these properties. Then the weight function h(t) is given by

$$h(t) = k(t)g(t). \qquad (4.6)$$

In the following, we choose G(t) to be the function

$$G(f) = \begin{cases} 1, & |f| \leq (f_c+f_T)/2, \\ 0 & |f| > (f_c+f_T)/2 \end{cases} \qquad (4.7a)$$

Then comparing with (3.6) and (3.7), we see that the corresponding weight function is

$$g(t) = \frac{\sin \pi(f_T+f_c)t}{\pi t} . \qquad (4.7b)$$

Noting that $G(f-z) = 0$ for $|f-z| > (f_T+f_c)/2$, (4.4) becomes

$$H(f) = \int_{f-(f_T+f_c)/2}^{f+(f_T+f_c)/2} K(z)dz. \qquad (4.8)$$

To find $H(f_o)$, $K(z)$ is integrated over an interval of length (f_T+f_c) with f_o as its mid-point. Note that any function $K(z)$ which is zero for $|z| > (f_T-f_c)/2 = \Delta f/2$ and is an even function of z with area 1 on $[-\Delta f/2, \Delta f/2]$ yields a satisfactory $H(f)$.

Filter 1. The Ormsby smoothing filter (p=1).

In (4.8) let

$$K(f) = K_1(f) = \begin{cases} 1/\Delta f & |f| \leq \Delta f/2, \\ 0, & |f| > \Delta f/2 \end{cases} \tag{4.9}$$

See Figure 4.1.

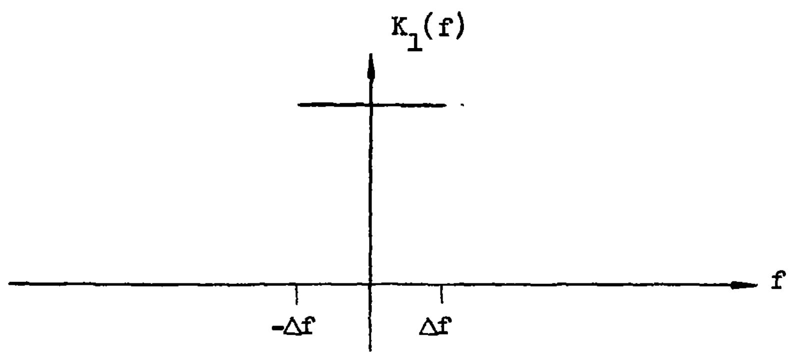


FIGURE 4.1

Then

$$k_1(t) = \int_{-\Delta f/2}^{\Delta f/2} (1/\Delta f) \exp(2\pi i f t) df$$

$$= 2/\Delta f \int_0^{\Delta f/2} \cos (2\pi ft) df$$

$$= 1/\pi\Delta ft \left[\sin 2\pi ft \right]_0^{\Delta f/2}$$

$$= \frac{\sin \pi\Delta ft}{\pi\Delta ft},$$

and with g(t) from (4.7b), we have from (4.6)

$$h_1(t) = k_1(t)g(t)$$

$$= \frac{\sin \pi\Delta ft \sin \pi(f_T+f_c)t}{\pi^2 \Delta ft^2} \qquad (4.10)$$

Changing to the angular frequency $w = 2\pi f$, $\Delta w = 2\pi\Delta f$, $w_T = 2\pi f_T$, $w_c = 2\pi f_c$, we have

$$h_1(t) = \frac{2 \sin \frac{\Delta wt}{2} \sin \frac{(w_T+w_c)t}{2}}{\pi\Delta wt^2}$$

and applying a well-known trigonometric identity

$$h_1(t) = \frac{\cos w_c t - \cos w_T t}{\pi\Delta wt^2} . \qquad (4.11)$$

This is the weight function given by Ormsby [14]for p=1. The corresponding transfer function, as a function of f, is

$$H_1(f) = \begin{cases} 1, & |f| \le f_c, \\ 0, & |f| > f_T, \\ (f+f_T)/\Delta f, & -f_T \le f < -f_c, \\ (f_T-f)/\Delta f, & f_c < f \le f_T. \end{cases}$$

$H_1(f)$ has a straight line roll-off (see Figure 4.2).

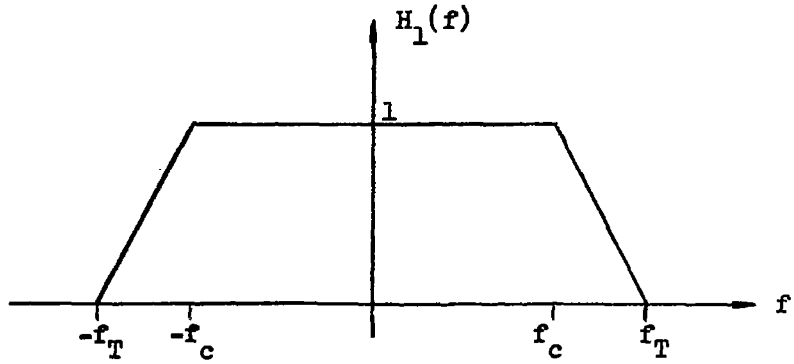


FIGURE 4.2

Note that $\frac{dH_1(f)}{df}$ is discontinuous at $\pm f_c$ and $\pm f_T$.

Filter 2. The Martin-Graham smoothing filter.

In (4.8) let

$$K(f) = K_2(f) = \begin{cases} (\pi/2\Delta f) \cos(\pi f/\Delta f), & |f| \le \Delta f/2, \\ 0, & |f| > \Delta f/2. \end{cases} \tag{4.12}$$

See Figure 4.3

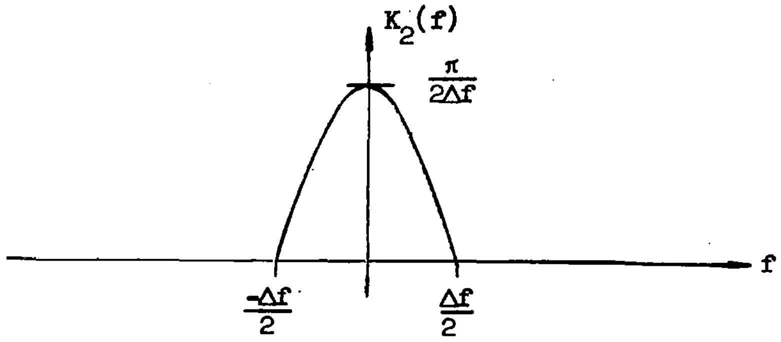


FIGURE 4.3

Then

$$k_2(t) = \int_{-\Delta f/2}^{\Delta f/2} (\pi/2\Delta f) \cos(\pi f/\Delta f) \exp(2\pi i f t) df$$

$$= (\pi/\Delta f) \int_0^{\Delta f/2} \cos(\pi f/\Delta f) \cos 2\pi f t \, df$$

$$= (\pi/\Delta f) \left[\frac{\sin((\pi/\Delta f) - 2\pi t)f}{2((\pi/\Delta f) - 2\pi t)} + \frac{\sin((\pi/\Delta f)+2\pi t)f}{2((\pi/\Delta f)+2\pi t)} \right]_0^{(\Delta f/2)}$$

$$= (1/2\Delta f) \left[\frac{\sin((\pi/2)-\pi\Delta f t)}{((1/\Delta f) - 2t)} + \frac{\sin((\pi/2) + \pi\Delta f t)}{((1/\Delta f) + 2t)} \right]$$

$$= (1/2\Delta f) \left[\frac{\cos \pi\Delta f t}{((1/\Delta f) - 2t)} + \frac{\cos \pi\Delta f t}{((1/\Delta f) + 2t)} \right]$$

$$= \frac{\cos \pi\Delta f t}{(1-4\Delta f^2 t^2)} .$$

Then with g(t) from (4.7b), we have from (4.6)

$$h_2(t) = k_2(t)g(t)$$

$$= \frac{\cos \pi\Delta f t \ \sin \pi(f_T+f_c)t}{\pi t(1-4\Delta f^2 t^2)} , \qquad (4.13)$$

where $\Delta f^2 = (\Delta f)^2$. We shall also use the notation $\Delta w^2 = (\Delta w)^2$.

Letting $w = 2\pi f$ in (4.13) gives

$$h_2(t) = \frac{\cos(\Delta wt/2) \ \sin((w_T+w_c)t/2)}{\pi t(1-\Delta w^2 t^2/\pi^2)}$$

and using a well-known trigonometric identity gives, after simplifying,

$$h_2(t) = \frac{\pi(\sin w_c t + \sin w_T t)}{2t(\pi^2 - \Delta w^2 t^2)} \qquad (4.14)$$

This is the form of the weight function given by Graham [13].

The form given by Martin [10], [11] is obtained from (4.13) by going to the <u>frequency ratio</u> $\tau = f/f_s$, $\tau_c = f_c/f_s$, $\tau_T = f_T/f_s$, $\tau_d = \Delta f/f_s$ (= 2h in Martin's notation), and computing

$$h_n = (1/f_s)h_2(-n/f_s)$$

$$= (1/f_s)\left[\frac{\cos \pi(\tau_d f_s)(-n/f_s) \quad \sin \pi f_s(\tau_c+\tau_T)(-n/f_s)}{\pi(-n/f_s)(1-4\tau_d^2 f_s^2 n^2/f_s^2)}\right]$$

$$= \frac{\cos n\pi\tau_d \ \sin n\pi(2\tau_c + \tau_d)}{n\pi(1-4\tau_d^2 n^2)} . \qquad (4.15)$$

The relation $\tau_T = \tau_c + \tau_d$ was used in obtaining the last line. This is a convenient expression for computing the weights h_n of the filter. The value of h_o is computed by using L'Hospital's rule,

and

$$h_o = 2\tau_c + \tau_d = (f_T + f_c)/f_s \tag{4.16}$$

The same procedure must be used for finding h_m if $m = 1/2\tau_d$ for then (4.15) assumes the indeterminate form 0/0. In this case, we have

$$h_m = (\tau_d/2) \cos (\pi\tau_c/\tau_d) = (\Delta f/2f_s) \cos (\pi f_c/\Delta f) \tag{4.17}$$

The transfer function of this filter, in terms of f, is

$$H_2(f) = \begin{cases} 1 & |f| \le f_c, \\ 0, & |f| > f_T, \\ (1+\cos\pi(f-f_c)/\Delta f), & f_c < f < f_T, \\ (1+\cos\pi(f+f_c)/\Delta f), & -f_T < f < -f_c. \end{cases} \tag{4.18}$$

See Figure 4.4.

Alternate expressions for the roll-off are

$$(1+\cos\pi(f-f_c)/\Delta f) = \cos^2\pi(f-f_c)/2\Delta f$$

and

$$(1+\cos\pi(f+f_c)/\Delta f) = \cos^2\pi(f+f_c)/2\Delta f.$$

Note that $H_2(f)$ has one continuous derivative, and $\dfrac{d^2H_2(f)}{df^2}$ is discontinuous at $\pm f_T$ and $\pm f_c$.

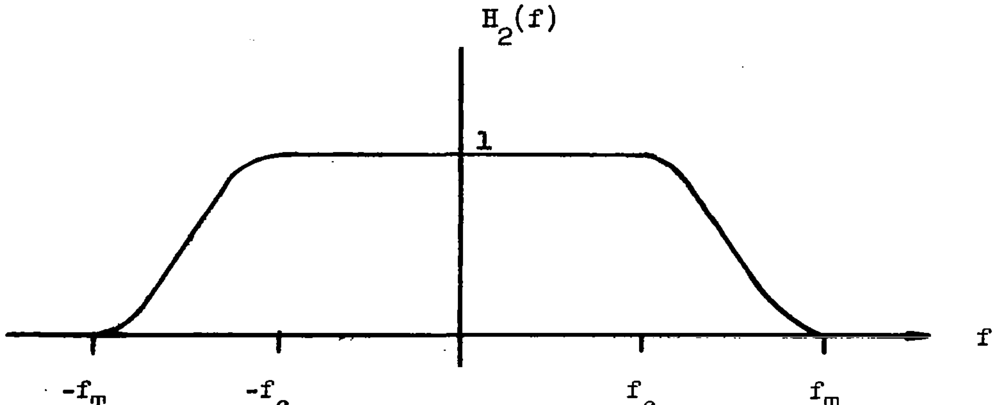


FIGURE 4.4

4.2 COMPARISON OF THE PERFORMANCE OF THE ORMSBY AND MARTIN-GRAHAM SMOOTHING FILTERS.

A comparison of the above filters can be drawn by expressing $H_{1,N}(f)$ and $H_{2,N}(f)$, the truncated Fourier series for $H_1(f)$ and $H_2(f)$, respectively, in integral form. We expand $K_1(f)$ and $K_2(f)$ in a Fourier series, then truncating these series gives:

$$H_{1,N}(f) = \int_{f-(f_T+f_c)/2}^{f+(f_T+f_c)/2} K_{1,N}(z)dz \tag{4.19}$$

$$H_{2,N}(f) = \int_{f-(f_T+f_c)/2}^{f+(f_T+f_c)/2} K_{2,N}(z)dz \tag{4.20}$$

Since $K_{1,N}(z)$ is the truncated series of a function with jump discontinuities at $\pm\ \Delta f/2$ [see (4.9)], the Gibbs' phenomenon is present. Hence overshoot is present near $\pm\ \Delta f/2$, the amplitude of which can not be reduced by increasing N. We can expect some relatively large oscillations to be present, at least for small values of N, in $H_{1,N}(f)$. $K_2(z)$ is continuous, and the amplitude of the oscillations of $K_{2,N}(z)$ decreases monotonically with increasing N. Hence we expect the Martin-Graham filter to perform better than the Ormsby (p=1) filter. The results of comparative programs where the truncated series (4.19) and (4.20) were computed at equidistant points indicate that this conclusion is true. For $\epsilon = .01$, over 50% more weights were required by the Ormsby filter.

4.3 SOME NEW SMOOTHING FILTERS

We shall give, without performing the details of integration, several new designs which are of some interest.

Filter 3. Let

$$K_3(f) = \begin{cases} (2/\Delta f)\ \cos^2(\pi f/\Delta f) & |f| \leq \Delta f/2 \\ 0 & |f| > 0 . \end{cases} \tag{4.21}$$

Then

$$k_3(t) = \frac{1}{1-\Delta f^2 t^2} \cdot \frac{\sin \pi \Delta f t}{\pi \Delta f t}$$

$$h_3(t) = k_3(t) g(t)$$

$$= \left(\frac{1}{1-\Delta f^2 t^2}\right)\left(\frac{\sin \pi\Delta f t \, \sin \pi(f_T + f_c)t}{\pi^2 \Delta f t^2}\right)$$

$$= \frac{1}{1-\Delta f^2 t^2} \cdot h_1(t) \qquad (4.22)$$

where $h_1(t)$ is the Ormsby weight function (4.10). The roll-off of $H_3(f)$ is given by

$$1/2\pi \sin 2\pi(f-f_c)/\Delta f + (f_T - f)/\Delta f \qquad f_c < f \leq f_T,$$

and $H_3(f)$ has two continuous derivatives. (see Figure 4.5)

Filter 4. Let

$$K_4(f) = \begin{cases} (3\pi/4\Delta f) \cos^3 (\pi f/\Delta f), & |f| \leq \Delta f/2, \\ 0 & |f| > \Delta f/2. \end{cases} \qquad (4.23)$$

Then

$$k_4(t) = \frac{9}{9-4\Delta f^2 t^2} \cdot \frac{\cos \pi\Delta f t}{(1-4\Delta f^2 t^2)},$$

and

$$h_4(t) = k_4(t) g(t)$$

$$= \frac{9}{9-4\Delta f^2 t^2} \left[\frac{\cos \pi\Delta f t}{1-4\Delta f^2 t^2} \cdot \frac{\sin \pi(f_T + f_c)t}{\pi t}\right]$$

$$= \frac{9}{9-4\Delta f^2 t^2} \quad h_2(t), \qquad (4.24)$$

where $h_2(t)$ is the Martin-Graham weight function (4.13). The transfer function $H_4(f)$ has three continuous derivatives and the roll-off is given by:

$$H_4(f) = \frac{9}{16} \cos \left(\frac{\pi(f-f_c)}{\Delta f}\right) - \frac{1}{16} \cos \left(\frac{3\pi(f-f_c)}{\Delta f}\right) + \frac{1}{2}$$

for $f_c < f \leq f_T$. This is shown in Figure 4.5.

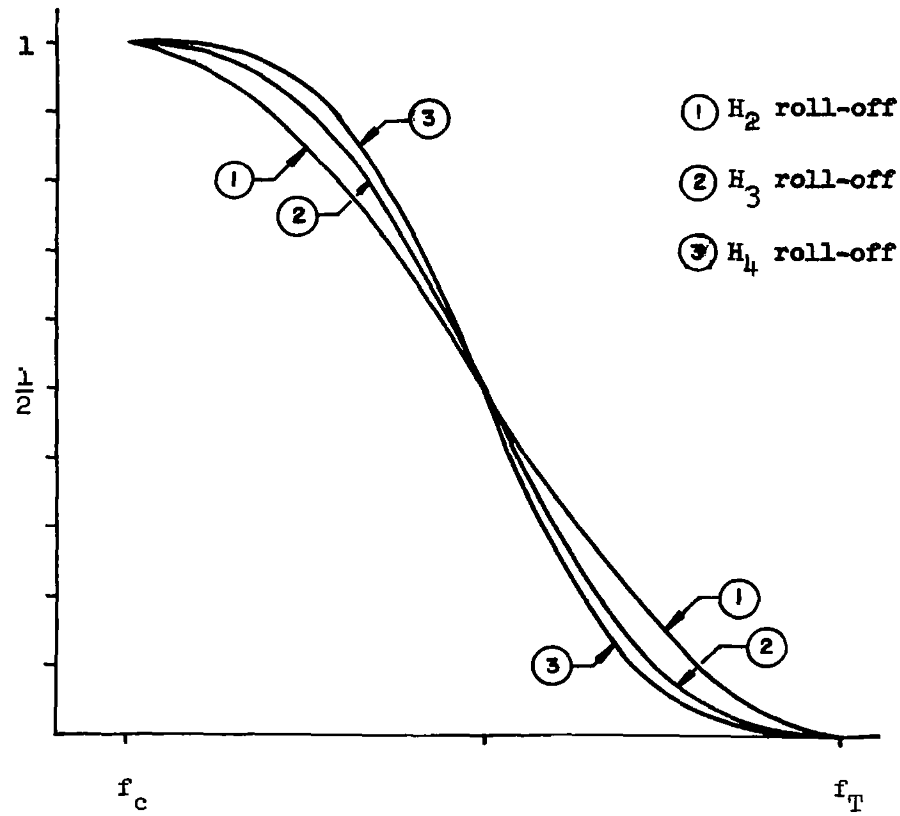


FIGURE 4.5

Filter 5. Let

$$K_5(f) = \begin{cases} (3/2\Delta f)\ (1-(4f^2/\Delta f^2)), & |f| \leq \Delta f/2, \\ 0\ , & |f| > \Delta f/2. \end{cases} \qquad (4.25)$$

This gives a weight function, where $\Delta f^3 = (\Delta f)^3$,

$$h_5(t) = \frac{3}{2\pi^4 \Delta f^3 t^4} [\sin \pi(f_T + f_c)t] \cdot [2 \sin \pi\Delta f t - 2\pi\Delta f t \cos \pi\Delta f t\ ]\ . \qquad (4.26)$$

The roll-off of $H_5(f)$, $f_c < f \leq f_T$, is a third degree polynomial and is essentially the same as that of $H_2(f)$.

Using the quantity ϵ defined by (3.42) as a measure of the performance of a filter to compare the above filters, one is led to the following conclusions:

1) The Martin-Graham filter gives $\epsilon = .01$ with smaller N than any of the others. In fact, out of numerous designs none has been found which gives $\epsilon = .01$ for smaller N than this filter. The performance of filter 5 is essentially the same, the ϵ values differing slightly in the third decimal place.
2) Filters 3 and 4 give values of $\epsilon \leq .005$ for smaller N than the Martin-Graham filter and filter 5.
3) In no case did the Ormsby filter perform as well as the other filters.

In comparison with the Martin-Graham filter, the only advantage filter 5 has is that no special evaluation for h_n, $n \neq 0$, is required; h_o is the same for all the above filters. In addition to the improved performance for $\epsilon \leq .005$, useable error bounds can be found for filters 3 and 4 without resorting to empirical methods.

4.4 SOME SMOOTHING ERROR BOUNDS

Except for filter 5, each of the above weight functions are of the form

$$h(t) = \frac{k(t)}{P(t)}$$

where k(t) is an expression containing sums and products of trigonometric functions of t and P(t) is a polynomial in t. The Fourier coefficients of H(f) computed from h(t) retain this character,

$$h_n = (1/f_s)h(-n/f_s) = (1/f_s)\,\frac{k(-n/f_s)}{P(-n/f_s)}.$$

Now the error as a function of f and N is

$$\epsilon(f,N) = H(f) - H_N(f)$$

$$= 2 \sum_{n=N+1}^{\infty} h_n \cos 2n\pi(f/f_s)$$

$$= (2/f_s) \sum_{n=N+1}^{\infty} \frac{k(-n/f_s)}{P(-n/f_s)} \cos 2n\pi(f/f_s) \; .$$

Letting $A = \max_{n,f} \left| k(-n/f_s) \cos 2n\pi\,(f/f_s) \right|$, we have

$$\epsilon = \max_f |\epsilon(f,N)| \leq (2A/f_s) \sum_{n=N+1}^{\infty} \left|\frac{1}{P(-n/f_s)}\right|. \qquad (4.27)$$

If $|P(t)| > 0$ for $t > (N/f_s)$, the sum in (4.27) can be approximated by

$$\left| \int_N^{\infty} \frac{dx}{P(-x/f_s)} \right| .$$

Martin-Graham bound

The above method gives (see Figure 4.6)

$$\epsilon \leq 1/\pi \log \frac{4N^2\Delta f^2}{4N^2\Delta f^2 - f_s^{\ 2}} \qquad (4.28)$$

For $\epsilon = .01$, the predicted value of N is

$$N \geq 2.85\ f_s/\Delta f \qquad (4.29)$$

and for $\epsilon = .005$

$$N \geq 4f_s/\Delta f = 4/\tau_d \qquad (4.30)$$

These values of N are much too large. It has been determined empirically that $N \geq 1.25\ f_s/\Delta f = 1.25/\tau_d$ gives $.005 < \epsilon < .01$.

Filter 3.

For this filter,

$$\epsilon \leq \frac{1}{\pi^2} \left\{ \log \left[\frac{N\Delta f + f_s}{N\Delta f - f_s} \right] - 2f_s/N\Delta f \right\}. \qquad (4.31)$$

For $\epsilon = .01$, the predicted value of N is

$$N \geq 2f_s/\Delta f = 2/\tau_d, \tag{4.32}$$

and for $N \geq 3f_s/\Delta f = 3/\tau_d$, $\epsilon < .003$ (see Figure 4.6)

Filter 4.

For this filter

$$\epsilon \leq \frac{1}{8\pi} \{9 \log [4\pi^2N^2\Delta f^2 - \pi^2 f_s^2] - 16 \log 2\pi N\Delta f - \log[4\pi^2N^2\Delta f^2 - 9\pi^2 f_s^2]\}. \tag{4.33}$$

For $\epsilon = .01$, the predicted value of N is the same as in (4.32).
For $N \geq 3f_s/\Delta f$, (4.33) gives $\epsilon < .0014$ (see Figure 4.6).

4.5 SMOOTHING FILTER CONSTRAINTS

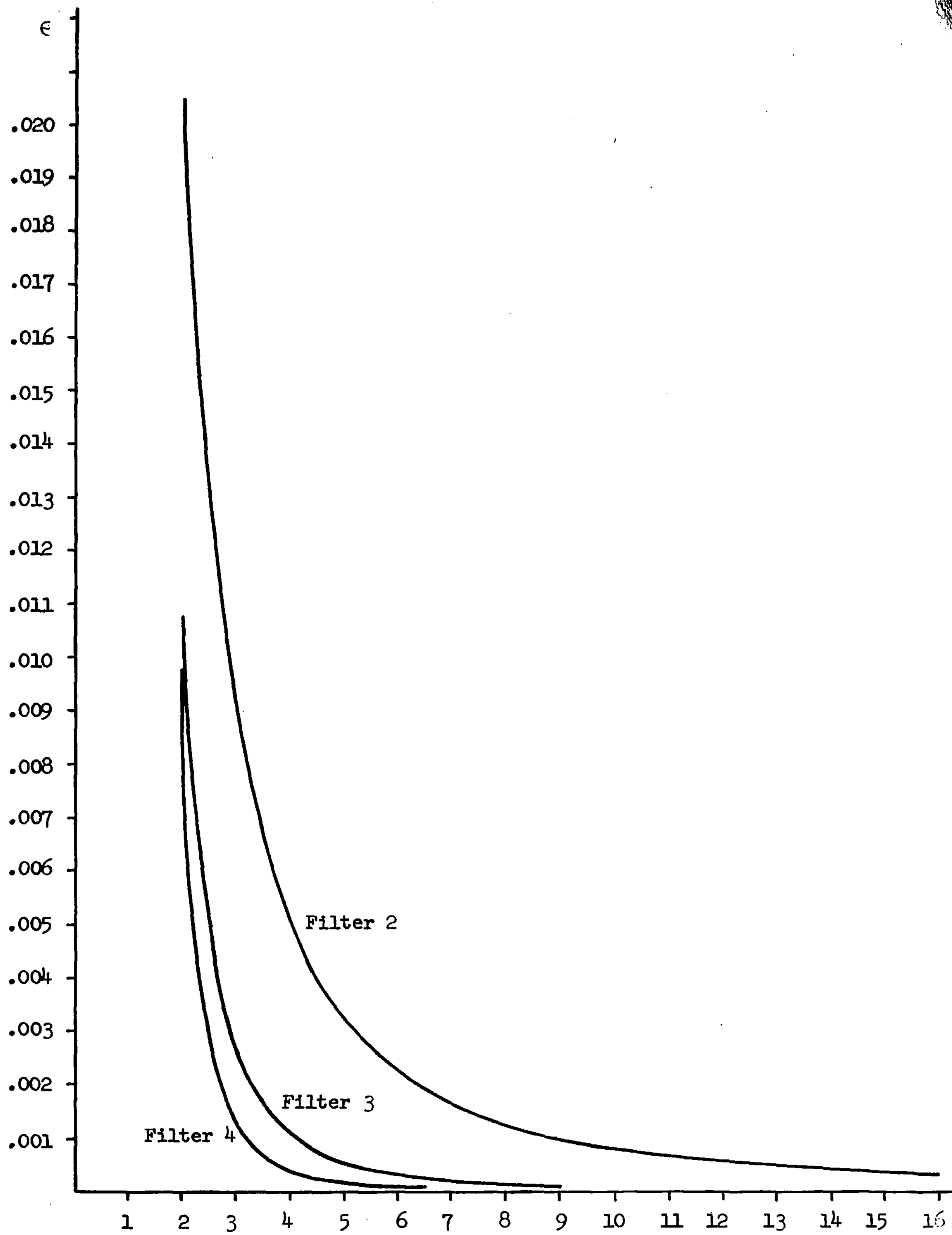
In general, a signal g(t) may have a polynomial content, and in such cases g(t) is not band-limited. Denoting the polynomial content of g(t) by P(t), if

$$g(t) = \bar{g}(t) + P(t) \tag{4.34}$$

where $\bar{g}(t)$ is a band-limited function, then the weights can be constrained so that the sampled values $P(m\Delta t)$, $\Delta t = 1/f_s$, are passed without error.

We recall that the output of a digital filter is given by

$$r_m = \sum_{n=-N}^{N} h_n g_{m+n}$$



ϵ
.020
.019
.018
.017
.016
.015
.014
.013
.012
.011
.010
.009
.008
.007
.006
.005
.004
.003
.002
.001
Filter 2
Filter 3
Filter 4
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
NΔf/f_s

Figure 4.6

Applying this to the sampled version of (4.34) gives

$$r_m = \sum_{n=-N}^{N} h_n [\bar{g}_{m+n} + P[(m+n)\Delta t]]$$

$$= \sum_{n=-N}^{N} h_n \bar{g}_{m+n} + \sum_{n=-N}^{N} h_n P[(m+n)\Delta t]. \qquad (4.35)$$

Since $\bar{g}(t)$ is band-limited, the first term on the right side of (4.35) poses no problems. We want the second term to be $P(m\Delta t)$. Assuming that $P(t)$ is of degree p,

$$P(t) = \sum_{j=0}^{p} a_j t^j. \qquad (4.36)$$

We want

$$P(m\Delta t) = \sum_{j=0}^{p} a_j (m\Delta t)^j$$

$$= \sum_{n=-N}^{N} h_n \sum_{j=0}^{p} a_j (m+n)^j \Delta t^j.$$

Interchanging the summation gives

$$\sum_{j=0}^{p} a_j (m\Delta t)^j = \sum_{j=0}^{p} a_j \sum_{n=-N}^{N} h_n (m+n)^j \Delta t^j. \qquad (4.37)$$

We see from (4.37) that it suffices to consider the k^{th} term

$$m^k \Delta t^k = \sum_{n=-N}^{N} h_n (m+n)^k \Delta t^k$$

or dividing by Δt^k,

$$m^k = \sum_{n=-N}^{N} h_n (m+n)^k.$$

Expanding $(m+n)^k$ and summing each term gives

$$m^k = m^k \sum_{n=-N}^{N} h_n + km^{k-1} \sum_{n=-N}^{N} nh_n + \cdots$$

$$+\binom{k}{r} m^{k-r} \sum_{n=-N}^{N} n^r h_n + \cdots + \sum_{n=-N}^{N} n^k h_n. \tag{4.38}$$

From (4.38) we see that it suffices to have

A:

$$\sum_{n=-N}^{N} h_n = 1 \tag{4.39}$$

B:

$$\sum_{n=-N}^{N} n^j h_n = 0, \qquad j = 1, 2, \ldots, p. \tag{4.40}$$

The transfer function of a digital smoothing filter which approximates smoothing filters of the types discussed in Section 4.1 is an even function of f and can be written in the form

$$H_n(f) = h_o + 2 \sum_{n=1}^{N} h_n \cos 2n\pi(f/f_s). \tag{4.41}$$

The weights are related by $h_n = h_{-n}$. Hence for odd integers j,

$$n^j h_n = - (-n)^j h_{-n}$$

or

$$n^j h_n + (-n)^j h_{-n} = 0$$

and

$$\sum_{n=-N}^{N} n^j h_n = 0 \tag{4.42}$$

Thus (4.40) is satisfied for all odd integers j without imposing any conditions on the h_n. If(4.39) is satisfied, the filter passes a first degree polynomial exactly. If, in addition, (4.40) is satisfied for j=2, the filter passes a third degree polynomial exactly, etc. Practical considerations usually limit j to 2, i.e., p=3.

The simplest way to satisfy (4.39) is to use new weights

$$\bar{h}_n = \frac{h_n}{\sum_{n=-N}^{N} h_n} \tag{4.43}$$

If N is chosen so that $.005 \leq \epsilon \leq .01$, the new weights usually do not change ϵ significantly.

For $j \geq 2$ the usual approach is to derive the constrained weights $\bar{h}_n$ so that the mean square error between the unconstrained transfer function $H_N(f)$ and the constrained transfer function

$$\bar{H}_N(f) = \bar{h}_o + 2 \sum_{n=0}^{N} \bar{h}_n \cos 2n\pi(f/f_s) \tag{4.44}$$

is minimized.

Note that (4.39) is equivalent to the condition

$$\bar{H}_N(0) = 1, \tag{4.45}$$

and (4.40) is equivalent to the conditions

$$\left. \frac{d^j \bar{H}_N(f)}{df^j} \right|_{f=0} = 0, \quad 1 \leq j \leq p. \tag{4.46}$$

Taking the case p=3 and using a Lagrangian multiplier, we wish to find weights $\bar{h}_n$ in terms of the h_n such that

$$R = \int_0^{f_s/2} [\bar{H}_N(f) - H_N(f)]^2 \, df + \lambda \sum_{n=1}^{N} n^2 \bar{h}_n$$

is minimized, i.e., $\frac{\partial R}{\partial \bar{h}_m} = 0$, $0 \leq m \leq N$, and such that $\bar{H}_N(f)$ satisfies (4.45) and (4.46) for p=3.

$$\frac{\partial R}{\partial \bar{h}_m} = 2 \int_0^{f_s/2} [\bar{H}_N(f) - H_N(f)] \frac{\partial \bar{H}_N(f)}{\partial \bar{h}_m} \, df + \lambda m^2. \tag{4.47}$$

The condition (4.45) is incorporated in the following way:

$$\bar{H}_N(0) = \bar{h}_o + 2 \sum_{n=1}^{N} \bar{h}_n = 1,$$

so

$$\bar{h}_o = 1 - 2 \sum_{n=1}^{N} \bar{h}_n. \tag{4.48}$$

Hence

$$\bar{H}_N(f) - H_N(f) = 1 + 2 \sum_{n=2}^{N} \bar{h}_n(\cos 2n\pi f/f_s - 1) - H_N(f).$$

Therefore
$$\frac{\partial \overline{H}_N}{\partial \overline{h}_m} = 2\,(\cos 2m\pi\ f/f_s - 1),$$

and
$$\frac{\partial R}{\partial \overline{h}_m} = 4 \int_0^{f_s/2} [1 + 2 \sum_{n=1}^{N} \overline{h}_n(\cos 2n\pi\ f/f_s - 1) - h_o - 2 \sum_{n=1}^{N} h_n \cos 2n\pi f/f_s]$$
$$[\cos 2m\pi f/f_s - 1]\,df \quad + \lambda m^2.$$

Let $\theta = 2\pi f/f_s$, then $df = (f_s/2\pi)d\theta$ and

$$\frac{\partial R}{\partial \overline{h}_m} = 2f_s/\pi \int_0^{\pi} [1-2 \sum_{n=1}^{N} \overline{h}_n - h_o + 2 \sum_{n=1}^{N} (\overline{h}_n - h_n) \cos n\theta]\,[\cos m\theta - 1]\,d\theta + \lambda m^2.$$

$$= 2f_s/\pi \int_0^{\pi} \{ [1-2 \sum_{n=1}^{N} \overline{h}_n - h_o][\cos m\theta - 1] + 2 \sum_{n=1}^{N} (\overline{h}_n - h_n) \cos n\theta \cos m\theta$$
$$-2 \sum_{n=1}^{N} (\overline{h}_n - h_n) \cos n\theta\}\, d\theta + \lambda m^2$$

$$= 2f_s/\pi\ \{-\pi[1 - 2\sum_{n=1}^{N} \overline{h}_n - h_o] + \pi\,(\overline{h}_m - h_m)\} + \lambda m^2.$$

Setting this equal to zero gives

$$2f_s\ [h_o - 1 + 2 \sum_{n=1}^{N} \overline{h}_n + \overline{h}_m - h_m] + \lambda m^2 = 0.$$

From (4.48) we see that we can replace $2 \sum_{n=1}^{N} \overline{h}_n - 1$ by $-\overline{h}_o$.

Thus

$$2f_s\{[h_o-\bar{h}_o + \bar{h}_m-h_m]\} + \lambda m^2 = 0.$$

Let $\delta = \bar{h}_o-h_o$, then

$$\bar{h}_m-h_m = \delta - \frac{\lambda m^2}{2f_s} . \qquad (4.49)$$

Summing both sides of (4.49) from 1 to N, then multiplying both sides by 2, and adding δ to both sides gives

$$\delta + 2\sum_{m=1}^{N}\bar{h}_m - 2\sum_{m=1}^{N}h_m = (2N+1)\ \delta - \frac{\lambda}{f_s}\sum_{m=1}^{N}m^2$$

or using (4.45) and reverting to the n subscript,

$$(2N+1)\ \delta - \frac{\lambda}{f_s}\sum_{n=1}^{N}n^2 = 1 - h_o - 2\sum_{n=1}^{N}h_n. \qquad (4.50)$$

Multiplying both sides of (4.49) by m^2, summing from 1 to N, using (4.46)--(or 4.40 with j=2)--and reverting to the n subscript gives

$$\delta\sum_{n=1}^{N}n^2 - \frac{\lambda}{2f_s}\sum_{n=1}^{N}n^4 = -\sum_{n=1}^{N}n^2h_n \qquad (4.51)$$

We solve (4.50) and (4.51) for δ and λ .

Let

$$Q_1 = 1 - h_o - 2\sum_{n=1}^{N}h_n$$

$$Q_2 = \sum_{n=1}^{N}n^2h_n$$

$$S_1 = \sum_{n=1}^{N} n^2$$

$$S_2 = \sum_{n=1}^{N} n^4 .$$

Then

$$\lambda = \frac{2f_s[S_1Q_1+(2N+1)Q_2]}{(2N+1)S_2-2S_1^2} \tag{4.52}$$

and

$$\delta = \frac{Q_1S_2+2S_1Q_2}{(2N+1)S_2-2S_1^2} . \tag{4.53}$$

Then

$$\bar{h}_o = h_o + \delta \tag{4.54a}$$

and from (4.49), for $n \geq 1$

$$\bar{h}_n = h_n + \delta - \frac{n^2}{2f_s} \lambda \tag{4.54b}$$

$$= h_n + \frac{Q_1S_2+2S_1Q_2-n^2[S_1Q_1+(2N+1)Q_2]}{(2N+1)S_2-2S_1^2} .$$

Note that

$$(2N+1)S_2-2S_1^2 = \frac{N(N+1)(2N-1)(2N+3)(2N+1)^2}{90}.$$

The constraint for p=1 is obtained by letting $\lambda = 0$ in (4.54b). Then we have

$$\bar{h}_n = h_n + \delta , \qquad n = 0,1, . . ., N$$

where

$$\delta = \frac{1-h_o-2\sum_{n=1}^{N} h_n}{2N+1} .$$

$$= \frac{1-H_N(0)}{2N+1}.$$

4.6 BAND-PASS FILTER

The ideal single band band-pass smoothing filter transfer function is

$$B_I(f) = \begin{cases} 1 & f_c \leq f \leq \bar{f}_c \\ 0 & 0 \leq f < f_c,\ f > \bar{f}_c. \\ B_I(-f) & f < 0 \end{cases} \qquad (4.55)$$

See Figure 4.7.

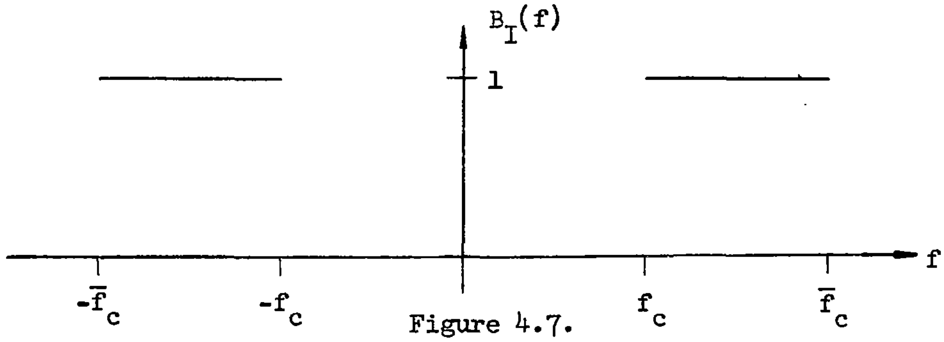


Figure 4.7.

Note that $B_I(f)$ can be written as the difference of two ideal smoothing filter transfer functions [see (3.6)] $H_1(f)$ and $H_2(f)$, where $H_2(f)$ has cut-off $\bar{f}_c$ and $H_1(f)$ has cut-off f_c. Then the weight function $b(t)$ is

$$b(t) = h_2(t) - h_1(t) \qquad (4.56)$$

where

$$h_1(t) \longleftrightarrow H_1(f),$$

$$h_2(t) \longleftrightarrow H_2(f),$$

and

$$b(t) \longleftrightarrow B_1(f).$$

A useable design is obtained by taking the difference of two low-pass smoothing filters of the types discussed in Section 4.1 and Section 4.3. The difference of two Martin-Graham filters, each with roll-off length Δf gives a satisfactory filter. The weight function of the resulting band-pass filter is then given by (4.56) with $h_1(t)$ and $h_2(t)$ the weights of the Martin-Graham filters. The weights of the corresponding digital filter are given by (3.40) and (4.56),

$$b_n = \frac{1}{f_s} b(\frac{-n}{f_s})$$

$$= \frac{1}{f_s} [h_2(\frac{-n}{f_s}) - h_1(\frac{-n}{f_s})]. \qquad (4.57)$$

Now suppose $B(f;f_o)$ is a band-pass smoothing filter with the mid-points of the "pass bands" at $\pm f_o$, "pass band" width $2\overline{\Delta f}$, and roll-off width Δf. For purposes of illustration, we assume that $B(f;f_o)$ has the Martin-Graham type roll-off [see (4.18)]. Let

$$H(f) = \begin{cases} 1 & 0 \leq f \leq \overline{\Delta f} \\ \frac{1}{2}[1+ \cos\frac{(f-\overline{\Delta f})\pi}{\Delta f}] & \overline{\Delta f} < f \leq \overline{\Delta f}+\Delta f \\ 0 & f > \overline{\Delta f}+\Delta f \\ H(-f) & f < 0 . \end{cases} \qquad (4.58)$$

See Figure 4.8.

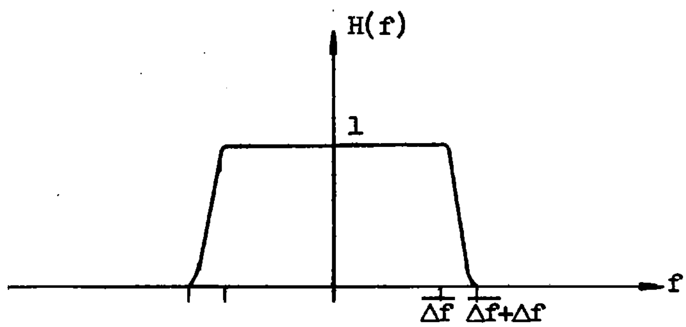


Figure 4.8.

Then H(f) is the transform of

$$h(t) = \frac{\sin 2\pi\overline{\Delta f} + \sin 2\pi(\Delta f+\overline{\Delta f})t}{2\pi t(1 - 4\Delta f^2 t^2)} . \qquad (4.59)$$

For $f \geq 0$

$$B(f;f_o) = H(f - f_o),$$

and for $f < 0$

$$B(f;f_o) = H(f + f_o).$$

Thus

$$B(f;f_o) = H(f - f_o)+H(f + f_o), \qquad (4.60)$$

(see Figure 4.9).

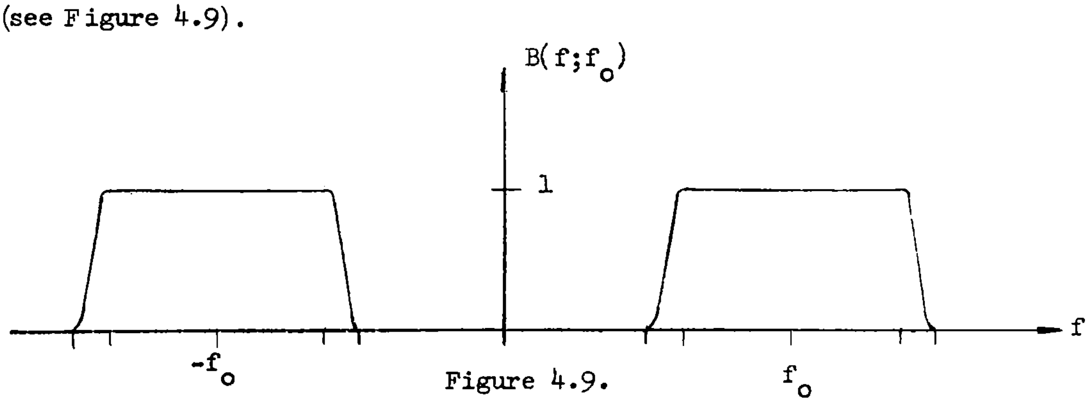


Figure 4.9.

Taking the inverse transform of each side and using the shift theorem (1.22) (or its generalized equivalent) gives

$$b(t;t_o) = h(t)(\exp(2\pi i f_o t) + \exp(-2\pi i f_o t))$$

$$= 2h(t)\cos 2\pi f_o t. \qquad (4.61)$$

The weights of the corresponding digital filter are given by

$$b_n(f_o) = 2h_n \cos 2\pi n \frac{f_o}{f_s}, \qquad (4.62)$$

where $h_n = \frac{1}{f_s} h(\frac{-n}{f_s})$.

For a given f_o, the weights can be computed from (4.62) more quickly than from (4.57). If several successive filtering operations are to be performed for a set of f_o values, say $f_1, f_2, \ldots, f_k$, then, using (4.62),

$$b_n(f_j) = 2h_n \cos 2n\pi \frac{f_j}{f_s}, \qquad j = 1,2, \ldots, k.$$

But in order to use (4.57) the functions $h_1(t)$ and $h_2(t)$ must be changed for each new value of f_j and the entire expression must be recomputed.

From (4.62) we see that the error ϵ' of a band-pass smoothing filter may be as much as twice the error ϵ of the smoothing filter whose transfer function is $H(f)$.

In a manner analogous to the ideal smoothing case in Section 3.1, the transfer function of a filter which will simultaneously "band-pass" filter and find the n^{th} derivative is

$$B^n(f) = (2\pi i f)^n B(f) \qquad (4.63)$$

where B(f) is the transfer function of a band-pass smoothing filter. Then if $b(t) \longleftrightarrow B(f)$,

$$b^{(n)}(t) = b^n(t) \longleftrightarrow B^n(f), \qquad (4.64)$$

and the weights [see the derivation of (4.3b)] are given by

$$b_k^n = (-1)^n (f_s)^n \frac{d^n b_k}{dk^n}, \qquad (4.65)$$

where $b_k = \frac{1}{f_s} b(\frac{-k}{f_s})$.

The weight function of a filter having several pass bands, each of equal pass width and roll-off width, can easily be found from (4.61). Let $\pm f_1, \pm f_2, \ldots, \pm f_k$ be the mid points of the pass bands, and denote the transfer function by $B(f;f_1,f_2,\ldots,f_k)$ (see Figure 4.10).

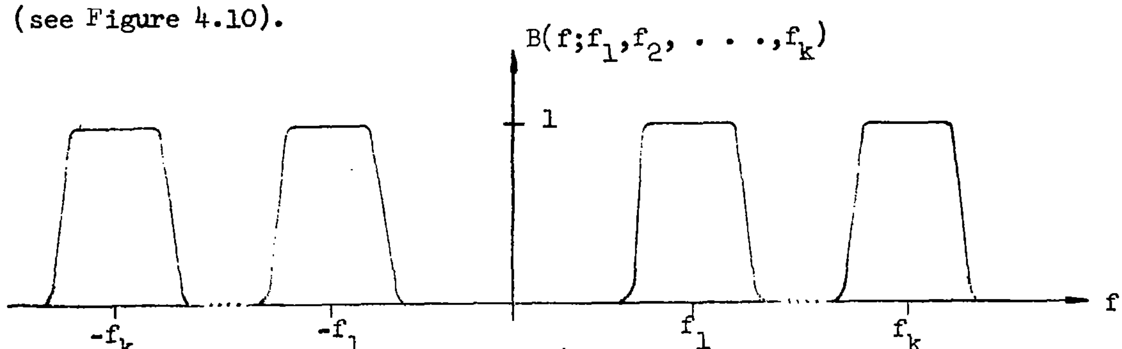


Figure 4.10.

Then the weight function is

$$b(t;f_1,f_2,\ldots,f_k) = 2\{\sum_{j=1}^{k} \cos 2\pi f_j t\}\, h(t). \qquad (4.66)$$

The weights are given by

$$b_n = \frac{1}{f_s} b(\frac{-n}{f_s}; f_1, f_2, \cdot \cdot \cdot, f_k)$$

$$= 2h_n \{ \sum_{j=1}^{k} \cos 2n\pi \frac{f_j}{f_s} \}. \qquad (4.67)$$

CHAPTER V

FILTERS FOR SMOOTHING AND DIFFERENTIATION

5.0 MARTIN-GRAHAM FILTERS

We shall call a filter a Martin-Graham filter if its transfer function either uses the Martin-Graham roll-off [see (4.18)] or is derivable from a transfer function having the Martin-Graham roll-off.

In Section 4.1, we discussed the Martin-Graham smoothing filter and found its weight function h(t) [see (4.13)] . From h(t) and the formula (3.40) for computing the weights of the approximating digital filter, we found the weights h_n [see (4.15)] which are used in the basic formula of digital filtering,

$$\bar{r}_m = \sum_{n=-N}^{N} h_n g_{m+n}, \tag{3.41}$$

where the g_j are the input data values and the $\bar{r}_j$ are the smoothed output values.

A Martin-Graham band-pass smoothing filter is easily obtained from the smoothing case and the discussion of Section 4.6.

In this chapter, we shall derive the weights of Martin-Graham filters for smoothing and differentiation. When referring to a set of data $\{g_m\}$, we assume that the data arises from a function g(t) such that

1) $g(t) = \bar{g}(t) + p(t)$, where p(t) is a polynomial in t,
2) $\bar{g}(t)$ satisfies conditions I - III of Section 4.0,
3) $g_m = g(\frac{m}{f_s})$ where f_s is greater than twice the highest frequency in $\bar{g}(t)$.

Let g_M be the first data value and $g_{\overline{M}}$ be the last. If p(t) is not identically zero for $\frac{M}{f_s} \leq t \leq \frac{\overline{M}}{f_s}$, then, in order to pass p(t) or differentiate it, constraints are necessary. Those for smoothing are in Section 4.5. A general procedure is given in Appendix A for the derivative cases, and the constraints for passing the first derivative of p(t) will be given in the next section.

5.1 SMOOTHING AND FIRST DERIVATIVE FILTER

We have shown the transfer function of a filter which will smooth and find the first derivative to be

$$Y^1(f) = (2\pi i f)H(f)$$

where H(f) is any smoothing filter transfer function [Put n=1 in (4.2)] . Note that $Y^1(f)$ inherits the cut-off, f_c, and termination, f_T, frequencies from H(f).

Putting n=1 in (4.3b), we obtain the weights of this filter in terms of the smoothing weights

$$y_k^1 = -f_s \frac{dh_k}{dk} \tag{5.1}$$

where $h(t) \longleftrightarrow H(f)$ and $h_k = \frac{1}{f_s} h(\frac{-k}{f_s})$.

The Martin-Graham smoothing filter weights given by (4.15) in terms of the frequency ratio, $\tau = \frac{w}{2\pi f_s} = \frac{f}{f_s}$, are

$$h_k = \frac{\cos k\pi\tau_d \sin k\pi(2\tau_c + \tau_d)}{k\pi(1-4\tau_d^2 k^2)}$$

$$= \frac{\sin 2\pi\tau_T k + \sin 2\pi\tau_c k}{2\pi k(1-4\tau_d^2 k^2)} ,$$

$$\tau_d = \frac{\Delta f}{f_s}, \quad \tau_c = \frac{f_c}{f_s}, \quad \tau_T = \frac{f_T}{f_s}.$$

Then

$$y_k^1 = -f_s \quad \frac{\tau_T \cos 2\pi\tau_T k + \tau_c \cos 2\pi\tau_c k}{k(1-4\tau_d^2k^2)} - \frac{h_k(1-12\tau_d^2k^2)}{k(1-4\tau_d^2k^2)}$$

$$= -f_s \quad \frac{\tau_T \cos 2\pi\tau_T k + \tau_c \cos 2\pi\tau_c k - h_k(1-12\tau_d^2k^2)}{k(1-4\tau_d^2k^2)} \tag{5.2}$$

Note that $y_{-k}^1 = -y_k^1$, and by applying L'Hospital's rule, $y_0^1 = 0$. Also note that for $k = \frac{1}{2\tau_d}$, y_k^1 must be computed by using L'Hospital's rule. [See Section 5.4.]

In a manner analogous to that of Section 4.5, we find that in order to pass exactly the derivative of P(t) of degree p the following conditions must be satisfied by the approximating filter transfer function

$$Y_N^1(f) = 2i \sum_{n=1}^{N} y_n^1 \sin 2n\pi f/f_s \tag{5.3}$$

(1) $Y_N^1(0) = 0$

(2) $\left.\frac{dy_N^1(f)}{df}\right|_{w=0} = i$

(3) $\frac{d^pY_N^1(f)}{df^p} = 0$ for $p > 1$.

Since $\frac{d^pY_N^1(f)}{df^p}$ is odd for all even $p \geq 0$, (1) and (3) are automatically satisfied for even integers $p \geq 0$. In particular, if p(t) is of degree 2, we need to satisfy only (2). The constrained weights $\bar{y}_k^1$ are given by

$$\bar{y}_k^1 = y_k^1 + \frac{kQ_1}{Q_2}, \qquad k \geq 1, \tag{5.4}$$

$$\bar{y}_k^1 = -\bar{y}_k^1,$$

where

$$Q_1 = \frac{f_s}{2} - \sum_{n=1}^{N} n y_n^1$$

$$Q_2 = \sum_{n=1}^{N} n^2 \,.$$

(See [14] for the derivation for p=4 from which the case p=2 follows easily.)

The constrained transfer function is

$$\bar{Y}_N^1(f) = 2i \sum_{n=1}^{N} \bar{y}_n^1 \sin \frac{2n\pi f}{f_s} \tag{5.5}$$

In order to smooth and differentiate a set of data ${}_1\{g_m\}$ where the polynomial content is of degree 2 or less, put $h_n = \bar{y}_n$ in (3.41). This gives

$$\bar{r}_m = \sum_{n=-N}^{N} \bar{y}_n^1 \, g_{m+n} \,. \tag{5.6}$$

If we let $\hat{h}_k = \frac{h_k}{f_s}$, then $\hat{y}_k^1 = \frac{y_k^1}{f_s}$ and $\hat{\bar{y}}_k^1 = \frac{\bar{y}_k^1}{f_s}$. Then

$$\bar{Y}_N^1(f) = 2if_s \sum_{n=1}^{N} \hat{\bar{y}}_k^1 \sin \frac{2n\pi f}{f_s} \tag{5.7}$$

and

$$\bar{r}_m = f_s \sum_{n=-N}^{N} \hat{\bar{y}}_n^1 \, g_{m+n} . \tag{5.8}$$

Using $\hat{y}^1_{-n} = -\hat{y}^1_n$, we have

$$\bar{r}_m = f_s \sum_{n=1}^{N} \hat{y}^1_n \, [g_{m+n} - g_{m-n}]. \tag{5.9}$$

Writing

$$Y^1(f) = 2\pi i f H(f) = 2\pi f H(f) \exp(i\pi/2),$$

we see that $Y^1(f)$ has a phase shift of $\pi/2 = 90^\circ$, i.e., $\theta(f) = \pi/2$. $2\pi f H(f)$ is shown in Figure 5.1 for the Martin-Graham smoothing and first derivative filter.

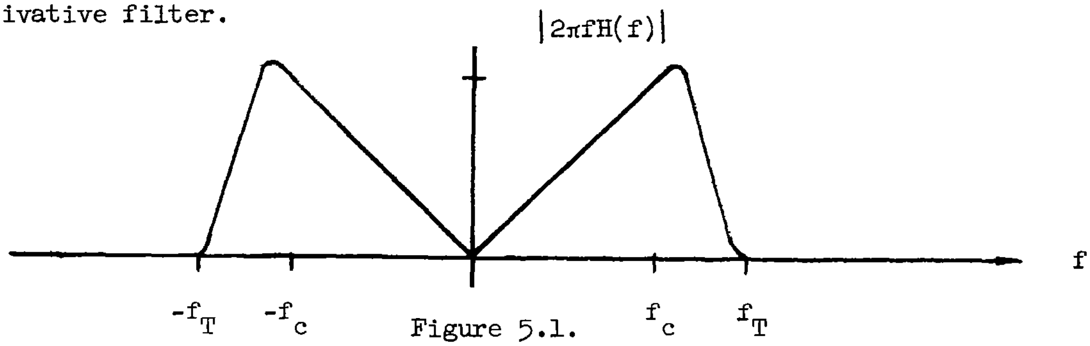


Figure 5.1.

Other first derivative filters with different roll-offs have been examined and in each case it was found that the Martin-Graham filter yielded the same or a more accurate result.

In an attempt to avoid the lengthy computation of (5.1) for the weights y^1_k, a "three-point derivative" of the smoothing weights h_k has been examined. Let

$$\theta_k = \frac{h_{k+1} - h_{k-1}}{\frac{2}{f_s}} \tag{5.10}$$

With H(f) the transform of the weight function h(t) from which the h_k are computed, we have

$$\theta_k = \frac{f_s}{2} [h_{k+1} - h_{k-1}]$$

$$= 1/2 \quad [h((-k-1)/f_s) - h((-k+1)/f_s)]$$

$$= 1/2 \int_{-f_s/2}^{f_s/2} H(f)\ [\exp(-2\pi i(k+1)f/f_s) - \exp(-2\pi i(k-1)f/f_s)]\ df$$

$$= 1/2 \int_{-f_s/2}^{f_s/2} H(f)\ \exp(-2\pi ikf/f_s)\ [\exp(-2\pi if/f_s) - \exp(2\pi if/f_s)]\ df$$

$$= \int_{-f_s/2}^{f_s/2} \left[\frac{\exp(2\pi if/f_s) - \exp(-2\pi if/f_s)}{2i}\right] [-iH(f)\exp(-2\pi ikf/f_s)]\ df$$

$$\theta_k = 1/f_s \int_{-f_s/2}^{f_s/2} \frac{-\sin 2\pi f/f_s}{2\pi f/f_s}\ [(2\pi if)H(f)]\ \exp(-2\pi ikf/f_s) df. \qquad (5.11)$$

The actual weights are

$$y_k^1 = 1/f_s \int_{-f_s/2}^{f_s/2} (2\pi if)H(f)\ \exp(-2\pi ikf/f_s)\ df \qquad (5.12)$$

Comparing (5.11) and (5.12), we see that if we define a weight $\gamma_k = -\theta_k$, then the transfer function of the γ_k is

$$\frac{\sin 2\pi f/f_s}{2\pi f/f_s}(2\pi i f)H(f)$$

which is the product of the desired transfer function and

$$\frac{\sin 2\pi f/f_s}{2\pi f/f_s} \cdot$$

Now

$$y_k^1 - \tilde{y}_k = 1/f_s \int_{-f_s/2}^{f_s/2} (2\pi i f)H(f)\left[1 - \frac{\sin 2\pi f/f_s}{2\pi f/f_s}\right] \exp(-2\pi i k f/f_s)\, df$$

and

$$1 - \frac{\sin 2\pi f/f_s}{2\pi f/f_s} \doteq 0$$

for $|f/f_s|$ small. If the cut-off f_c is small, then H(f) in the above integral becomes zero for f/f_s relatively small. Then the $\tilde{y}_k$ are good approximations of the y_k^1. It has been found empirically that for filters such that $f_c/f_s \leq .1$, the $\tilde{y}_k$ give an acceptable output.

5.2 BAND-PASS SMOOTHING AND FIRST DERIVATIVE FILTER

We have shown that the transfer function of a band-pass filter which will smooth and find the first derivative to be

$$B^1(f) = 2\pi i f B(f)$$

where B(f) is any band-pass smoothing filter transfer function [put n=1 in (4.63)] . Note that $B^1(f)$ has the same cutoff and termination frequencies as B(f). B(f) may be designed by either of the methods discussed in Section 4.6.

Putting n=1 in (4.65), we obtain the weights of this filter in terms of the band-pass smoothing weights

$$b_k^1 = -f_s \frac{db_k}{d_k}$$

where $b(t) \longleftrightarrow B(f)$ and $b_k = \frac{1}{f_s} b(\frac{-k}{f_s})$.

If the b_k are obtained by taking the difference [see (4.57)] of the weights of two low pass filters, say h_k' and h_k'', then

$$b_k^1 = -f_s \left\{ \frac{dh_k''}{dk} - \frac{dh_k'}{dk} \right\} . \tag{5.13}$$

When the b_k are obtained by the second method [see (4.62)], we have

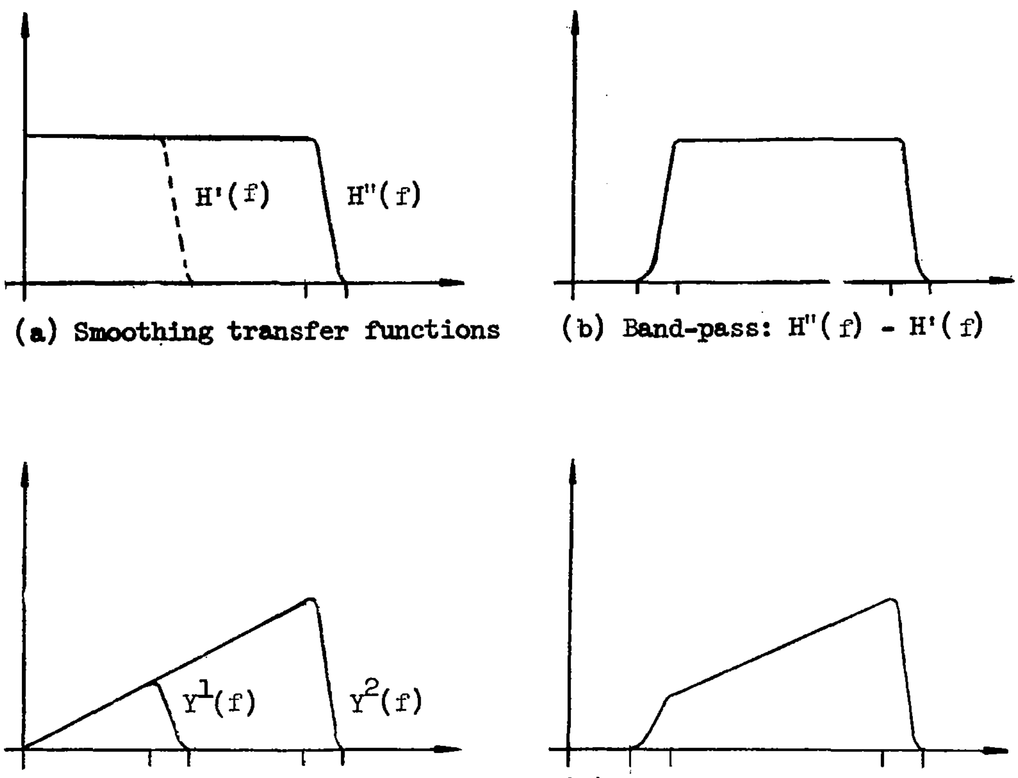
$$b_k^1 = -2f_s \frac{d}{dk} \left\{ h_k \cos 2k\pi \frac{f_o}{f_s} \right\} \tag{5.14}$$

$$= -2f_s \left\{ \frac{-h_k 2\pi f_o}{f_s} \sin 2k\pi \frac{f_o}{f_s} + \frac{dh_k}{dk} \cos 2k\pi \frac{f_o}{f_s} \right\}$$

$$= 4\pi f_o h_k \sin 2k\pi \frac{f_o}{f_s} - 2f_s \frac{dh_k}{dk} \cos 2k\pi \frac{f_o}{f_s} .$$

To obtain a Martin-Graham filter of this type by the first method, we simply select two Martin-Graham smoothing filters with transfer-weight functions $h'(t) \longleftrightarrow H'(f)$ and $h''(t) \longleftrightarrow H''(f)$ and compute the weights b_k^1 by (5.13). To use the second method, the appropriate

Martin-Graham filter with $h(t) \longleftrightarrow H(f)$ is selected and the weights b_k^1 are computed by (5.14). These weights are used for the h_k in (3.41). Note that a factor of f_s can be removed from the sum (3.41) in a manner analogous to the first derivative case [see (5.7) and (5.8)].



(a) Smoothing transfer functions

(b) Band-pass: $H''(f) - H'(f)$

(c) Derivative transfer functions derived from H' and H".

(d) Band-pass derivative: $Y^2(f) - Y^1(f)$

Figure 5.2.

5.3 SMOOTHING AND SECOND DERIVATIVE FILTER

Letting n=2 in (4.2), we find that the transfer function of a filter which will smooth and find the second derivative is

$$Y^2(f) = -(2\pi f)^2 H(f) \tag{5.15}$$

where H(f) is any smoothing filter transfer function.

Putting n=2 in (4.3b), we find that the weights of the filter in terms of the smoothing weights are

$$y_k^2 = f_s^2 \frac{d^2 h_k}{dk^2} \tag{5.16}$$

where $h(t) \longleftrightarrow H(f)$ and $h_k = \frac{1}{f_s} h(\frac{-k}{f_s})$.

Using the Martin-Graham smoothing weights given by (4.15) in terms of $\tau = \frac{w}{2\pi f_s} = \frac{f}{f_s}$ and (5.16) gives

$$y_k^2 = \frac{f_s^2}{k(1-4\tau_d^2k^2)} \left\{ 24\tau_d^2 k h_k + \frac{2y_k^1}{f_s}(1-12\tau_d^2k^2) - 2\pi(\tau_T^2 \sin 2\pi\tau_T k + \tau_c^2 \sin 2\pi\tau_c k) \right\} \tag{5.17}$$

where y_k^1 is given by equation (5.2). This gives the weights to be used in (3.41). Note that a factor of f_s^2 may be removed in this case.

For k=0, using L'Hospital's rule gives

$$y_o^2 = f_s^2 \left[8\tau_d^2(\tau_T+\tau_c) - \frac{4\pi^2}{3} (\tau_T^3+\tau_c^3) \right]. \tag{5.18}$$

For $k = \frac{1}{2\tau_d}$, L'Hospital's rule must be used to compute h_k. [See Section 5.4.]

A constraint is developed in Appendix A to improve the fit of the approximating transfer function at some specific frequency ratio $\bar{\tau}$.

5.4 FIRST AND SECOND DERIVATIVE WEIGHTS FOR $1/2\tau_d$ AN INTEGER

In cases where τ_d is such that $1/2\tau_d$ is an integer, say m, then h_m, h_{-m}, y^1_m, y^1_{-m}, y^2_m, and y^2_{-m} assume the indeterminate form 0/0 and these weights must be computed by using L'Hospital's rule. The value of h_m in this case is given by (4.17) and h_{-m} is obtained from $h_m = h_{-m}$.

Application of L'Hospital's rule to the first derivative weight expression y^1_m when $m = 1/2\tau_d$ yields

$$y^1_m = (f_s/2)\{\pi\tau_d(\tau_d+2\tau_c)\sin(\pi\tau_c/\tau_d) + (3\tau_d^2/2)\cos(\pi\tau_c/\tau_d)\} \quad (5.19)$$

The first derivative weight function is odd, and hence we have

$$y^1_{-m} = -y^1_m \ .$$

Application of L'Hospital's rule to the second derivative weight expression $y^2_m = 1/2\tau_d$ yields

$$y^2_m = f_s^2\{3\pi\tau_d^2(\tau_d+2\tau_c)\sin(\pi\tau_c/\tau_d) + (7\tau_d^3-2\pi^2[\ \tau_c\tau_d(\tau_c+\tau_d)+\tau_d^3/3\ ]\ )\ \cos(\pi\tau_c/\tau_d)\} \quad (5.20)$$

The second derivative weight function is even, and hence we have

$$y^2_{-m} = y^2_m$$

CHAPTER VI

FILTERS FOR INTEGRATION AND INTERPOLATION

6.0 INTEGRATING FILTERS

Let $A \exp(2\pi ift)$ be a component of an input to a filter. Assuming that the constant of integration is zero, the indefinite integral of this component is $(2\pi if)^{-1}A \exp(2\pi ift)$. If this is to be the output of the filter, then, using (3.5), we find that the transfer function must be

$$X(f) = (2\pi if)^{-1} \tag{6.0}$$

Suppose that $g(t)$ is the input to a filter, and that $g(t) \longleftrightarrow G(f)$. Letting $k'(t) = g(t)$, assuming that the constant of integration is zero, and that $k(t)$ satisfies conditions sufficient for the Fourier integral theorem to hold, we have

$$k(t) \longleftrightarrow (2\pi if)^{-1}G(f). \tag{6.1}$$

If we also smooth, we have

$$\bar{k}(t) \longleftrightarrow (2\pi if)^{-1}H(f)G(f) \tag{6.2}$$

where $H(f)$ is the smoothing filter transfer function. From (6.2) we see that the transfer function of a filter which will simultaneously smooth and give the indefinite integral is

$$Y^{(-1)}(f) = (2\pi if)^{-1}H(f) \tag{6.3}$$

Note that the smoothed output, $\overline{g}(t)$, of the smoothing filter acting alone on $g(t)$ is the inverse transform of $H(f)G(f)$, and that

$$\overline{k}(t) = \int_a^t \overline{g}(\beta)d\beta \tag{6.4}$$

where we assume $\overline{k}(a) = 0$.

For the transfer functions, $H_j(f)$, $j=1,2, \ldots,5$, of the smoothing filters discussed in Chapter IV, $Y^{(-1)}(f)$ has an infinite discontinuity at $f=0$. Hence, in order to approximate $Y^{(-1)}(f)$ with a truncated Fourier series, we must modify $Y^{(-1)}(f)$ on an interval containing zero. To avoid some integrals which cannot be evaluated in closed form, we shall consider only the case $j=1$, i.e., an Ormsby type filter.

Let $\Delta f > 0$ and

$$Y^{(-1)}(f) = (2\pi i)^{-1} \begin{cases} f(\Delta f)^{-2} & , \quad |f| < \Delta f \ , \\ f^{-1} & , \quad \Delta f \le f \le f_c , \\ \\ \dfrac{f_T - f}{f_c \Delta f} & , \quad f_c < f \le f_T \ , \\ 0 & , \quad |f| > f_T , \end{cases} \tag{6.5}$$

and $Y^{(-1)}(-f) = -Y^{(-1)}(f)$ for $f < 0$. See Figure 6.2

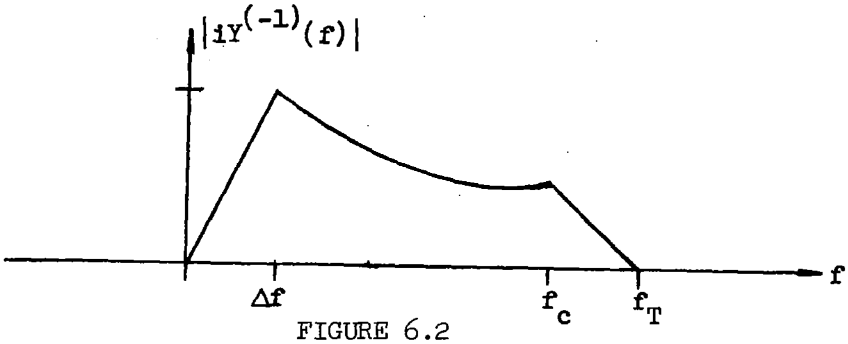


FIGURE 6.2

The weights in terms of the frequency ratio $\tau = \frac{f}{f_s}$, $\tau_d = \frac{\Delta f}{f_s}$, $\tau_c = \frac{f_c}{f_s}$, $\tau_T = \frac{f_T}{f_s}$, are

$$y_n^{(-1)} = \frac{1}{2\pi^2 \tau_d f_s} \left[\frac{\cos 2n\pi\tau_d}{n} - \frac{\sin 2n\pi\tau_d}{2\pi\tau_d n^2} - \frac{\tau_d \cos 2n\pi\tau_c}{\tau_c n} + \frac{(\sin 2n\pi\tau_T - \sin 2n\pi\tau_c)}{2\pi\tau_c n^2} - 2\pi\tau_d \left[\mathrm{Si}\,(2n\pi\tau_c) - \mathrm{Si}(2n\pi\tau_d)\right] \right], \tag{6.6}$$

where

$$\mathrm{Si}(x) = \int_0^x \frac{\sin y}{y}\, dy = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{(2k-1)}}{(2k-1)!(2k-1)} .$$

<u>A definite integral</u>

From (6.4) we have

$$\bar{k}(t+a) - \bar{k}(t-a) = \int_{t-a}^{t+a} \bar{g}(\beta)\, d\beta,$$

and by (6.2) and the First Shifting Theorem,

$$\bar{k}(t+a) - \bar{k}(t-a) \longleftrightarrow (2\pi if)^{-1}G(f)H(f)[\exp(2\pi iaf)\ \exp(-2\pi iaf)]$$

or

$$\bar{k}(t+a) - \bar{k}(t-a) \longleftrightarrow \frac{\sin 2\pi af}{\pi f} H(f)G(f). \tag{6.7}$$

Thus, if H(f) is a smoothing filter transfer function, the transfer function of a filter which will simultaneously smooth and give the integral of the input over [t-a,t+a] is

$$Y^{(-1)}(f) = \frac{\sin 2\pi af}{\pi f}H(f). \tag{6.8}$$

Let

$$x(t) = \begin{cases} 1 & |t| \leq a \\ 0 & |t| > a\ , \end{cases} \tag{6.9}$$

then

$$x(t) \longleftrightarrow X(f) = \frac{\sin 2\pi af}{\pi f}\ . \tag{6.10}$$

Applying the convolution theorem gives

$$\begin{aligned} y^{(-1)}(t) &= \int_{-\infty}^{\infty} h(z)x(t-z)dz \\ &= \int_{t-a}^{t+a} h(z)dz \\ &= \int_{-a}^{a} h(t-z)dz \end{aligned} \tag{6.11}$$

where $y^{(-1)}(t) \longleftrightarrow Y^{(-1)}(f)$ and $h(t) \longleftrightarrow H(f)$.

By (3.40), the weights of the corresponding filter are

$$y_n^{(-1)} = \frac{1}{f_s} \int_{\frac{-n}{f_s}-a}^{\frac{-n}{f_s}+a} h(z)dz. \tag{6.12}$$

Choosing $h_1(t)$, the Ormsby smoothing filter weight function, we have

$$\begin{aligned} y^{(-1)}(t) = \frac{1}{\pi\Delta f} \Bigg\{ & \frac{t(\sin 2\pi f_c t \sin 2\pi f_c a - \sin 2\pi f_T t \sin 2\pi f_T a)}{\pi(t^2-a^2)} \\ + & \frac{a(\cos 2\pi f_c t \cos 2\pi f_c a - \cos 2\pi f_T t \cos 2\pi f_T a)}{\pi(t^2-a^2)} \\ - & f_c[\mathrm{Si}(2\pi f_c[t+a]) - \mathrm{Si}(2\pi f_c[t-a])] \\ + & f_T[\mathrm{Si}(2\pi f_T[t+a]) - \mathrm{Si}(2\pi f_T[t-a])] \Bigg\}. \end{aligned} \tag{6.13}$$

Using the frequency ratio $\tau = \frac{f}{f_s}$, letting $a = \frac{b}{f_s}$, and computing the weights by (3.40), we have

$$\begin{aligned} y_n^{(-1)} = \frac{1}{\pi^2 \tau_d f_s} \Bigg\{ & \frac{n(\sin 2n\pi\tau_c \sin 2\pi\tau_c b - \sin 2n\pi\tau_T \sin 2\pi\tau_T b)}{(n^2-b^2)} \\ + & \frac{b(\cos 2n\pi\tau_c \cos 2\pi\tau_c b - \cos 2n\pi\tau_T \cos 2\pi\tau_T b)}{(n^2-b^2)} \\ - & \pi\tau_c[\mathrm{Si}(2\pi\tau_c[n+b]) - \mathrm{Si}(2\pi\tau_c[n-b])] \\ + & \pi\tau_T[\mathrm{Si}(2\pi\tau_T[n+b]) - \mathrm{Si}(2\pi\tau_T[n-b])] \Bigg\}. \end{aligned} \tag{6.14}$$

6.1 INTERPOLATING FILTERS

If $g(t)$ is a function with Fourier transform $G(f)$, then by replacing t_o by $-t_o$ in (1.21) we obtain

$$g(t+t_o) \longleftrightarrow G(f)\exp(2\pi i t_o f) \tag{6.15}$$

From this and (3.5), we see that the transfer function of a filter with output $g(t+t_o)$ is

$$\frac{G(f)\exp(2\pi i t_o f)}{G(f)} = \exp(2\pi i t_o f)$$

Suppose that $g(t)$ is band-limited, and let $h(t) \longleftrightarrow H(f)$ be the weight and transfer functions of a filter, then

$$g(t+t_o)*h(t) \longleftrightarrow G(f)\exp(2\pi i t_o f)H(f) \tag{6.16}$$

But applying (1.21) to $h(t)$ as we did to $g(t)$ above, we have that

$$h(t+t_o) \longleftrightarrow H(f)\exp(2\pi i t_o f). \tag{6.17}$$

Hence

$$g(t)*h(t+t_o) \longleftrightarrow G(f)\exp(2\pi i t_o f)H(f) \tag{6.18}$$

Comparing (6.16) and (6.18), we have

$$g(t)*h(t+t_o) = g(t+t_o)*h(t) \tag{6.19}$$

From this we see that the operations of filtering and shifting the output by a constant t_o can be accomplished by shifting the weight function by t_o. Letting $\bar{g}(t)$ denote the smoothed output of the filter with weight function h(t), we have

$$\bar{g}(t) = g(t)*h(t)$$

and

$$\bar{g}(t+t_o) = g(t)*h(t+t_o) \tag{6.19}$$

Now $\bar{g}(t+t_o)$ is the output of a filter with transfer function $\exp(2\pi i t_o f)H(f)$, and from (6.17) we see that the corresponding weight function is $h(t+t_o)$. Then from (3.40), the weights of the filter are

$$\bar{h}_n = (1/f_s)h(-n/f_s + t_o) \tag{6.20}$$

The corresponding digital filter has for its output

$$\bar{g}'_m = \sum_{n=-N}^{N} \bar{h}_n g_{m+n} \tag{6.21}$$

where $\bar{g}'_m$ is an approximation of $\bar{g}(m/f_s+t_o)$, that is, it is an interpolated value of $\bar{g}(t)$ between $\bar{g}_m$ and $\bar{g}_{m+1}$ for $0 < t_o < 1/f_s$.

Note that (6.21) uses only the assumed known sample values g_n of the input g(t). The weights $\bar{h}_n$ are computed from the known weight function. Also, it is important to note that the weights are no longer either even or odd functions of n, that is, $\bar{h}_{-n} \neq \bar{h}_n$, and $\bar{h}_{-n} \neq -\bar{h}_n$.

Choosing h(t) to be the appropriate Martin-Graham weight function, and using (6.20), we may compute weights to simultaneously smooth and interpolate; smooth, differentiate, and interpolate; or band-pass filter and interpolate.

In the first two cases, if the cutoff frequency can be chosen greater than the largest frequency f_β present in the data, that is, if $f_\beta \leq f_c < f_s/2$, then choosing $f_T = f_s/2$ to maximize the roll-off length, we obtain filters which interpolate for raw data values between known data values in the first case, and which differentiate and interpolate without smoothing in the second case.

There is a relation between the weights for interpolating with $t_o > 0$ and the weights for interpolating with $t_o' = -t_o$ which is sometimes useful. Suppose the weight function h(t) of the original filter is an even function of t. Then

$$\bar{h}_n = (1/f_s)h(-n/f_s + t_o) = (1/f_s)h(n/f_s - t_o) = \bar{h}'_{-n} \qquad (6.22)$$

where the $\bar{h}'_{-n}$ are the weights for interpolation with t_o replaced by $-t_o$. When $\bar{h}_{-N}, \bar{h}_{-N+1}, \ldots, \bar{h}_{-1}, \bar{h}_o, \bar{h}_1, \ldots, \bar{h}_{N-1}, \bar{h}_N$ are the weights for interpolating with $t_o > 0$, then the weights for interpolating with t_o replaced by $-t_o$ are $\bar{h}'_{-N} = \bar{h}_N, \bar{h}'_{-N+1} = \bar{h}_{N-1}, \ldots,$ $\bar{h}'_{-1} = \bar{h}_1, \bar{h}'_o = \bar{h}_o, \bar{h}'_1 = \bar{h}_{-1}, \ldots, \bar{h}'_{N-1} = \bar{h}_{-N+1}, \bar{h}'_N = \bar{h}_{-N}$.

For h(t) an odd function of t, we obtain the relation

$$\bar{h}_n = -\bar{h}'_{-n}. \qquad (6.23)$$

As was the case previously, when using Martin-Graham filters for filtering and interpolating, there are values of m, τ_d, and t_o which make the denominator of the weight expressions zero. In these cases, the weights $\bar{h}_m$ require special attention. Letting $t_o = \Phi/f_s$ in the weight expressions for these filters, we see that this will be true when m is an integer such that

$$m-\Phi = 0, \text{ or } m-\Phi = \pm (1/2\tau_d) \qquad (6.24)$$

Using the Martin-Graham smoothing weight function (4.13), the weight equation (6.20), and replacing t_o by Φ/f_s, we find that the weights for smoothing and interpolating are

$$\bar{h}_n = \frac{\cos(\pi\tau_d(n-\Phi)) \sin(\pi(2\tau_c+\tau_d)(n-\Phi))}{\pi(n-\Phi)(1 - 4\tau_d^2(n-\Phi)^2)} \qquad (6.25)$$

We now see that no special evaluation of this expression when (6.24) holds is necessary. That is, the values of m-Φ given in (6.24) are those which make the denominator of (6.25) zero and we have already determined what the value of this expression is in this case. These are given by (4.16) for $m-\Phi = 0$ and by (4.17) for $m-\Phi = \pm (1/2\tau_d)$. Hence, for $m = \Phi$,

$$\bar{h}_m = 2\tau_c + \tau_d ,$$

and for $m = \Phi \pm (1/2\tau_d)$,

$$\bar{h}_m = (\tau_d/2) \cos(\pi\tau_c/\tau_d).$$

Similar reasoning applies to the derivative filters. In Chapter VII, a sample program and some tabulated results for smoothing and interpolation; smoothing, first derivative, and interpolation; and smoothing, second derivative, and interpolation are given. Values of Φ used there are .25 and .5. This corresponds to interpolation for values one-quarter and one-half the length of the sampling interval from known values, respectively. A check for the special cases in (6.24) is included in the program.

CHAPTER VII

APPLICATIONS

7.0 EDITING AND DETERMINATION OF DIGITAL FILTER PARAMETERS.

In order to apply a digital filter to a set of data $\{g_m\}$, we assume that the data values are obtained by taking equally spaced samples of a function g(t) which satisfies the three conditions of section 5.0. A variety of problems may arise from the methods used to obtain the samples, and editing may be necessary. Common problems are missing values and "bad" values, i.e., values grossly in error. Since these can affect the output considerably, it is important to replace them in some manner. The common practice is to consider the "bad" values as missing values and then replace each missing value by linear interpolation between the nearest data values on each side of the missing value. (See [11]).

Next, the following parameters must be determined:

A. The largest frequency, f_β, which is present in the data. This is commonly found by visually determining the shortest period in the data.

B. The sampling frequency, f_s, which must be at least $2f_\beta$.

C. The cut-off frequency, f_c, which is chosen to be at least as great as the highest frequency of interest present in the data.

D. The termination frequency, f_T. This should be chosen such that either, (1) no frequencies present in the data are in the interval (f_c,f_T) or, (2) frequencies appearing in (f_c,f_T) have no significant amplitude.

E. The value of N and hence the number of weights, 2N+1, of the filter.

From the above, the corresponding frequency ratios may be found from $\tau = \frac{f}{f_s}$. That is, $\tau_c = \frac{f_c}{f_s}$, $\tau_T = \frac{f_T}{f_s}$, $\tau_d = \frac{\Delta f}{f_s}$.

7.1 EMPIRICAL ERROR BOUNDS FOR MARTIN-GRAHAM FILTERS

Empirical error bounds are found by recovering the digital filter's transfer function, i.e., computing

$$H_N(f_j) = \sum_{n=-N}^{N} h_n \exp(2n\pi i f_j/f_s)$$

$j=1,2, \ldots,k$, for various values of the parameters of Section 7.0. The recovered values are then compared with the designed or ideal transfer function values at the f_j. An expression for the error ϵ is then determined in terms of N and the other parameters.

The following error bounds were obtained by transfer function recoveries and comparison with bounds obtained by the method of Section 4.4.

I. Martin-Graham smoothing filter.

For a maximum error ϵ [see (3.43)] of about .01, take

$$N \geq \frac{1.25}{\tau_d} = \frac{1.25 f_s}{\Delta f} \tag{7.1}$$

This gives a maximum error of 1% (ϵ referred to unity) between the actual transfer function and the designed transfer function. Note that the error does not change with τ_c, τ_d held constant.

The bound given by the method of Section 4.4 was compared with the results of computation with τ_c values ranging from .025 to .2, τ_d values ranging from .021 to .11, and N values up to 100. It was found to be about 5 times too large. Hence, in terms of the frequency ratio,

$$\epsilon \doteq \frac{1}{5\pi} \log \frac{4N^2\tau_d^{\ 2}}{4N^2\tau_d^{\ 2}-1} \tag{7.2}$$

where "log" denotes the natural logarithm.

II. Martin-Graham first derivative filter.

Comparison of recoveries for $f \geq f_T$, i.e., where $Y^1(f)$ is ideally zero, and the bound obtained by the method of Section 4.4 yielded, over the same range of frequencies ratio and N values given above, the expression

$$\epsilon' \doteq \frac{f_s}{4} \left[(\tau_c + \tau_T) \log \frac{4N^2\tau_d^{\,2}}{4N^2\tau_d^{\,2} - 1} + \frac{2}{\pi N(4N^2\tau_d^{\,2} - 1)} \right] . \qquad (7.3)$$

III. Martin-Graham second derivative filter.

As above, the following expression was found

$$\epsilon'' \doteq \frac{f_s^{\,2}}{2} \left[\pi(\tau_c^2 + \tau_T^2) \log \frac{4N^2\tau_d^{\,2}}{4N^2\tau_d^{\,2} - 1} + \frac{\tau_c + \tau_T}{N(4N^2\tau_d^{\,2} - 1)} \right]. \qquad (7.4)$$

IV. Martin-Graham band-pass filters.

The error can be as much as the sum of the errors in the low-pass filters from which the band-pass filter is derived (see Section 4.6). Hence, in band-pass smoothing the error may be twice that obtained with a low-pass smoothing filter having the same roll-off length Δf.

The values of ϵ' given by (7.3) become too large for small τ_d, but are still useable for $\tau_d = .021$. The values of ϵ'' given by (7.4) are too small for large τ_d and small τ_c. The actual value may be as much as $\frac{4}{3}\epsilon''$ for τ_d values from .07 to .11 and τ_c values of .025 to .07. However, it is still useable. ϵ' and ϵ'' are values for the error on the rejection band $|f| \geq f_T$ ($|\tau| \geq \tau_T$). The error on the pass-band $|f| \leq f_c$ ($|\tau| \leq \tau_c$) is essentially the same. For the first derivative filter, the amplitude at f_c ideally is $2\pi f_c = 2\pi f_s \tau_c$. For an error of 1 % of $2\pi f_c$, we need

$$\epsilon' = .01(2\pi f_s \tau_c)$$
$$= (.02)\pi\tau_c f_s$$
$$= (.08)(\pi\tau_c)\frac{f_s}{4}.$$

Comparing with (7.3), we see that N must be taken such that

$$(.08)\pi\tau_c \doteq (\tau_c + \tau_T) \log \frac{4N^2\tau_d^{\,2}}{4N^2\tau_d^{\,2}-1} + \frac{2}{\pi N(2N^2\tau_d^{\,2}-1)} . \tag{7.5}$$

For the second derivative, the amplitude at f_c ideally is $4\pi^2 f_c^{\,2} = 4\pi^2 f_s \tau_c^{\,2}$. Similar to the above, we find that for an error of 1% of $4\pi^2 f_c^{\,2}$, we need to take N such that

$$(.08)\pi^2\tau_c^{\,2} \doteq \pi(\tau_c^{\,2} + \tau_T^{\,2}) \log \frac{4N^2\tau_d^{\,2}}{4N^2\tau_d^{\,2}-1} + \frac{\tau_c + \tau_T}{N(4N^2\tau_d^{\,2}-1)} \tag{7.6}$$

7.2 SAMPLE PROGRAM AND RESULTS FOR THE MARTIN-GRAHAM SMOOTHING AND DERIVATIVE FILTERS

When the data has been edited and the parameters of Section 7.0 have been determined, the filtering can be performed. The weights of the filter are computed from the appropriate weight expression and (3.40). If the data has a polynomial content, then these weights are constrained appropriately (see Section 4.5, Section 5.1, and Appendix A). Finally, the output of the filter is computed using (3.45).

As an example, we take as the input function

$$g(t) = a_1 \cos 2\pi f t_1 t + a_2 \sin 2\pi f_2 t + a_3 \cos 2\pi f_3 t + a_4 \tag{7.7}$$

Using the Martin-Graham filters, in this section we shall perform the operations of smoothing; smoothing and finding the first derivative; and smoothing and finding the second derivative. The same function will be used in the next sections for interpolation and integration. The time-sampled version of g(t) is

$$g_n = g(n/f_s) = a_1 \cos 2\pi f_1 n/f_s + a_2 \sin 2\pi f_2 n/f_s + a_3 \cos 2\pi f_3 n/f_s + a_4,$$

and going to the frequency ratio, $\tau = f/f_s$, we have

$$g_n = a_1 \cos 2\pi n\tau_1 + a_2 \sin 2\pi n\tau_2 + a_3 \cos 2\pi n\tau_3 + a_4. \qquad (7.8)$$

The following program was run in extended precision (10 digits) on the IBM 1130 computer. The program is sectioned by comment cards which state what each part of the program computes. Table 7.1 gives the frequencies used for the various runs. In each run, the input component with frequency f_3 is to be removed by the filters. Table 7.2 gives the frequency ratios, coefficients of the terms in (7.8), coefficients of the terms of the desired output, and the parameter values for each run. The value of N used, and hence the number of weights for each run, is given by the last two digits in the run number. That is, Run 2.20 reads, "Run 2 with N = 20". The symbolism selected for the program is as follows:

FS: The sampling rate f_s.

HO: The center smoothing weight, h_o.

DDHO: The center smoothing and second derivative weight, y_o^2.

H(I): The smoothing weights h_i, $i \neq 0$.

DH(I): The smoothing and first derivative weights y_i^1, $i \neq 0$.

DDH(I): The smoothing and second derivative weights y_i^2, $i \neq 0$.

TF1: The recovered transfer function for smoothing.

TF2: The recovered transfer function for smoothing and the first derivative (divided by 2π).

TF3: The recovered transfer function for smoothing and the second derivative (divided by $4\pi^2$).

Z(I): The input samples g_i.

R(I): The desired smoothed output (used for headings for both desired and actual outputs).

DR(I): The desired smoothed first derivative output (used for headings for both desired and actual outputs).

DDR(I): The desired smoothed second derivative output (used for headings for both desired and actual outputs).

The a_i, τ_i, etc. are denoted by AA, RA, etc., AB, RB, etc. in the program.

The following weight properties are used in the program:

1) Smoothing: $h_{-n} = h_n$,

2) First derivative: $y^1_{-n} = -y^1_n$,

3) Second derivative: $y^2_{-n} = y^2_n$.

The results for each run follow the program. In each run, one term of the input is to be removed by filtering. The desired output is obtained by taking the coefficient of that term in (7.8) to be zero. Some of the frequencies were chosen near cut-off and termination of the filters. This is where the largest error is obtained in the transfer functions. See Figure 7.1 and Figure 7.2 for graphs of the recovered transfer functions, the input, and the smoothing filter output. The output of the smoothing filter is so near the desired output that they coincide in the scale of the figure.

Tabular values for Run 3.20, Run 4.30 and Run 5.30 are used in the next section on interpolation and are not used in this section. The values for Run 1.20 and Run 2.30 are used in both the sections.

FREQ.	RUN 1.20	RUN 2.30	RUN 3.20	RUN 4.30	RUN 5.30
f_1	.5	.5	.5	.5	.5
f_2	.9	2.0	.9	2.0	.9
f_3	2.0	4.0	2.0	4.0	2.0
f_s	10.0	10.0	10.0	10.0	10.0
f_c	1.0	2.0	1.0	2.0	1.0
Δf	.6	.6	.6	.6	.4
f_T	1.6	2.6	1.6	2.6	1.4

Table 7.1

PROGRAM SYMBOLS AND PARAMETER VALUES

FREQ. RATIO	PROGRAM SYMBOLS	PARAMETER VALUES RUN 1.20	RUN 2.30	RUN 3.20	RUN 4.30	RUN 5.30
τ_c	TC	.1	.2	.1	.2	.1
τ_T	RT	.16	.26	.16	.26	.14
τ_d	TD	.06	.06	.06	.06	.04
τ_1	RA	.05	.05	.05	.05	.05
τ_2	RB	.09	.2	.09	.2	.09
τ_3	RC	.2	.4	.2	.4	.2
a_1	AA	1.0	1.0	1.0	1.0	1.0
a_2	AB	1.0	2.0	2.0	2.0	1.0
a_3	AC	.5	1.5	.5	1.5	.5
a_4	AD	.5	1.0	.5	1.0	.5
	BA*	1.0	1.0	1.0	1.0	1.0
	BB	1.0	2.0	1.0	2.0	1.0
	BC	0.0	0.0	0.0	0.0	0.0
	BD	.5	1.0	.5	1.0	.5
Φ	TX	.25	.25	.5	.5	.5

Table 7.2

*BA, BB, BC, and BD are the coefficients for the desired outputs.

```
C         SAMPLE PROGRAM, WEIGHTS, RECOVERY, FILTERING OF DATA,
C         SMOOTHING, FIRST AND SECOND DERIVATIVES
C         IBM 1130 FORTRAN IV LANGUAGE
      DIMENSION H(30),DH(30),DDH(30),Z(101)
      DIMENSION R(40),DR(40),DDR(40)
    1 FORMAT (4F10.0)
    2 FORMAT(A4,A4)
    3 FORMAT(1H1,5X,A4,A4,30H  RECOVERED TRANSFER FUNCTIONS/)
    4 FORMAT(5X,1HF,7X,3HTF1,13X,3HTF2,13X,3HTF3)
    5 FORMAT(/////6X,A4,A4,28H  INPUT ON RANGE OF INTEREST/)
    6 FORMAT(19X,1HT,8X,5HZI(T))
    7 FORMAT(1H1,5X,A4,A4,16H  DESIRED OUTPUT/)
    8 FORMAT(5X,1HT,7X,4HR(T),11X,5HDR(T),10X,6HDDR(T))
    9 FORMAT(1H1,5X,A4,A4,15H  ACTUAL OUTPUT/)
   10 FORMAT(1X,F7.3,3E15.7)
   11 FORMAT(16X,F7.3,E15.7)
      P=3.14159
C
C         READ PROBLEM PARAMETERS
C
   12 READ(2,2) RUN,XNUM
      READ(2,1) XN,TC,TD
      READ(2,1) RA,RB,RC,FS
      READ(2,1) AA,AB,AC,AD
      READ(2,1) BA,BB,BC,BD
      N=XN
C
C         COMPUTATION OF THE UNCONSTRAINED WEIGHTS
C         THE FACTORS -FS AND FS**2 OF THE FIRST AND SECOND
C         DERIVATIVE WEIGHTS WILL BE INTRODUCED LATER
C
C         COMPUTATION OF THE CENTER WEIGHTS
C
      HO=2.*TC+TD
      RT=TC+TD
      DDHO=8.*(TD**2*(RT+TC)-P**2*(RT**3+TC**3)/6.)
C
C         COMPUTATION OF THE REMAINING WEIGHTS
C
      DO 13 I=1,N
      X=I
      H(I)=SIN(HO*X*P)/(X*P)
      H(I)=H(I)*COS(TD*X*P)/(1.-4.*TD**2*X**2)
      DH(I)=RT*COS(2.*RT*X*P)+TC*COS(2.*TC*X*P)
      DH(I)=DH(I)-H(I)*(1.-12.*TD**2*X**2)
      DH(I)=DH(I)/(X*(1.-4.*TD**2*X**2))
      DDH(I)=-2.*DH(I)*(1.-12.*TD**2*X**2)+24.*TD**2*X*H(I)
      DDH(I)=DDH(I)-2.*P*TC**2*SIN(2.*TC*P*X)
      DDH(I)=DDH(I)-2.*P*RT**2*SIN(2.*RT*P*X)
   13 DDH(I)=DDH(I)/(X*(1.-4.*TD**2*X**2))
C
C         COMPUTATION OF CONSTRAINED SMOOTHING WEIGHTS
C         DERIVATIVE WEIGHTS ARE NOT CONSTRAINED
C
```

```
      SA=0
      DO 14 I=1,N
   14 SA=SA+H(I)
      FN=2*N+1
      SA=1.-(HO+2.*SA)
      SA=SA/FN
      DO 15 I=1,N
   15 H(I)=H(I)+SA
      HO=HO+SA
C
C        RECOVERY OF ALL TRANSFER FUNCTIONS
C
      WRITE(3,3) RUN,XNUM
      WRITE(3,4)
      ZA=100.*TC-1.
      ZB=ZA+6.
      ZC=ZB+100.*TD
      ZD=ZC+6.
      DO 26 K=1,57
      TF1=0.
      TF2=0.
      TF3=0.
      X=K
      IF(X-ZA) 16,17,17
   16 Y=K-1
      Y=.01*Y
      GO TO 24
   17 IF(X-ZB) 18,19,19
   18 Y=Y+.005
      GO TO 24
   19 IF(X-ZC) 20,21,21
   20 Y=Y+.01
      GO TO 24
   21 IF(X-ZD) 22,23,23
   22 Y=Y+.005
      GO TO 24
   23 Y=Y+.01
   24 CONTINUE
      DO 25 I=1,N
      X=I
      X=2.*X*P*Y
      TF1=TF1+2.*H(I)*COS(X)
      TF2=TF2+2.*DH(I)*SIN(X)
   25 TF3=TF3+2.*DDH(I)*COS(X)
      TF1=HO+TF1
      TF2=-FS*TF2/(2.*P)
      TF3=FS**2*(DDHO+TF3)/(4.*P**2)
      Y1=Y*FS
   26 WRITE(3,10) Y1,TF1,TF2,TF3
C
C        GENERATION OF SAMPLE INPUT DATA
C
      MA=N+1
      M=2*N+40
```

```
      DO 27 I=1,M
      T=I-MA
      CA=COS(2.*P*RA*T)
      S= SIN(2.*P*RB*T)
      CC=COS(2.*P*RC*T)
   27 Z(I)=AA*CA+AB*S+AC*CC+AD
C
C         WRITE INPUT ON THE RANGE OF INTEREST
C
      WRITE(3,5) RUN,XNUM
      WRITE(3,6)
      DO 28 I=1,40
      T=I-1
      Y=T/FS
      J=I+20
   28 WRITE(3,11) Y,Z(J)
C
C         COMPUTATION OF DESIRED OUTPUTS
C
      WRITE(3,7) RUN,XNUM
      WRITE(3,8)
      DO 29 I=1,40
      T=I-1
      CA=COS(2.*P*RA*T)
      S=SIN(2.*P*RB*T)
      CC=COS(2.*P*RC*T)
      R(I)=BA*CA+BB*S+BC*CC+BD
      DDR(I)=-4.*(P*FS)**2*(BA*RA**2*CA+BB*RB**2*S+BC*RC**2*CC)
      CA=SIN(2.*P*RA*T)
      S= COS(2.*P*RB*T)
      CC=SIN(2.*P*RC*T)
      DR(I)=-2.*P*FS*(BA*RA*CA-BB*RB*S+BC*RC*CC)
      Y=T/FS
   29 WRITE(3,10) Y,R(I),DR(I),DDR(I)
C
C         COMPUTATION OF THE ACTUAL OUTPUT
C
      WRITE(3,9) RUN,XNUM
      WRITE(3,8)
      DO 31 K=1,40
      MB=K-1
      MC=N+1
      SA=0.
      SB=0.
      SC=0.
      T=MB
      T=T/FS
      DO 30 I=1,N
      KA=MC-I
      KB=I+MB
      KC=MC+I+MB
      SA=SA+H(KA)*Z(KB)+H(I)*Z(KC)
      SB=SB-DH(KA)*Z(KB)+DH(I)*Z(KC)
   30 SC=SC+DDH(KA)*Z(KB)+DDH(I)*Z(KC)
```

```
      KD=MC+MB
      SA=HO*Z(KD)+SA
C
C        THE FACTORS -FS AND FS**2 ARE INTRODUCED HERE
C
      SB=-FS*SB
      SC=FS**2*(DDHO*Z(KD)+SC)
   31 WRITE(3,10) T,SA,SB,SC
      GO TO 12
   32 CALL EXIT
      END
```

RUN 1.20 RECOVERED TRANSFER FUNCTIONS

F	TF1	TF2	TF3
0.000	0.1000000E 01	0.0000000E 00	-0.1311312E-02
0.100	0.9979708E 00	0.1064420E 00	-0.1094588E-01
0.200	0.9951461E 00	0.2027441E 00	-0.3935141E-01
0.300	0.9968977E 00	0.2944890E 00	-0.8744310E-01
0.400	0.1002811E 01	0.3948321E 00	-0.1586992E 00
0.500	C.1005624E 01	0.5038771E 00	-0.2532428E 00
0.600	0.1000225E 01	0.6070451E 00	-0.3641147E 00
0.700	0.9928568E 00	0.6978028E 00	-0.4871102E 00
0.750	0.9926648E 00	0.7428995E 00	-0.5559792E 00
0.800	0.9962421E 00	0.7916221E 00	-0.6327757E 00
0.850	0.1002821E 01	0.8453967E 00	-0.7189744E 00
0.900	0.1009538E 01	0.9027563E 00	-0.8136282E 00
0.950	0.1011476E 01	0.9585889E 00	-0.9120918E 00
1.000	0.1002304E 01	0.1004446E 01	-0.1005558E 01
1.100	0.9256323E 00	0.1025274E 01	-0.1127162E 01
1.200	0.7502907E 00	0.9021662E 00	-0.1081105E 01
1.300	0.5007748E 00	0.6448855E 00	-0.8382299E 00
1.400	0.2500291E 00	0.3447629E 00	-0.4841259E 00
1.500	0.7292437E-01	0.1126013E 00	-0.1697427E 00
1.600	-0.3242715E-02	0.1952202E-02	-0.2021546E-02
1.650	-0.1137795E-01	-0.1382274E-01	0.2441104E-01
1.700	-0.8413781E-02	-0.1354452E-01	0.2445817E-01
1.750	-0.1139287E-02	-0.5759337E-02	0.1072304E-01
1.800	0.5260366E-02	0.2660113E-02	-0.5231769E-02
1.850	0.7969893E-02	0.7606103E-02	-0.1545016E-01
1.900	0.6575732E-02	0.7890945E-02	-0.1675595E-01
2.000	-0.2177529E-02	-0.3734135E-04	-0.4022046E-03
2.100	-0.5887906E-02	-0.5253134E-02	0.1261440E-01
2.200	-0.5841164E-03	-0.1658775E-02	0.4961693E-02
2.300	0.4631496E-02	0.3213084E-02	-0.8373012E-02
2.400	0.2607105E-02	0.2442886E-02	-0.7743576E-02
2.500	-0.2762178E-02	-0.1478645E-02	0.3784080E-02
2.600	-0.3631360E-02	-0.2490295E-02	0.8551128E-02
2.700	0.6433748E-03	0.1206043E-03	0.7060655E-03
2.800	0.3587548E-02	0.2025167E-02	-0.7317515E-02
2.900	0.1261190E-02	0.7712017E-03	-0.4210880E-02
3.000	-0.2625765E-02	-0.1304567E-02	0.4600845E-02
3.100	-0.2540651E-02	-0.1199737E-02	0.6161725E-02
3.200	0.1093053E-02	0.5405387E-03	-0.1195917E-02
3.300	0.2954239E-02	0.1225777E-02	-0.6406264E-02
3.400	0.5538661E-03	0.9670234E-04	-0.2188057E-02
3.500	-0.2492955E-02	-0.9674295E-03	0.4918593E-02
3.600	-0.1872954E-02	-0.4999767E-03	0.4641884E-02
3.700	0.1361831E-02	0.5857003E-03	-0.2248612E-02
3.800	0.2536092E-02	0.6647356E-03	-0.5593535E-02
3.900	0.7613562E-04	-0.2048472E-03	-0.7416088E-03
4.000	-0.2406083E-02	-0.6356356E-03	0.5042111E-02
4.100	-0.1394519E-02	-0.9081311E-04	0.3403408E-02
4.200	0.1572296E-02	0.4840050E-03	-0.3103943E-02
4.300	0.2227681E-02	0.2544710E-03	-0.4999321E-02
4.400	-0.3086174E-03	-0.3063226E-03	0.3360198E-03
4.500	-0.2359755E-02	-0.3104730E-03	0.5054315E-02
4.600	-0.1006163E-02	0.1595017E-03	0.2348410E-02

4.700	0.1772230E-02	0.3071723E-03	-0.3751266E-02
4.800	0.1984118E-02	-0.5821355E-04	-0.4346328E-02
4.900	-0.6548489E-03	-0.2841886E-03	0.1405878E-02
5.000	-0.2343314E-02	-0.1845173E-07	0.5127026E-02

RUN 1.20 INPUT ON RANGE OF INTEREST

T	ZI(T)
0.000	0.2000000E 01
0.100	0.2141391E 01
0.200	0.1809336E 01
0.300	0.1675391E 01
0.400	0.1734038E 01
0.500	0.1309020E 01
0.600	0.9680893E-01
0.700	-0.1221256E 01
0.800	-0.1695813E 01
0.900	-0.1226329E 01
1.000	-0.5877891E 00
1.100	-0.3593391E 00
1.200	-0.2317749E 00
1.300	0.3840030E 00
1.400	0.1343508E 01
1.500	0.1809017E 01
1.600	0.1331661E 01
1.700	0.4959050E 00
1.800	0.2199592E 00
1.900	0.6369729E 00
2.000	0.1048940E 01
2.100	0.9681456E 00
2.200	0.7791751E 00
2.300	0.1109043E 01
2.400	0.1807841E 01
2.500	0.2000006E 01
2.600	0.1189845E 01
2.700	-0.6648854E-01
2.800	-0.8388498E 00
2.900	-0.9339736E 00
3.000	-0.9510520E 00
3.100	-0.1265121E 01
3.200	-0.1398078E 01
3.300	-0.6797079E 00
3.400	0.7135751E 00
3.500	0.1808997E 01
3.600	0.1961560E 01
3.700	0.1559595E 01
3.800	0.1386260E 01
3.900	0.1542770E 01

RUN 1.20 DESIRED OUTPUT

T	R(T)	DR(T)	DDR(T)
0.000	0.1500000E 01	0.5654862E 01	-0.9869587E 01
0.100	0.1986882E 01	0.3803755E 01	-0.2652090E 02
0.200	0.2213843E 01	0.5611491E 00	-0.3691872E 02
0.300	0.2079900E 01	-0.3250332E 01	-0.3752652E 02
0.400	0.1579532E 01	-0.6592364E 01	-0.2768897E 02
0.500	0.8090205E 00	-0.8519679E 01	-0.9881665E 01
0.600	-0.5770258E-01	-0.8465039E 01	0.1100223E 02
0.700	-0.8167500E 00	-0.6412636E 01	0.2911167E 02
0.800	-0.1291302E 01	-0.2906222E 01	0.3939568E 02
0.900	-0.1380833E 01	0.1110859E 01	0.3911847E 02
1.000	-0.1087789E 01	0.4574855E 01	0.2866559E 02
1.100	-0.5138531E 00	0.6614497E 01	0.1139459E 02
1.200	0.1727297E 00	0.6801981E 01	-0.7420417E 01
1.300	0.7885156E 00	0.5265874E 01	-0.2222074E 02
1.400	0.1189006E 01	0.2632792E 01	-0.2886447E 02
1.500	0.1309017E 01	-0.1822216E 00	-0.2587040E 02
1.600	0.1177144E 01	-0.2269908E 01	-0.1482174E 02
1.700	0.9004082E 00	-0.3013099E 01	0.1905620E 00
1.800	0.6244733E 00	-0.2275657E 01	0.1390524E 02
1.900	0.4824740E 00	-0.4355370E 00	0.2158624E 02
2.000	0.5489405E 00	0.1747413E 01	0.2054288E 02
2.100	0.8136265E 00	0.3386321E 01	0.1099689E 02
2.200	0.1183676E 01	0.3763698E 01	-0.3976525E 01
2.300	0.1513559E 01	0.2575110E 01	-0.1941627E 02
2.400	0.1653344E 01	0.4225793E-01	-0.3004919E 02
2.500	0.1500006E 01	-0.3141522E 01	-0.3197752E 02
2.600	0.1035324E 01	-0.6017803E 01	-0.2394987E 02
2.700	0.3380115E 00	-0.7658254E 01	-0.7814574E 01
2.800	-0.4343325E 00	-0.7456880E 01	0.1199203E 02
2.900	-0.1088467E 01	-0.5328023E 01	0.2976937E 02
3.000	-0.1451052E 01	-0.1747550E 01	0.4028182E 02
3.100	-0.1419645E 01	0.2377005E 01	0.4035951E 02
3.200	-0.9935802E 00	0.5968716E 01	0.2987515E 02
3.300	-0.2751891E 00	0.8096265E 01	0.1179374E 02
3.400	0.5590838E 00	0.8245612E 01	-0.8721251E 01
3.500	0.1308997E 01	0.6465511E 01	-0.2586990E 02
3.600	0.1807033E 01	0.3343007E 01	-0.3496410E 02
3.700	0.1964092E 01	-0.1825450E 00	-0.3382345E 02
3.800	0.1790780E 01	-0.3108737E 01	-0.2339037E 02
3.900	0.1388281E 01	-0.4672874E 01	-0.7379217E 01

RUN 1.20 ACTUAL OUTPUT

T	R(T)	DR(T)	DDR(T)
0.000	0.1504535E 01	0.5672180E 01	-0.1003143E 02
0.100	0.1997006E 01	0.3810961E 01	-0.2674775E 02
0.200	0.2227905E 01	0.5542731E 00	-0.3717136E 02
0.300	0.2093550E 01	-0.3272280E 01	-0.3776334E 02
0.400	0.1588283E 01	-0.6626684E 01	-0.2786724E 02
0.500	0.8108793E 00	-0.8560511E 01	-0.9959752E 01
0.600	-0.6214910E-01	-0.8504871E 01	0.1104908E 02
0.700	-0.8261282E 00	-0.6444131E 01	0.2927188E 02
0.800	-0.1304341E 01	-0.2923855E 01	0.3962049E 02
0.900	-0.1395387E 01	0.1109594E 01	0.3934507E 02
1.000	-0.1100108E 01	0.4588866E 01	0.2884398E 02
1.100	-0.5201376E 00	0.6639421E 01	0.1149700E 02
1.200	0.1736555E 00	0.6831545E 01	-0.7405311E 01
1.300	0.7944491E 00	0.5293857E 01	-0.2229047E 02
1.400	0.1196451E 01	0.2654761E 01	-0.2899620E 02
1.500	0.1315645E 01	-0.1680404E 00	-0.2602011E 02
1.600	0.1182057E 01	-0.2262730E 01	-0.1494237E 02
1.700	0.9028076E 00	-0.3010334E 01	0.1226907E 00
1.800	0.6233748E 00	-0.2274032E 01	0.1388026E 02
1.900	0.4782478E 00	-0.4324276E 00	0.2157488E 02
2.000	0.5444044E 00	0.1752765E 01	0.2051726E 02
2.100	0.8125589E 00	0.3392249E 01	0.1093810E 02
2.200	0.1187912E 01	0.3766631E 01	-0.4081605E 01
2.300	0.1521807E 01	0.2571003E 01	-0.1957196E 02
2.400	0.1662799E 01	0.2825790E-01	-0.3023803E 02
2.500	0.1508456E 01	-0.3165883E 01	-0.3215459E 02
2.600	0.1041303E 01	-0.6050140E 01	-0.2405959E 02
2.700	0.3396478E 00	-0.7693564E 01	-0.7819777E 01
2.800	-0.4391971E 00	-0.7488450E 01	0.1209409E 02
2.900	-0.1100232E 01	-0.5349008E 01	0.2995409E 02
3.000	-0.1466836E 01	-0.1752902E 01	0.4051224E 02
3.100	-0.1434570E 01	0.2388952E 01	0.4059167E 02
3.200	-0.1003779E 01	0.5995729E 01	0.3005731E 02
3.300	-0.2794016E 00	0.8132917E 01	0.1187637E 02
3.400	0.5605205E 00	0.8284772E 01	-0.8762754E 01
3.500	0.1315625E 01	0.6500052E 01	-0.2601960E 02
3.600	0.1817954E 01	0.3367375E 01	-0.3517496E 02
3.700	0.1976637E 01	-0.1711107E 00	-0.3404369E 02
3.800	0.1800806E 01	-0.3109664E 01	-0.2358241E 02
3.900	0.1392695E 01	-0.4682742E 01	-0.7520319E 01

RUN 2.30 RECOVERED TRANSFER FUNCTIONS

F	TF1	TF2	TF3
0.000	0.1000000E 01	0.0000000E 00	-0.5835911E-02
0.100	0.9990799E 00	0.1001930E 00	-0.8069360E-02
0.200	0.9993164E 00	0.1997105E 00	-0.3547989E-01
0.300	0.1001067E 01	0.3001236E 00	-0.9505551E-01
0.400	0.1000089E 01	0.4003138E 00	-0.1611912E 00
0.500	0.9988623E 00	0.4994860E 00	-0.2439548E 00
0.600	0.1000642E 01	0.6000685E 00	-0.3628578E 00
0.700	0.1000775E 01	0.7006694E 00	-0.4943077E 00
0.800	0.9988035E 00	0.7992988E 00	-0.6339328E 00
0.900	0.9999858E 00	0.8996904E 00	-0.8095680E 00
1.000	0.1001316E 01	0.1001210E 01	-0.1006855E 01
1.100	0.9990952E 00	0.1099378E 01	-0.1205643E 01
1.200	0.9991569E 00	0.1198866E 01	-0.1435493E 01
1.300	0.1001651E 01	0.1301818E 01	-0.1698287E 01
1.400	0.9997751E 00	0.1400028E 01	-0.1959239E 01
1.500	0.9981467E 00	0.1497367E 01	-0.2240645E 01
1.600	0.1001773E 01	0.1602381E 01	-0.2568333E 01
1.700	0.1001004E 01	0.1701849E 01	-0.2895519E 01
1.750	0.9981318E 00	0.1747196E 01	-0.3054255E 01
1.800	0.9964056E 00	0.1793940E 01	-0.3222829E 01
1.850	0.9980397E 00	0.1846379E 01	-0.3411917E 01
1.900	0.1002675E 01	0.1904661E 01	-0.3620667E 01
1.950	0.1006047E 01	0.1961278E 01	-0.3830481E 01
2.000	0.1000527E 01	0.2000879E 01	-0.4006828E 01
2.100	0.9296782E 00	0.1952890E 01	-0.4095702E 01
2.200	0.7532773E 00	0.1657016E 01	-0.3643926E 01
2.300	0.5003579E 00	0.1150323E 01	-0.2652270E 01
2.400	0.2464820E 00	0.5920952E 00	-0.1418133E 01
2.500	0.6976558E-01	0.1745989E 00	-0.4318729E 00
2.600	0.8834037E-04	-0.4770605E-03	-0.4970335E-02
2.650	-0.5385111E-02	-0.1465639E-01	0.3295904E-01
2.700	-0.2528228E-02	-0.6543208E-02	0.1714038E-01
2.750	0.1466907E-02	0.4777730E-02	-0.7799526E-02
2.800	0.2877506E-02	0.8649288E-02	-0.1751051E-01
2.850	0.1536750E-02	0.4309431E-02	-0.9930184E-02
2.900	-0.6649908E-03	-0.2629067E-02	0.3551889E-02
3.000	-0.1271256E-02	-0.3980459E-02	0.7884209E-02
3.100	0.1174813E-02	0.4522064E-02	-0.6962965E-02
3.200	0.1751021E-03	0.1374801E-03	-0.1050756E-02
3.300	-0.9239686E-03	-0.3733852E-02	0.5517542E-02
3.400	0.4036508E-03	0.2293624E-02	-0.2429389E-02
3.500	0.4551198E-03	0.1702539E-02	-0.2622343E-02
3.600	-0.5630087E-03	-0.3119310E-02	0.3316056E-02
3.700	-0.3001862E-04	0.5204254E-03	0.2529160E-04
3.800	0.4488020E-03	0.2477033E-02	-0.2605461E-02
3.900	-0.2276487E-03	-0.2106855E-02	0.1393238E-02
4.000	-0.2248208E-03	-0.8951134E-03	0.1168418E-02
4.100	0.2968816E-03	0.2569953E-02	-0.1730209E-02
4.200	0.1809746E-04	-0.8623488E-03	0.5417549E-04
4.300	-0.2370779E-03	-0.1864251E-02	0.1318255E-02
4.400	0.1030652E-03	0.2077714E-02	-0.6394788E-03
4.500	0.1345563E-03	0.4006935E-03	-0.6029474E-03
4.600	-0.1337616E-03	-0.2291661E-02	0.7678554E-03

4.700	-0.5112099E-04	0.1153040E-02	0.1630979E-03
4.800	0.1197832E-03	0.1465714E-02	-0.5972868E-03
4.900	0.9760027E-05	-0.2141209E-02	-0.3583842E-04
5.000	-0.1094264E-03	-0.1822910E-06	0.4454670E-03

RUN 2.30 INPUT ON RANGE OF INTEREST

T	ZI(T)
0.000	0.1500021E 01
0.100	0.7375208E 00
0.200	0.1830090E 01
0.300	-0.2998620E 00
0.400	-0.2424638E 01
0.500	0.2500011E 01
0.600	0.1997600E 01
0.700	0.3226885E 01
0.800	0.1096962E 01
0.900	-0.1164579E 01
1.000	0.3500000E 01
1.100	0.2639645E 01
1.200	0.3448110E 01
1.300	0.8757551E 00
1.400	-0.1806630E 01
1.500	0.2499990E 01
1.600	0.1379579E 01
1.700	0.2051303E 01
1.800	-0.5210227E 00
1.900	-0.3066717E 01
2.000	0.1499978E 01
2.100	0.7375435E 00
2.200	0.1830061E 01
2.300	-0.2997712E 00
2.400	-0.2424694E 01
2.500	0.2499964E 01
2.600	0.1997619E 01
2.700	0.3226855E 01
2.800	0.1097054E 01
2.900	-0.1164631E 01
3.000	0.3499957E 01
3.100	0.2639671E 01
3.200	0.3448087E 01
3.300	0.8758545E 00
3.400	-0.1806676E 01
3.500	0.2499953E 01
3.600	0.1379608E 01
3.700	0.2051281E 01
3.800	-0.5209245E 00
3.900	-0.3066766E 01

RUN 2.30 DESIRED OUTPUT

T	R(T)	DR(T)	DDR(T)
0.000	0.2000000E 01	0.2513272E 02	-0.9869587E 01
0.100	0.3853168E 01	0.6795659E 01	-0.3097555E 03
0.200	0.2984591E 01	-0.2217934E 02	-0.1936235E 03
0.300	0.4122205E 00	-0.2287444E 02	0.1798363E 03
0.400	-0.5930976E 00	0.4778507E 01	0.2973196E 03
0.500	0.9999907E 00	0.2199113E 02	0.1663130E-02
0.600	0.2593093E 01	0.4778758E 01	-0.2973186E 03
0.700	0.1587798E 01	-0.2287429E 02	-0.1798390E 03
0.800	-0.9845724E 00	-0.2217950E 02	0.1936208E 03
0.900	-0.1853174E 01	0.6795397E 01	0.3097566E 03
1.000	-0.2122949E-04	0.2513271E 02	0.9872940E 01
1.100	0.1951048E 01	0.8737512E 01	-0.2909814E 03
1.200	0.1366572E 01	-0.1848603E 02	-0.1776569E 03
1.300	-0.7633362E 00	-0.1779140E 02	0.1914360E 03
1.400	-0.1211142E 01	0.1075390E 02	0.3034205E 03
1.500	0.9999641E 00	0.2827431E 02	0.5073595E-02
1.600	0.3211115E 01	0.1075467E 02	-0.3034173E 03
1.700	0.2763381E 01	-0.1779092E 02	-0.1914441E 03
1.800	0.6334746E 00	-0.1848648E 02	0.1776488E 03
1.900	0.4892945E-01	0.8736775E 01	0.2909845E 03
2.000	0.1999957E 01	0.2513273E 02	-0.9862882E 01
2.100	0.3853157E 01	0.6796182E 01	-0.3097535E 03
2.200	0.2984628E 01	-0.2217901E 02	-0.1936290E 03
2.300	0.4122592E 00	-0.2287474E 02	0.1798308E 03
2.400	-0.5931057E 00	0.4778004E 01	0.2973217E 03
2.500	0.9999535E 00	0.2199113E 02	0.8317557E-02
2.600	0.2593085E 01	0.4779262E 01	-0.2973166E 03
2.700	0.1587837E 01	-0.2287398E 02	-0.1798445E 03
2.800	-0.9845349E 00	-0.2217983E 02	0.1936153E 03
2.900	-0.1853186E 01	0.6794873E 01	0.3097586E 03
3.000	-0.6375927E-04	0.2513269E 02	0.9879656E 01
3.100	0.1951033E 01	0.8738004E 01	-0.2909793E 03
3.200	0.1366603E 01	-0.1848573E 02	-0.1776623E 03
3.300	-0.7633061E 00	-0.1779173E 02	0.1914306E 03
3.400	-0.1211160E 01	0.1075339E 02	0.3034226E 03
3.500	0.9999163E 00	0.2827430E 02	0.1183093E-01
3.600	0.3211097E 01	0.1075519E 02	-0.3034151E 03
3.700	0.2763411E 01	-0.1779059E 02	-0.1914494E 03
3.800	0.6335058E 00	-0.1848678E 02	0.1776434E 03
3.900	0.4891466E-01	0.8736282E 01	0.2909866E 03

RUN 2.30 ACTUAL OUTPUT

T	R(T)	DR(T)	DDR(T)
0.000	0.1998525E 01	0.2514377E 02	-0.9792138E 01
0.100	0.3853364E 01	0.6805032E 01	-0.3103277E 03
0.200	0.2984187E 01	-0.2219441E 02	-0.1939563E 03
0.300	0.4108270E 00	-0.2287275E 02	0.1800844E 03
0.400	-0.5941806E 00	0.4780036E 01	0.2976198E 03
0.500	0.9996534E 00	0.2200541E 02	-0.1595349E 00
0.600	0.2594722E 01	0.4790205E 01	-0.2981915E 03
0.700	0.1588983E 01	-0.2288864E 02	-0.1805052E 03
0.800	-0.9843770E 00	-0.2217853E 02	0.1935356E 03
0.900	-0.1852824E 01	0.6794853E 01	0.3097560E 03
1.000	0.7791440E-03	0.2514376E 02	0.9473095E 01
1.100	0.1953407E 01	0.8744890E 01	-0.2920075E 03
1.200	0.1368009E 01	-0.1850490E 02	-0.1783759E 03
1.300	-0.7633923E 00	-0.1779494E 02	0.1914036E 03
1.400	-0.1211522E 01	0.1074929E 02	0.3035731E 03
1.500	0.9996268E 00	0.2828213E 02	-0.1561212E 00
1.600	0.3212040E 01	0.1075998E 02	-0.3041426E 03
1.700	0.2763229E 01	-0.1781050E 02	-0.1918297E 03
1.800	0.6318294E 00	-0.1848930E 02	0.1779497E 03
1.900	0.4711614E-01	0.8734235E 01	0.2914379E 03
2.000	0.1998482E 01	0.2514379E 02	-0.9785417E 01
2.100	0.3853352E 01	0.6805556E 01	-0.3103256E 03
2.200	0.2984224E 01	-0.2219408E 02	-0.1939618E 03
2.300	0.4108657E 00	-0.2287305E 02	0.1800790E 03
2.400	-0.5941886E 00	0.4779534E 01	0.2976218E 03
2.500	0.9996162E 00	0.2200541E 02	-0.1528632E 00
2.600	0.2594714E 01	0.4790708E 01	-0.2981894E 03
2.700	0.1589022E 01	-0.2288834E 02	-0.1805107E 03
2.800	-0.9843394E 00	-0.2217886E 02	0.1935301E 03
2.900	-0.1852835E 01	0.6794329E 01	0.3097580E 03
3.000	0.7366263E-03	0.2514375E 02	0.9479817E 01
3.100	0.1953392E 01	0.8745383E 01	-0.2920054E 03
3.200	0.1368040E 01	-0.1850460E 02	-0.1783813E 03
3.300	-0.7633623E 00	-0.1779526E 02	0.1913982E 03
3.400	-0.1211540E 01	0.1074878E 02	0.3035752E 03
3.500	0.9995790E 00	0.2828213E 02	-0.1493541E 00
3.600	0.3212022E 01	0.1076049E 02	-0.3041405E 03
3.700	0.2763259E 01	-0.1781018E 02	-0.1918351E 03
3.800	0.6318605E 00	-0.1848960E 02	0.1779443E 03
3.900	0.4710128E-01	0.8733744E 01	0.2914400E 03

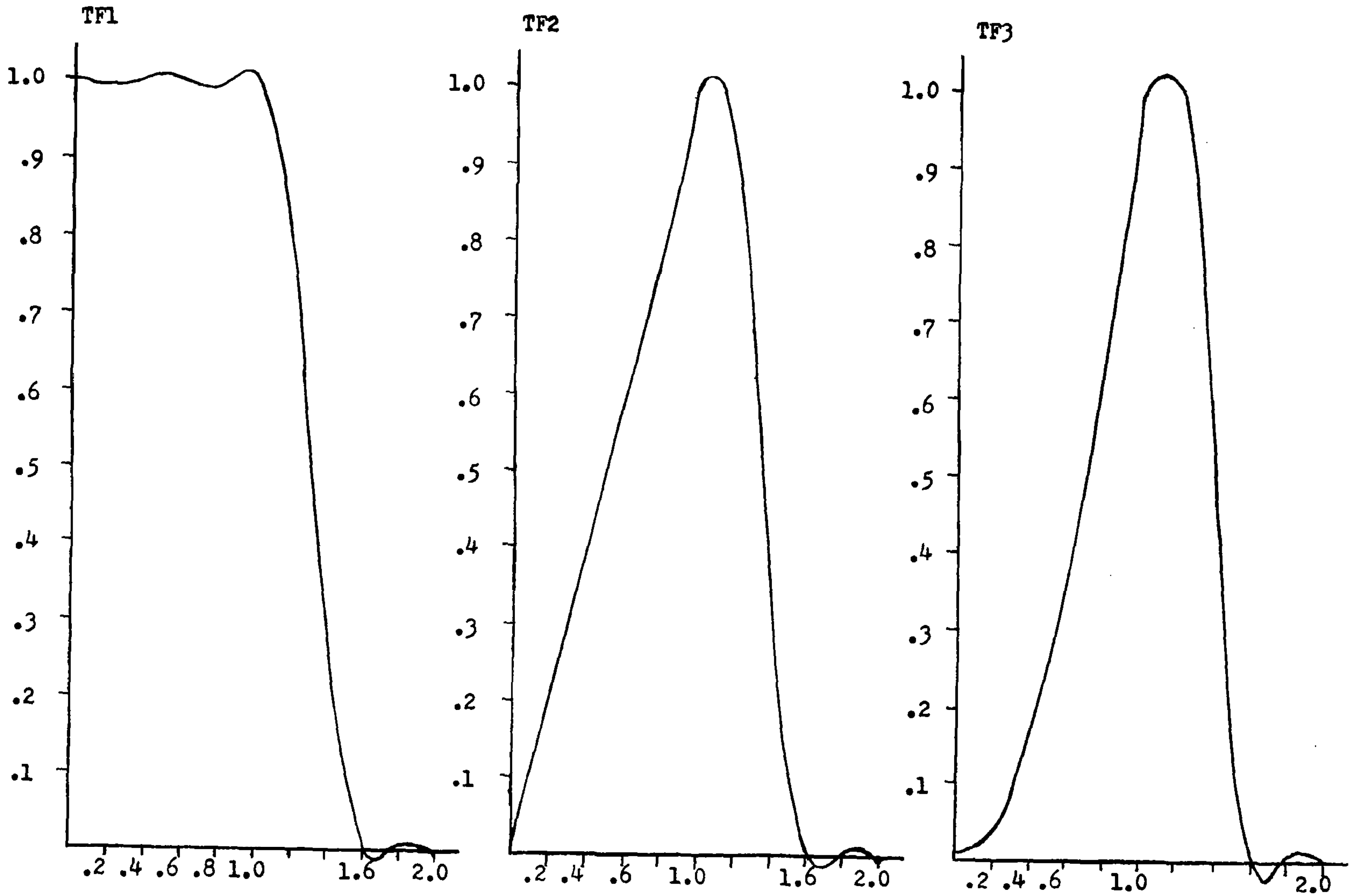


Figure 7.1. Run 1.20 transfer functions

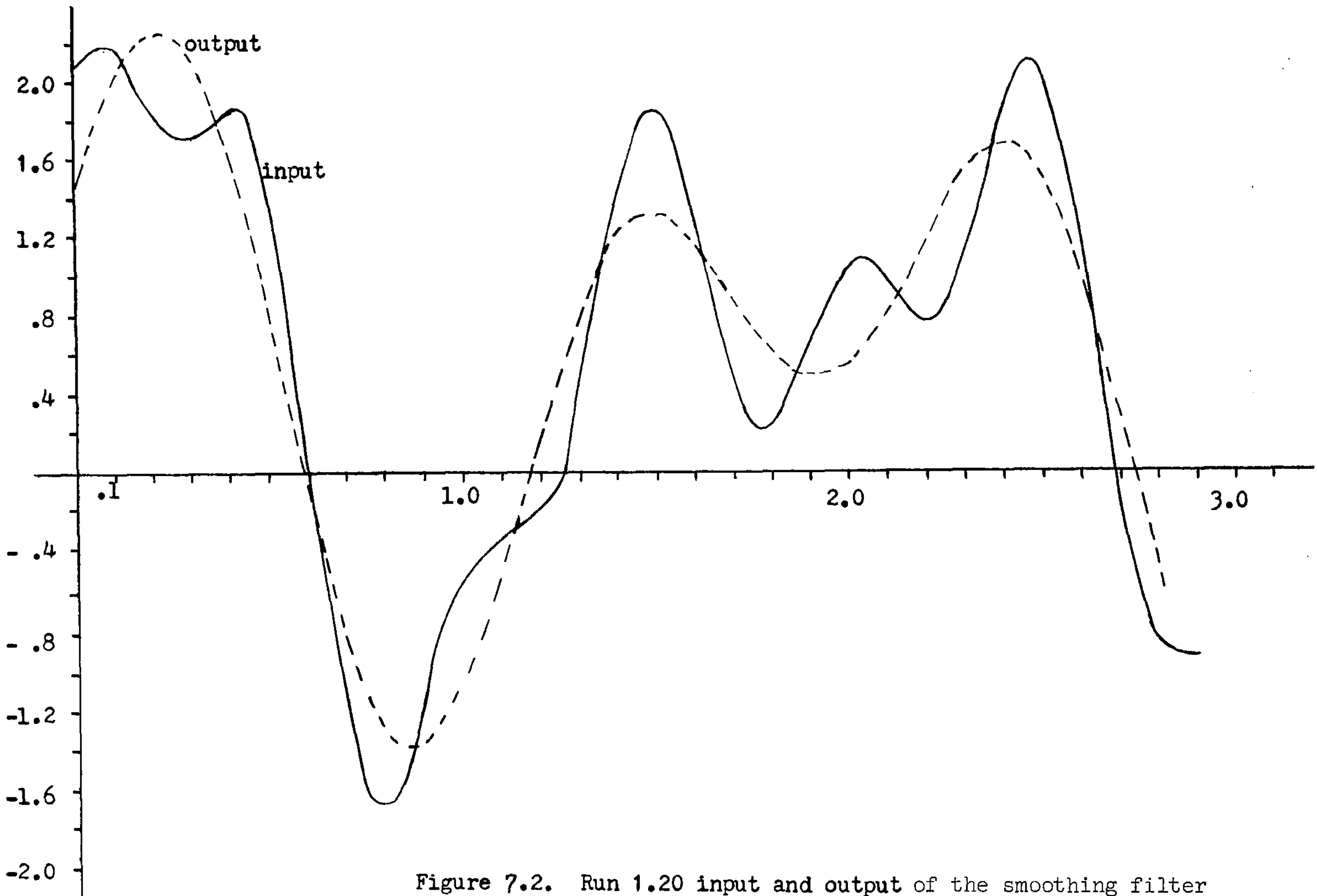


Figure 7.2. Run 1.20 input and output of the smoothing filter

7.3 SAMPLE PROGRAM AND RESULTS FOR THE MARTIN-GRAHAM INTERPOLATING FILTERS

The program of the preceding section yields a program for interpolation when suitably modified. The necessary changes made to obtain the program in this section include:

1. Provision has been made for the special cases given in (6.24). However, the weight subscript values which satisfy $m-\Phi = \pm 1/2\tau_d$ are computed externally and read into the program. This is done because of errors introduced by truncation in the computer which cause $m-\Phi \pm 1/2\tau_d$ to be non-zero when it should be zero. This is essentially a programming problem, and it could be handled by choosing a small $\epsilon > 0$ and using the special formulas when $|m - \Phi \pm 1/2\tau_d| < \epsilon$. The value of ϵ chosen will depend on the particular computer being used. Of course, we could avoid this problem by using Filter 5 of Chapter IV. As noted there, the performance of this filter is essentially the same as the Martin-Graham filter and no special evaluation is necessary except when $m - \Phi = 0$ (this includes $\Phi = 0$).
2. Statements for computing the weights h_o, y_o^1, y_o^2, h_m, y_m^1, and y_m^2 when (6.24) holds are included. We chose to compute these in every run whether needed or not, and they are designated HO, DHO, DDHO, HM, DHM, and DDHM in the program.
3. The loop for computing the weights for interpolation uses exactly the same weight expressions as the loop in the first program. In this case, the loop's upper index is 2N+1

instead of N because the symmetry of the weights is lost in interpolation. The shift of Φ units is provided by subtracting Φ from m(TX from X in the program). This is what allows use of the same weight expressions. Note that taking $\Phi = 0$ (TX = 0) gives the weights for filtering without interpolation, and hence this loop can be used for computing both ordinary filtering weights and weights for filtering and interpolation. The loop also contains statements to handle the use of the special weight expressions. If (6.24) is not satisfied for any subscript values m, then NA and NB must be read in as values which will make (N+NA-I+1) and(N+NB-I+1) non-zero for all values of I. I has maximum value 2N+1, and hence any integer greater than N+1 will suffice for NA and NB in this case. When (6.24) holds for certain subscript values, then NA is to be the negative value for which (6.24) holds and NB the positive value for which (6.24) holds. This is necessary to determine the correct sign for the first derivative weight DHM in each case. In Run 5.30, NA = -12 and NB = 13.

4. The transfer functions for interpolation are all complex functions and the recovery of these functions has been omitted.
5. Printing of the input has been omitted.
6. The same loop for computing the desired output has been used with the argument being shifted by Φ.
7. The loop for computing the actual output was modified to allow for the unsymmetric nature of the weights used in interpolation.

The desired output and actual output is listed for five runs. The parameter values used for each run are given in Tables 7.1 and 7.2. In each run, the input component with frequency f_3 is to be removed by the filters. The Φ values used are .25 and .5, so that the values interpolated for are one-quarter and one-half the length of the sampling interval to the right of the center input value. That is, letting the output of the filter without interpolation be

$$r_j = \sum_{n=-N}^{N} h_n g_{j+n},$$

and the output with interpolation be

$$\bar{r}_j = \sum_{n=-N}^{N} \bar{h}_n g_{j+n},$$

then $\bar{r}_j$ is Φf_s units to the right of r_j.

To interpolate for values Φf_s units to the left of r_j, the weight relations (6.22) and (6.23) may be used to eliminate recomputation of the weights. This can be accomplished in the sample program by using the following loop for computing the actual output.

```
      DO 30 I=1,NN
      J=I+K-1
      IF(TX) 40,41,40
   40 I1 = NN-I+1
      SA1 = SA1 + H(I1)*Z(J)
      SB1 = SB1 - DH(I1)*Z(J)
      SC1 = SC1 + DDH(I1)*Z(J)
   41 SA = SA + H(I)*Z(J)
```

```
      SB = SB + DH(I)*Z(J)
   30 SC = SC + DDH(I)*Z(J)
      SB1 = -FS*SB1
      SB = -FS*SB
      SC1 = SC1*FS**2
      SC = SC*FS**2
```

Provisions for initializing and printing SA1, SB1, and SC1 must also be made. The IF statement is included to eliminate duplicate outputs when using the loop for filtering without interpolation ($\Phi = 0$).

```
      SAMPLE PROGRAM, FILTERING WITH INTERPOLATION
      SMOOTHING, FIRST AND SECOND DERIVATIVES

    DIMENSION H(61),DH(61),DDH(61),Z(101)
    DIMENSION R(40),DR(40),DDR(40)
  1 FORMAT (4F10.0)
  2 FORMAT(A4,A4)
  3 FORMAT (3I5)
  7 FORMAT(1H1,5X,A4,A4,16H  DESIRED OUTPUT/)
  8 FORMAT(5X,1HT,7X,4HR(T),11X,5HDR(T),10X,6HDDR(T))
  9 FORMAT(1H1,5X,A4,A4,15H  ACTUAL OUTPUT/)
 10 FORMAT(1X,F7.3,3E15.7)
    P=3.14159

      READ PROBLEM PARAMETERS

 12 READ(2,2) RUN,XNUM
    READ(2,1) XN,TC,TD,TX
    READ(2,1) RA,RB,RC,FS
    READ(2,1) AA,AB,AC,AD
    READ(2,1) BA,BB,BC,BD

      NA AND NB ARE SUBSCRIPTS OF THOSE WEIGHTS FOR WHICH
      SPECIAL EVALUATION IS NECESSARY

    READ(2,3) NA,NB
    N=XN

      COMPUTATION OF THE UNCONSTRAINED WEIGHTS
      THE FACTORS -FS AND FS**2 OF THE FIRST AND SECOND
      DERIVATIVE WEIGHTS WILL BE INTRODUCED LATER

    RT=TC+TD
    HO=2.*TC+TD
    DHO=0.
    DDHO=8.*TD**2*(RT+TC)-4.*P**2/3.*(RT**3+TC**3)
    HM=TD/2.*COS(P*TC/TD)

      THE MINUS SIGN IS NECESSARY IN DHM BECAUSE WE HAVE
      REMOVED A FACTOR OF -FS FROM EACH DERIVATIVE WEIGHT

    DHM=-(P*TD*(TD+2.*TC)*SIN(P*TC/TD)+3.*TD**2*COS(P*TC/TD))/2.
    DDHM=(7.*TD**3-2.*P**2*(TC*TD*(TC+TD)+TD**3/3.))*COS(P*TC/TD)
    DDHM=DDHM+3.*P*TD**2*(TD+2.*TC)*SIN(P*TC/TD)
    NN=2*N+1
    DO 13 I=1,NN
    X=I-N-1
    X=X-TX
    IF(X) 51,50,51
 50 H(I)=HO
    DH(I)=DHO
    DDH(I)=DDHO
    GO TO 13
 51 IF(N+NA-I+1) 53,52,53
```

```
 52 H(I)=HM
    DH(I)=-DHM
    DDH(I)=DDHM
    GO TO 13
 53 IF(N+NB-I+1) 55,54,55
 54 H(I)=HM
    DH(I)=DHM
    DDH(I)=DDHM
    GO TO 13
 55 H(I)=SIN(HO*X*P)/(X*P)
    H(I)=H(I)*COS(TD*X*P)/(1.-4.*TD**2*X**2)
    DH(I)=RT*COS(2.*RT*X*P)+TC*COS(2.*TC*X*P)
    DH(I)=DH(I)-H(I)*(1.-12.*TD**2*X**2)
    DH(I)=DH(I)/(X*(1.-4.*TD**2*X**2))
    DDH(I)=-2.*DH(I)*(1.-12.*TD**2*X**2)+24.*TD**2*X*H(I)
    DDH(I)=DDH(I)-2.*P*TC**2*SIN(2.*TC*P*X)
    DDH(I)=DDH(I)-2.*P*RT**2*SIN(2.*RT*P*X)
    DDH(I)=DDH(I)/(X*(1.-4.*TD**2*X**2))
 13 CONTINUE

        GENERATION OF SAMPLE INPUT DATA

    MA=N+1
    M=2*N+40
    DO 27 I=1,M
    T=I-MA
    CA=COS(2.*P*RA*T)
    S= SIN(2.*P*RB*T)
    CC=COS(2.*P*RC*T)
 27 Z(I)=AA*CA+AB*S+AC*CC+AD

        COMPUTATION OF DESIRED OUTPUTS

    WRITE(3,7) RUN,XNUM
    WRITE(3,8)
    DO 29 I=1,40
    T=I-1
    T=T+TX
    CA=COS(2.*P*RA*T)
    S=SIN(2.*P*RB*T)
    CC=COS(2.*P*RC*T)
    R(I)=BA*CA+BB*S+BC*CC+BD
    DDR(I)=-4.*(P*FS)**2*(BA*RA**2*CA+BB*RB**2*S+BC*RC**2*CC)
    CA=SIN(2.*P*RA*T)
    S= COS(2.*P*RB*T)
    CC=SIN(2.*P*RC*T)
    DR(I)=-2.*P*FS*(BA*RA*CA-BB*RB*S+BC*RC*CC)
    Y=T/FS
 29 WRITE(3,10) Y,R(I),DR(I),DDR(I)

        COMPUTATION OF THE ACTUAL OUTPUT

    WRITE(3,9) RUN,XNUM
    WRITE(3,8)
```

```
      DO 31 K=1,40
      MB=K-1
      SA=0.
      SB=0.
      SC=0.
      T=MB
      T=(T+TX)/FS
      DO 30 I=1,NN
      J=I+K-1
      SA=SA+H(I)*Z(J)
      SB=SB+DH(I)*Z(J)
   30 SC=SC+DDH(I)*Z(J)

         THE FACTORS -FS AND FS**2 ARE INTRODUCED HERE

      SB=-FS*SB
      SC=SC*FS**2
   31 WRITE(3,10) T,SA,SB,SC
      PAUSE
      GO TO 12
   32 CALL EXIT
      END
```

RUN 1.20 DESIRED OUTPUT

T	R(T)	DR(T)	DDR(T)
0.025	0.1637818E 01	0.5351961E 01	-0.1434482E 02
0.125	0.2073327E 01	0.3097759E 01	-0.2988599E 02
0.225	0.2216199E 01	-0.3775347E 00	-0.3806872E 02
0.325	0.1987057E 01	-0.4170801E 01	-0.3600096E 02
0.425	0.1406460E 01	-0.7237290E 01	-0.2382530E 02
0.525	0.5934738E 00	-0.8702560E 01	-0.4723590E 01
0.625	-0.2653625E 00	-0.8126870E 01	0.1601407E 02
0.725	-0.9675943E 00	-0.5640479E 01	0.3257205E 02
0.825	-0.1351528E 01	-0.1907884E 01	0.4035715E 02
0.925	-0.1341003E 01	0.2068331E 01	0.3737358E 02
1.025	-0.9648516E 00	0.5244037E 01	0.2480251E 02
1.125	-0.3454269E 00	0.6839658E 01	0.6609569E 01
1.225	0.3400074E 00	0.6562417E 01	-0.1169485E 02
1.325	0.9129402E 00	0.4677525E 01	-0.2475618E 02
1.425	0.1245775E 01	0.1908088E 01	-0.2900906E 02
1.525	0.1296586E 01	-0.8033547E 00	-0.2373833E 02
1.625	0.1116132E 01	-0.2596151E 01	-0.1124212E 02
1.725	0.8255349E 00	-0.2961104E 01	0.3948057E 01
1.825	0.5722126E 00	-0.1894210E 01	0.1654082E 02
1.925	0.4784087E 00	0.1128211E 00	0.2218744E 02
2.025	0.5988859E 00	0.2241276E 01	0.1887761E 02
2.125	0.9013744E 00	0.3619314E 01	0.7590124E 01
2.225	0.1276106E 01	0.3613879E 01	-0.8006871E 01
2.325	0.1571517E 01	0.2047774E 01	-0.2271295E 02
2.425	0.1644850E 01	-0.7276661E 00	-0.3144828E 02
2.525	0.1411572E 01	-0.3928616E 01	-0.3088420E 02
2.625	0.8777371E 00	-0.6574944E 01	-0.2053925E 02
2.725	0.1446102E 00	-0.7793754E 01	-0.2993331E 01
2.825	-0.6164962E 00	-0.7095955E 01	0.1685274E 02
2.925	-0.1211989E 01	-0.4539264E 01	0.3324809E 02
3.025	-0.1482024E 01	-0.7258547E 00	0.4134038E 02
3.125	-0.1347767E 01	0.3366154E 01	0.3866184E 02
3.225	-0.8354293E 00	0.6666752E 01	0.2589256E 02
3.325	-0.6962841E-01	0.8327211E 01	0.6663772E 01
3.425	0.7619902E 00	0.7966818E 01	-0.1353896E 02
3.525	0.1462207E 01	0.5778070E 01	-0.2903459E 02
3.625	0.1879593E 01	0.2458884E 01	-0.3565577E 02
3.725	0.1949135E 01	-0.1006309E 01	-0.3198193E 02
3.825	0.1706126E 01	-0.3648266E 01	-0.1971893E 02
3.925	0.1269598E 01	-0.4803969E 01	-0.3112830E 01

RUN 1.20 ACTUAL OUTPUT

T	R(T)	DR(T)	DDR(T)
0.025	0.1645196E 01	0.5365969E 01	-0.1444303E 02
0.125	0.2085801E 01	0.3100372E 01	-0.3002462E 02
0.225	0.2232048E 01	-0.3900173E 00	-0.3825466E 02
0.325	0.2002062E 01	-0.4198776E 01	-0.3623316E 02
0.425	0.1416376E 01	-0.7276670E 01	-0.2404929E 02
0.525	0.5965244E 00	-0.8745769E 01	-0.4843942E 01
0.625	-0.2684384E 00	-0.8165518E 01	0.1606258E 02
0.725	-0.9752920E 00	-0.5667804E 01	0.3277412E 02
0.825	-0.1362462E 01	-0.1919801E 01	0.4064327E 02
0.925	-0.1353011E 01	0.2073037E 01	0.3768262E 02
1.025	-0.9743336E 00	0.5263395E 01	0.2510072E 02
1.125	-0.3488280E 00	0.6868403E 01	0.6856093E 01
1.225	0.3436755E 00	0.6593228E 01	-0.1156772E 02
1.325	0.9214365E 00	0.4703687E 01	-0.2480119E 02
1.425	0.1255661E 01	0.1925944E 01	-0.2920605E 02
1.525	0.1305589E 01	-0.7938837E 00	-0.2399836E 02
1.625	0.1123397E 01	-0.2592850E 01	-0.1147678E 02
1.725	0.8303442E 00	-0.2960987E 01	0.3771034E 01
1.825	0.5736964E 00	-0.1894297E 01	0.1641314E 02
1.925	0.4769812E 00	0.1149186E 00	0.2210921E 02
2.025	0.5972460E 00	0.2246207E 01	0.1886403E 02
2.125	0.9030666E 00	0.3624881E 01	0.7628645E 01
2.225	0.1282765E 01	0.3615464E 01	-0.7978300E 01
2.325	0.1581786E 01	0.2040599E 01	-0.2276006E 02
2.425	0.1655958E 01	-0.7459885E 00	-0.3157423E 02
2.525	0.1421372E 01	-0.3957067E 01	-0.3102409E 02
2.625	0.8848663E 00	-0.6609727E 01	-0.2062175E 02
2.725	0.1473961E 00	-0.7829313E 01	-0.2992994E 01
2.825	-0.6199315E 00	-0.7125638E 01	0.1692924E 02
2.925	-0.1221819E 01	-0.4556328E 01	0.3340488E 02
3.025	-0.1495326E 01	-0.7254911E 00	0.4159112E 02
3.125	-0.1359797E 01	0.3384594E 01	0.3898044E 02
3.225	-0.8425124E 00	0.6699155E 01	0.2618887E 02
3.325	-0.7065820E-01	0.8366367E 01	0.6832387E 01
3.425	0.7665408E 00	0.8005024E 01	-0.1354440E 02
3.525	0.1471727E 01	0.5808911E 01	-0.2918478E 02
3.625	0.1893066E 01	0.2477922E 01	-0.3589650E 02
3.725	0.1963911E 01	-0.1000987E 01	-0.3227905E 02
3.825	0.1718232E 01	-0.3655303E 01	-0.2004334E 02
3.925	0.1276141E 01	-0.4818810E 01	-0.3403159E 01

RUN 2.30 DESIRED OUTPUT

T	R(T)	DR(T)	DDR(T)
0.025	0.2614950E 01	0.2365615E 02	-0.1074349E 03
0.125	0.3923879E 01	-0.1202200E 01	-0.3249451E 03
0.225	0.2378444E 01	-0.2594291E 02	-0.1051014E 03
0.325	-0.9553063E-01	-0.1745135E 02	0.2503517E 03
0.425	-0.3845928E 00	0.1171776E 02	0.2532060E 03
0.525	0.1539565E 01	0.2077077E 02	-0.9681983E 02
0.625	0.2617318E 01	-0.2902285E 01	-0.3120498E 03
0.725	0.9686020E 00	-0.2629146E 02	-0.9118839E 02
0.825	-0.1470662E 01	-0.1641430E 02	0.2639228E 03
0.925	-0.1590414E 01	0.1403904E 02	0.2651079E 03
1.025	0.6210957E 00	0.2414919E 02	-0.8775341E 02
1.125	0.2076119E 01	0.1202526E 01	-0.3067084E 03
1.225	0.8576506E 00	-0.2186224E 02	-0.9009484E 02
1.325	-0.1140519E 01	-0.1209428E 02	0.2606635E 03
1.425	-0.8515008E 00	0.1782711E 02	0.2578161E 03
1.525	0.1696458E 01	0.2703466E 02	-0.9836530E 02
1.625	0.3382679E 01	0.2902889E 01	-0.3196036E 03
1.725	0.2267513E 01	-0.2151360E 02	-0.1040111E 03
1.825	0.2346264E 00	-0.1313154E 02	0.2470904E 03
1.925	0.3543106E 00	0.1550563E 02	0.2459161E 03
2.025	0.2614910E 01	0.2365633E 02	-0.1074285E 03
2.125	0.3923881E 01	-0.1201650E 01	-0.3249451E 03
2.225	0.2378488E 01	-0.2594273E 02	-0.1051078E 03
2.325	-0.9550111E-01	-0.1745178E 02	0.2503477E 03
2.425	-0.3846126E 00	0.1171733E 02	0.2532099E 03
2.525	0.1539530E 01	0.2077093E 02	-0.9681349E 02
2.625	0.2617323E 01	-0.2901757E 01	-0.3120499E 03
2.725	0.9686465E 00	-0.2629131E 02	-0.9119481E 02
2.825	-0.1470634E 01	-0.1641474E 02	0.2639188E 03
2.925	-0.1590438E 01	0.1403859E 02	0.2651118E 03
3.025	0.6210549E 00	0.2414934E 02	-0.8774702E 02
3.125	0.2076117E 01	0.1203044E 01	-0.3067084E 03
3.225	0.8576876E 00	-0.2186209E 02	-0.9010120E 02
3.325	-0.1140498E 01	-0.1209472E 02	0.2606596E 03
3.425	-0.8515309E 00	0.1782668E 02	0.2578201E 03
3.525	0.1696412E 01	0.2703483E 02	-0.9835886E 02
3.625	0.3382674E 01	0.2903430E 01	-0.3196036E 03
3.725	0.2267549E 01	-0.2151342E 02	-0.1040174E 03
3.825	0.2346486E 00	-0.1313196E 02	0.2470865E 03
3.925	0.3542845E 00	0.1550521E 02	0.2459201E 03

RUN 2.30 ACTUAL OUTPUT

T	R(T)	DR(T)	DDR(T)
0.025	0.2614879E 01	0.2366695E 02	-0.1075791E 03
0.125	0.3925231E 01	-0.1207670E 01	-0.3255331E 03
0.225	0.2378706E 01	-0.2596372E 02	-0.1051581E 03
0.325	-0.9571261E-01	-0.1744320E 02	0.2505466E 03
0.425	-0.3845009E 00	0.1172585E 02	0.2533687E 03
0.525	0.1540655E 01	0.2077860E 02	-0.9725229E 02
0.625	0.2620083E 01	-0.2913330E 01	-0.3129780E 03
0.725	0.9703921E 00	-0.2631990E 02	-0.9160369E 02
0.825	-0.1469351E 01	-0.1641507E 02	0.2637756E 03
0.925	-0.1589010E 01	0.1403777E 02	0.2649786E 03
1.025	0.6231883E 00	0.2414815E 02	-0.8839926E 02
1.125	0.2079479E 01	0.1183964E 01	-0.3077504E 03
1.225	0.8595701E 00	-0.2189610E 02	-0.9051335E 02
1.325	-0.1139556E 01	-0.1209786E 02	0.2606240E 03
1.425	-0.8508899E 00	0.1782594E 02	0.2578949E 03
1.525	0.1697390E 01	0.2703660E 02	-0.9872289E 02
1.625	0.3384626E 01	0.2889903E 01	-0.3203056E 03
1.725	0.2267904E 01	-0.2153983E 02	-0.1040709E 03
1.825	0.2340952E 00	-0.1312619E 02	0.2473931E 03
1.925	0.3536089E 00	0.1551381E 02	0.2462870E 03
2.025	0.2614839E 01	0.2366713E 02	-0.1075727E 03
2.125	0.3925233E 01	-0.1207120E 01	-0.3255331E 03
2.225	0.2378750E 01	-0.2596355E 02	-0.1051645E 03
2.325	-0.9568308E-01	-0.1744362E 02	0.2505426E 03
2.425	-0.3845207E 00	0.1172542E 02	0.2533726E 03
2.525	0.1540620E 01	0.2077876E 02	-0.9724594E 02
2.625	0.2620088E 01	-0.2912802E 01	-0.3129780E 03
2.725	0.9704366E 00	-0.2631974E 02	-0.9161012E 02
2.825	-0.1469323E 01	-0.1641552E 02	0.2637716E 03
2.925	-0.1589034E 01	0.1403732E 02	0.2649825E 03
3.025	0.6231474E 00	0.2414830E 02	-0.8839286E 02
3.125	0.2079477E 01	0.1184483E 01	-0.3077504E 03
3.225	0.8596071E 00	-0.2189595E 02	-0.9051971E 02
3.325	-0.1139536E 01	-0.1209831E 02	0.2606201E 03
3.425	-0.8509201E 00	0.1782550E 02	0.2578989E 03
3.525	0.1697344E 01	0.2703676E 02	-0.9871645E 02
3.625	0.3384621E 01	0.2890443E 01	-0.3203055E 03
3.725	0.2267940E 01	-0.2153966E 02	-0.1040773E 03
3.825	0.2341173E 00	-0.1312661E 02	0.2473892E 03
3.925	0.3535827E 00	0.1551339E 02	0.2462909E 03

RUN 3.20 DESIRED OUTPUT

T	R(T)	DR(T)	DDR(T)
0.050	0.1766679E 01	0.4938876E 01	-0.1866949E 02
0.150	0.2141117E 01	0.2313379E 01	-0.3278050E 02
0.250	0.2194795E 01	-0.1336816E 01	-0.3856261E 02
0.350	0.1871746E 01	-0.5044983E 01	-0.3382819E 02
0.450	0.1218520E 01	-0.7779930E 01	-0.1951801E 02
0.550	0.3749803E 00	-0.8754983E 01	0.5394089E 00
0.650	-0.4630277E 00	-0.7666565E 01	0.2075845E 02
0.750	-0.1098110E 01	-0.4788715E 01	0.3547091E 02
0.850	-0.1386567E 01	-0.8941116E 00	0.4062941E 02
0.950	-0.1277845E 01	0.2974423E 01	0.3501531E 02
1.050	-0.8264314E 00	0.5811985E 01	0.2058021E 02
1.150	-0.1728700E 00	0.6944924E 01	0.1818384E 01
1.250	0.4999934E 00	0.6220047E 01	-0.1563247E 02
1.350	0.1021921E 01	0.4032777E 01	-0.2672656E 02
1.450	0.1284444E 01	0.1187430E 01	-0.2854307E 02
1.550	0.1269343E 01	-0.1365276E 01	-0.2114330E 02
1.650	0.1048102E 01	-0.2830577E 01	-0.7490256E 01
1.750	0.7531204E 00	-0.2817090E 01	0.7538406E 01
1.850	0.5302667E 00	-0.1452336E 01	0.1873035E 02
1.950	0.4881806E 00	0.6690398E 00	0.2221362E 02
2.050	0.6606031E 00	0.2687022E 01	0.1670003E 02
2.150	0.9938517E 00	0.3763516E 01	0.3906191E 01
2.250	0.1363534E 01	0.3363824E 01	-0.1198093E 02
2.350	0.1615299E 01	0.1442648E 01	-0.2562756E 02
2.450	0.1616731E 01	-0.1525188E 01	-0.3225165E 02
2.550	0.1303869E 01	-0.4680505E 01	-0.2916398E 02
2.650	0.7073371E 00	-0.7040914E 01	-0.1666674E 02
2.750	-0.5065416E-01	-0.7806685E 01	0.1976008E 01
2.850	-0.7881384E 00	-0.6616079E 01	0.2149321E 02
2.950	-0.1314759E 01	-0.3670047E 01	0.3619575E 02
3.050	-0.1487195E 01	0.3137219E 00	0.4170975E 02
3.150	-0.1251760E 01	0.4304714E 01	0.3631849E 02
3.250	-0.6611172E 00	0.7259899E 01	0.2149681E 02
3.350	0.1400939E 00	0.8428938E 01	0.1471941E 01
3.450	0.9564504E 00	0.7571182E 01	-0.1805476E 02
3.550	0.1597300E 01	0.5018523E 01	-0.3163064E 02
3.650	0.1929902E 01	0.1565717E 01	-0.3568807E 02
3.750	0.1914219E 01	-0.1777057E 01	-0.2959067E 02
3.850	0.1609163E 01	-0.4092371E 01	-0.1577006E 02
3.950	0.1148966E 01	-0.4829101E 01	0.1083350E 01

RUN 3.20 ACTUAL OUTPUT

T	R(T)	DR(T)	DDR(T)
0.050	0.1774376E 01	0.4950708E 01	-0.1873413E 02
0.150	0.2153616E 01	0.2313020E 01	-0.3287326E 02
0.250	0.2210266E 01	-0.1354320E 01	-0.3875856E 02
0.350	0.1885964E 01	-0.5079988E 01	-0.3413317E 02
0.450	0.1227372E 01	-0.7825891E 01	-0.1981448E 02
0.550	0.3769149E 00	-0.8801300E 01	0.3959359E 00
0.650	-0.4670542E 00	-0.7703970E 01	0.2080606E 02
0.750	-0.1106445E 01	-0.4811996E 01	0.3563456E 02
0.850	-0.1397744E 01	-0.9009312E 00	0.4083441E 02
0.950	-0.1289674E 01	0.2984905E 01	0.3525874E 02
1.050	-0.8353605E 00	0.5837741E 01	0.2086821E 02
1.150	-0.1754956E 00	0.6979118E 01	0.2073010E 01
1.250	0.5044458E 00	0.6252705E 01	-0.1554569E 02
1.350	0.1031031E 01	0.4056003E 01	-0.2686711E 02
1.450	0.1294705E 01	0.1199263E 01	-0.2882790E 02
1.550	0.1278515E 01	-0.1362002E 01	-0.2143419E 02
1.650	0.1055399E 01	-0.2832199E 01	-0.7724943E 01
1.750	0.7578907E 00	-0.2821063E 01	0.7334094E 01
1.850	0.5317099E 00	-0.1456047E 01	0.1854444E 02
1.950	0.4867821E 00	0.6689462E 00	0.2210497E 02
2.050	0.6590886E 00	0.2692149E 01	0.1672224E 02
2.150	0.9956966E 00	0.3770613E 01	0.4005763E 01
2.250	0.1370227E 01	0.3365614E 01	-0.1193893E 02
2.350	0.1625346E 01	0.1432769E 01	-0.2572307E 02
2.450	0.1627319E 01	-0.1548013E 01	-0.3243235E 02
2.550	0.1312906E 01	-0.4712880E 01	-0.2932142E 02
2.650	0.7135680E 00	-0.7077912E 01	-0.1675848E 02
2.750	-0.4877054E-01	-0.7843221E 01	0.1918299E 01
2.850	-0.7923162E 00	-0.6645688E 01	0.2146376E 02
2.950	-0.1324989E 01	-0.3684793E 01	0.3626463E 02
3.050	-0.1500422E 01	0.3196643E 00	0.4193747E 02
3.150	-0.1263242E 01	0.4330678E 01	0.3664585E 02
3.250	-0.6673232E 00	0.7298474E 01	0.2176670E 02
3.350	0.1400626E 00	0.8470385E 01	0.1567866E 01
3.450	0.9619329E 00	0.7607867E 01	-0.1812337E 02
3.550	0.1607544E 01	0.5045543E 01	-0.3179290E 02
3.650	0.1943788E 01	0.1579341E 01	-0.3592174E 02
3.750	0.1929044E 01	-0.1779031E 01	-0.2992190E 02
3.850	0.1620995E 01	-0.4107951E 01	-0.1617132E 02
3.950	0.1155057E 01	-0.4851237E 01	0.7379718E 00

RUN 4.30 DESIRED OUTPUT

T	R(T)	DR(T)	DDR(T)
0.050	0.3163258E 01	0.1984135E 02	-0.1953862E 03
0.150	0.3793120E 01	-0.9192650E 01	-0.3091631E 03
0.250	0.1707112E 01	-0.2735415E 02	-0.6979695E 01
0.350	-0.4481194E 00	-0.1056570E 02	0.2958880E 03
0.450	-0.1914259E-01	0.1722981E 02	0.1840956E 03
0.550	0.2019128E 01	0.1722997E 02	-0.1840929E 03
0.650	0.2448128E 01	-0.1056545E 02	-0.2958891E 03
0.750	0.2929105E 00	-0.2735416E 02	0.6976321E 01
0.850	-0.1793112E 01	-0.9192911E 01	0.3091621E 03
0.950	-0.1163274E 01	0.1984118E 02	0.1953889E 03
1.050	0.1187863E 01	0.2082440E 02	-0.1758874E 03
1.150	0.2011112E 01	-0.6339902E 01	-0.2915764E 03
1.250	0.2929174E 00	-0.2291128E 02	0.6974684E 01
1.350	-0.1356097E 01	-0.4967608E 01	0.3048484E 03
1.450	-0.3320337E 00	0.2343547E 02	0.1871862E 03
1.550	0.2331974E 01	0.2343595E 02	-0.1871780E 03
1.650	0.3356110E 01	-0.4966835E 01	-0.3048515E 03
1.750	0.1707140E 01	-0.2291127E 02	-0.6984689E 01
1.850	-0.1109660E-01	-0.6340641E 01	0.2915733E 03
1.950	0.8120834E 00	0.2082396E 02	0.1758955E 03
2.050	0.3163224E 01	0.1984168E 02	-0.1953808E 03
2.150	0.3793136E 01	-0.9192128E 01	-0.3091652E 03
2.250	0.1707158E 01	-0.2735414E 02	-0.6986441E 01
2.350	-0.4481015E 00	-0.1056620E 02	0.2958859E 03
2.450	-0.1917173E-01	0.1722950E 02	0.1841009E 03
2.550	0.2019098E 01	0.1723028E 02	-0.1840875E 03
2.650	0.2448146E 01	-0.1056495E 02	-0.2958912E 03
2.750	0.2929568E 00	-0.2735417E 02	0.6969575E 01
2.850	-0.1793097E 01	-0.9193434E 01	0.3091600E 03
2.950	-0.1163308E 01	0.1984085E 02	0.1953944E 03
3.050	0.1187828E 01	0.2082470E 02	-0.1758819E 03
3.150	0.2011123E 01	-0.6339408E 01	-0.2915785E 03
3.250	0.2929561E 00	-0.2291129E 02	0.6968016E 01
3.350	-0.1356089E 01	-0.4968124E 01	0.3048464E 03
3.450	-0.3320733E 00	0.2343516E 02	0.1871917E 03
3.550	0.2331934E 01	0.2343627E 02	-0.1871725E 03
3.650	0.3356118E 01	-0.4966319E 01	-0.3048535E 03
3.750	0.1707179E 01	-0.2291125E 02	-0.6991364E 01
3.850	-0.1108587E-01	-0.6341135E 01	0.2915713E 03
3.950	0.8120483E 00	0.2082366E 02	0.1759009E 03

RUN 4.30 ACTUAL OUTPUT

T	R(T)	DR(T)	DDR(T)
0.050	0.3163427E 01	0.1984857E 02	-0.1956055E 03
0.150	0.3794153E 01	-0.9212233E 01	-0.3096701E 03
0.250	0.1706817E 01	-0.2737617E 02	-0.6916901E 01
0.350	-0.4479947E 00	-0.1055026E 02	0.2961674E 03
0.450	-0.1877488E-01	0.1724259E 02	0.1842178E 03
0.550	0.2020320E 01	0.1722880E 02	-0.1845075E 03
0.650	0.2450356E 01	-0.1059710E 02	-0.2966246E 03
0.750	0.2938679E 00	-0.2739075E 02	0.6799791E 01
0.850	-0.1791801E 01	-0.9193114E 01	0.3092147E 03
0.950	-0.1161902E 01	0.1983878E 02	0.1953192E 03
1.050	0.1189780E 01	0.2080999E 02	-0.1764403E 03
1.150	0.2013714E 01	-0.6381557E 01	-0.2923832E 03
1.250	0.2938597E 00	-0.2295365E 02	0.6801010E 01
1.350	-0.1355187E 01	-0.4968807E 01	0.3049777E 03
1.450	-0.3314098E 00	0.2343695E 02	0.1872595E 03
1.550	0.2332868E 01	0.2342993E 02	-0.1875356E 03
1.650	0.3357515E 01	-0.4996419E 01	-0.3054297E 03
1.750	0.1706830E 01	-0.2293906E 02	-0.6919042E 01
1.850	-0.1137338E-01	-0.6326196E 01	0.2919293E 03
1.950	0.8117026E 00	0.2084062E 02	0.1761607E 03
2.050	0.3163394E 01	0.1984890E 02	-0.1956001E 03
2.150	0.3794168E 01	-0.9211710E 01	-0.3096722E 03
2.250	0.1706863E 01	-0.2737616E 02	-0.6923659E 01
2.350	-0.4479768E 00	-0.1055076E 02	0.2961652E 03
2.450	-0.1880399E-01	0.1724228E 02	0.1842232E 03
2.550	0.2020291E 01	0.1722911E 02	-0.1845021E 03
2.650	0.2450374E 01	-0.1059660E 02	-0.2966267E 03
2.750	0.2939142E 00	-0.2739076E 02	0.6793033E 01
2.850	-0.1791785E 01	-0.9193637E 01	0.3092126E 03
2.950	-0.1161935E 01	0.1983845E 02	0.1953247E 03
3.050	0.1189745E 01	0.2081029E 02	-0.1764349E 03
3.150	0.2013724E 01	-0.6381064E 01	-0.2923852E 03
3.250	0.2938985E 00	-0.2295367E 02	0.6794329E 01
3.350	-0.1355179E 01	-0.4969322E 01	0.3049757E 03
3.450	-0.3314494E 00	0.2343664E 02	0.1872650E 03
3.550	0.2332828E 01	0.2343024E 02	-0.1875301E 03
3.650	0.3357524E 01	-0.4995904E 01	-0.3054318E 03
3.750	0.1706868E 01	-0.2293905E 02	-0.6925717E 01
3.850	-0.1136279E-01	-0.6326689E 01	0.2919273E 03
3.950	0.8116673E 00	0.2084033E 02	0.1761662E 03

RUN 5.30 DESIRED OUTPUT

T	R(T)	DR(T)	DDR(T)
0.050	0.1766679E 01	0.4938876E 01	-0.1866949E 02
0.150	0.2141117E 01	0.2313379E 01	-0.3278050E 02
0.250	0.2194795E 01	-0.1336816E 01	-0.3856261E 02
0.350	0.1871746E 01	-0.5044983E 01	-0.3382819E 02
0.450	0.1218520E 01	-0.7779930E 01	-0.1951801E 02
0.550	0.3749803E 00	-0.8754983E 01	0.5394089E 00
0.650	-0.4630277E 00	-0.7666565E 01	0.2075845E 02
0.750	-0.1098110E 01	-0.4788715E 01	0.3547091E 02
0.850	-0.1386567E 01	-0.8941116E 00	0.4062941E 02
0.950	-0.1277845E 01	0.2974423E 01	0.3501531E 02
1.050	-0.8264314E 00	0.5811985E 01	0.2058021E 02
1.150	-0.1728700E 00	0.6944924E 01	0.1818384E 01
1.250	0.4999934E 00	0.6220047E 01	-0.1563247E 02
1.350	0.1021921E 01	0.4032777E 01	-0.2672656E 02
1.450	0.1284444E 01	0.1187430E 01	-0.2854307E 02
1.550	0.1269343E 01	-0.1365278E 01	-0.2114330E 02
1.650	0.1048102E 01	-0.2830577E 01	-0.7490256E 01
1.750	0.7531204E 00	-0.2817090E 01	0.7538406E 01
1.850	0.5302667E 00	-0.1452336E 01	0.1873035E 02
1.950	0.4881806E 00	0.6690398E 00	0.2221362E 02
2.050	0.6606031E 00	0.2687022E 01	0.1670003E 02
2.150	0.9938517E 00	0.3763516E 01	0.3906191E 01
2.250	0.1363534E 01	0.3363824E 01	-0.1198093E 02
2.350	0.1615299E 01	0.1442648E 01	-0.2562756E 02
2.450	0.1616731E 01	-0.1525188E 01	-0.3225165E 02
2.550	0.1303869E 01	-0.4680505E 01	-0.2916398E 02
2.650	0.7073371E 00	-0.7040914E 01	-0.1666674E 02
2.750	-0.5065416E-01	-0.7806685E 01	0.1976008E 01
2.850	-0.7881384E 00	-0.6616079E 01	0.2149321E 02
2.950	-0.1314759E 01	-0.3670047E 01	0.3619575E 02
3.050	-0.1487195E 01	0.3137219E 00	0.4170975E 02
3.150	-0.1251760E 01	0.4304714E 01	0.3631849E 02
3.250	-0.6611172E 00	0.7259899E 01	0.2149681E 02
3.350	0.1400939E 00	0.8428938E 01	0.1471941E 01
3.450	0.9564504E 00	0.7571182E 01	-0.1805476E 02
3.550	0.1597300E 01	0.5018523E 01	-0.3163064E 02
3.650	0.1929902E 01	0.1565717E 01	-0.3568807E 02
3.750	0.1914219E 01	-0.1777057E 01	-0.2959067E 02
3.850	0.1609163E 01	-0.4092371E 01	-0.1577006E 02
3.950	0.1148966E 01	-0.4829101E 01	0.1083350E 01

RUN 5.30 ACTUAL OUTPUT

T	R(T)	DR(T)	DDR(T)
0.050	0.1762292E 01	0.4933204E 01	-0.1840920E 02
0.150	0.2139609E 01	0.2320372E 01	-0.3255046E 02
0.250	0.2196410E 01	-0.1324026E 01	-0.3844223E 02
0.350	0.1874763E 01	-0.5029287E 01	-0.3381265E 02
0.450	0.1221371E 01	-0.7757496E 01	-0.1957212E 02
0.550	0.3778123E 00	-0.8724501E 01	0.4111782E 00
0.650	-0.4592694E 00	-0.7636938E 01	0.2052792E 02
0.750	-0.1093578E 01	-0.4773226E 01	0.3516295E 02
0.850	-0.1382604E 01	-0.8975653E 00	0.4034260E 02
0.950	-0.1275110E 01	0.2959144E 01	0.3484286E 02
1.050	-0.8237377E 00	0.5793915E 01	0.2052532E 02
1.150	-0.1685263E 00	0.6925314E 01	0.1810723E 01
1.250	0.5059884E 00	0.6195520E 01	-0.1564504E 02
1.350	0.1027705E 01	0.4005633E 01	-0.2673257E 02
1.450	0.1288331E 01	0.1167402E 01	-0.2851989E 02
1.550	0.1271353E 01	-0.1371376E 01	-0.2112144E 02
1.650	0.1049060E 01	-0.2827500E 01	-0.7529090E 01
1.750	0.7528389E 00	-0.2815910E 01	0.7435587E 01
1.850	0.5275501E 00	-0.1457683E 01	0.1864270E 02
1.950	0.4827871E 00	0.6627890E 00	0.2222530E 02
2.050	0.6544383E 00	0.2686408E 01	0.1681328E 02
2.150	0.9895827E 00	0.3767041E 01	0.4052523E 01
2.250	0.1362264E 01	0.3365698E 01	-0.1185485E 02
2.350	0.1616205E 01	0.1443381E 01	-0.2551871E 02
2.450	0.1618903E 01	-0.1517107E 01	-0.3215389E 02
2.550	0.1307665E 01	-0.4659297E 01	-0.2912905E 02
2.650	0.7134020E 00	-0.7012595E 01	-0.1677362E 02
2.750	-0.4319079E-01	-0.7784130E 01	0.1713680E 01
2.850	-0.7815321E 00	-0.6606293E 01	0.2115982E 02
2.950	-0.1310492E 01	-0.3670035E 01	0.3589899E 02
3.050	-0.1484557E 01	0.3082342E 00	0.4149155E 02
3.150	-0.1249042E 01	0.4291059E 01	0.3615935E 02
3.250	-0.6578129E 00	0.7232847E 01	0.2139175E 02
3.350	0.1429601E 00	0.8391574E 01	0.1461235E 01
3.450	0.9581022E 00	0.7536420E 01	-0.1794703E 02
3.550	0.1598452E 01	0.4997765E 01	-0.3146130E 02
3.650	0.1931647E 01	0.1558773E 01	-0.3556241E 02
3.750	0.1916124E 01	-0.1778140E 01	-0.2956319E 02
3.850	0.1609352E 01	-0.4091518E 01	-0.1580219E 02
3.950	0.1146293E 01	-0.4822620E 01	0.1058495E 01

7.4 SAMPLE PROGRAM AND RESULTS FOR INDEFINITE INTEGRATION WITH SMOOTHING

The following program for indefinite integration with smoothing was run with the input given by (7.7) and parameter values as follow: $a_1 = 1.5$, $a_2 = 2.0$, $a_3 = 1.5$, $a_4 = 0$, $f_1 = 0.7$, $f_2 = 0.9$, $f_3 = 2.0$, $f_s = 10$, $f_c = 1.0$, and $\Delta f = 0.6$ (Δf both the inner and outer roll-off length). In terms of the frequency ratio, the input frequencies are .07, .09, and 0.2. Also $\tau_c = 0.1$, $\tau_d = .06$, and $\tau_T = .16$.

N was taken to be 25, and hence 2N+1 = 51 weights were used. The number of terms used in computing the sine integral was 25--which is too many for small values of the argument. For large values of the argument, the first terms of the series may become large enough to cause loss of significance, and computation of the sine integral in this case should be approached with caution.

A transfer function recovery is provided in this case. This may be compared with the designed transfer function of this filter in Section 6.0.

```
      INDEFINITE INTEGRATION WITH SMOOTHING

  DIMENSION TERMA(50),TERMB(50),H(30),Z(101)
1 FORMAT (4F10.0)
2 FORMAT (15X,F7.3,E20.7)
3 FORMAT(1H1,37HINDEFINITE INTEGRATION WITH SMOOTHING/)
4  FORMAT (20X,23HF      TRANSFER FUNCTION/)
5 FORMAT(/////1X,32HDESIRED OUTPUT AND ACTUAL OUTPUT/)
6 FORMAT (17X,1HT,8X,14HDESIRED OUTPUT,6X,13HACTUAL OUTPUT/)
  P=3.14159

      READ PROBLEM PARAMETERS

  READ(2,1) XM,XN,TC,TD
  READ(2,1) RA,RB,RC,FS
  READ(2,1) AA,AB,AC
  READ(2,1) BA,BB,BC
  M=XM
  N=XN
  RT=TC+TD

      COMPUTATION OF SINE INTEGRAL

  TERMA(1)=1.
  TERMB(1)=1.
  DO 9 I=1,N
  X=I
  XA=2.*X*P*TC
  XB=2.*X*P*TD
  DO 7 K=1,M
  Y=K
  J=K+1
  Y=(2.*Y-1.)/(2.*Y*(2.*Y+1.)**2)
  TERMA(J)=-XA**2*TERMA(K)*Y
7 TERMB(J)=-XB**2*TERMB(K)*Y
  SA=0.
  SB=0.
  DO 8 J=1,M
  SA=SA+TERMA(J)
8 SB=SB+TERMB(J)

      COMPUTATION OF THE FILTER WEIGHTS

  A=2.*P*TD*(XB*SB-XA*SA)
  A=A+COS(2.*P*X*TD)/X-SIN(2.*P*X*TD)/(2.*P*TD*X**2)
  A=A+(SIN(2.*P*X*RT)-SIN(2.*P*X*TC))/(2.*P*TC*X**2)
9 H(I)=A-TD*COS(2.*P*X*TC)/(TC*X)

      TRANSFER FUNCTION RECOVERY

  WRITE(3,3)
  WRITE(3,4)
  DO 11 K=1,51
  HX=0.
```

```
      Y=K-1
      Y=.01*Y
      DO 10 I=1,N
      X=I
   10 HX=HX+2.*H(I)*SIN(2.*P*X*Y)
      HX=HX/(2.*P**2*TD*FS)
      Y=Y*FS
   11 WRITE(3,2) Y,HX

          GENERATION OF SAMPLE INPUT DATA

      MA=N+1
      MB=2*N+40
      DO 12 I=1,MB
      T=I-MA
      CA=COS(2.*P*RA*T)
      S= SIN(2.*P*RB*T)
      CC=COS(2.*P*RC*T)
   12 Z(I)=AA*CA+AB*S+AC*CC

          COMPUTATION OF DESIRED AND ACTUAL OUTPUTS

      WRITE(3,5)
      WRITE(3,6)
      DO 14 K=1,40
      MA=K-1
      MB=N+1
      SA=0.
      T=MA
      CA=SIN(2.*P*RA*T)
      S= COS(2.*P*RB*T)
      CC=SIN(2.*P*RC*T)
      W=(1./(2.*P*FS))*((BA*CA/RA)-(BB*S/RB)+(BC*CC/RC))
      T=T/FS
      DO 13 I=1,N
      KA=MB-I
      KB=I+MA
      KC=MA+MB+I
   13 SA=SA-H(KA)*Z(KB)+H(I)*Z(KC)
      SA=SA/(2.*P**2*TD*FS)
   14 WRITE(3,16) T,W,SA
   15 PAUSE
   16 FORMAT (13X,F7.3,2E20.7)
      CALL EXIT
      END
```

INDEFINITE INTEGRATION WITH SMOOTHING

F	TRANSFER FUNCTION
0.000	0.0000000E 00
0.100	-0.4835658E-01
0.200	-0.8780948E-01
0.300	-0.1277766E 00
0.400	-0.1789851E 00
0.500	-0.2279376E 00
0.600	-0.2476159E 00
0.700	-0.2315462E 00
0.800	-0.2007518E 00
0.900	-0.1757450E 00
1.000	-0.1565062E 00
1.100	-0.1339059E 00
1.200	-0.1067456E 00
1.300	-0.7933023E-01
1.400	-0.5228427E-01
1.500	-0.2565890E-01
1.600	-0.5420293E-02
1.700	0.2290246E-02
1.800	0.6325266E-03
1.900	-0.1412288E-02
2.000	-0.2060364E-03
2.100	0.9969801E-03
2.200	0.8562479E-04
2.300	-0.7656668E-03
2.400	-0.3707978E-04
2.500	0.6189613E-03
2.600	0.1383451E-04
2.700	-0.5182999E-03
2.800	-0.1719720E-05
2.900	0.4456929E-03
3.000	-0.4721969E-05
3.100	-0.3915387E-03
3.200	0.7956620E-05
3.300	0.3501991E-03
3.400	-0.9256311E-05
3.500	-0.3181395E-03
3.600	0.9363038E-05
3.700	0.2930673E-03
3.800	-0.8737031E-05
3.900	-0.2734517E-03
4.000	0.7676857E-05
4.100	0.2582632E-03
4.200	-0.6377806E-05
4.300	-0.2468168E-03
4.400	0.4938406E-05
4.500	0.2386498E-03
4.600	-0.3382480E-05
4.700	-0.2334196E-03
4.800	0.1717076E-05
4.900	0.2308705E-03
5.000	0.1579199E-07

DESIRED OUTPUT AND ACTUAL OUTPUT

T	DESIRED OUTPUT	ACTUAL OUTPUT
0.000	-0.3536779E 00	-0.3514901E 00
0.100	-0.1534097E 00	-0.1485978E 00
0.200	0.1121917E 00	0.1181381E 00
0.300	0.3746589E 00	0.3802788E 00
0.400	0.5604481E 00	0.5649213E 00
0.500	0.6122803E 00	0.6152744E 00
0.600	0.5068678E 00	0.5080646E 00
0.700	0.2635254E 00	0.2626033E 00
0.800	-0.5927271E-01	-0.6217347E-01
0.900	-0.3788076E 00	-0.3828688E 00
1.000	-0.6104846E 00	-0.6146804E 00
1.100	-0.6913374E 00	-0.6950833E 00
1.200	-0.5978873E 00	-0.6010847E 00
1.300	-0.3531308E 00	-0.3556196E 00
1.400	-0.2054097E-01	-0.2175848E-01
1.500	0.3132720E 00	0.3139244E 00
1.600	0.5623014E 00	0.5648551E 00
1.700	0.6645101E 00	0.6683750E 00
1.800	0.5981959E 00	0.6026798E 00
1.900	0.3868218E 00	0.3914806E 00
2.000	0.9117493E-01	0.9553805E-01
2.100	-0.2086029E 00	-0.2054478E 00
2.200	-0.4357007E 00	-0.4349084E 00
2.300	-0.5374080E 00	-0.5396085E 00
2.400	-0.4981004E 00	-0.5028978E 00
2.500	-0.3410508E 00	-0.3473235E 00
2.600	-0.1190831E 00	-0.1256373E 00
2.700	0.1026215E 00	0.9682514E-01
2.800	0.2660714E 00	0.2621591E 00
2.900	0.3364188E 00	0.3356144E 00
3.000	0.3097564E 00	0.3127672E 00
3.100	0.2109084E 00	0.2172432E 00
3.200	0.8255693E-01	0.9059381E-01
3.300	-0.3031365E-01	-0.2251394E-01
3.400	-0.9537795E-01	-0.8934363E-01
3.500	-0.1024980E 00	-0.9927360E-01
3.600	-0.6495364E-01	-0.6530841E-01
3.700	-0.1235776E-01	-0.1659110E-01
3.800	0.2197467E-01	0.1457962E-01
3.900	0.1462374E-01	0.5923016E-02

BIBLIOGRAPHY

1. Tolstov, Georgi P., Fourier Series, Prentice-Hall, Englewood Cliffs, N. J., 1962.
2. Titchmarsh, Edward Charles, Introduction to the Theory of Fourier Integrals, The Clarendon Press, Oxford, 1937.
3. Lighthill, M., Introduction to Fourier Analysis and Generalized Functions, Cambridge (Eng.), University Press, 1958.
4. Campbell, G. A., and Foster, R. M., Fourier Integrals for Practical Applications, Van Nostrand, Princeton, N. J., 1947.
5. Friedman, Avner, Generalized Functions and Partial Differential Equations, Prentice-Hall, Englewood Cliffs, N. J., 1963.
6. Schwartz, Laurent, Theorie des distributions, Volumes 1 and 2, Hermann et Cie, Paris, 1950-1.
7. Temple, George, "Generalised functions", Proc. Roy. Soc. A, 228, 175-90, 1955.
8. Gel'fand, I. M., and Shilov, G. E., Generalized Functions, Academic Press, N. Y., 1964.
9. Papoulis, Athanasios, The Fourier Integral and Its Applications, McGraw-Hill, N. Y., 1962.
10. Martin, Marcel A., "Frequency Domain Applications in Data Processing", GE Technical Information Series, 57SD340, May, 1957.
11. Martin, Marcel A., "Digital Filters for Data Processing", GE Technical Information Series, 62SD484, October, 1962.
12. Carslaw, Horatio Scott, Introduction to the Theory of Fourier Series and Integrals, Dover, N. Y., 1930.
13. Graham, Ronald J., "Determination and Analysis of Numerical Smoothing Weights", NASA Technical Report, R-179, 1963.
14. Ormsby, Joseph F. A., "Design of Numerical Filters with Applications to Missile Data Processing", Journal of the Association for Computing Machinery, Vol. 8, No. 3, July, 1961.
15. Wiley, C. R., Jr., Advanced Engineering Mathematics, McGraw-Hill, N. Y., 1960.
16. Friant, R. J., Jr., "Practical Digital Data Smoothing", GE Technical Information Series, R58EMH29, June, 1958.
17. Fleck, J. T., and Fryer, W. D., "An Exploration of Numerical Filtering Techniques", Cornell Aeronautical Laboratory, Inc., Report No. XA-869-P-1.
18. Anders, Edward B., et al., "Digital Filters", NASA CR-136, National Aeronautics and Space Administration, Washington, December, 1964.

APPENDIX A

CONSTRAINTS

In order to develop constraints on the weights h_k such that the recovered transfer function $\bar{H}$ has an exact fit at some specified frequency $\bar{F}$ we need to consider two separate cases.* The first is

when H is of the form $H(r) = h_o + 2\sum_{n=1}^{N} h_n \cos 2\pi nr$, $r = \frac{f}{f_s} = \frac{w}{2\pi f_s}$.

The second is when H(r) is of the form $H(r) = 2i\sum_{n=1}^{N} h_n \sin 2\pi nr$.

A.1 Constraints at one point

Case I. Suppose $\bar{H}(r) = \bar{h}_o + 2\sum_{n=1}^{N} \bar{h}_n \cos 2\pi nr$,

then

$$\bar{H}'(r) = -4\pi \sum_{n=1}^{N} n\bar{h}_n \sin 2\pi nr.$$

We wish to impose the following constraints:

$$\bar{H}(\bar{r}) = F(\bar{r}),$$

$$\bar{H}'(\bar{r}) = F'(\bar{r}),$$

i.e.,

$$\bar{h}_o + 2\sum_{n=1}^{N} \bar{h}_n \cos 2\pi n\bar{r} - F(\bar{r}) = 0,$$

* This is a reprint of Appendix A of NASA CR-136. The symbol r is used here to denote the frequency ratio f/f_s. Also, the symbol F is used here to denote a function of r.

$$4\pi \sum_{n=1}^{N} n\bar{h}_n \sin 2\pi n\bar{r} + F'(\bar{r}) = 0.$$

In order to minimize the error between H and $\bar{H}$ under the above constraints we define

$$R = \int_0^{\frac{1}{2}} [\bar{H}(r) - H(r)]^2 dr + \alpha[4\pi \sum_{n=1}^{N} n\bar{h}_n \sin 2\pi n\bar{r} + F'(\bar{r})].$$

Since

$$\bar{h}_o = F(\bar{r}) - 2\sum_{n=1}^{N} \bar{h}_n \cos 2\pi n\bar{r},$$

$$R = \int_0^{\frac{1}{2}} [F(\bar{r}) + 2\sum_{n=1}^{N} \bar{h}_n(\cos 2\pi nr - \cos 2\pi n\bar{r}) - h_o - 2\sum_{n=1}^{N} h_n \cos 2\pi nr]^2 dr$$

$$+ \alpha[4\pi \sum_{n=1}^{N} n\bar{h}_n \sin 2\pi n\bar{r} + F'(\bar{r})].$$

$$\frac{\partial R}{\partial \bar{h}_k} = 2\int_0^{\frac{1}{2}} [F(\bar{r}) + 2\sum_{n=1}^{N} \bar{h}_n(\cos 2\pi nr - \cos 2\pi n\bar{r}) - h_o - 2\sum_{n=1}^{N} h_n \cos 2\pi nr]$$

$$[\cos 2\pi kr - \cos 2\pi k\bar{r}]dr + \alpha[4\pi k \sin 2\pi k\bar{r}].$$

Let $\frac{\partial R}{\partial \bar{h}_k} = 0,\ k = 1, \ldots, N$, then

$$-\frac{1}{2} F(\bar{r})\cos 2\pi k\bar{r} + \frac{\bar{h}_k}{2} + \sum_{n=1}^{N} \bar{h}_n \cos 2\pi n\bar{r} \cos 2\pi k\bar{r} + \frac{h_o}{2}\cos 2\pi k\bar{r} - \frac{h_k}{2}$$

$$= -\alpha[\pi k \sin 2\pi k\bar{r}]$$

$$\frac{1}{2}(\bar{h}_k - h_k) + [\sum_{n=1}^{N} \bar{h}_n \cos 2\pi n\bar{r} - \frac{F(\bar{r})}{2}]\cos 2\pi k\bar{r}$$

$$+ \frac{h_o}{2}\cos 2\pi k\bar{r} = - \alpha[\pi k \sin 2\pi k\bar{r}].$$

Let $\delta = (\bar{h}_o - h_o)$, then

$$(\bar{h}_k - h_k) = \delta \cos 2\pi k\bar{r} - 2\alpha[\pi k \sin 2\pi k\bar{r}]. \qquad (A.1)$$

Multiply (A.1) by ($2 \cos 2\pi k\bar{r}$). Summing from 1 to N gives

$$2\sum_{k=1}^{N}(\bar{h}_k - h_k)\cos 2\pi k\bar{r} = 2\delta\sum_{k=1}^{N}\cos^2 2\pi k\bar{r} - 4\alpha\sum_{k=1}^{N}\pi k \cos 2\pi k\bar{r} \sin 2\pi k\bar{r},$$

adding $(\bar{h}_o - h_o)$ to both sides gives

$$(\bar{h}_o - h_o) + 2\sum_{k=1}^{N}(\bar{h}_k - h_k)\cos 2\pi k\bar{r}$$

$$= \delta + 2\delta\sum_{k=1}^{N}\cos^2 2\pi k\bar{r} - 4\alpha\sum_{k=1}^{N}\pi k \cos 2\pi k\bar{r} \sin 2\pi k\bar{r}.$$

Let

$$\Delta_1 = h_o + 2\sum_{n=1}^{N} h_n \cos 2\pi n\bar{r} - F(\bar{r}).$$

Hence

$$\Delta_1 = (h_o - \bar{h}_o) + 2\sum_{n=1}^{N}(h_n - \bar{h}_n)\cos 2\pi n\bar{r},$$

so

$$\Delta_1 = 4\alpha\sum_{n=1}^{N}\pi n \cos 2\pi n\bar{r} \sin 2\pi n\bar{r} - \delta - 2\delta\sum_{n=1}^{N}\cos^2 2\pi n\bar{r}.$$

Now multiply (A.1) by $2\kappa\sin 2\pi k\bar{r}$. Summing from 1 to N gives

$$2\sum_{k=1}^{N} k(\bar{h}_k - h_k)\sin 2\pi k\bar{r} = 2\,\delta\sum_{k=1}^{N} k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r} - 4\alpha\sum_{k=1}^{N}\pi k^2\sin^2 2\pi k\bar{r}.$$

Let

$$\Delta_2 = -4\pi\sum_{n=1}^{N} nh_n \sin 2\pi n\bar{r} - F'(\bar{r}).$$

Hence

$$\Delta_2 = 4\pi\sum_{n=1}^{N} n(\bar{h}_n - h_n)\sin 2\pi n\bar{r}.$$

So

$$\Delta_2 = 4\pi\,\delta\sum_{k=1}^{N} k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r} - 8\pi^2\alpha\sum_{k=1}^{N} k^2\sin^2 2\pi k\bar{r}.$$

Let $Q_1 = 2\sum_{k=1}^{N}\cos^2 2\pi k\bar{r}$,

$$Q_2 = 4\pi\sum_{k=1}^{N} k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r},$$

$$Q_3 = 8\pi^2\sum_{k=1}^{N} k^2\sin^2 2\pi k\bar{r}.$$

Then

$$\Delta_1 = Q_2\alpha - (1+Q_1)\,\delta\ ,$$

and

$$\Delta_2 = -Q_3\alpha + Q_2\,\delta\ .$$

Solving we find that

$$\delta = \frac{\Delta_1 Q_3 + \Delta_2 Q_2}{Q_2^2 - (1+Q_1)Q_3}, \tag{A.2}$$

$$\alpha = \frac{\Delta_1 Q_2 + \Delta_2 (1+Q_1)}{Q_2^2 - (1+Q_1)Q_3} \;. \tag{A.3}$$

Therefore the constrained weights are

$$\bar{h}_o = h_o + \delta \quad ,$$

$$\bar{h}_k = h_k + \delta \;\; \cos 2\pi k\bar{r} - \alpha 2\pi k \sin 2\pi k\bar{r}, \quad k \geq 1,$$

where δ and α are as defined in (A.2) and (A.3).

Case II. Suppose $\bar{H}(r) = 2i \sum_{n=1}^{N} \bar{h}_n \sin 2\pi nr,$

then

$$\bar{H}'(r) = 4\pi i \sum_{n=1}^{N} n\bar{h}_n \cos 2\pi nr.$$

We wish to impose the following constraints

$$\bar{H}(\bar{r}) = F(\bar{r}),$$

$$\bar{H}'(\bar{r}) = F'(\bar{r}),$$

i.e.,

$$2 \sum_{n=1}^{N} \bar{h}_n \sin 2\pi n\bar{r} - \frac{F(\bar{r})}{i} = 0,$$

and

$$4\pi \sum_{n=1}^{N} n\bar{h}_n \cos 2\pi n\bar{r} - \frac{F'(\bar{r})}{i} = 0.$$

In order to minimize the error between H and $\bar{H}$ under the above conditions we define

$$R = \int_0^{\frac{1}{2}} [\bar{H}(r) - H(r)]^2 dr + \alpha[4\pi \sum_{n=1}^{N} n\bar{h}_n \cos 2\pi n\bar{r} - \frac{F'(\bar{r})}{i}].$$

Since

$$\bar{h}_1 = \frac{\frac{F(\bar{r})}{i} - 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}}{2\sin 2\pi\bar{r}},$$

$$R = \int_0^{\frac{1}{2}} [\frac{\sin 2\pi r}{\sin 2\pi\bar{r}} [\frac{F(\bar{r})}{i} - 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}] + 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi nr$$

$$- 2\sum_{n=1}^{N} h_n \sin 2\pi nr]^2 dr + \alpha[4\pi \sum_{n=1}^{N} n\bar{h}_n \cos 2\pi n\bar{r} - \frac{F(\bar{r})}{i}]$$

$$\frac{\partial R}{\partial \bar{h}_k} = 2\int_0^{\frac{1}{2}} [\frac{\sin 2\pi r}{\sin 2\pi\bar{r}}[\frac{F(\bar{r})}{i} - 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}] + 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi nr$$

$$-2\sum_{n=1}^{N} h_n \sin 2\pi nr][\frac{-2\sin 2\pi r \sin 2\pi k\bar{r}}{\sin 2\pi\bar{r}} + 2\sin 2\pi kr]dr + 4\pi\alpha k\cos 2\pi k\bar{r}.$$

Let $\frac{\partial R}{\partial \bar{h}_k} = 0$, $k = 2, \ldots, N$.

Now $$\int_0^{\frac{1}{2}} \frac{\sin 2\pi r}{\sin 2\pi\bar{r}} [\frac{F(r)}{i} - 2\sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}][\sin 2\pi kr - \frac{\sin 2\pi r \sin 2\pi kr}{\sin 2\pi\bar{r}}]dr$$

$$= - \frac{F(\bar{r})}{4i} \frac{\sin 2\pi k\bar{r}}{\sin^2 2\pi\bar{r}} + \frac{\sin 2\pi k\bar{r}}{2\sin^2 2\pi\bar{r}} \sum_{n=2}^{N} \bar{h}_n \sin 2\pi n\bar{r}$$

$$\text{Also} \int_0^{\frac{1}{2}} \sum_{n=2}^{N} [\bar{h}_n - h_n] \sin 2\pi n r [\sin 2\pi k r - \frac{\sin 2\pi r \sin 2\pi k \bar{r}}{\sin 2\pi \bar{r}}] dr = \frac{1}{4}[\bar{h}_k - h_k],$$

$$\int_0^{\frac{1}{2}} [h_1 \sin^2 2\pi r] \frac{\sin 2\pi k \bar{r}}{\sin 2\pi \bar{r}} dr = \frac{h_1}{4} \frac{\sin 2\pi k \bar{r}}{\sin 2\pi \bar{r}} .$$

Hence

$$\frac{1}{4} [2 \sum_{n=2}^{N} \bar{h}_n \sin 2\pi n \bar{r} - \frac{F(\bar{r})}{i}] \frac{\sin 2\pi k \bar{r}}{\sin^2 2\pi \bar{r}} + \frac{1}{4} [\bar{h}_k - h_k] + \frac{h_1}{4} \frac{\sin 2\pi k \bar{r}}{\sin 2\pi \bar{r}}$$

$$= - \frac{\alpha}{4} \pi k \cos 2\pi k \bar{r}$$

$$(h_1 - \bar{h}_1) \frac{\sin 2\pi \bar{r} k}{\sin 2\pi \bar{r}} + (\bar{h}_k - h_k) = - \alpha \pi k \cos 2\pi k \bar{r}.$$

Let $\delta = \bar{h}_1 - h_1$, then

$$\bar{h}_k - h_k = \delta \frac{\sin 2\pi \bar{r} k}{\sin 2\pi \bar{r}} - \alpha \pi k \cos 2\pi k \bar{r}. \tag{A.4}$$

Multiplying (A.4) by $2\sin 2\pi k\bar{r}$ and summing from 2 to N gives

$$2 \sum_{k=2}^{N} (\bar{h}_k - h_k) \sin 2\pi k \bar{r} = 2\delta \sum_{k=2}^{N} \frac{\sin^2 2\pi k \bar{r}}{\sin 2\pi \bar{r}} - \alpha 2 \sum_{k=2}^{N} k \cos 2\pi k \bar{r} \sin 2\pi k \bar{r}.$$

Adding $2(\bar{h}_1 - h_1) \sin 2\pi\bar{r}$ to both sides yields

$$2 \sum_{k=1}^{N} (\bar{h}_k - h_k) \sin 2\pi k \bar{r} = 2\delta \sum_{k=1}^{N} \frac{\sin^2 2\pi k \bar{r}}{\sin 2\pi \bar{r}} - 2\alpha \sum_{k=2}^{N} k \cos 2\pi k \bar{r} \sin 2\pi k \bar{r}.$$

Let $\Delta_1 = 2 \sum_{k=1}^{N} h_k \sin 2\pi k \bar{r} - F(\bar{r}).$

Since

$$F(\bar{r}) = 2\sum_{k=1}^{N}\bar{h}_k \sin 2\pi k\bar{r},$$

$$\Delta_1 = 2\sum_{k=1}^{N}(h_k-\bar{h}_k)\sin 2\pi k\bar{r},$$

or

$$\Delta_1 = 2\alpha\sum_{k=2}^{N} k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r} - 2\,\delta\sum_{k=1}^{N}\frac{\sin^2 2\pi k\bar{r}}{\sin 2\pi\bar{r}}\,.$$

Multiplying (A.4) by $4\pi k\cos 2\pi k\bar{r}$ and summing from 2 to N gives

$$4\pi\sum_{k=1}^{N}(\bar{h}_k-h_k)k\cos 2\pi k\bar{r} = 4\pi\delta\sum_{k=1}^{N}\frac{k\sin 2\pi k\bar{r}\,\cos 2\pi k\bar{r}}{\sin 2\pi\bar{r}} - 4\alpha\sum_{k=2}^{N}\pi^2k^2\cos^2 2\pi k\bar{r},$$

adding $4\pi(\bar{h}_1-h_1)\cos 2\pi\bar{r}$ to both sides of the above equation gives

$$4\pi\sum_{k=1}^{N}(\bar{h}_k-h_k)k\cos 2\pi k\bar{r} = 4\pi\delta\sum_{k=1}^{N}\frac{k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r}}{\sin 2\pi\bar{r}} - 4\alpha\sum_{k=2}^{N}\pi^2k^2\cos^2 2\pi k\bar{r}.$$

Let $\Delta_2 = 4\pi\sum_{k=1}^{N}kh_k\cos 2\pi k\bar{r} - F'(\bar{r})$.

Since $F'(\bar{r}) = 4\pi\sum_{k=1}^{N}k\bar{h}_k\cos 2\pi k\bar{r}$.

$$\Delta_2 = 4\pi\sum_{k=1}^{N}(h_k-\bar{h}_k)k\cos 2\pi k\bar{r}.$$

Hence

$$\Delta_2 = 4\pi\delta\sum_{k=1}^{N}\frac{k\cos 2\pi k\bar{r}\,\sin 2\pi k\bar{r}}{\sin 2\pi\bar{r}} - 4\alpha\pi^2\sum_{k=2}^{N}k^2\cos^2 2\pi k\bar{r}.$$

Let

$$Q_1 = 2\sum_{k=1}^{N} \frac{\sin^2 2\pi k\bar{r}}{\sin 2\pi\bar{r}},$$

$$Q_2 = 2\sum_{k=1}^{N} k\cos 2\pi k\bar{r} \sin 2\pi k\bar{r},$$

$$Q_3 = \frac{2\pi Q_2}{\sin 2\pi\bar{r}}$$

$$Q_4 = 4\pi^2 \sum_{k=1}^{N} k^2 \cos^2 2\pi k\bar{r}.$$

Then

$$\Delta_1 = \alpha(Q_2 - \cos 2\pi\bar{r} \sin 2\pi\bar{r}) - \delta Q_2,$$

and

$$\Delta_2 = -\alpha(Q_4 - 4\pi^2 \cos^2 2\pi\bar{r}) + \delta Q_3.$$

Solving for δ and α we find that

$$\delta = \frac{\Delta_1(Q_4 - 4\pi^2\cos^2 2\pi\bar{r}) + \Delta_2(Q_2 - \cos 2\pi\bar{r} \sin 2\pi\bar{r})}{Q_3(Q_2 - \cos 2\pi\bar{r} \sin 2\pi\bar{r}) - Q_2(Q_4 - 4\pi^2 \cos^2 2\pi\bar{r})} \tag{A.5}$$

$$\alpha = \frac{\Delta_1 Q_3 - \Delta_2 Q_2}{Q_3(Q_2 - \cos 2\pi\bar{r} \sin 2\pi\bar{r}) - Q_2(Q_4 - 4\pi^2\cos^2 2\pi\bar{r}))}. \tag{A.6}$$

Therefore the constrained weights are

$$\bar{h}_1 = h_1 + \delta$$

$$\bar{h}_k = h_k + \delta \frac{\sin 2\pi\bar{r}k}{\sin 2\pi\bar{r}} - \alpha\pi k \cos 2\pi k\bar{r}, \quad k \geq 2.$$

APPENDIX B

DETERMINATION OF DIGITAL FILTER WEIGHTS FOR A FILTER WHOSE GAIN AND PHASE FUNCTIONS ARE GIVEN AT A FINITE NUMBER OF POINTS

The procedure discussed in Chapter III for obtaining the weights of a digital filter assumes that the transfer function is given for all values of the frequency f. In some applications, the values of $H(f) = A(f)\exp(i\Phi(f))$ are known at only a finite number of points. In particular, the known values are sometimes the values of $A(f)$ and $\Phi(f)$ at a finite number of points on $[0, f_s/2]$. In this case, the filter weights must be determined by other means.

The method given here is a simple extension of harmonic analysis as presented in most advanced engineering mathematics and numerical analysis books to complex-valued functions.

Let $H(f)$ be a complex-valued function which is periodic with period f_s, and suppose that the values of $H(f)$ are known at $M + 1$ equally spaced points on $[-f_s/2, f_s/2]$, say

$$f_j = -f_s/2 + j(f_s/M) \quad , \; j=0,1,2,. . .,M.$$

We wish to approximate $H(f)$ by a finite trigonometric sum of the form

$$\sum_{n=-N_1}^{N_2} h_n \exp(2n\pi i f_j/f_s) \tag{B.1}$$

where the h_n are to be chosen such that

$$R = \sum_{j=0}^{M-1} \left[H(f_j) - \sum_{n=-N_1}^{N_2} h_n \exp(2n\pi i f_j/f_s)\right]^2 \tag{B.2}$$

is a minimum. This, of course, is minimization in the least squares sense. A necessary condition for R minimum is

$$\frac{\partial R}{\partial h_k} = 0 \tag{B.3}$$

for each k, $-N_1 \leq k \leq N_2$. For each k

$$\frac{\partial R}{\partial h_k} = 2 \sum_{j=0}^{M-1} \left[H(f_j) - \sum_{n=-N_1}^{N_2} h_n \exp(2n\pi i f_j/f_s)\right] \exp(2k\pi i f_j/f_s)$$

Setting $\frac{\partial R}{\partial h_k} = 0$ gives $N_1 + N_2 + 1$ equations

$$\sum_{j=0}^{M-1} H(f_j) \exp(2k\pi i f_j/f_s) - \sum_{n=-N_1}^{N_2} h_n \exp(2(n+k)\pi i f_j/f_s) = 0 \tag{B.4}$$

in $N_1 + N_2 + 1$ unknowns, the h_n's.

From (B.4) we have

$$\sum_{j=0}^{M-1} H(f_j) \exp(2k\pi i f_j/f_s) - \sum_{n=-N_1}^{N_2} \left[h_n \sum_{j=0}^{M-1} \exp(2(n+k)\pi i f_j/f_s)\right] = 0 \tag{B.5}$$

For each n,

$$\sum_{j=0}^{M-1} \exp(2(n+k)\pi i f_j/f_s) = \sum_{j=0}^{M-1} \exp(2(n+k)\pi i\{-f_s/2+j(f_s/M)\}/f_s)$$

$$= \sum_{j=0}^{M-1} \exp(2(n+k)\pi i(j/M-\tfrac{1}{2})$$

$$= \sum_{j=0}^{M-1} \exp(2(n+k)j\pi i/M)\ \exp(-(n+k)\pi i)$$

$$= (-1)^{n+k} \sum_{j=0}^{M-1} \exp(2(n+k)\pi ji/M)$$

for $n = -k$,

$$\sum_{j=0}^{M-1} \exp(2(n+k)\pi ij/M) = \sum_{j=0}^{M-1} 1 = M.$$

Suppose $n \neq -k$. Employing the identity

$$\sum_{i=0}^{n} z^i = \frac{1 - z^{n+1}}{1 - z},$$

$$\sum_{j=0}^{M-1} \exp(2(n+k)\pi ji/M) = \sum_{j=0}^{M-1} \{\exp(2(n+k)\pi i/M)\}^j$$

$$= \frac{1-\{\exp(2(n+k)\pi i/M\}^M}{1-\exp(2(n+k)\pi i/M)}$$

$$= \frac{1-\exp(2(n+k)\pi i)}{1-\exp(2(n+k)\pi i/M)}$$

$$= \frac{1-\cos 2(n+k)\pi - \sin 2(n+k)\pi}{1-\exp(2(n+k)\pi i/M)}$$

$$= 0$$

if $(n+k)/M$ is not an integer. This condition is always satisfied if $M > n+k$ or synonymously if

$$M > \max \{ N_1 + N_2, 2N_1, 2N_2\} = \max \{2N_1, 2N_2\} \qquad (B.6)$$

If condition (B.6) holds, then each of the equations (B.5) reduces to

$$\sum_{j=0}^{M-1} H(f_j) \exp(2k\pi i f_j/f_s) - Mh_{-k} = 0,$$

or

$$h_{-k} = 1/M \sum_{j=0}^{M-1} H(f_j)\exp(2k\pi i f_j/f_s)$$

Hence, replacing k by -k,

$$h_k = 1/M \sum_{j=0}^{M-1} H(f_j)\exp(-2k\pi i f_j/f_s) \qquad (B.7)$$

Note that condition (B.6) requires that the number of intervals of equal length into which the period of H(f) is divided is greater than twice the larger of the integers N_1 and N_2, or synonymously, the number $M + 1$ of equally spaced points is greater than or equal to $\max \{2N_1, 2N_2\}$.

The following discussion shows that the h_n's which give the least squares minimization are those computed by the trapezoidal rule from the formula (3.37) for the Fourier coefficients of H(f). Equation (B.7) can be written as

$$h_k = (1/M)\left[H(f_o)\ \exp(-2k\pi i f_o/f_s) + \sum_{j=1}^{M-1} H(f_j)\ \exp(-2k\pi i f_j/f_s)\right]$$

$$= (1/M)\left[\ H(-f_s/2)\ \exp(k\pi i) + \sum_{j=1}^{M-1} H(f_j)\ \exp(2k\pi i f_j/f_s)\right] \qquad \text{(B.8)}$$

$$= (1/M)\left[(\tfrac{1}{2})H(-f_s/2)\ \exp(k\pi i) + \sum_{j=1}^{M-1} H(f_j)\ \exp(-2k\pi i f_j/f_s) + (\tfrac{1}{2})H(f_s/2)\ \exp(-k\pi i)\right]$$

The last equality of (B.8) is possible since $H(-f_s/2) = H(f_s/2)$ and $\exp(k\pi i) = \exp(-k\pi i)$.

By applying the trapizoidal rule to (3.37), we have

$$1/f_s \int_{-f_s/2}^{f_s/2} H(f)\ \exp(-2k\pi i f/f_s)\ df \doteq$$

$$(1/f_s)\left\{\frac{f_s/M}{2}\left[H(-f_s/2)\ \exp\{(-2k\pi i)(f_s/2)/f_s\} + 2\sum_{j=1}^{M-1} H(-f_s/2 + j\frac{f_s}{M})\ \exp\{(-2k\pi i)(-f_s/2+j(f_s/M))/f_s\} + H(f_s/2)\ \exp\{(-2k\pi i)(f_s/2)/f_s\}\right]\right\}$$

$$= (1/M)[(\tfrac{1}{2})H(-f_s/2)\ \exp(k\pi i) + \sum_{j=1}^{M-1} H(f_j)\ \exp(-2k\pi i f_j/f_s)$$

$$+ (\tfrac{1}{2})H(f_s/2)\ \exp(-k\pi i)] \qquad (B.9)$$

which is identical to (B.8). Hence the coefficients for the least squares minimization can be computed by applying the trapezoidal rule to (3.37).

Writing H(f) in polar form, we have

$$H(f) = A(f)\ \exp(i\Phi(f))$$

where A(f) and Φ(f) are real. These are called the gain and phase functions, respectively, of the filter. In practice, the gain A(f) and phase Φ(f) are specified on $[0,f_s/2]$. Now a necessary and sufficient condition for the weights of a filter to be real is that

$$H(-f) = H^*(f)$$

where $H^*(f)$ denotes the complex conjugate of H(f). If A(f) is extended such that it is an even function on $[-f_s/2,f_s/2]$, then

$$H(-f) = A(-f)\ \exp(i\Phi(-f)) = A(f)\ \exp(-i\Phi(f)) = H^*(f)$$

and the corresponding weights are real. The formula for the weights can be written in a more useful form in this case. From (3.37), we have

$$h_n = 1/f_s \int_{-f_s/2}^{f_s/2} H(f)\ \exp(-2n\pi i f/f_s)\ df$$

$$= 1/f_s \int_{-f_s/2}^{0} H(f)\ \exp(-2n\pi if/f_s)df + 1/f_s \int_{0}^{f_s/2} H(f)\ \exp(-2n\pi if/f_s)df$$

$$= -1/f_s \int_{0}^{-f_s/2} H(f)\ \exp(-2n\pi if/f_s)df\ +\ 1/f_s \int_{0}^{f_s/2} H(f)\ \exp(-2n\pi if/f_s)df$$

$$= 1/f_s \int_{0}^{f_s/2} H(-f)\ \exp(2n\pi if/f_s)df\ +\ 1/f_s \int_{0}^{f_s/2} H(f)\ \exp(-2n\pi if/f_s)df$$

$$= 1/f_s \int_{0}^{f_s/2} H^*(f)\ \exp(2n\pi if/f_s)df\ +\ 1/f_s \int_{0}^{f_s/2} H(f)\ \exp(-2n\pi if/f_s)df$$

$$= 1/f_s \int_{0}^{f_s/2} [\ H^*(f)\ \exp(-2n\pi if/f_s)\ +\ H(f)\ \exp(-2n\pi if/f_s)]\,df$$

$$= 2/f_s \int_{0}^{f_s/2} \mathrm{Re}[H(f)\ \exp(-2n\pi if/f_s)]df,$$ where $\mathrm{Re}[\ H(f)\ \exp(-2n\pi if/f_s)]$

denotes the real part of $H(f)\ \exp(-2n\pi if/f_s)$.

Hence,

$$h_n = 2/f_s \int_{0}^{f_s/2} \mathrm{Re}[\ A(f)\ \exp(i\Phi(f))\ \exp(-2n\pi if/f_s)\ ]\ df$$

$$= 2/f_s \int_{0}^{f_s/2} \mathrm{Re}[A(f)\ \exp(i\Phi(f) - 2n\pi if/f_s)]\ df$$

$$= 2/f_s \int_0^{f_s/2} A(f)\cos[2n\pi f/f_s - \Phi(f)]\,df \qquad \text{(B.10)}$$

Now the h_n's which give the least squares minimization may be computed by applying the trapezoidal rule to (B.10)

Subdivide the closed interval $[0, f_s/2]$ into $N \geq M$ subintervals of equal length, and let $f_j = j(f_s/N)$, $j = 0,1,2,\ldots,N$. Then $f_{j+1} - f_j = f_s/N$, $j = 0,1,\ldots,N-1$. By applying the trapezoidal rule to (B.10),

$$\int_0^{f_s/2} A(f)\cos[2n\pi f/f_s - \Phi(f)]\,df$$

$$= f_s/2N \;\{A(0)\cos\Phi(0) + 2\sum_{j=1}^{N-1} A(f_j)\cos[2n\pi f_j/f_s - \Phi(f_j)] + A(f_s/2)\cos[n\pi - \Phi(f_s/2)]\}$$

So that

$$h_n = 2/f_s \int_0^{f_s/2} A(f)\cos[2n\pi f/f_s - \Phi(f)]\,df \qquad \text{(B.11)}$$

$$\doteq 1/N\;\{A(0)\cos\Phi(0) + 2\sum_{j=1}^{N-1} A(f_j)\cos[2n\pi f_j/f_s - \Phi(f_j)] + A(f_s/2)\cos[n\pi - \Phi(f_s/2)]\}$$

The function $\Phi(f)$ is odd, and hence we must have $\Phi(0) = 0$. Using this and applying a trigonometric identity to the last term, we have

$$h_n = (1/N) \{A(0) + 2 \sum_{j=1}^{N-1} A(f_j) \cos [2n\pi f_j/f_s - \Phi(f_j)]$$

$$+ (-1)^n (f_s/2) \cos (\Phi(f_s/2))\}$$

This yields the weights to be used in (3.41) to give the output of a digital filter which approximates the original filter.

APPENDIX C

DETERMINATION OF FREQUENCY CHARACTERISTICS IN SAMPLED DATA

By

Edward B. Anders

Given a set of tabulated data which is periodic and admits a finite trigonometric expansion, one may determine the frequencies present in the data and the coefficients of these frequency components by using the following theorems. The procedure is extremely simple and is based on a simple numerical integration procedure--the trapezoidal rule.

<u>Theorem 1:</u> Let

$$h\,(t) = a_o + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi}{f_s} nt + b_n \sin \frac{2\pi}{f_s} nt) \tag{C.1}$$

where f_s is the fundamental period of $h(t)$. If $h(t)$ is sampled at the $N + 1$ equally spaced points of $[-\frac{f_s}{2}, \frac{f_s}{2}]$ including the end points, then

$$\frac{1}{2}h(-\frac{f_s}{2}) + \sum_{P=-\frac{N}{2}+1}^{\frac{N}{2}-1} h(\frac{f_s}{N}P) + \frac{1}{2}h(\frac{f_s}{2}) = \begin{cases} N(a_o + \sum_{\beta=1}^{\infty} a_{\beta N}), & N \text{ even} \\ N(a_o + \sum_{\beta=1}^{\infty} (-1)^{\beta} a_{\beta N}), & N \text{ odd} \end{cases} \tag{C.2}$$

Note: $\sum_{k=\beta}^{\epsilon}$ means $k=\beta, \beta+1, \ldots, \beta+n$ where $\beta+n \leq \epsilon$, $\beta+n+1 > \epsilon$.

Proof: Since sin x is odd, all terms of (C.1) containing $\sin \frac{2\pi}{f_s}nt$ vanish. Thus, we are concerned only with terms containing $\cos\frac{2\pi}{f_s}nt$, $n=0,1,\ldots$. If N is even, then P is an integer and for integral β, we obtain

$$a_{\beta N}\cos \frac{2\pi}{f_s}\beta N(\frac{f_s}{N}P) = a_{\beta N}\cos 2\pi\beta P = a_{\beta N}$$

If N is odd, P is an odd multiple of $\frac{1}{2}$, say $P = m(\frac{1}{2})$, m odd, and for integral β, we obtain

$$a_{\beta N}\cos \frac{2\pi}{f_s}\beta N(\frac{f_s}{N}P) = a_{\beta N}\cos \pi\beta m = (-1)^{\beta}a_{\beta N}$$

To complete the proof we must consider two cases:

I) When N is even, $n \neq \beta N$, $\beta=0,1,\ldots$.

II) When N is odd and $n \neq \beta N$, $\beta=0,1,\ldots$.

Case I: Consider the set of points $0, \frac{f_s}{N}, 2\frac{f_s}{N}, \ldots, \frac{N}{2}(\frac{f_s}{N})$. Substituting these $\frac{N}{2}+1$ points into a term of (C.1) where $n \neq \beta N$, $\beta=0,1,\ldots$, multiplying the first and last such quantity by $\frac{1}{2}$ and adding we obtain

$$\frac{1}{2}a_n \cos \frac{2\pi}{f_s}n(0) + a_n \sum_{m=1}^{\frac{N}{2}-1} \cos \frac{2\pi}{f_s}n\frac{f_s}{N} + \frac{1}{2}a_n \cos \frac{2\pi}{f_s}n\frac{f_s}{N}\frac{N}{2}$$

$$= a_n\left(\frac{1}{2} + \sum_{m=1}^{\frac{N}{2}} \cos \frac{2\pi}{f_s} n \frac{f_s}{N} m - \frac{1}{2} \cos \pi n\right)$$

$$= a_n\left(\frac{\sin(\frac{N}{2}+\frac{1}{2})\frac{2\pi n}{N}}{2 \sin \frac{\pi n}{N}} - \frac{1}{2}\cos \pi n\right)$$

$$= a_n\left(\frac{\sin \pi n \cos \frac{\pi n}{N} + \cos \pi n \sin \frac{\pi n}{N}}{2 \sin \frac{\pi n}{N}} - \frac{1}{2}\cos \pi n\right) = 0.$$

<u>Case II</u>: Consider the $\frac{N+1}{2}$ points $\frac{1}{2}\frac{f_s}{N}, \frac{3}{2}\frac{f_s}{N}, \ldots, \frac{N}{2}\frac{f_s}{N}$.

Substituting as in Case I but multiplying only the last by $\frac{1}{2}$ and adding we obtain, for $n \neq \beta N$, $\beta = 0,1,\ldots,$

$$a_n \sum_{P=\frac{1}{2}}^{\frac{N}{2}-1} \cos \frac{2\pi}{f_s} n \frac{f_s}{N} P + \frac{1}{2} a_n \cos \pi n$$

$$= a_n\left(\frac{1}{2} + \sum_{m=1}^{N} \cos \frac{\pi n}{N} m - \frac{1}{2} - \sum_{m=1}^{\frac{N-1}{2}} \cos \frac{2\pi n}{N} m - \frac{1}{2} \cos \pi n\right)$$

$$= a_n \left(\frac{\sin(N+\frac{1}{2})\frac{\pi n}{N}}{2 \sin \frac{\pi n}{2N}} - \frac{\sin \pi n}{2 \sin \frac{\pi n}{N}} - \frac{1}{2} \cos \pi n\right)$$

$$= a_n\left(\frac{\sin \pi n \cos \frac{\pi n}{2N} + \cos \pi n \sin \frac{\pi n}{2N}}{2 \sin \frac{\pi n}{2N}} - \frac{1}{2} \cos \pi n \right)$$

$$= 0.$$

Since the cosine function is even, the theorem is proved.

Theorem 2: If h(t) is as in Theorem 1, using N + 1 equally spaced points of the interval $[-\frac{f_s}{2} + \frac{f_s}{4N}, \frac{f_s}{2} + \frac{f_s}{4N}]$ including the end points of the interval, then

$$\frac{1}{2} h(-\frac{f_s}{2} + \frac{f_s}{4N}) + \sum_{P=-\frac{N}{2}+1}^{\frac{N}{2}-1} h(\frac{f_s}{N}P + \frac{f_s}{4N}) + \frac{1}{2}h(\frac{f_s}{2} + \frac{f_s}{4N}) \qquad (C.3)$$

$$= \begin{cases} N [a_o + \sum_{\beta=1}^{\infty} (-1)^{\frac{\beta}{2}} a_{\beta N} + \sum_{\beta=1}^{\infty} (-1)^{\frac{\beta-1}{2}} b_{\beta N}] , & (-1)^{\epsilon} \text{real}, N \text{ even}, \\ N [a_o + \sum_{\beta=1}^{\infty} (-1)^{\frac{\beta}{2}} a_{\beta N} + \sum_{\beta=1}^{\infty} (-1)^{\frac{\beta+1}{2}} b_{\beta N}], & (-1)^{\epsilon} \text{real}, N \text{ odd}. \end{cases}$$

Proof: $h(t+\frac{f_s}{4N}) = a_o + \sum_{n=1}^{\infty} [a_n \cos \frac{2\pi}{f_s} n(t+\frac{f_s}{4N}) + b_n \sin \frac{2\pi}{f_s} n(t+\frac{f_s}{4N})]$

$$= \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n(\cos \frac{2\pi}{f_s}nt \cos \frac{\pi n}{2N} - \sin \frac{2\pi}{f_s}nt \sin \frac{\pi n}{2N}) \qquad (C.4)$$

$$+ b_n(\sin \frac{2\pi}{f_s}nt \cos \frac{\pi n}{2N} + \cos \frac{2\pi}{f_s}nt \sin \frac{\pi n}{2N})] .$$

When $n=\beta N$, $\beta=0,1, . . . , N$ even, we obtain

$$a_{\beta N}\cos\frac{2\pi}{f_s}\beta N\left(\frac{f_s}{N}P+\frac{f_s}{4N}\right)$$

$$= a_{\beta N}\left(\cos 2\pi\beta P\cos\beta\frac{\pi}{2} - \sin 2\pi\beta P\sin\beta\frac{\pi}{2}\right)$$

$$= \begin{cases} 0 & , \beta \text{ odd} \\ (-1)^{\frac{\beta}{2}}\ a_{\beta N} & , \beta \text{ even} \end{cases}$$

and

$$b_{\beta N}\sin\frac{2\pi}{f_s}\beta N\left(\frac{f_s}{N}P+\frac{f_s}{4N}\right)$$

$$= b_{\beta N}\left(\sin 2\pi\beta P\cos\beta\frac{\pi}{2} + \sin\beta\frac{\pi}{2}\cos 2\pi\beta P\right)$$

$$= \begin{cases} 0 & , \beta \text{ even} \\ (-1)^{\frac{\beta-1}{2}}\ b_{\beta N}, & \beta \text{ odd.} \end{cases}$$

When $n=\beta N$, $\beta=0,1, . . . ,N$, N odd, then $P = m(\frac{1}{2})$, m odd, and we obtain

$$a_{\beta N}\cos\frac{2\pi}{f_s}\beta N\left(\frac{f_s}{N}P+\frac{f_s}{4N}\right) = \begin{cases} 0 & , \beta \text{ odd} \\ (-1)^{\frac{\beta}{2}}\ a_{\beta N}, & \beta \text{ even} \end{cases}$$

and

$$b_{\beta N}\sin\frac{2\pi}{f_s}\beta N\left(\frac{f_s}{N}P+\frac{f_s}{4N}\right) = \begin{cases} 0 & , \beta \text{ even} \\ (-1)^{\frac{\beta+1}{2}}\ b_{\beta N}, & \beta \text{ odd} \end{cases}$$

Again, to complete the proof there are two cases. These cases are as in Theorem 1. In either case, since sin x is an odd function, those terms of the last member of (C.4) containing $\sin \frac{2\pi}{f_s}nt$ vanish. Since the other two terms contain only one factor which depends on t, namely $\cos \frac{2\pi}{f_s}nt$, we show as in Theorem 1 that they vanish when $n \neq \beta N$, $\beta=0,1, \ldots$.

Thus, if given a set of data which represents a band-limited function or a function which can be considered as band-limited by assuming all coefficients for $n > N$ to be insignificant, we can determine all coefficients by use of the above two theorems. If

$$h(t) = a_o + \sum_{n=1}^{N} (a_n \cos \frac{2\pi}{f_s}nt + b_n \sin \frac{2\pi}{f_s}nt),$$

we can find a_o by using any number of equally spaced points greater than N + 1. For simplicity, we illustrate with N + 2 points. Thus,

$$\frac{1}{2}h(-\frac{f_s}{2}) + \sum_{P=-\frac{N+1}{2}+1}^{\frac{N+1}{2}-1} h(\frac{f_s}{N}P) + \frac{1}{2}h(\frac{f_s}{2}) = (N+1)(a_o + \sum_{\beta=1}^{\infty} a_{\beta(N+1)}).$$

Since $a_n = 0$ for $n > N$, we obtain $(N+1)a_o$. To minimize the number of points required one usually uses a number such as 2N rather than N+2. The remaining coefficients can be found one by one beginning with N+1 points and dropping one point each time. It is recommended that the a_i, $i=0,1, \ldots, N$ be calculated first and then the b_i, $i=1,2, \ldots, N$.

Due to the assumed periodicity, the shift of $\frac{f_s}{4N}$ will not require going outside of $[-\frac{f_s}{2}, \frac{f_s}{2}]$ for points. We must only be careful as to which points will use $\frac{1}{2}$ as weights.

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