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DIMENSION OF AMALGAMATED GRAPHS AND TREES

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INTRODUCTION

An observation on the intuitive encoding of graphs led the author to the concept of *G, H-separation*. A mapping of this type is a kind of characterization of the structure of the graph H not belonging to the subgraph G of H. This concept turned out to be fruitful to estimate the dimension of subgraphs of a given graph (chapter III). This result again is essential for the following considerations (chapters III and IV) which deal with the decomposition and composition of graphs via subgraphs. Using the technique of amalgamation estimates for the dimension of a "complex" graph can be achieved by its components. However, for lack of examples of graphs for which the dimension is known this technique cannot be demonstrated impressively in the very general case. Still we can get quite good estimates if we assume a more special situation in which all graphs to be amalgamated are full subgraphs of the same graph. This recognition will be applied (in chapter V) to trees which can be regarded as amalgams of paths. The derived estimate then is simply based on calculating some invariants for the tree namely the number of vertices of a fixed degree resp. the maximum degree of the vertices in all spheres around a certain "central" point.

I. BASIC DEFINITIONS

1.1. Conventions and notations. Throughout this paper, a graph $G = (V(G), E(G))$ is a finite undirected and simple graph. $E(G)$ is also used to denote the symmetric antireflexive binary relation $\{(x, y) \mid \{x, y\} \in E(G)\}$. The *complete* and *discrete* graph with n vertices is denoted by K_n resp. D_n, $n \in \mathbb{N}$. $K_{n,m}$ denotes the *complete bipartite graph*, $K_{n,m} = D_n \oplus D_m$ (cf. 1.4). The *path* and *cycle* of length n, i.e. with n edges, is denoted by P_n resp. C_n. Define $P_0 := K_1 := D_1$.

1.2. Numbers.

r^+	least integer greater than or equal to r; $\log_n^+ m := (\log_n m)^+$
$\boldsymbol{k}$	least prime power greater than or equal to k

$\lvert G \rvert$	cardinality of the graph G: $\lvert G \rvert := \lvert V(G) \rvert$
$\delta_G(x)$	degree of vertex x in G
$\Delta(G)$	maximum degree in G
$d_G(x, y)$	distance of vertices x and y in G
diam G	diameter of G
$\chi(G)$	(vertex) chromatic number of G; $\chi(\emptyset) = 0$
$\chi^e(G)$	edge chromatic number of G

1.3. Subsets and subgraphs. The *k-sphere* $S_k(x)$ of $x \in V(G)$ in G is defined by $S_k(x) := \{y \in V(G) \mid d_G(x, y) = k\}$, $k \in \mathbb{N}$. For $X \subseteq V(G)$, $S_k(X) := \bigcup_{x \in X} S_k(x)$.

A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G) \cap V(H)^{(2)}$ ($X^{(2)}$ denotes the set of all subsets of X containing just two elements). If $E(H) = E(G) \cap V(H)^{(2)}$, H is called *full* (or *induced, spanned*) *subgraph* of G. $H \subseteq G$ and $H \leqq G$ indicate that H is a subgraph resp. full subgraph of G. For $X \subseteq V(G)$ resp. $H \subseteq G$ let $G[X]$ resp. $G[H]$ be the full subgraph of G induced by X resp. $V(H)$. The *neighbourhood* $N_G(x)$ of x in G is the full subgraph of G induced by the "closed" 1-sphere of x, i.e. $N_G(x) = G[\{x\} \cup S_1(x)]$.

For $X \subseteq V(G)$ resp. $H \subseteq G$, $N_G(x)$ resp. $N_G(H)$ is induced by $X \cup S_1(X)$ resp. $V(H) \cup S_1(V(H))$.

1.4. Operations. The *complement graph* of a graph G is denoted by G^c. For two graphs G, H, the *intersection* $G \cap H$ is defined by $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$.

The *sum* $\sum_{i=1}^{n} G_i$ of a family $\{G_1, G_2, \ldots, G_n\}$ of graphs is the disjoint union of the family. The *strong sum* $\bigoplus_{i=1}^{n} G_i$ is obtained from $\sum_{i=1}^{n} G_i$ by adding all edges between G_i and G_j, $i \neq j$.

For $G \subseteq H$, $H - G$ is induced by $V(H) - V(G)$ in H: $H - G = H[V(H) - V(G)]$. For $G \subseteq H$, $H \div G$ has those vertices that belong to G and those edges that belong to H but not to G: $V(H \div G) = V(G)$ abd $E(H \div G) = E(H[G]) - E(G)$.

1.5. Products. The *direct* (or *categorial*) *product* $G \times H$, *weak* (or *cartesian*) *product* $G \square H$ and *strong product* $G \boxtimes H$ of graphs G, H is defined by:

$$
\begin{aligned}
V(G \times H) &= V(G \square H) = V(G \boxtimes H) = V(G) \times V(H), \\
E(G \times H) &= \{\{(x_1, x_2), (y_1, y_2)\} \mid \{x_1, y_1\} \in E(G) \text{ and } \{x_2, y_2\} \in E(H)\}, \\
E(G \square H) &= \{\{(x_1, x_2), (y_1, y_2)\} \mid \text{either } \{x_1, y_1\} \in E(G) \text{ and } x_2 = y_2 \text{ or } x_1 = y_1 \\
&\qquad \text{and } \{x_2, y_2\} \in E(H)\}, \\
E(G \boxtimes H) &= E(G \times H) \cup E(G \square H).
\end{aligned}
$$

Since all products are associative, we can define the direct product $\mathop{\times}_{i=1}^{n} G_i$, weak product $\mathop{\square}_{i=1}^{n} G_i$ and strong product $\mathop{\boxtimes}_{i=1}^{n} G_i$ of a family $\{G_1, G_2, \ldots, G_n\}$ of graphs.

If $G_i = G$ for all $i = 1, 2, \ldots, n$, we shortly write $G^n, G^{\Box n}, G^{\boxtimes n}$ for the n-th direct, weak resp. strong power.

1.6. Mappings. A *homomorphism* $f: G \to H$ is a mapping $f: V(G) \to V(H)$ that preserves edges, i.e. $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. In the case that H is a complete graph, f is a *colouring* of G. If $f: G \to H$ is injective and $f(G) \leqq H$, i.e. the image of G is a full subgraph of H, then f is called *embedding*.

1.7. Dimension (cf. [6]). The *dimension* of a graph G, denoted by $\dim G$, is the least natural number n such that G can be embedded into $\mathbb{N}^n$ where $\mathbb{N}$ denotes the complete graph with vertex-set $\mathbb{N}$. An embedding $u: G \to \mathbb{N}^n$ is called (n-*ary*) *encoding* of G and satisfies $\{x, y\} \in E(G)$ iff $u_i(x) \neq u_i(y)$ for all $i = 1, \ldots, n$. Obviously, $\dim G$ equals the least natural number n such that there exists an embedding $u: G \to \mathbb{N}^n$.

Since discrete subgraphs of G correspond to complete subgraphs of G^c, it is easy to see that $\dim G$ equals the minimum number of equivalences $E_1, \ldots, E_n$ with the following properties:

(i) $\bigcup_{i=1}^{n} E_i = E(G^c)$ and

(ii) $\bigcap_{i=1}^{n} E_i = \Delta$, the trivial equivalence.

If we do not require u to be injective, i.e. if we drop (ii), we can possibly do with one coordinate less; such a minimum will be denoted by idim G (*intersection dimension*). We define idim $K_n = 0$, $n \in \mathbb{N}$.

The concatenation of two (encoding) vectors u, v is simply denoted by uv.

II. SOME FACTS

Proofs of the following two propositions can be found e.g. in [6]. 2.1 and 2.2 are of more fundamental importance whereas 2.3 states some concrete results.

2.1. Proposition. $G \leqq H \Rightarrow \dim G \leqq \dim H$. □

2.2. Proposition. *Let* $x^1, \ldots, x^k, y^1, \ldots, y^k$ *be vertices in* G *with*

(i) $\forall_i$: $\{x^i, y^i\} \in E(G)$ *and*

(ii) $\forall i < j$: $\{x^i, y^j\} \notin E(G)$.

Then $\dim G \geqq \log_2^+ k$.

2.3. Proposition.

(i) $\dim K_n = 1$ *for* $n \geqq 1$.

(ii) $\dim K_1 + K_n = n$ *for* $n \geqq 2$.

(iii) $\dim D_n = 2$ *for* $n \geqq 2$.

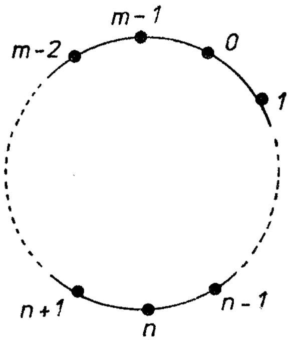
(iv) $\dim P_n = \log_2^+ n$ *for* $n \geqq 3$.

(v) $\dim C_{2n+2} = 1 + \log_2^+ n \leqq \dim C_{2n+1} \leqq 2 + \log_2^+ n$ *for* $n \geqq 2$.

(vi) *If* C'_m *is a cycle of length* $m \geqq 5$ *with one additional chord, then* $\log_2^+(m-2) \leqq \dim C'_m \leqq 2 + \log_2^+(m-2)$.

(vii) *For the cartesian cube* $Q_n := K_2^{\square n}$ *one has* $\dim Q_2 = 2$ *and* $\dim Q_n = n - 1$ *for* $n \geqq 3$.

Proof. (i)–(v) see e.g. in [6], (vii) see [4]. Ad (vi): W.l.o.g. let $V(C'_m) = \{0, 1, \ldots, m-1\}$ and for $0 < n < m - 2$, $E(C'_m) = \{\{i, i+1\} \mid i = 0, 1, \ldots, m-1\} \cup \{\{0, m-1\}, \{n, m-1\}\}$.



Let u be an encoding of P_{m-2} with all coordinates in $\{0, 1, 2\}$ (cf. [6]). Then v is an encoding of C'_m if one puts:

$$v(x) = \begin{cases} u(x)\,\alpha\alpha, & \text{if } x \in \{1, 2, \ldots, m-3\} - \{n\} \\ & \text{where } \alpha := x \bmod 2, \\ u(x)\,22, & \text{if } x \in \{0, m-2\}, \\ u(n)\,33, & \text{if } x = n, \\ 44\ldots401, & \text{if } x = m-1. \end{cases}$$

□

We now want to find a relation between the dimension of two graphs G, H, where G is a (not necessarily full) subgraph of H. To this end we introduce a new type of mapping:

2.4. Definition. For $G \subseteq H$, the mapping $\sigma: V(G) \to \mathbb{N}^n$, is called G, H-*separation*, indicated by $\sigma: G \to^H \mathbb{N}^n$, iff it satisfies

(i) $\{x, y\} \in E(G) \Rightarrow \forall i: \sigma_i(x) \neq \sigma_i(y)$ and

(ii) $\{x, y\} \in E(H \div G) \Rightarrow \exists_i: \sigma_i(x) = \sigma_i(y)$.

For $G \nleqq H$ we define the G, H-*separation number* $\sigma(G, H)$ to be the minimum number n such that there exists a G, H-separation $\sigma: G \to^H \mathbb{N}^n$. For $G \leqq H$ let $\sigma(G, H) = 0$.

Equivalent to the existence of a G, H-separation is the existence of a family of equivalences $E_1, \ldots, E_n$ on $V(G)$ such that

(i) $\bigcup_{i=1}^{n} E_i \supseteq E(H) - E(G)$ and

(ii) $E(G) \cap \bigcup_{i=1}^{n} E_i = \emptyset$.

Hence, $\sigma(G, H)$ is the minimum number of such a family (cf. 1.7).

Let us now state some properties of separations:

A) If $G \leqq G' \subseteq H$ with $V(G) \neq V(G') = V(H)$ and $E(G') = E(G) \cup \{\{x, y\} \mid x \in V(H), y \in V(H - G)\}$, then $G \nleqq H$ implies $\sigma(G, H) = \sigma(G', H)$ since $E(H) - E(G') = E(H \div G)$.

B) Let $Z_1, \ldots, Z_k$ be the components of $H \div G$, then $\sigma(G, H) = \max\{\sigma(H \div Z_i, H) \mid i = 1, \ldots, k\}$.

C) $\sigma(G, H) \leqq \operatorname{idim} G$, since $\bigcup_{i=1}^{n} E_i = E(G^c)$ implies $\bigcup_{i=1}^{n} E_i \supseteq E(H \div G)$.

D) $\sigma(G, K_n) = \operatorname{idim} G$.

E) $\sigma(G, H) \leqq \chi^e(H \div G)$: For each pairset $\{x, y\}$ of $V(G) \times V(G)$, let $\{x, y\}$ be a class of the i-th equivalence iff $\{x, y\} \in E(G)$ coloured i. All other classes are singletons.

F) $\sigma(D_n, H) = 1$ iff $D_n \nleqq H$. □

2.5. Proposition. $G \subseteq H \Rightarrow \dim G \leqq \dim H + \sigma(G, H)$.

Proof. If u is an encoding of H and $\sigma\colon G \to^H \mathbb{N}^n$ is a G, H-separation. then the concatenation $u\sigma$ is obviously an encoding of G. □

Example. If G is an arbitrary subgraph of $\mathop{\square}\limits_{i=1}^{m} P_{k_i}$ (cartesian product of paths of lengths $k_1, \ldots, k_m$), then $\dim G \leqq 2m - 1 + \sum_{i=1}^{m} \log_2^+ k_i$. This follows from 3.7 and the fact that $\sigma(G, \mathop{\square}\limits_{i=1}^{m} P_{k_i}) \leqq m$ for each $G \subseteq \mathop{\square}\limits_{i=1}^{m} P_{k_i}$, since if the vertices of $\mathop{\square}\limits_{i=1}^{m} P_{k_i}$ are numbered one-to-one, σ is a G, $\mathop{\square}\limits_{i=1}^{m} P_{k_i}$ – separation if one puts for $x = (x_1, \ldots, x_m) \in V(\mathop{\square}\limits_{i=1}^{m} P_{k_i})$ $(x_i \in \{0, 1, \ldots, k_i\}, i = 1, \ldots, m)$:

$$\sigma_i(x) := \begin{cases} \text{number of } x\,, & \text{if} \quad x_i = 0 \quad \text{or} \quad \{x, x - e_i\} \in E(G)\,, \\ \sigma_i(x - e_i)\,, & \text{else}\,, \end{cases}$$

where $e_i = (0, \ldots, 0, \underset{\substack{\uparrow \\ i}}{1}, 0, \ldots, 0)$. □

2.6. Proposition.

(i) $\Delta(G^c) = 1 \Rightarrow \dim G = 2$.

(ii) $\Delta(G^c) > 1 \Rightarrow \dim G \leqq \chi^e(G^c) \leqq \Delta(G^c) + 1$.

If $K_3 \nleqq G^c$ *then* $\Delta(G^c) \leqq \dim G = \chi^e(G^c) \leqq \Delta(G^c) + 1$.

Proof. (cf. [9, 1.5] resp. [6, 2.3]).

This follows from 2.5, the Vizing Theorem, 2.4.E and the fact that the equivalences induced by the construction in 2.4.E already satisfy 1.7 (ii) if $\Delta(G^c) > 1$. If $K_3 \nleqq G^c$ then each equivalence class contains at most two vertices of G. □

2.7. Lemma.

(i) $\sigma(G + H, G \oplus H) = \max\{\chi(G), \chi(H)\}$.

(ii) $\sigma(\sum_{i=1}^{n} G_i, \bigoplus_{i=1}^{n} G_i) \leqq 1 + (\chi - 1) \log_\chi^+ n$, *where* $\chi := \max_{1 \leqq i \leqq n} \chi(G_i)$.

Proof. Ad (i): $\sigma(G + H, G \oplus H) = \min\{n \mid \bigcup_{i=1}^{n} E_i \supseteq E(G \oplus H) - E(G + H), \bigcup_{i=1}^{n} E_i \cap E(G + H) = \emptyset \ (E_i \text{ equivalences})\} = \min\{n \mid \exists \text{ homomorphism } f\colon G + H \to K_n\} = \max\{\chi(G), x(H)\}$.

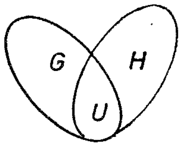
Ad (ii): $\sigma(\sum_{i=1}^{n} G_i \quad \bigoplus_{i=1}^{n} G_i) = \sigma(\sum_{i=1}^{n} K_{\chi(G_i)}, \quad \bigoplus_{i=1}^{n} K_{\chi(G_i)}) \leqq 1 + (\chi - 1) . \log_{\chi}^{+} n$ according to Corollary 2.5 in [3] and 2.4.C. □

Now it is easy to see that the following is true (Thm. 2.6 in [3]):

2.8. Theorem. $\dim \sum_{i=1}^{n} G_i \leqq \max_{1 \leqq i \leqq n} \dim G_i + \sigma(\sum_{i=1}^{n} G_i, \bigoplus_{i=1}^{n} G_i) \leqq \max_{1 \leqq i \leqq n} \dim G_i + 1 + (\chi - 1) . \log_{\chi}^{+} n$ *where* $\chi := \max_{1 \leqq i \leqq n} \chi(G_i)$. □

III. SIMPLE AMALGAMATIONS

3.1. Definition. Let G, H, U be graphs such that $U = G \cap H$ is proper full subgraph of both G and H. Then the *amalgam of G and H in U* is denoted by $\mathfrak{A}(G, H{:}U)$.



The restriction on U to be a proper subgraph (i.e. $V(G) \neq V(U) \neq V(H)$) is to avoid trivial cases and is tacitly assumed in the whole chapter.

A first approach towards the estimation of the dimension of an amalgam by its "components" is the following: First encode all vertices belonging to one component, G, then encode those belonging to the second component, H, and finally take care of non-adjacent vertices belonging to different parts of the amalgam, $G - U$ resp. $H - U$.

3.2. Proposition.

$$\dim \mathfrak{A}(G, H : U) \leqq \dim G + \dim H + \max \{\chi(G - U), \chi(H - U)\}.$$

Proof. For $\chi := \max \{\chi(G - U), \chi(H - U)\}$ let $f: G - U \to X_\chi$, $g: H - U \to K_\chi$ be homomorphisms, let u_G and u_H be encodings of G resp. H and let $\alpha: V(G - U) \to \mathbb{N}_k$, $\beta: V(H - U) \to \mathbb{N}_k$ and $\gamma: V(U) \to \mathbb{N}_k$ ($\mathbb{N}_k = \{n \in \mathbb{N} \mid n \geqq k\}$) be enumerations, where k is sufficiently large, i.e. larger than χ and all coordinates of the encodings u_G and u_H. Then u defines an encoding of $\mathfrak{A}(G, H : U)$ by

$$u(x) = \begin{cases} u_G(x)\, \alpha(x) \ldots \alpha(x) f(x) \ldots f(x), & \text{if} \quad x \in V(G - U), \\ \beta(x) \ldots \beta(x)\, u_H(x)\, g(x)\, (g(x) + 1) \ldots (g(x) + \chi - 1), & \text{if} \quad x \in V(H - U), \\ u_G(x)\, u_H(x)\, \gamma(x) \ldots \gamma(x), & \text{if} \quad x \in V(U), \end{cases}$$

where addition is modulo χ. □

An improvement of the estimate is attainable if one applies the technique of separation:

3.3. Proposition. *Let* G, H, U, K *be graphs,* $U = G \cap H$ *proper full subgraph* *of both G and H and let $h_G: G \to K$, $h_H: H \to K$ be injective homomorphisms which equal on U (i.e. $h(x) := h_G(x) = h_H(x)$ for each $x \in V(U)$). Then*

$$\dim \mathfrak{A}(G, H : U) \leqq \dim K + \sigma(h_G(G), K) + \sigma(h_H(H), K) + \\ + \max \{\chi(G - U), \chi(H - U)\} .$$

Proof. Let u_K be an encoding of K, $\sigma_1: h_G(G) \to^K \mathbb{N}^{\sigma(h_G(G),K)}$ and $\sigma_2: h_H(H) \to^K \to^K \mathbb{N}^{\sigma(h_H(H),K)}$ be a $h_G(G)$, K-separation resp. $h_H(H)$, K-separation and let $f: G - U \to K_\chi$ and $g: H - U \to K_\chi$ be homomorphisms where $\chi := \max \{\chi(G - U), \chi(H - U)\}$. Let $\alpha: V(G - U) \to \mathbb{N}_k$, $\beta: V(H - U) \to \mathbb{N}_k$ and $\gamma: V(U) \to \mathbb{N}_k$ be enumerations as in 3.2 (or anything else to fill the "gaps" of the encodings; essential point is that these coordinates do not destroy adjacency), k suff. large. Then u defines an encoding of $\mathfrak{A}(G, H : U)$ by

$$u(x) = \begin{cases} u_K(h_G(x))\, \sigma_1(x)\, \alpha(x) \ldots \alpha(x) f(x) \ldots f(x)\,, & \text{if } \ x \in V(G - U)\,, \\ u_K(h_H(x))\, \beta(x) \ldots \beta(x)\, \sigma_2(x)\, g(x)\, (g(x) + 1) \ldots (g(x) + \chi - 1)\,, & \\ & \text{if } \ x \in V(H - U)\,, \\ u_K(h(x))\, \sigma_1(x)\, \sigma_2(x)\, \gamma(x) \ldots \gamma(x)\,, & \text{if } \ x \in V(U)\,, \end{cases}$$

where addition is modulo χ. □

Observe that $h(H)$ need not be a full subgraph of K. In 3.2 and 3.3 the last χ coordinates result from the fact that no vertex of $G - U$ is adjacent to any vertex of $H - U$ whence we can replace χ in the estimate by

$$\sigma((G - U) + (H - U), (G - U) \oplus (H - U)) .$$

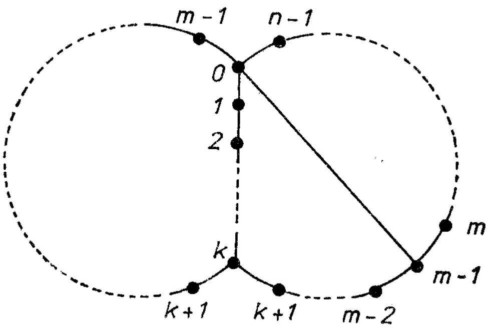
Corollary. *Let G, H, U be graphs, $U = G \cap H$ be a proper full subgraph of both G and H and let $h: G \to H$ be an injective homomorphism (i.e. G is isomorphic to a subgraph of H) with $h(u) = u$ for each $u \in V(U)$. Then*

$$\dim \mathfrak{A}(G, H : U) \leqq \dim H + \sigma(h(G), H) + \chi(H - U) . \qquad \square$$

Example. Let $3 \leqq m < n$ and $0 \leqq k \leqq m - 2$. Then

$$\dim \mathfrak{A}(C_m, C_n : P_k) \leqq 5 + \log_2^+ (n - 2) .$$

Proof. Let $K = C_n'$ be the cycle C_n with one additional edge $\{0, m - 1\}$; let the



vertices of $U = P_k$ be numbered $\{0, 1, \ldots, k\}$. Then $\sigma(G, K) = \sigma(C_m, C'_n) = 0$, $\sigma(H, K) = \sigma(C_n, C'_n) = 1$ and $\max\{\chi(C_m - P_k), \chi(C_n - P_k)\} = 2$. According to 2.3 (vi) we have $\dim C'_n \leqq 2 + \log_2^+ (n - 2)$ which completes the estimate. □

If we do not only demand that U is a common full subgraph of G and H but also that all its adjacent vertices belong to both G and H then we can slightly improve the estimate:

3.4. Proposition. *Let G, H, U, K be graphs, $U = G \cap H$ be proper full subgraph of G and H and let $h_G: G \to K$, $h_H: H \to K$ be injective homomorphisms which equal on U (i.e. $h(x) := h_G(x) = h_H(x)$ for each $x \in V(U)$) such that $N_K(h(U)) \cap \cap K[V(h_G(G)) \cup V(h_H(H))]$ is a full subgraph of both $h_G(G)$ and $h_H(H)$. Then*

$$\dim \mathfrak{A}(G, H : U) \leqq \dim K + \max\{\sigma(G, K), \sigma(H, K)\} + \\ + \max\{\chi(G - U), \chi(H - U)\}.$$

Proof. Here we demand in addition to 3.3 that

(i) $\{h_G(x) \mid x \in V(G - U),\ \{x, u\} \in E(G)\} =$
$= \{h_H(x) \mid x \in V(H - U),\ \{x, u\} \in E(H)\}$ for each $u \in V(U)$;

(ii) $E(K[V(N_K(h(U)) \cap (V(h_G(G)) \cup V(h_H(H)))]) - E(h_G(G)) - E(h_H(H)) = \emptyset$.

One can apply the same encoding instruction as in 3.3 except that in this situation σ_1 and σ_2 can both be concatenated simultaneously to u_K which is obvious from (i), (ii) above. □

Corollary. *Let all assumptions from 3.4 be satisfied and suppose in addition that $h_G(G)$ and $h_H(H)$ be full subgraphs of K. Then* $\dim \mathfrak{A}(G, H : U) \leqq \dim K + \max\{\chi(G - U), \chi(H - U)\}$. □

Further improvements of the estimates are possible if we restrict the relations among the two structures G and H. In the case where h in 3.3 Corollary is an embedding, we get $\dim \mathfrak{A}(G, H : U) \leqq \dim H + \chi(H - U)$, since $\sigma(h(G), H) = 0$. But this can be done better:

3.6. Theorem. *Let G, H, U be graphs, $U = G \cap H$ be proper full subgraph of both G and H, let $h: G \to H$ be an embedding with $h(u) = u$ for each $u \in V(U)$ and let $H' := H[N_H(h(G)) - U]$. Then* $\dim \mathfrak{A}(G, H : U) \geqq \dim H$ *and*

(i) $\chi(H') > 1 \Rightarrow \dim \mathfrak{A}(G, H : U) \leqq \dim H + \chi(H') - 1$;

(ii) $\chi(H') = 1 \Rightarrow \dim \mathfrak{A}(G, H : U) \leqq \operatorname{idim} H + 1$.

Proof. Let u_H be an encoding of H, $f: H' \to K_{\chi(H')}$ be a homomorphism and $\alpha: V(H - H') \to \mathbb{N}_k$ be an enumeration with k sufficiently large (cf. 3.2).

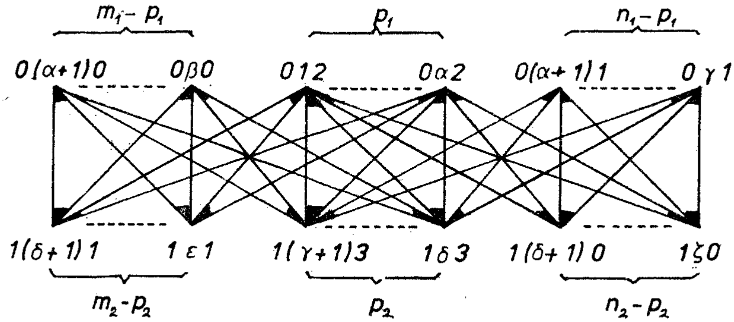
If $\chi(H') > 1$, then u defines an encoding of $\mathfrak{A}(G, H : U)$ by

$$u(x) = \begin{cases} u_H(h(x))\, f(x) \ldots f(x)\,, & \text{if} \quad x \in V(G - U)\,, \\ u_H(x)\,(f(x) + 1)\,(f(x) + 2) \ldots (f(x) + \chi(H') - 1)\,, & \text{if} \quad x \in V(H')\,, \\ u_H(x)\,\alpha(x) \ldots \alpha(x)\,, & \text{if} \quad x \in V(H - H')\,, \end{cases}$$

addition modulo $\chi(H')$, since for $x \in V(G - U)$, $y \in V(H - U)$, either $\{x, y\} \notin E(H)$, whence the first dim H coordinates meet in one coordinate, or $\{x, y\} \in E(H)$ implying $f(x) \neq f(y)$. If $\chi(H') = 1$, then $\{x, y\} \in E(H)$ with $x \in V(G - U)$, $y \in V(H - U)$ can never occur, whence additional coordinates obtained by a colouring (with one colour) are superfluous. On the other hand, an additional coordinate is necessary to distinguish between the encodings of x in $G - U$ and $h(x)$ in $H - U$. □

Examples. (1) Let $0 < p_1 < m_1 \leqq n_1$ and $0 < p_2 < m_2 \leqq n_2$, then

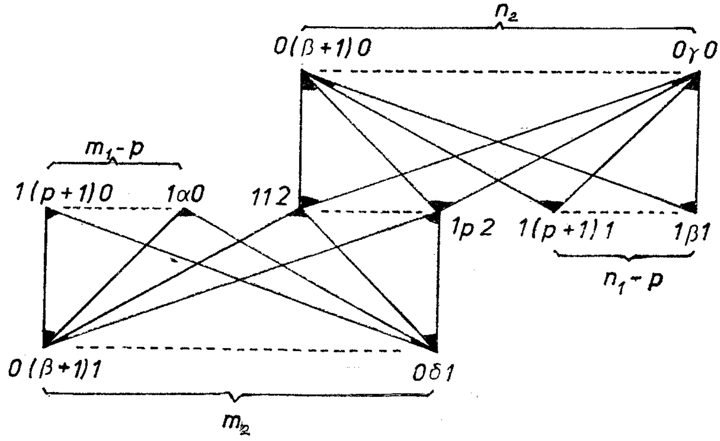
$$\dim \mathfrak{A}(K_{m_1,m_2}, K_{n_1,n_2} : K_{p_1,p_2}) \leqq \dim K_{n_1,n_2} + \chi(K_{n_1-p_1,n_2-p_2}) - 1 = 3$$



($\alpha := p_1$, $\beta := m_1$, $\gamma := n_1$, $\delta := n_1 + p_2$, $\varepsilon := n_1 + m_2$, $\zeta := n_1 + n_2$).

(2) Let $0 < p < m_1 \leqq n_1$ and $0 < m_2 \leqq n_2$, then

$$\dim \mathfrak{A}(K_{m_1,m_2}, K_{n_1,n_2} : D_p) \leqq \dim K_{n_1,n_2} + \chi(K_{n_1-p,n_2}) - 1 = 3$$



($\alpha := m_1$; $\beta := n_1$; $\gamma := n_1 + n_2$; $\delta := n_1 + m_2$). □

Now we will demonstrate how the technique of amalgamation works for weak and strong products of paths.

3.7. Proposition. *Let be* $n \geqq 2$, $k_1, \ldots, k_n \geqq 1$. *Then*

$$\sum_{i=1}^{n-1} \log_2 (1 + k_i) + \log_2 k_n \leqq \dim \mathop{\square}_{i=1}^{n} P_{k_i} \leqq n - 1 + \sum_{i=1}^{n} \log_2^+ k_i .$$

Proof. Let $B_n(x_1, \ldots, x_n)$ be the full subgraph of $\mathop{\square}_{i=1}^{n} P_{k_i}$ induced by $\mathop{\times}_{i=1}^{n} X_i$, where $x_i \in \{a_i, \bar{a}_i\}$, $a_i \geqq 0$, and

$$X_i = \begin{cases} \{0, 1, \ldots, a_i\}, & \text{if } x_i = a_i , \\ \{a_i, a_i + 1, \ldots, 2a_i\}, & \text{if } x_i = \bar{a}_i . \end{cases}$$

Let us note some properties of B_n which are needed in the sequel:

(i) $B_n(1, 1, \ldots, 1) \simeq K_2^{\square n} = Q_n$, cf. 2.3 (vii);

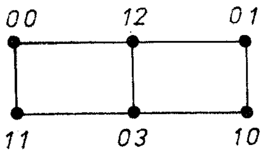
(ii) $B_n(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \cap B_n(x_1, \ldots, x_{i-1}, \bar{a}_i, x_{i+1}, \ldots, x_n) \simeq$
$\simeq B_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$;

(iii) $B_n(x_1, \ldots, x_{i-1}, 2a_i, x_{i+1}, \ldots, x_n) = \mathfrak{A}(B_n(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n),$
$B_n(x_1, \ldots, x_{i-1}, \bar{a}_i, x_{i+1}, \ldots, x_n) : B_n(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \cap$
$\cap B_n(x_1, \ldots, x_{i-1}, \bar{a}_i, x_{i+1}, \ldots, x_n))$, $a_i \geqq 1$;

(iv) $B_n(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) - (B_n(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \cap$
$\cap B_n(x_1, \ldots, x_{i-1}, \bar{a}_i, x_{i+1}, \ldots, x_n)) = B_n(x_1, \ldots, x_{i-1}, a_i - 1, x_{i+1}, \ldots, x_n)$,
$a_i \geqq 1$;

(v) $\chi(B_n(k_1, \ldots, k_n)) = 2$, since $f: \mathop{\square}_{i=1}^{n} P_{k_i} \to K_2$, defined by $f((v_1, \ldots, v_n)) :=$
$:= \sum_{i=1}^{n} v_i \bmod 2$, $(v_1, \ldots, v_n) \in V(\mathop{\square}_{i=1}^{n} P_{k_i})$, is a homomorphism.

First, let $n = 2$. $\dim B_2(1, 2) = 2$:



Suppose, $\alpha_1 > 0$, $\alpha_2 > 1$:

$$\dim B_2(2^{\alpha_1}, 2^{\alpha_2}) = \dim \mathfrak{A}(B_2(2^{\alpha_1}, 2^{\alpha_2 - 1}),$$

$$B_2(2^{\alpha_1}, \overline{2^{\alpha_2 - 1}}) : B_2(2^{\alpha_1}, 2^{\alpha_2 - 1}) \cap B_2(2^{\alpha_1}, \overline{2^{\alpha_2 - 1}})) \leqq$$

$$\leqq \dim B_2(2^{\alpha_1}, 2^{\alpha_2 - 1}) + 2 - 1 \leqq \dim B_2(2^{\alpha_1}, 2) + \alpha_2 - 1 ,$$

$$= \dim \mathfrak{A}(B_2(2^{\alpha_1 - 1}, 2),\ B_2(\overline{2^{\alpha_1 - 1}}, 2) : B_2(2^{\alpha_1 - 1}, 2) \cap B_2(\overline{2^{\alpha_1 - 1}}, 2)) \leqq$$

$$\leqq \dim B_2(2^{\alpha_1 - 1}, 2) + \alpha_2 - 1 + 2 - 1 \leqq \ldots$$

$$\ldots \leqq \dim B_2(1, 2) + \alpha_2 - 1 + \alpha_1 = 1 + \alpha_1 + \alpha_2 ,$$

consequently: $\dim P_{k_1} \square P_{k_2} \leqq 1 + \log_2^+ k_1 + \log_2^+ k_2$.

Now, assume $n > 2$.

$$\dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_n}) = \dim \mathfrak{A}(B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 2^{\alpha_n - 1}),$$

$$B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, \overline{2^{\alpha_n - 1}}) : B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 2^{\alpha_n - 1}) \cap B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, \overline{2^{\alpha_n - 1}})) \leqq$$

$$\leqq \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 2^{\alpha_n - 1}) + 2 - 1 \leqq \ldots \leqq \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 1) +$$
$$+ \alpha_n \leqq \ldots \leqq \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-2}}, 1, 1) + \alpha_{n-1} +$$
$$+ \alpha_n \leqq \ldots \leqq \dim B_n(1, 1, \ldots, 1) + \sum_{i=1}^{n} \alpha_i = \dim Q_n + \sum_{i=1}^{n} \alpha_i ,$$

consequently: $\dim \mathop{\square}\limits_{i=1}^{n} P_{k_i} \leqq n - 1 + \sum_{i=1}^{n} \log_2^+ k_i$.

For the lower estimate apply Proposition 2.2.

Let $d_1, \ldots, d_m$, $m := \prod_{i=1}^{n-1} (1 + k_i)$, be all $(n-1)$-tuples of $\mathop{\times}\limits_{i=1}^{n-1} \{0, 1, \ldots, k_i\}$. Put $x^{i+jm} := d_i j$, $y^{i+jm} := d_i(j+1)$ for $i = 1, \ldots, m$, $j = 0, \ldots, k_n - 1$. Obviously, the vertices $x^1, \ldots, x^{mk_n}, y^1, \ldots, y^{mk_n}$ satisfy

$$\{x^i, y^i\} \in E(\mathop{\square}\limits_{i=1}^{n} P_{k_i}) \quad \text{for all} \quad i = 1, \ldots, mk_n ,$$

$$\{x^i, y^j\} \notin E(\mathop{\square}\limits_{i=1}^{n} P_{k_i}) \quad \text{for all} \quad i < j ,$$

whence

$$\dim \mathop{\square}\limits_{i=1}^{n} P_{k_i} \geqq \log_2^+ mk_n = \log_2^+ k_n \cdot \prod_{i=1}^{n-1} (1 + k_i) \geqq$$
$$\geqq \sum_{i=1}^{n-1} \log_2 (1 + k_i) + \log_2 k_n . \qquad \square$$

In the same fashion an estimate for the strong product of paths can be attained.

3.8. Proposition. *Let be* $n \geqq 2$, $k_1, \ldots, k_n \geqq 1$. *Then*

(a) $\dim \mathop{\boxtimes}\limits_{i=1}^{n} P_{k_i} \leqq 1 + (2^{n-1} - 1) \cdot \sum_{i=1}^{n} (\min \{1, \log_2^+ k_i\} + (2 + (1/(2^{n-1} - 1))) \cdot$
$\cdot \max \{0, \log_2^+ k_i - 1\})$,

(b) (i) $\dim \mathop{\boxtimes}\limits_{i=1}^{n} P_{k_i} \geqq \sum_{i=1}^{n} \log_2 k_i$.

(ii) *Let* $p := |\{k_i | k_i > 1\}|$. *Then* $\dim \mathop{\boxtimes}\limits_{i=1}^{n} P_{k_i} \geqq 2^{n-p} \cdot (2^p - 1)$.

(iii) *If* $k_i > 2$ *for some* i, *then* $\dim \mathop{\boxtimes}\limits_{i=1}^{n} P_{k_i} \geqq 2^n$.

Proof. Let be $B_n(x_1, \ldots, x_n) \leqq \mathop{\boxtimes}\limits_{i=1}^{n} P_{k_i}$ defined as in 3.7.

(i) $B_n(1, 1, \ldots, 1) \simeq K_2^{\boxtimes n} \simeq K_{2^n}$.

(ii)–(iv) the same as in 3.7.

(v) $\chi(B_n(k_1, \ldots, k_n)) = 2^n$, since $f\colon \mathop{\boxtimes}\limits_{i=1}^{n} P_{k_i} \to K_{2^n}$, defined by $f((v_1, \ldots, v_n)) :=$
$:= \sum_{i=1}^{n} (v_i \bmod 2) \cdot 2^{i-1}$, $(v_1, \ldots, v_n) \in V(\mathop{\boxtimes}\limits_{i=1}^{n} P_{k_i})$, is a homomorphism.

$$\dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_n}) = \dim \mathfrak{A}(B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 2^{\alpha_n - 1}),$$
$$B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, \overline{2^{\alpha_n - 1}}) : B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 2^{\alpha_n - 1}) \cap$$
$$\cap B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, \overline{2^{\alpha_n - 1}})) \leqq \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 2^{\alpha_n - 1}) +$$
$$+ 2^n - 1 \leqq \ldots \leqq \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 2) + (\alpha_n - 1)(2^n - 1) =$$
$$= \dim \mathfrak{A}(B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 1), B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, \overline{1}) : B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 1) \cap$$
$$\cap B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, \overline{1}) + (\alpha_n - 1)(2^n - 1) \leqq$$
$$\leqq \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 1) + \chi(B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 0)) - 1 + (\alpha_n - 1)(2^n - 1) =$$
$$= \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 1) + 2^{n-1} - 1 + (\alpha_n - 1)(2^n - 1) =$$
$$= \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-1}}, 1) + \min\{1, \log_2^+ k_n\} . (2^{n-1} - 1) +$$
$$+ \max\{0, \log_2^+ k_n - 1)\} . (2^n - 1) \leqq \ldots$$
$$\ldots \leqq \dim B_n(2^{\alpha_1}, \ldots, 2^{\alpha_{n-2}}, 1, 1) + (\min\{1, \log_2^+ k_{n-1}\} +$$
$$+ \min\{1, \log_2^+ k_n\}) . (2^{n-1} - 1) + (\max\{0, \log_2^+ k_{n-1} - 1\} +$$
$$+ \max\{0, \log_2^+ k_n - 1\}) . (2^n - 1) \leqq \ldots \leqq \dim B_n(1, 1, \ldots, 1) +$$
$$+ \sum_{i=1}^{n} \min\{1, \log_2^+ k_i\} . (2^{n-1} - 1) + \sum_{i=1}^{n} \max\{0, \log_2^+ k_i - 1\} . (2^n - 1) =$$
$$= 1 + (2^{n-1} - 1) \sum_{i=1}^{n} \left(\min\{1, \log_2^+ k_i\} + \left(2 + \frac{1}{2^{n-1} - 1}\right) \max\{0, \log_2^+ k_i - 1\}\right).$$

For the lower estimate apply Proposition 2.2.

Let $x^i := (x_1^i, \ldots, x_n^i)$, $y^i := (y_1^i, \ldots, y_n^i) \in V(\boxtimes_{i=1}^{n} P_{k_i})$ $(i = 1, \ldots, \prod_{i=1}^{n} k_i)$ where $y_j^i := x_j^i + 1$ $(j = 1, \ldots, n)$ be numbered in the following way:

$$x^1 := (0, 0, \ldots, 0),$$

followed by those x^i with $\sum_{j=1}^{n} x_j^i = 1$ (of course, such that $y^i \in V(\boxtimes_{j=1}^{n} P_{k_i})$), those with $\sum_{j=1}^{n} x_j^i = 2$, and so on.

It is easy to see that all vertices are numbered such that $\{x^i, y^i\} \notin E(\boxtimes_{i=1}^{n} P_{k_i})$ whenever $i < j$, because all y^j adjacent to x^i fulfil $\sum_{l=1}^{r} y_l^j \leqq n + \sum_{l=1}^{n} x_l^i$ and therefore $j \leqq i$. W.l.o.g. let be $k_1, \ldots, k_p > 1$, $p \leqq n$. Then $K_1 + K_{2^{n-p}(2^p - 1)} \leqq \boxtimes_{i=1}^{n} P_{k_i}$ induced by

$$\{(\underbrace{2, 2, \ldots, 2}_{p}, 1, \ldots, 1)\} \cup \bigcup_{i=1}^{p} \{(\varepsilon_1, \ldots, \varepsilon_{i-1}, 0, \varepsilon_{i+1}, \ldots, \varepsilon_n) \mid \varepsilon_i \in \{0, 1\}\}.$$

If $k_i > 2$ for some i, then $K_1 + K_{2^n} \leqq \boxtimes_{i=1}^{n} P_{k_i}$ induced by

$$\{(0, 0, \ldots, 0, \underset{\substack{\uparrow \\ i}}{3}, 0, \ldots, 0)\} \cup \{(\varepsilon_1, \ldots, \varepsilon_n) \mid \varepsilon_i \in \{0, 1\}\}.$$

□

In the special case where $k_1 = \ldots = k_n = 2$, we get

$$2^n - 1 \leqq \dim P_2^{\boxtimes n} \leqq 1 + n \,.\, (2^{n-1} - 1)\,.$$

The first lower estimate of the proof above also holds for $\dim \bigtimes_{i=1}^{n} P_{k_i}$. On the other hand, since $P_{k_i} \leqq K_3^{\log_2^+ k_i}$ if $k_i > 2$, $\dim \bigtimes_{i=1}^{n} P_{k_i} \leqq \sum_{i=1}^{n} \log_2^+ k_i$. Thus

3.9. Proposition. *Let be $n \geqq 2$, $k_1, \ldots, k_n \geqq 3$. Then*

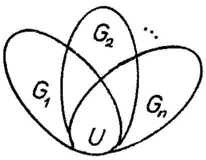
$$\sum_{i=1}^{n} \log_2 k_i \leqq \dim \bigtimes_{i=1}^{n} P_{k_i} \leqq \sum_{i=1}^{n} \log_2^+ k_i\,. \qquad \square$$

In [2, 6.1] we have proved that, for $k_1 = k_2 = \ldots = k_n = 2$, there holds: $n \leqq \leqq \dim P_2^n \leqq n + 1$. (The upper estimate follows from the fact that P_2^n is the sum of 2^{n-1} complete bipartite graphs which can be encoded in principal according to 2.8 but with one coordinate less since all summands are isomorphic to subgraphs of one of them (cf. [3, 2.6 Remark])).

IV. MULTIPLE AMALGAMATIONS

In the last chapter we have developed estimates for the case where two graphs are glued together. Now we shall investigate the situation where several graphs are amalgamated via the same subgraph, and then investigate the case where graphs are amalgamated in different subgraphs.

4.1. Definition. Let $G_1, G_2, \ldots, G_n$, $n \geqq 3$, and U be graphs such that $U = G_i \cap G_j$, $i \neq j$, is proper full subgraph of G_i, $i = 1, \ldots, n$. Then the *amalgam of* $G_1, \ldots, G_n$ *in* U is denoted by $\mathfrak{A}(G_1, \ldots, G_n : U)$.



In analogy to Proposition 3.2 using Lemma 2.7, we get the following rough estimate:

4.2. Proposition.

$$\dim \mathfrak{A}(G_1, \ldots, G_n : U) \leqq \sum_{i=1}^{n} \dim G_i + \sigma\Big(\sum_{i=1}^{n} (G_i - U),\ \bigoplus_{i=1}^{n} (G_i - U)\Big) \leqq$$

$$\leqq \sum_{i=1}^{n} \dim G_i + 1 + (\chi - 1) \log_\chi^+ n\,, \quad \textit{where} \quad \chi := \max_{1 \leqq i \leqq n} \chi(G_i - U)\,. \qquad \square$$

The generalization of Proposition 3.3 and improvement of Proposition 4.2 now reads:

4.3. Proposition. *Let $G_1, \ldots, G_n, U, K$ be graphs, $U = G_i \cap G_j$, $i \neq j$, be proper full subgraph of G_i $(i = 1, \ldots, n)$ and let $h_i\colon G_i \to K$ be injective homomorphisms which equal on U (i.e. $h_i(x) = h_j(x)$ for each $x \in V(U)$). Then*

$$\dim \mathfrak{A}(G_1, \ldots, G_n : U) \leqq$$
$$\leqq \dim K + \sum_{i=1}^{n} \sigma(h_i(G_i), K) + \sigma\big(\sum_{i=1}^{u} (G_i - U), \bigoplus_{i=1}^{n} (G_i - U)\big). \qquad \square$$

4.4. Proposition. *Let $G_1, \ldots, G_n$, $n \geqq 3$, U and K be graphs, $U = G_i \cap G_j$, $i \neq j$, be proper full subgraphs of G_i $(i = 1, \ldots, n)$ and let $h_i\colon G_i \to K$ be injective homomorphisms which equal on U (i.e. $h(x) := h_i(x) = h_j(x)$ for all $i, j \in \{1, \ldots, n\}$ and $x \in V(U)$) such that $N_K(h(U)) \cap K[\bigcup_{i=1}^{n} V(h_i(G_i))]$ is full subgraph of $h_i(G_i)$, $i = 1, \ldots, n$. Then*

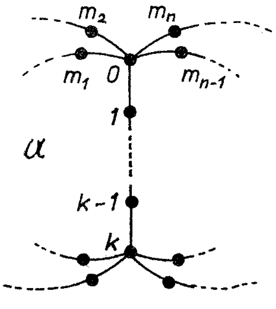
$$\dim \mathfrak{A}(G_1, \ldots, G_n : U) \leqq$$
$$\leqq \dim K + \max_{1 \leqq i \leqq n} \sigma(h_i(G_i), K) + \sigma\big(\sum_{i=1}^{n} (G_i - U), \bigoplus_{i=1}^{n} (G_i - U)\big).$$

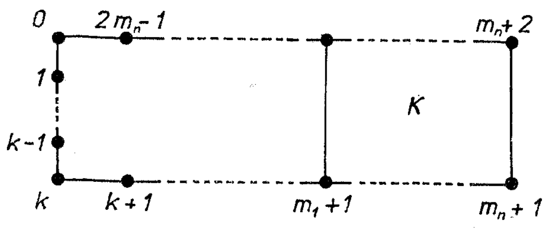
Proof. This is obvious from the fact that $V(U) \cap V(K \div G_i) = \emptyset$ $(i = 1, \ldots, n)$. $\square$

4.5. Corollary. *Let all assumptions of 4.4 be satisfied and suppose in addition that $h_i(G_i)$, $i = 1, \ldots, n$, be full subgraphs of K. Then*

$$\dim \mathfrak{A}(G_1, \ldots, G_n : U) \leqq \dim K + \sigma\big(\sum_{i=1}^{n} (G_i - U), \bigoplus_{i=1}^{n} (G_i - U)\big). \qquad \square$$

Example: Let $3 \leqq m_1 \leqq m_2 \leqq \ldots \leqq m_n$ and $0 \leqq k \leqq 2m_1 - 4$, $\mathfrak{A} := \mathfrak{A}(C_{2m_1}, C_{2m_2}, \ldots, C_{2m_n} : P_k)$.





$K \subseteq P_1 \square P_{m_n - 1}$ with $V(K) = \{0, 1, \ldots, 2m_n - 1\}$ and $E(K) = \{\{i, i+1\} \mid i = 0, 1, \ldots, 2m_n - 2\} \cup \{\{m_i + 1, \ m_i + 2\} \mid i = 1, \ldots, n\} \cup \{\{0, 2m_n - 1\}\}$.

Then $\sigma(K, P_1 \square P_{m_n - 1}) = \sigma(C_{2m_i}, P_1 \square P_{m_n - 1}) = 1$, $i = 1, \ldots, n$ (2.4.E), thus

$$\dim K \leqq \dim P_1 \square P_{m_n - 1} + \sigma(K, P_1 \square P_{m_n - 1}) \leqq 2 + \log_2^+ (m_n - 1)$$

and

$$\dim \mathfrak{A} \leqq \dim K + 1 + 1 + \log_2^+ n = 4 + \log_2^+ (m_n - 1) + \log_2^+ n. \qquad \square$$

The best results can be attained in the case that all G_i are full subgraphs of say G_n, since then each G_i can be encoded in the first dim G_n coordinates by the encoding of the image of G_i in G_n and one of the last coordinates can be dropped:

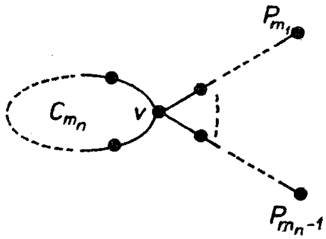
4.6. Theorem. *Let $G_1, \ldots, G_n$, $n \geqq 3$, and U be graphs, $U = G_i \cap G_j$, $i \neq j$, be proper full subgraph of G_i, $i = 1, \ldots, n$, let $h_i\colon G_i \to G_n$, $i = 1, \ldots, n-1$, be embeddings with $h(u) := h_i(u) = u$ for each $u \in V(U)$, $i = 1, \ldots, n-1$, and let $G'_n := G[\bigcup_{i=1}^{n-1} V(N_{G_n}(h_i(G_i))) - V(U)]$. Then*

(i) $\chi(G'_n) = 1 \Rightarrow \dim \mathfrak{A}(G_1, \ldots, G_n : U) \leqq 1 + \operatorname{idim} G_n$;

(ii) $\chi(G'_n) > 1 \Rightarrow \dim \mathfrak{A}(G_1, \ldots, G_n : U) \leqq \dim G_n + \sigma(\sum_{i=1}^{n-1} (G_i - U) + G'_n, \bigoplus_{i=1}^{n-1} (G_i - U) \oplus G'_n) - 1 \leqq \dim G_n + (\chi(G_n) - 1) \log^+_{\chi(G_n')} n$. □

Examples. (1) Let $1 \leqq m_1, \ldots, m_{n-1} \leqq m_n - 2$ and v be an end-vertex of $P_{m_1}, \ldots, P_{m_{n-1}}$. Then

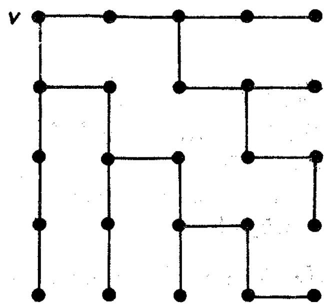
$$\dim \mathfrak{A}(P_{m_1}, \ldots, P_{m_{n-1}}, C_{m_n} : v) \leqq \dim C_{m_n} + \log_2^+ n .$$



(2) Let $1 \leqq p < m_1, \ldots, m_{n-1} \leqq m_n$ and $m := m_n - p$. Then

$$\dim \mathfrak{A}(K_{m_1}, \ldots, K_{m_n} : K_p) \leqq 1 + (m - 1) \log_m^+ n .$$

(3) Let $T \subseteq P_4 \square P_4$ be the opposite tree and let v be the vertex top left. It is easy to see that $\sigma(T, P_4 \square P_4) = 1$.



Thus

$$\dim \mathfrak{A}(\underbrace{T, \ldots, T}_{n\text{-times}} : v) \leqq \dim T + \log_2^+ n \leqq \dim T + \log_2^+ n \leqq 5 + \log_2^+ n .$$

According to Proposition 5.1 we have

$$\dim \mathfrak{A}(T, \ldots, T : v) \geqq \log_2^+ (18n + 1) \geqq 4 + \log_2^+ n . \qquad \square$$

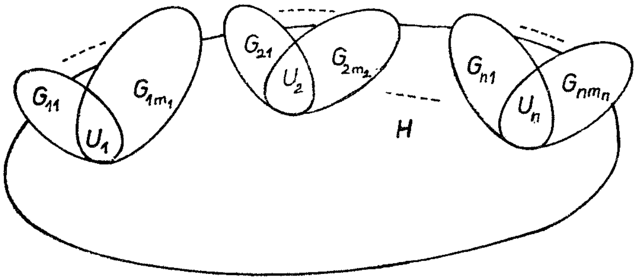
4.7. Corollary. *Let $U \leqq G$ and $\chi(G) > 1$. Then*

$$\dim \mathfrak{A}(\underbrace{G, \ldots, G}_{n\text{-times}} : U) \leqq \dim G + (\chi(G - U) - 1) \log_{\chi(G-U)}^+ n . \qquad \square$$

Let us now turn to the problem that graphs are glued togethet in various subgraphs and examine when this can be done in one step.

4.8. Definition. Let G_{ij}, U_i $(i = 1, \ldots, n \geqq 2; j = 1, \ldots, m_i \geqq 1)$ and H be graphs such that $U_i = G_{ij_1} \cap G_{ij_2} = G_{ij_1} \cap H$, $j_1 \neq j_2$, is proper full subgraph of G_{ij} $(j = 1, \ldots, m_i)$ and H $(i = 1, \ldots, n)$. Then the *amalgam of* $G_{i1}, \ldots, G_{im_i}$ *and* H *in* U_i $(i = 1, \ldots, n)$ is denoted by

$$\mathfrak{A} := \mathfrak{A}(G_{11}, \ldots, G_{1m_1} : U_1 \,.\, G_{21}, \ldots, G_{2m_2} : U_2, \ldots, G_{n1}, \ldots, G_{nm_n} : U_n \,.\, H) .$$



If we want to amalgamate all graphs in one step we have to guarantee that the graphs except H are not "too large". Else we have to apply 4.2–4.4 several times (at most n times).

4.9. Proposition. *Let G_{ij}, U_i $(i = 1, \ldots, n \geqq 2;\ j = 1, \ldots, m_i \geqq 1)$, H and K be graphs such that, for each $i = 1, \ldots, n$, $U_i = G_{ij_1} \cap G_{ij_2} = G_{ij_1} \cap H$, $j_1 \neq j_2$, is proper full subgraph of $G_{ij}(j = 1, \ldots, m_i)$ and H. Let $h_{ij}: G_{ij} \to K$ be injective homomorphisms which equal on U_i* (i.e. $h_i(u) := h_{ij_1}(u) = h_{ij_2}(u)$ *for each $u \in V(U_i)$) and let $h: H \to K$ be an injective homomorphism with $h|_{U_i} = h_i$. Moreover, let for all i, j, r, s, $i \neq j$, be satisfied:*

$$(N_K(h_{ir}(G_{ir} - U_i)) - h_i(U_i)) \cap$$
$$\cap\, (N_K(h_{js}(G_{js} - U_j)) - h_j(U_j)) = \emptyset = h_i(U_i) \cap h_j(U_j) .$$

With

$$\chi := \max_{1 \leq i \leq n} \{\chi(N_K(h(H))),\ \chi(N_K(\bigcup_{j=1}^{m_i} h_{ij}(G_{ij} - U_i)) - h_i(U_i))\},$$

we get

$$\dim \mathfrak{A} \leqq \dim K + \sigma(h(H), K) + \max_{1 \leqq i \leqq n} \sum_{j=1}^{m_i} \sigma(h_{ij}(G_{ij}), K) +$$

$$+ \max_{1 \leqq i \leqq n} \sigma(H + \sum_{j=1}^{m_i} (G_{ij} - U_i),\ H \oplus \bigoplus_{j=1}^{m_i} (G_{ij}(G_{ij} - U_i)) \leqq$$

$$\leqq \dim K + \sigma(h(H), K) + \max_{1 \leqq i \leqq n} \sum_{j=1}^{m_i} \sigma(h_{ij}(G_{ij}), K) +$$

$$+ 1 + (\chi - 1) \log_\chi^+ (1 + \max_{1 \leqq i \leqq n} m_i).$$

If in addition, for each $i = 1, \ldots, n$, $N_K(h_i(U_i)) \cap K[\bigcup_{j=1}^{m_i} V(h_{ij}(G_{ij})) \cup V(h(H))]$ *is full subgraph of* $h_i(G_{ij})$ $(j = 1, \ldots, m_i)$ *and* $h(H)$, *then*

$$\dim \mathfrak{A} \leqq \dim K + \max \{\sigma(h(H), K), \sigma(h_{ij}(G_{ij}), K) \mid 1 \leqq i \leqq n,\ 1 \leqq j \leqq m_i\} +$$

$$+ 1 + (\chi - 1) \log_\chi^+ (1 + \max_{1 \leqq i \leqq n} m_i). \qquad \square$$

If we furthermore restrict the relation between all G_{ij} and H, we get:

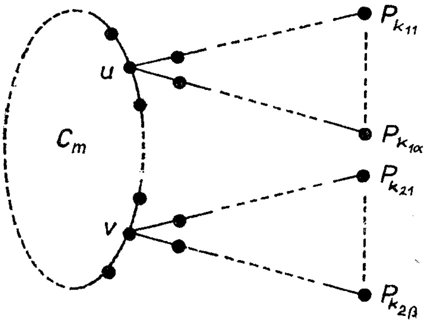
4.10. Theorem. *Let* G_{ij}, U_i $(i = 1, \ldots, n \geqq 2;\ j = 1, \ldots, m_i \geqq 1)$ *and* H *be graphs such that* $U_i = G_{ij_1} \cap G_{ij_2} = G_{ij_1} \cap H$, $j_1 \neq j_2$, *is proper full subgraph of* G_{ij} $(j = 1, \ldots, m_i)$ *and* H $(i = 1, \ldots, n)$ *and let* $h_{ij}: G_{ij} \to H$ *be embeddings which equal on* U_i (*i.e.* $h_i(u) := h_{ij_1}(u) = h_{ij_2}(u)$ *for each* $u \in V(U_i)$).

Let $H' := H[\bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} V(N_H(h_{ij}(G_{ij} - U_i)) - U_i)]$ *and let be satisfied for all* $i \neq j$:

$$(\bigcup_{p=1}^{m_i} V(N_H(h_{ip}(G_{ip} - U_i)) - U_i)) \cap (\bigcup_{p=1}^{m_i} V(N_H(h_{jp}(G_{jp} - U_j)) = \emptyset.$$

Then (i) $\chi(H') > 1$: $\dim \mathfrak{A} \leqq \dim H + (\chi(H') - 1) \log^+_{\chi(H')} (1 + \max_{1 \leqq i \leqq n} m_i)$;
(ii) $\chi(H') = 1$: $\dim \mathfrak{A} \leqq 1 + \text{idim}\, H$. $\square$

Example.



$H = C_m$, $m \geqq 6$, $G_{ij} = P_{k_{ij}}$, $k_{ij} \geqq 1$, $i = 1$ and $j = 1, \ldots, \alpha$ resp. $i = 2$ and $j = 1, \ldots, \beta$, w.l.o.g. $\alpha \geqq \beta$, let $d := d_H(u, v)$ $(\leqq m/2)$ and $\varkappa_i := \max_j k_{ij}$, $i = 1, 2$.

Assume either $\varkappa_1 + \varkappa_2 + d + 3 \leqq m$ or $\varkappa_1 + 1 \leqq d$ and $\varkappa_2 + d + 1 \leqq m$. Then $\log_2^+ (\sum_{j=1}^{\alpha} k_{1j} + \sum_{j=1}^{\beta} k_{2j} + m - \alpha - \beta - \gamma) \leqq \dim \mathfrak{A}(P_{k_{11}}, \ldots, P_{k_{1\alpha}} : u \,.\, P_{k_{21}}, \ldots$ $\ldots, P_{k_{2\beta}} : v \,.\, C_m) \leqq \dim C_m + \log_2^+ (\alpha + 1)$, where $\gamma = 3$ if $d = 1$, $\gamma = 4$ if $d > 1$. In the case when $m = 2n + 4$, $n \geqq 1$; $\alpha = \beta = 2^k - 1$, $k \geqq 1$; $d = 1$ and $k_{ij} = n$ for all i, j, then:

$$k + 1 + \log_2^+ n \leqq$$

$$\leqq \dim \mathfrak{A}(\underbrace{P_n, \ldots, P_n}_{\alpha\text{-times}} : u \,.\, \underbrace{P_n, \ldots, P_n}_{\alpha\text{-times}} : v \,.\, C_{2n+4}) \leqq k + 1 + \log_2^+ (n + 1)\,. \qquad \square$$

V. TREES

In this chapter we want to apply the techniques developed in III and IV to find upper bounds for the dimension of trees. Before we start with the investigation we state a lower bound which is useful to get an idea how good the upper bounds are, at least in special cases.

In the sequel T shall always designate a (finite) tree with diam $T \geqq 2$.

5.1. Proposition. $\dim T \geqq \log_2^+ (|T| + 1 - n_1(T))$

Proof. Let P_k be a maximal path in T, i.e. $k = \operatorname{diam} T$. Set

$$T_{-1} := \emptyset\,, \quad T_0 := P_k\,,$$
$$T_{i+1} := \{x \in V(T) \mid \exists y \in T_i : \{x, y\} \in E(T)\} - T_{i-1} \quad (i = 1, k - 1)\,,$$
$$T^{(i)} := \bigcup_{j=0}^{i} T_j \quad (i = 1, \ldots, k)\,.$$

Let the vertices $t_0, t_1, \ldots, t_k$ of T_0 be numbered such that $\{t_{i-1}, t_i\} \in E(T)$ $(i = 1, \ldots, k)$. Put $x^i := t_i$, $y^i := t_{i-1}$, $i = 1, \ldots, k$. Hence (Prop. 2.2) $\dim T_0 = \dim P_k = \log_2^+ k = \log_2^+ (|T_0| + 1 - n_1(T_0))$. For $T^{(1)}$ we have $|T^{(1)}| = k + 1 + |T_1|$, $n_1(T^{(1)}) = 2 + |T_1|$, and therefore $\dim T^{(1)} \geqq \log_2^+ (|T^{(1)}| + 1 - n_1(T^{(1)})) = \log_2^+ k = \dim T_0$. Suppose the statement is true for $T^{(1)}, \ldots, T^{(i)}$, $i \geqq 1$. If $T_{i+1} = \emptyset$, then $T^{(i)} = T$ and the proof is completed. Otherwise we have $T_i = T_i^1 \cup T_i^2$, where $T_i^1 = \{x \in T_i \mid \delta_T(x) = 1\}$. Let $T_i^2 = \{t_1, \ldots, t_p\}$, $p = |T_i^2| = |T_i| - |T_i^1|$. Put $x^{\alpha+j} = t_j$, $y^{\alpha+j} = y \in T_{i+1}$ if $\{t_j, y\} \in E(T)$, $j = 1, \ldots, p$, where $\alpha = |T^{(i)}| + 1 - n_1(T^{(i)})$. Now, $|T^{(i+1)}| = |T^{(i)}| + |T_{i+1}|$, $n_1(T^{(i+1)}) = n_1(T^{(i)}) - |T_i| + |T_i^1| + |T_{i+1}|$, $\alpha + p = |T^{(i)}| + 1 - n_1(T^{(i)}) + p = |T^{(i+1)}| + 1 - n_1(T^{(i+1)})$ and $\dim T^{(i+1)} \geqq \log_2^+ (\alpha + p)$. $\square$

Poljak and Pultr proved in [10, 2.1]

$$\dim T \geqq \log_2^+ \mu(T)$$

where $\mu(T)$ is the matching number of T. Since $\mu(T) \leqq |T| - n_1(T)$, the above bound is stronger.

Let us now turn to estimate from above. The most important tools will be Theorem 4.6 and Theorem 4.10.

Using Theorem 4.6 we get a first quite rough result:

5.2. Proposition. $\dim T \leqq \log_2^+ \operatorname{diam} T + \sum_{i=3}^{\infty} n_i(T)(\log_2^+ i - 1)$.

Proof. If we start the encoding procedure by a maximal path P (of length diam T) we ensure that all paths that can be amalgamated to P in a vertex of P are subgraphs of P(in the sense of Thm. 4.6). If, in addition, we make sure that all paths which are amalgamated in subsequent steps to the already existing subtree of T are as large as possible we maintain the "subgraph property", since we started with a maximal path. Thus we can apply Thm. 4.6 at each step of amalgamation. Since the number of paths that can be amalgamated in a certain vertex v is determined by $\delta_T(v) \geqq 3$ the upper bound increases at each step by

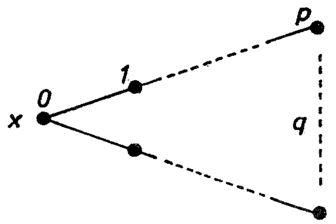
$$\log_2^+ \frac{\delta_T(v)}{2} = \log_2^+ \delta_T(v) - 1 .$$

Hence we get the following estimate for dim T:

$$\dim T \leqq \log_2^+ \operatorname{diam} T + \sum_{\substack{v \in T \\ \delta_T(v) \geqq 3}} (\log_2^+ \delta_T(v) - 1) \leqq$$

$$\leqq \log_2^+ \operatorname{diam} T + \sum_{i=3}^{\infty} n_i(T) \,.\, (\log_2^+ i - 1) \,. \qquad \square$$

Example. Let $S_{p,q}$ be the star consisting of q paths of length p, $q \geqq 3$, $p \geqq 2$.



Obviously, $S_{p,q} = \mathfrak{A}(\underbrace{P_p, \ldots, P_p}_{q\text{-times}} : x)$, diam $S_{p,q} = 2p$, $n_q(S_{p,q}) = 1$ and $n_i(S_{p,q}) = 0$ for $i \geqq 3$, $i \neq q$. Thus, by 5.1 and 5.2:

$$\log_2^+ (q(p-1) + 2) \leqq \dim S_{p,q} \leqq \log_2^+ p + \log_2^+ q \,.$$

This estimate can be derived, of course, directly from Thm. 4.6, the above proposition, however, allows to give an estimate simply by calculating some numbers for T.

$\square$

An approach towards a more refined estimate is the application of Theorem 4.10. Similarly as in 5.2, we start with a maximal path P and fix the midpoint x of P if P

is even and one of the two central points if P is odd. In a first step we amalgamate to P in x all those $\delta_T(x) - 2$ paths that start in x, are pairwise disjoint and disjoint to P except in x, and are as long as possible. Because of the choice of x and the maximal length of P we can apply Thm. 4.6 which says that the dimension increases after the amalgamation by at most

$$\log_2^+ \frac{\delta_T(x)}{2} = \log_2^+ \delta_T(x) - 1 .$$

We continue by working off simultaneously all vertices y in $S_1(x)$ satisfying $\delta_T(y) > 2$, i.e. we amalgamate all $\delta_T(y) - 2$ paths starting in y and not yet being amalgamated. According to Theorem 4.10 the dimension increases by at most $\log_2^+ \max\{1, \max\{\delta_T(y) \mid d_T(x, y) = 1\} - 1\}$.

This procedure is repeated with the 2-sphere $S_2(x)$ if $S_2(x) \neq \emptyset$ and so on.

Hence we get the following

5.3. Proposition. *Let P be a maximal path in T, i.e. of length* diam T, *and let x be its central point if P's length is even and anyone of its two central points else. Define* $\Delta_0 := \delta_T(x)$,

$$\Delta_k := \begin{cases} \max\{\delta_T(y) \mid d_T(x, y) = k\} - 1 , & \text{if this set is non-empty,} \\ 1 , & \text{else} , \end{cases} \quad k \geqq 1 .$$

Then

$$\dim T \leqq \log_2^+ \operatorname{diam} T + \sum_{k=0}^{\infty} \log_2^+ \Delta k - 1 . \qquad \square$$

The concept of "sphere-wise" encoding is too rigid. There is no reason for not amalgamating simultaneously in different spheres around x. It is therefore easy to see that the following is true; instead of a proof we will give an algorithm for encoding.

5.4. Proposition. *Let P be a maximal path in T, i.e. of length* diam T, *and let x be its central point if P's length is even and anyone of its central points else. Define*

$$\tilde{S}_0(x) := \{x\} , \quad \Delta_0 := \delta_T(x) ,$$
$$\tilde{S}_k(x) := \{y \in V(T) \mid \delta_T(y) > 2 \text{ and if there is a } z \in V(T) \text{ with } d_T(x, z) < d_T(x, y) \text{ then either } \delta_T(z) = 2 \text{ or } z \in \tilde{S}_j(x) \text{ for some } j < k\} ,$$

$$\Delta_k := \begin{cases} \max \delta_T(y) \mid y \in \tilde{S}_k(x)\} - 1 , & \text{if } \tilde{S}_k(x) \neq \emptyset , \\ 1 , & \text{else} , \end{cases} \quad k \geqq 1$$

Then

$$\dim T \leqq \log_2^+ \operatorname{diam} T + \sum_{k=0}^{\infty} \log_2^+ \Delta_k - 1 . \qquad \square$$

Before we develop the algorithm we have to introduce a certain set of 0, 1-vectors which is needed in the encoding procedure.

Let $t_p(i, j)$, $p \geqq 1$, $i \geqq 0$, $j \in \{0, 1\}$, be recursively defined in the following way: $t_1(0, 0) = t_1(1, 1) = 0$, $t_1(0, 1) = t_1(1, 0) = 1$,

$$t_{p+1}(i, j) = \begin{cases} t_p(i, j)\, t_1(0, j), & \text{if} \quad 0 \leqq i < 2^p, \\ t_p(i - 2^p, j)\, t_1(1, j), & \text{if} \quad 2^p \leqq i < 2^{p+1}. \end{cases}$$

5.5. Algorithm.

A1: Let P_{m_0}, $m_0 = \operatorname{diam} T$, be a maximal path in T. Let $V(P_{m_0}) = \{x_{00}, x_{01}, \ldots, x_{0m_0}\}$ be numbered such that $\{x_{0j}, x_{0j+1}\} \in E(T)$. Encode P_{m_0} by u.

A2: Put $x = x_{0\alpha}$ where $\alpha = m_0/2$ if m_0 is even and $\alpha = (m_0 - 1)/2$ if m_0 is odd. Set $k := 1$. If $\delta_T(x) = 2$, set $\mathfrak{A}_0 := P_{m_0}$ and continue with A3. Let be P_{m_i}, $i = 1, \ldots, \delta_T(x) - 2$, be all the paths starting in x, that are pairwise disjoint and disjoint with P_{m_0} except in x, and have maximal length. Since P_{m_0} is maximal in T and x is in the center of P_{m_0}, $m_i \leqq m_0/2$ resp. $m_i \leqq (m_0 - 1)/2$ for all $i \geqq 1$. Let $P_{m_i} = \{x_{m_i0}, x_{m_i1}, \ldots, x_{m_iq_i}\}$ where $x_{m_i0} = x$ and $\{x_{m_ij}, x_{m_ij+1}\} \in E(T)$ for all i, j. Then u_0 is an encoding of $\mathfrak{A}_0 := \mathfrak{A}(P_{m_0}, P_{m_1}, \ldots, P_{m\delta_T(x)-2}:x)$ if one puts:

$$u_0(y) = \begin{cases} u(x)\, 22 \ldots 2, & \text{if} \quad y = x, \\ u(x_{0,\alpha-j})\, t_p(m_i, j \bmod 2), & \text{if} \quad y = x_{m_ij}, \quad i, j \neq 0 \quad \text{and} \quad m_i \quad \text{even} \\ u(x_{0,\alpha+j})\, t_p(m_i, j \bmod 2), & \text{if} \quad y = x_{m_ij}, \quad i, j \neq 0 \quad \text{and} \quad m_i \quad \text{odd}, \\ u(x_{0j})\, t_p(0, j \bmod 2), & \text{if} \quad y = x_{0j}, \end{cases}$$

where $p := \log_2^+ \delta_T(x) - 1$.

A3: Set $\tilde{S}_k(x) = \{x_{k1}, x_{k2}, \ldots, x_{kr_k}\}$. Set $i = 1$ and $p := \log_2^+ \Delta_k$.

A4: Let be $P_{m_j}^{(i)}$, $j = 1, \ldots, \delta_T(x_{ki}) - 2$, be all paths starting in x_{ki}, being pairwise disjoint and disjoint with $\mathfrak{A}_{k-1}$ except in x_{ki} and having maximal length. Set $z_0 = x_{ki}$ and denote $z_j \in V(\mathfrak{A}_{k-1})$ such that $d_T(x, z_j) = j + d_T(x, z_0)$ and $\{z_{j-1}, z_j\} \in E(T)$ for all $j \geqq 1$. Set $P_{m_j}^{(i)} = \{y_{m_j0}, y_{m_j1}, \ldots, y_{m_jq_j}\}$ where $y_{m_j0} = z_0$ and $\{y_{m_js}, y_{m_js+1}\} \in E(T)$ for all j, s. Define

$$u_k(y) = \begin{cases} u_{k-1}(z_0)\, \beta\beta \ldots \beta, & \text{if} \quad y = z_0, \quad \text{where} \quad \beta := 2 + (d_T(x, z_0) \bmod 2), \\ u_{k-1}(z_j)\, t_p(0, j \bmod 2), & \text{if} \quad y = z_j, \quad j \neq 0, \\ u_{k-1}(z_j)\, t_p(m_s, j \bmod 2), & \text{if} \quad y = y_{m_sj}, \quad j \neq 0. \end{cases}$$

Set $i \leftarrow i + 1$. If $i \leqq r_k$, repeat A4.

A5: For all vertices y from $\mathfrak{A}_{k-1}$ for which u_k is not yet defined, set $u_k(y) = u_{k-1}(y)\, \beta\beta \ldots \beta$, where $\beta = 2 + (d_T(x, y) \bmod 2)$.
Now u_k is an encoding of

$$\mathfrak{A}_k := \mathfrak{A}(P_{m_1}^{(1)}, \ldots, P_{m_{s_1}}^{(1)}: x_{k_1} \ldots\ldots P_{m_1}^{(r_k)}, \ldots, P_{m_{s_{r_k}}}^{(r_k)} : x_{kr_k} \,.\, \mathfrak{A}_{k-1})$$

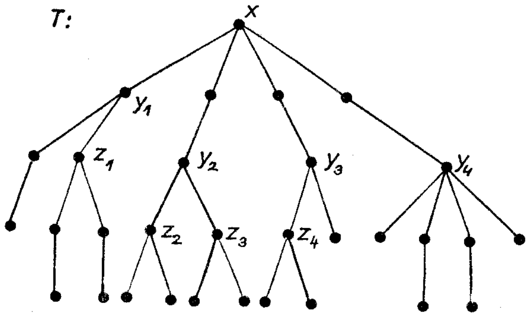
and it holds:

$$\dim \mathfrak{A}_k \leqq \dim \mathfrak{A}_{k-1} + \log_2^+ \Delta_k .$$

Set $k \leftarrow k + 1$. If $\tilde{S}_k(x) \neq \emptyset$, continue with A3.
STOP. □

Example.

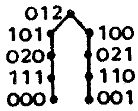
T:



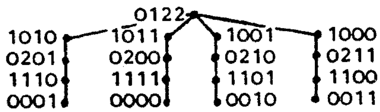
$$\tilde{S}_1(x) = \{y_1, y_2, y_3\} \Rightarrow \Delta_1 = 4\,,$$

$$\tilde{S}_2(x) = \{z_1, z_2, z_3, z_4\} \Rightarrow \Delta_2 = 2 \Rightarrow 5 \leqq \dim T \leqq 7$$

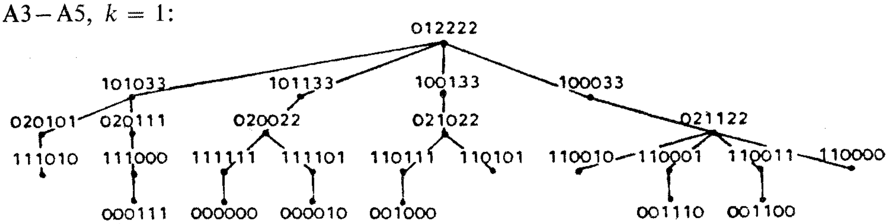
A1:



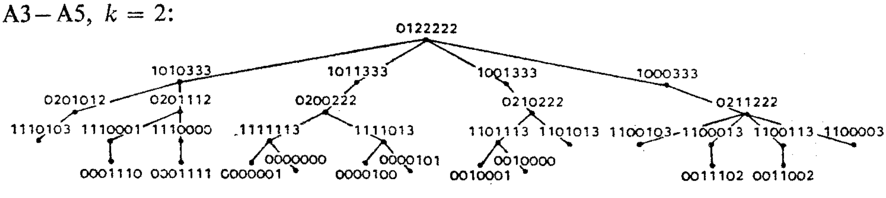
A2:



A3 – A5, $k = 1$:

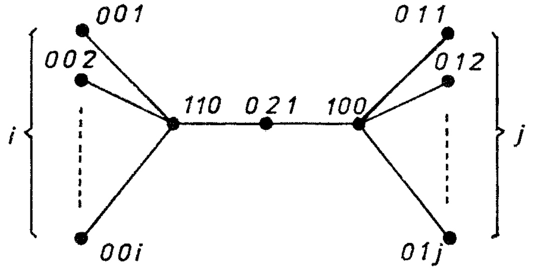


A3 – A5, $k = 2$:



□

Remark. Let r be such that $\tilde{S}_r(x) \neq \emptyset$ and $\tilde{S}_s(x) = \emptyset$ for all $s > r$. If $\tilde{S}_r(x)$ contains only those vertices which are adjacent exclusively to end-vertices of T, then one can put $\Delta_r = 2$ in the above formula, since it is possible to encode all the end-vertices by only one additional coordinate (cf. [10, 1.5]). Example:



□

Finally, we want to apply Proposition 5.3 to the following type of symmetrical tree T with diam $T = 2n$, in which all vertices of a fixed sphere around the root x have same degree:

$$\delta_T(x) = 2^{\alpha_0+1}, \quad \alpha_0 \geqq 0; \quad \Rightarrow \Delta_0 = 2^{\alpha_0+1},$$
$$\delta_T(y) = 2^{\alpha_k} + 1, \quad \alpha_k \geqq 0, \quad \text{if} \quad d_T(x, y) = k \in \{1, \ldots, n-1\}; \quad \Rightarrow \Delta_k = 2^{\alpha_k},$$
$$(\delta_T(y) = 1, \quad \text{if} \quad d_T(x, y) = n).$$

Define $m_0 := 1$,

$$m_k := 2^{1+\alpha_0+\alpha_1+\ldots+\alpha_{k-1}} \quad (k = 1, \ldots, n).$$

Then T has $\sum_{k=0}^{n} m_k$ vertices, m_n of which have degree 1. This implies the lower bound according to Proposition 5.1

$$\dim T \geqq \log_2^+ (1 + \sum_{k=0}^{n-1} m_k) \geqq 2 + \sum_{k=0}^{n-2} \alpha_k,$$

and the upper bound according to Proposition 5.3

$$\dim T \leqq \log_2^+ \operatorname{diam} T + \sum_{k=0}^{n-1} \log_2^+ \Delta_k - 1 = 1 + \log_2^+ n + \sum_{k=0}^{n-1} \alpha_k. \qquad \square$$

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