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Dimension-theoretical structure of locally compact groups

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This paper is devoted to the study of dimension-theoretical structure of locally compact groups and their factor spaces. Montgomery-Zippin [9] proved that every finite-dimensional, locally compact group is a generalized Lie group and finally Yamabe [17] proved that every locally compact group is also a generalized Lie group. These are the most important results not only for the group-theoretical structure of locally compact groups but also for the dimension-theoretical structure of such groups. Montgomery [7] had proved also, before his fundamental theorem cited above was established, that the invariance theorem of a domain is true in finite-dimensional, locally connected, locally compact, separable metric groups. P. Alexandroff conjectured that the covering dimension of any locally compact group coincides with its inductive dimension. Recently this conjecture has been solved in the affirmative by Pasynkov [15]. His result will be generalized in § 2, after some preliminaries of § 1, for factor spaces of finite-dimensional locally compact groups by connected compact subgroups. It will also be proved that $\dim G = \dim H + \dim G/H$, where dim denotes the covering dimension, for any locally compact group G and any closed subgroup H of it. Montgomery-Zippin [8], Yamanoshita [18] and others have considered the dimension of factor spaces of locally compact groups and obtained the equality for some special cases. Our theorem seems to be a complete answer for the problem concerning the covering dimension of factor spaces of locally compact groups. In § 3 the decomposition theorem for locally compact groups will be proved. Both Pasynkov's theorem cited above and the author's decomposition theorem show that there are some analogy between the dimension-theoretical structure of locally compact groups and that of Euclidean spaces. In § 4 we shall point out a difference between the two by proving that the invariance theorem of a domain is not true in any finite-dimensional, locally compact, metric group which is not locally connected. Combining this with Montgomery's invariance theorem mentioned above, we know that a finite-dimensional, locally compact, metric group is locally connected (or equivalently a Lie group) if and only if the invariance theorem is valid in it.

In this paper a topological group means a T_1-group. Hence a locally compact group and its factor space by a closed subgroup are always normal

Hausdorff spaces (cf. Lemma 1.1 below). A homomorphism means a continuous one and an isomorphism means a homeomorphic one. A coset and a factor space mean respectively a left coset and a left factor space. Throughout this paper n and m denote integers which are not less than -1. We use three notions of dimension of a normal Hausdorff space R defined as follows. R has the covering dimension $\leqq n$, $\dim R \leqq n$, if every finite open covering of R can be refined by an open covering whose order is at most $n+1$, where the order of a covering is the greatest number r such that r elements of it have a non-empty intersection. $\operatorname{Ind} R$ and $\operatorname{ind} R$ denote respectively the large and the small inductive dimension of R: For the empty set ϕ, let $\operatorname{Ind}\phi = \operatorname{ind}\phi = -1$. We call $\operatorname{Ind} R \leqq n$, if for any pair $F \subset D$ of a closed set F and an open set D there exists an open set E with $F \subset E \subset D$ such that $\operatorname{Ind}(\bar{E}-E) \leqq n-1$. We call $\operatorname{ind} R \leqq n$, if for any point x of R and any open set D with $x \in D$ there exists an open set E with $x \in E \subset D$ such that $\operatorname{ind}(\bar{E}-E) \leqq n-1$. When d is any one of dim, Ind, ind, we call $\mathrm{d}\, R = n$, if $\mathrm{d}\, R \leqq n$ is true and $\mathrm{d}\, R \leqq n-1$ is false. It is well known that for any separable metric space R the equalities $\dim R = \operatorname{Ind} R = \operatorname{ind} R$ are valid [**6**].

§ 1. Preliminaries.

LEMMA 1.1. *Let G be a locally compact group and H a closed subgroup of G. Then the factor space $G/H = K$ is paracompact.*

PROOF. Let $\mathfrak{V} = \{V\}$ be the system of all neighborhoods of the identity of G. For any point k of K let

$$U_V(k) = \rho(V \cdot \rho^{-1}(k)),$$

where $\rho : G \to G/H = K$ is the natural projection. Then $\{U_V ; V \in \mathfrak{V}\}$ forms a uniform structure which agrees with the preasigned natural topology of K. This is verified by a straight-forward computation and its proof is left to the reader. Let V be a compact neighborhood of the identity, k an arbitrary point of K and g an element of G with $\rho(g) = k$. Since $U_V(k) = \rho(V \cdot \rho^{-1}(k)) = \rho(VgH) = \rho(Vg)$, $U_V(k)$ is compact. Thus K is uniformly locally compact. Hence K is paracompact by Morita [**11**].

REMARK 1.2. If a topological space R admits a locally finite open covering $\{D_\delta ; \delta \in \Delta\}$ such that $\bar{D}_\delta$ is compact for every $\delta \in \Delta$, then R is the sum of mutually disjoint open sets each of which is σ-compact (i.e. expressible as the sum of a countable number of compact sets).

PROOF. Since $\{D_\delta ; \delta \in \Delta\}$ is locally finite, $\{\bar{D}_\delta ; \delta \in \Delta\}$ is locally finite. Suppose that $\{\bar{D}_\delta ; \delta \in \Delta\}$ is not star-finite. ($\{\bar{D}_\delta ; \delta \in \Delta\}$ is called star-finite if for any $\delta \in \Delta$, the number of indices δ' with $\bar{D}_\delta \frown \bar{D}_{\delta'} \neq \phi$ is finite.) Then it is easy to find an index $\delta \in \Delta$ and sequences $\{\delta_i ; \delta_i \in \Delta\}$ and $\{p_i\}$ such that i)

$p_i \in \bar{D}_\delta \cap \bar{D}_{\delta_i}$, ii) $\delta_i \neq \delta_j$ if $i \neq j$, iii) $p_i \neq p_j$ if $i \neq j$. Since $\bar{D}_\delta$ is compact, there exists an accumulating point p of $\{p_i\}$. Then $\{\bar{D}_\delta; \delta \in \Delta\}$ cannot be locally finite at p, which is a contradiction. The assertion of the remark is a trivial consequence of the star-finiteness of $\{\bar{D}_\delta; \delta \in \Delta\}$.

LEMMA 1.3 (Montgomery-Zippin [**10**, Theorem, p. 237]). *A locally compact group G with* $\dim G = n$ *has a small neighborhood of the identity which is the direct product of a local Lie group L with* $\dim L = n$ *and a compact group N with* $\dim N = 0$.

LEMMA 1.4. *Let G be a non-empty locally compact group and N a compact normal subgroup of G with* $\dim N = 0$. *If the factor group G/N is a Lie group with* $\dim G/N = n$, *then there exists a neighborhood of the identity which is the direct product of a local Lie group L with* $\dim L = n$ *and N.*

PROOF. Let f be the natural projection of G onto G/N and $g^*(t)$ an arbitrary one-parameter subgroup of G/N. Then there exists a one-parameter subgroup $g(t)$ of G such that $f(g(t)) = g^*(t)$ by Montgomery-Zippin [**10**, Theorem 1, p. 192]. Hence the method of the proof of Pontrjagin [**16**, Theorem 69] can be applied with no modification and we have the lemma.

LEMMA 1.5 (Nagami [**14**] or C. H. Dowker [**3**]). *If every point of a paracompact Hausdorff space R has its neighborhood whose covering dimension is at most n, then we have* $\dim R \leqq n$.

LEMMA 1.6.[1] *Let G be a non-empty locally compact group with* $\dim G = n$ *and H a closed subgroup of G with* $\dim H = m$.[2] *Then every point of G/H has a neighborhood which is homeomorphic to the direct product of an $(n-m)$-dimensional Euclidean cube and a compact Hausdorff space whose covering dimension is* 0.

PROOF. It suffices to construct a neighborhood of $\rho(e)$ satisfying the conditions of the lemma, where ρ is the natural projection of G onto G/H and e is the identity of G. Let D be an arbitrary open neighborhood of $\rho(e)$. By Lemma 1.3 $\rho^{-1}(D)$ contains a neighborhood of e which is the direct product of a connected local Lie group L_1 with $\dim L_1 = n$ and a compact subgroup N with $\dim N = 0$ such that i) $L_1 = L_1^{-1}$, ii) $\overline{(L_1N)^2}$ is compact. Let $P = H \cap N$ and $G_0 = (L_1N)^\infty$ $(= \bigcup_{i=1}^{\infty} (L_1N)^i)$. Then G_0 is an open subgroup of G and N is a normal subgroup of G_0. Hence $H_0 = H \cap G_0$ is a relatively open subgroup of H and P is a normal subgroup of H_0. By Lemmas 1.1 and 1.5 we have

$$\dim H_0 = \dim H = m\,.$$

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1) This lemma generalizes the last half of Montgomery-Zippin [**10**, Theorem, p. 239], the first half of which will also be generalized in Theorem 2.1 below.

2) Since H is closed, we have $n \geqq m$ at once.

Since $(\overline{L_1N})^2$ is compact, both H_0 and H_0N are σ-compact and locally compact. Hence by Pontrjagin [16, G), § 20] H_0/P is isomorphic to H_0N/N. Let ρ_1 be the natural projection of G_0 onto G_0/N. Since H_0N is closed in G_0 and $H_0N/N = G_0/N - \rho_1(G_0 - H_0N)$, we know that H_0N/N is a closed subgroup of G_0/N. Since G_0/N is evidently a Lie group, H_0N/N is a Lie group and hence H_0/P is so.

By Lemma 1.3 there exists a connected, compact, local Lie group M_1 with $\dim M_1 = m$ such that

i) M_1P is a relative neighborhood of the identity in H_0,

ii) M_1P is the direct product of M_1 and P,

iii) $M_1P \subset L_1N$.

Since M_1 is connected and N is totally disconnected, we have $M_1 \subset L_1$. Since M_1 is compact, M_1 is a closed local subgroup of L_1. Therefore there exist subsets L of L_1 and M of M_1 with $\dim L = n$ and $\dim M = m$ such that we can introduce into L a canonical coordinate system of the second kind which has the following properties:

i) L is the totality of points whose coordinates are of the form

$$(t_1, \cdots, t_n), \qquad |t_i| \leqq 1,$$

ii) M is the totality of points whose coordinates are of the form

$$(0, \cdots, 0, t_{n-m+1}, \cdots, t_n), \qquad |t_i| \leqq 1.$$

Let Λ be the totality of points whose coordinates are of the form

$$(t_1, \cdots, t_{n-m}, 0, \cdots, 0), \qquad |t_i| \leqq 1.$$

Here we notice that every element of LN can be expressed as $\lambda\mu\nu$, $\lambda \in \Lambda$, $\mu \in M$, $\nu \in N$, and every element of MP can be so as μp, $\mu \in M$, $p \in P$. We continue to use this notion in the following of the present proof. Let W_1 be the totality of points whose coordinates are of the form

$$(t_1, \cdots, t_n), \qquad |t_i| \leqq \varepsilon, \quad 0 < \varepsilon < 1,$$

such that $W_1^{-1}W_1 \subset L$. We set $W = W_1N$. Let $\lambda_1\mu_1\nu_1$ and $\lambda_2\mu_2\nu_2$ be two points of W which are contained in the same coset by H; then there exists a point μp of MP such that $\lambda_1\mu_1\nu_1 = \lambda_2\mu_2\nu_2\mu p$. Hence we have $\lambda_1 = \lambda_2$. Since $\mu_1\nu_1 = \mu_2\nu_2\mu p = \mu_2\mu\nu_2 p$, we have $\mu_1 = \mu_2\mu$ and hence $\nu_1 = \nu_2 p$. Conversely if $\lambda_1\mu_1\nu_1$ and $\lambda_2\mu_2\nu_2$ of W satisfy $\lambda_1 = \lambda_2$ and $\nu_2^{-1}\nu_1 \in P$, then these two points are contained in the same coset by H. Thus we know that $\lambda_1\mu_1\nu_1$ and $\lambda_2\mu_2\nu_2$ of W fall in the same coset by H if and only if $\lambda_1 = \lambda_2$ and $\nu_2^{-1}\nu_1 \in P$.

Let f be the natural projection of N onto N/P and g the mapping of $\rho(W)$ onto the product space $(W_1 \cap \Lambda) \times (N/P)$ defined in such a way that

$$g(\rho(\lambda\mu\nu)) = (\lambda, f(\nu));$$

then it can easily be seen by the above observation that g is a homeomorphism. Let A_ε be the totality of points whose coordinates are of the form

$$(t_1, \cdots, t_{n-m}, 0, \cdots, 0), \qquad |t_i| \leqq \varepsilon .$$

Then $\rho(A_\varepsilon N)$ is a neighborhood of $\rho(e)$ which is homeomorphic to $A_\varepsilon \times (N/P)$. N/P is a compact Hausdorff space with $\dim N/P = 0$ by Pontrjagin [**16**, A), § 48] and the lemma is proved.

LEMMA 1.7. *Let G be a non-empty, locally compact, projective limit of Lie groups with* $\dim G = n$, *and H a compact subgroup of G. Then the factor space $G/H = K$ is the projective limit of $(n-m)$-manifolds K_α, $\alpha \in A$, accompanied by the mappings $\omega_{\alpha\beta} : K_\alpha \to K_\beta$, $\beta < \alpha$, which are open continuous and locally homeomorphic.*

PROOF. Let G_α, $\alpha \in A_1$, be Lie groups and $\pi_{\alpha\beta} : G_\alpha \to G_\beta$, $\beta < \alpha$, open homomorphism of G_α onto G_β such that the projective limit of $\{G_\alpha, \pi_{\alpha\beta} ; \alpha \in A_1\}$ is G. Let π_α, $\alpha \in A_1$, be (open) homomorphism of G onto G_α. Let N_α^β, $\beta < \alpha$, be the kernel of $\pi_{\alpha\beta}$ and N_α the kernel of π_α. Since G is locally compact, we can assume without loss of generality that every N_α and every N_α^β are compact. We set

$$\dim G_\alpha = n(\alpha) .$$

Since any small $n(\alpha)$-cell in G_α can be lifted to G by Montgomery-Zippin [**10**, p. 194], we have

$$n(\alpha) \leqq \dim G = n .$$

Let

$$\max \{n(\alpha) ; \alpha \in A_1\} = n_1$$

and α_0 an element of A_1 such that $\dim G_{\alpha_0} = n_1$. Let β be an arbitrary index with $\alpha_0 < \beta$. Since G_{α_0} and G_β are Lie groups, it is well known that $\dim G_\beta = \dim G_{\alpha_0} + \dim N_\beta^{\alpha_0}$. Thus we have $\dim G_\beta = n_1$ and $\dim N_\beta^{\alpha_0} = 0$. Since $N_\beta^{\alpha_0}$ is a 0-dimensional compact Lie group, it is a finite group. Since N_{α_0} is isomorphic to the projective limit of $\{N_\beta^{\alpha_0}, \pi_{\gamma\beta} ; \alpha_0 \leqq \beta < \gamma\}$, we have $\dim N_{\alpha_0} = 0$. Therefore there exists a neighborhood of the identity of G which is the direct product of a local Lie group L with $\dim L = n_1$ and N_{α_0} by Lemma 1.4. By Morita [**12**] we have $\dim LN_{\alpha_0} \leqq \dim L + \dim N_{\alpha_0} = n_1$. Since G is, by Lemma 1.1, paracompact, we have $n = \dim G \leqq \dim LN_{\alpha_0} = n_1$ by Lemma 1.5. Therefore we have $n = n_1$. Let

$$A = \{\alpha ; \alpha_0 \leqq \alpha\} ;$$

then we have $\dim G_\alpha = n$ and $\dim N_\alpha = 0$ for any $\alpha \in A$ and G is the projective limit of

$$\{G_\alpha ; \alpha \in A\} .$$

Let $\dim H = m$. Then $m \leqq n$. Let H_α, $\alpha \in A$, be the image of H under π_α. Since N_α is compact, HN_α is a closed subgroup of G. Since $H_\alpha = G_\alpha - \pi_\alpha$

$(G-HN_\alpha)$, H_α is a closed subgroup of G_α. Hence we know that H_α is a Lie group.

Let p_α be the natural projection of H onto $H/H\cap N_\alpha$ and define $p_{\alpha\beta}$: $H/H\cap N_\alpha \to H/H\cap N_\beta$, $\beta<\alpha$, as follows:

$$p_{\alpha\beta}(r_\alpha)=p_\beta p_\alpha^{-1}(r_\alpha), \qquad r_\alpha \in H/H\cap N_\alpha .$$

Let

$$q_\alpha(r_\alpha)=\pi_\alpha p_\alpha^{-1}(r_\alpha), \qquad r_\alpha \in H/H\cap N_\alpha .$$

Since H is compact, q_α is an isomorphism of $H/H\cap N_\alpha$ onto H_α. Thus we obtain the following diagram:

$$\begin{array}{ccc} & H & \\ {\scriptstyle p_\alpha}\swarrow & & \searrow{\scriptstyle \pi_\alpha} \\ H/H\cap N_\alpha & \xrightarrow{q_\alpha} & H_\alpha=HN_\alpha/N_\alpha \\ {\scriptstyle p_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \pi_{\alpha\beta}} \\ H/H\cap N_\beta & \xrightarrow{q_\beta} & H_\beta=HN_\beta/N_\beta \end{array}$$

It is almost evident that $\{H/H\cap N_\alpha, p_{\alpha\beta}\,;\alpha\in A\}$ forms a spectrum. Let $\tilde{H}$ be the projective limit of $\{H/H\cap N_\alpha, p_{\alpha\beta}\,;\alpha\in A\}$ and define $p: H\to\tilde{H}$ as follows:

$$p(h)=\langle p_\alpha(h)\,;\alpha\in A\rangle, \qquad h\in H.$$

Then p is an isomorphism of H onto $\tilde{H}$ by Gleason [4]. Since $q_\alpha p_\alpha=\pi_\alpha$, we know that H[3] is the projective limit of $\{H_\alpha, \pi_{\alpha\beta}\}$. Since $\dim H\cap N_\alpha=0$, we have

$$\dim H_\alpha=\dim H=m, \qquad \alpha\in A,$$

as in the preceding argument.

Let ρ_α be the natural projection of G_α onto $K_\alpha=G_\alpha/H_\alpha$. For any pair $\beta<\alpha$ define $\omega_{\alpha\beta}: K_\alpha\to K_\beta$ in such a way that

$$\omega_{\alpha\beta}(k_\alpha)=\rho_\beta\pi_{\alpha\beta}\rho_\alpha^{-1}(k_\alpha), \qquad k_\alpha\in K_\alpha .$$

Then we obtain the following diagram:

$$\begin{array}{ccc} G_\alpha & \xrightarrow{\rho_\alpha} & K_\alpha=G_\alpha/H_\alpha \\ {\scriptstyle \pi_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \omega_{\alpha\beta}} \\ G_\beta & \xrightarrow{\rho_\beta} & K_\beta=G_\beta/H_\beta \end{array}$$

Let g_α and g'_α be arbitrary elements of $\rho_\alpha^{-1}(k_\alpha)$; then $g_\alpha^{-1}g'_\alpha\in H_\alpha$ and hence $\pi_{\alpha\beta}(g_\alpha)^{-1}\cdot\pi_{\alpha\beta}(g'_\alpha)\in\pi_{\alpha\beta}(H_\alpha)=H_\beta$, which implies $\rho_\beta\pi_{\alpha\beta}(g_\alpha)=\rho_\beta\pi_{\alpha\beta}(g'_\alpha)$. Thus $\omega_{\alpha\beta}$ is a mapping of K_α into K_β. It is almost evident that i) $\omega_{\alpha\beta}$ is an open con-

3) When H is not compact but σ-compact, we can also conclude that H is the projective limit of Lie groups, since q_α are also isomorphisms in this case by virtue of Pontrjagin [**16**, G), § 20].

tinuous onto mapping, ii) $\{K_\alpha, \omega_{\alpha\beta}; \alpha \in A\}$ forms a spectrum, iii) the equality

$$\omega_{\alpha\beta}\rho_\alpha = \rho_\beta \pi_{\alpha\beta}$$

holds for any pair $\beta < \alpha$.

Let ρ be the natural projection of G onto G/H and define mappings $\omega_\alpha : G/H = K \to G_\alpha/H_\alpha = K_\alpha$, $\alpha \in A$, as follows:

$$\omega_\alpha(k) = \rho_\alpha \pi_\alpha \rho^{-1}(k), \qquad k \in K.$$

Let g and g' be arbitrary elements of G with $g^{-1}g' \in H$; then $\pi_\alpha(g)^{-1} \cdot \pi_\alpha(g') \in H_\alpha$ and hence ω_α is well defined. Thus we have the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & K = G/H \\ \pi_\alpha \downarrow & & \downarrow \omega_\alpha \\ G_\alpha & \xrightarrow{\rho_\alpha} & K_\alpha = G_\alpha/H_\alpha \\ \pi_{\alpha\beta} \downarrow & & \downarrow \omega_{\alpha\beta} \\ G_\beta & \xrightarrow{\rho_\beta} & K_\beta = G_\beta/H_\beta \end{array}$$

It is almost evident that i) ω_α is an open continuous onto mapping for any $\alpha \in A$, ii) the equality

$$\omega_\beta = \omega_{\alpha\beta}\omega_\alpha$$

holds for any pair $\beta < \alpha$.

Let $\tilde{K}$ be the projective limit of $\{K_\alpha, \omega_{\alpha\beta}; \alpha \in A\}$ and let

$$\omega(k) = \langle \omega_\alpha(k); \alpha \in A \rangle, \qquad k \in K.$$

Since $\omega_{\alpha\beta}\omega_\alpha(k) = \omega_\beta(k)$ for any pair $\beta < \alpha$, ω is a mapping of K into $\tilde{K}$. Let us prove that ω is a homeomorphism of K onto $\tilde{K}$. Since ω_α is continuous for any $\alpha \in A$, ω is evidently continuous.

To prove that ω is one-to-one, let k and k_1 be different elements of K. Let g and g_1 be elements of G such that $gH = \rho^{-1}(k)$ and $g_1H = \rho^{-1}(k_1)$; then $g^{-1}g_1 \notin H$. Since H is the projective limit of $\{H_\alpha\}$, there exists an index α with $\pi_\alpha(g^{-1}g_1) \notin H_\alpha$. Then $\pi_\alpha(g)H_\alpha \cap \pi_\alpha(g_1)H_\alpha = \phi$ and $\rho_\alpha\pi_\alpha(g) \neq \rho_\alpha\pi_\alpha(g_1)$. Hence $\omega_\alpha(k) \neq \omega_\alpha(k_1)$ and we have $\omega(k) \neq \omega(k_1)$. Therefore ω is one-to-one.

To prove that ω is onto, let $\tilde{\omega}_\alpha$, $\alpha \in A$, be the projections of $\tilde{K}$ onto K_α and $\tilde{k}$ an arbitrary element of $\tilde{K}$. For any $\alpha \in A$, $\pi_\alpha^{-1}\rho_\alpha^{-1}\tilde{\omega}_\alpha(\tilde{k})$ is a coset of G by the compact subgroup HN_α and hence a compact subset of G. It is obvious that $\{\pi_\alpha^{-1}\rho_\alpha^{-1}\tilde{\omega}_\alpha(\tilde{k}); \alpha \in A\}$ is the family of compact subsets of G which has the finite intersection property. Hence $\cap \{\pi_\alpha^{-1}\rho_\alpha^{-1}\tilde{\omega}_\alpha(\tilde{k}); \alpha \in A\}$ is not empty and contains a point g. The image of $\rho(g)$ under ω is $\tilde{k}$. Therefore ω is onto.

To show that ω is open let k be an arbitrary point of K, D an arbitrary open neighborhood of k and g an element of G with $\rho(g) = k$. Then there exist an index α and an open neighborhood D_1 of g such that $\pi_\alpha^{-1}\pi_\alpha(D_1) \subset \rho^{-1}(D)$. $E = \tilde{\omega}_\alpha^{-1}\rho_\alpha\pi_\alpha(D_1)$ is an open neighborhood of $\omega(k)$. We have $\pi_\alpha^{-1}\rho_\alpha^{-1}\tilde{\omega}_\alpha(E)$

$=\pi_\alpha^{-1}\rho_\alpha^{-1}\rho_\alpha\pi_\alpha(D_1)=\pi_\alpha^{-1}(\pi_\alpha(D_1)H_\alpha)=\pi_\alpha^{-1}(\pi_\alpha(D_1)\pi_\alpha(H))=\pi_\alpha^{-1}\pi_\alpha(D_1H)=D_1HN_\alpha$ $=(\pi_\alpha^{-1}\pi_\alpha(D_1))H\subset\rho^{-1}(D)H=\rho^{-1}(D)$. Let g_1 be an element of G with $\rho(g_1)\in\omega^{-1}(E)$. Then $\rho_\alpha\pi_\alpha\rho^{-1}\rho(g_1)=\rho_\alpha\pi_\alpha(g_1H)=\rho_\alpha(\pi_\alpha(g_1)H_\alpha)=\rho_\alpha\pi_\alpha(g_1)$ is an element of $\tilde{\omega}_\alpha(E)$ $=\tilde{\omega}_\alpha\tilde{\omega}_\alpha^{-1}\rho_\alpha\pi_\alpha(D_1)=\rho_\alpha\pi_\alpha(D_1)$. Hence g_1 is an element of $\pi_\alpha^{-1}\rho_\alpha^{-1}\rho_\alpha\pi_\alpha(D_1)=D_1HN_\alpha$. Therefore we have $\omega^{-1}(E)\subset\rho(D_1HN_\alpha)\subset D$ and know that ω is open.

By the above observation ω is a homeomorphism of $G/H=K$ onto $\tilde{K}$. Recall that $\dim H_\alpha=m$ for any $\alpha\in A$. Hence we have

$$\dim K_\alpha=n-m$$

for any $\alpha\in A$.

Finally let us show that the local restriction of $\omega_{\alpha\beta}$ is a homeomorphism. Let $\beta<\alpha$ be an arbitrary ordered pair and k_α an arbitrary element of K_α. Let k be an element of K with $\omega_\alpha(k)=k_\alpha$ and g an element of G with $\rho(g)$ $=k$. Let LN_β be a neighborhood of the identity of G which is the direct product of a local Lie group L with $\dim L=n$ and N_β such that L is the totality of points of the form $(t_1,\cdots,t_n)$, $|t_i|\leqq 1$, in some canonical coordinate system Σ of the second kind. For any number δ with $0<\delta<1$, let Λ_δ be the totality of points of the form

$$(t_1,\cdots,t_{n-m},0,\cdots,0)\,,\qquad |t_i|\leqq\delta\,,$$

in Σ and M_δ the totality of points of the form

$$(0,\cdots,0,t_{n-m+1},\cdots,t_n)\,,\qquad |t_i|\leqq\delta\,,$$

in Σ. Let P_β be the intersection of H and N_β. By the same argument as in the proof of Lemma 1.5 there exists a positive number $\varepsilon<1$ such that

i) $M_\varepsilon P_\beta$ is a relative neighborhood of the identity in H which is the direct product of M_ε and P_β,

ii) $\Lambda_\varepsilon M_\varepsilon N_\beta\frown H=M_\varepsilon P_\beta$,

iii) $(\Lambda_\varepsilon M_\varepsilon N_\beta)^{-1}\Lambda_\varepsilon M_\varepsilon N_\beta\subset LN_\beta$.

Let λ_1 and λ_2 be two elements of Λ_ε and consider two elements $\rho_\alpha\pi_\alpha(g\lambda_1)$ and $\rho_\alpha\pi_\alpha(g\lambda_2)$ of K_α. Suppose that $\omega_{\alpha\beta}\rho_\alpha\pi_\alpha(g\lambda_1)=\omega_{\alpha\beta}\rho_\alpha\pi_\alpha(g\lambda_2)$. Then $\rho_\beta\pi_{\alpha\beta}\pi_\alpha(g\lambda_1)$ $=\rho_\beta\pi_{\alpha\beta}\pi_\alpha(g\lambda_2)$ and hence $\rho_\beta\pi_\beta(g\lambda_1)=\rho_\beta\pi_\beta(g\lambda_2)$. We have $\pi_\beta(g\lambda_2)^{-1}\cdot\pi_\beta(g\lambda_1)$ $=\pi_\beta(\lambda_2^{-1}\lambda_1)\in H_\beta$ and hence $\lambda_2^{-1}\lambda_1\in HN_\beta$. On the other hand $\lambda_2^{-1}\lambda_1\in\Lambda_\varepsilon^{-1}\Lambda_\varepsilon\subset L$ and hence $\lambda_2^{-1}\lambda_1\in HN_\beta\frown LN_\beta\subset M_\varepsilon N_\beta$. Therefore there exist an element μ of M_ε and an element ν of N_β such that $\lambda_1=\lambda_2\mu\nu$. Since this expression is unique, we have $\lambda_1=\lambda_2$. Thus we can conclude by the compactness of $g\Lambda_\varepsilon$ that $\rho_\alpha\pi_\alpha(g\Lambda_\varepsilon)$ and $\rho_\beta\pi_\beta(g\Lambda_\varepsilon)$ are the homeomorphic image of $g\Lambda_\varepsilon$ under the mapping $\rho_\alpha\pi_\alpha$ and $\rho_\beta\pi_\beta$ respectively and that $\rho_\beta\pi_\beta(g\Lambda_\varepsilon)$ is the homeomorphic image of $\rho_\alpha\pi_\alpha(g\Lambda_\varepsilon)$ under the mapping $\omega_{\alpha\beta}$. Let Λ'_ε be the totality of points of the form $(t_1,\cdots,t_{n-m},0,\cdots,0)$, $|t_i|<\varepsilon$, in Σ. If we replace Λ_ε with Λ'_ε, then the above statements with this replacement is also valid. Since $g\Lambda'_\varepsilon$ is homeomorphic to an $(n-m)$-Euclidean space and K_α and K_β are $(n-m)$-manifolds, we know that $\rho_{\alpha}\pi_{\alpha}(g\Lambda'_{\varepsilon})$ and $\rho_{\beta}\pi_{\beta}(g\Lambda'_{\varepsilon})$ are open sets of K_{α} and K_{β} respectively, by a famous Brouwer's invariance theorem of a domain. $\rho_{\alpha}\pi_{\alpha}(g\Lambda'_{\varepsilon})$ contains $\rho_{\alpha}\pi_{\alpha}(g)=\omega_{\alpha}\rho(g)=\omega_{\alpha}(k)=k_{\alpha}$. Thus the proof is completely finished

COROLLARY 1.8.[4] *A σ-compact closed subgroup of a locally compact group which is the projective limit of Lie groups is also the projective limit of Lie groups.*

Cf. the footnote 3).

COROLLARY 1.9.[4] *A locally compact group G has a σ-compact open subgroup which is the projective limit of Lie groups.*

PROOF. By Glushkov [5] there exists an open subgroup G_1 of G which is the projective limit of Lie groups. Let U be a symmetric open neighborhood of the identity of G such that i) $\overline{U^2}$ is compact and ii) $\overline{U^2}\subset G_1$. Then U^{∞} is a σ-compact open subgroup with $U^{\infty}\subset G_1$. By Corollary 1.8 U^{∞} is also the projective limit of Lie groups and the corollary is proved.

COROLLARY 1.10. *Let G be a locally compact group with* $\dim G=n$ *and H a connected compact subgroup of G with* $\dim H=m$. *Then $K=G/H$ is the projective limit of $(n-m)$-manifolds K_{α} accompanied with projections $\omega_{\alpha\beta}$ which are open continuous and locally topological.*

PROOF. By Glushkov [5] there exists an open subgroup G_0 of G which is the projective limit of Lie groups. Decompose G into cosets $g_{\xi}G_0$, $\xi\in\Xi$, such that $G=\cup\{g_{\xi}G_0\,;\,\xi\in\Xi\}$ and $g_{\xi_1}G_0\cap g_{\xi_2}G_0=\phi$ for any ξ_1 and ξ_2 of Ξ with $\xi_1\neq\xi_2$. For any $g\in G$ and any $g_0\in G_0$ we have $gG_0\supset gg_0H$ by virtue of the connectedness of H. Therefore $\rho(g_{\xi_1}G_0)\cap\rho(g_{\xi_2}G_0)=\phi$ whenever $\xi_1\neq\xi_2$, where ρ is the natural projection of G onto $G/H=K$. If we set

$$\varphi_{\xi}(k)=\rho(g_{\xi}\cdot\rho^{-1}(k)), \qquad k\in\rho(G_0),$$

we have a mapping φ_{ξ} of $\rho(G_0)$ into $\rho(g_{\xi}G_0)$. By a straight-forward argument it can easily be seen that φ_{ξ} is a homeomorphism of $\rho(G_0)$ onto $\rho(g_{\xi}G_0)$. Thus K is the sum of mutually disjoint open sets $\rho(g_{\xi}G_0)$, $\xi\in\Xi$, any of which is homeomorphic to $\rho(G_0)$.

By Lemma 1.7 we can consider G_0/H as the projective limit of $(n-m)$-manifolds K_{α}^0, $\alpha\in A$, accompanied with open continuous mappings $\omega_{\alpha\beta}^0:K_{\alpha}^0\to K_{\beta}^0$, $\beta<\alpha$, which are locally topological. For any $\xi\in\Xi$ and any $\alpha\in A$, let K_{α}^{ξ} be a copy of K_{α}^0 (as a topological space) and $\varphi_{\alpha}^{\xi}:K_{\alpha}^0\to K_{\alpha}^{\xi}$ a copy-mapping. For any $\xi\in\Xi$ and any pair $\beta<\alpha$ let $\omega_{\alpha\beta}^{\xi}:K_{\alpha}^{\xi}\to K_{\beta}^{\xi}$ be a mapping defined by

$$\omega_{\alpha\beta}^{\xi}=\varphi_{\beta}^{\xi}\omega_{\alpha\beta}^0(\varphi_{\alpha}^{\xi})^{-1}.$$

Then it is evident that $\{K_{\alpha}^{\xi},\omega_{\alpha\beta}^{\xi}\,;\,\alpha\in A\}$ forms a spectrum. For any α let K_{α} be the disjoint sum of K_{α}^{ξ}, $\xi\in\Xi$, whose topology is defined as follows:

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4) Corollaries 1.8 and 1.9 were proved by Pasynkov [15].

A subset D_α of K_α is open if and only if $D_\alpha \cap K_\alpha^\xi$ is open for every $\xi \in \Xi$. For any pair $\beta < \alpha$ let $\omega_{\alpha\beta}$: $K_\alpha \to K_\beta$ be a mapping defined as follows: The restriction of $\omega_{\alpha\beta}$ to K_α^ξ coincides with $\omega_{\alpha\beta}^\xi$ for any ξ. It is almost evident that $\{K_\alpha, \omega_{\alpha\beta}; \alpha \in A\}$ forms a spectrum which has the following properties:

i) For any $\alpha \in A$, K_α is an $(n-m)$-manifold.

ii) For any pair $\beta < \alpha$, $\omega_{\alpha\beta}$ is an open continuous mapping which is locally topological.

iii) The projective limit of $\{K_\alpha\}$ is homeomorphic to G.

Thus the corollary is essentially proved.

LEMMA 1.11 (Pasynkov's criterion [**15**, Lemma 3]). *Let a locally compact Hausdorff space K be the projective limit of the spectrum $\{K_\alpha, \omega_{\alpha\beta}\}$ which satisfies the following conditions:*

i) *For any α, the sum theorem for the large inductive dimension is valid.*

ii) *For any α,* $\operatorname{Ind} K_\alpha \leqq r$.

iii) *For any pair $\beta < \alpha$, $\omega_{\alpha\beta}$ is locally topological.*

iv) *K is covered by a countable number of compact sets F_i, $i = 1, 2, \cdots$, with* $\operatorname{Ind} F_i \leqq r$ *for any i.*

Then we have $\operatorname{Ind} K \leqq r$.

§ 2. Dimension of factor spaces.

THEOREM 2.1. *Let G be a locally compact group and H a closed subgroup of G. Then*

$$\dim G = \dim H + \dim G/H.$$ [5]

PROOF. First we consider the case when $\dim G < \infty$. Let $\dim G = n$ and $\dim H = m$. By Lemma 1.6 an arbitrary point k of G/H has a neighborhood U which is homeomorphic to the direct product of an $(n-m)$-dimensional Euclidean cube E and a compact Hausdorff space C with $\dim C = 0$. By Morita [**12**] we have $\dim E \times C \leqq n-m$. On the other hand $E \times C$ contains a closed subset which is homeomorphic to E. Hence we have $\dim E \times C \geqq n-m$. Since G/H is paracompact by Lemma 1.1, we have $\dim G/H \leqq n-m$ by Lemma 1.5. Since $E \times C$ is compact and hence U is closed in G/H, we have $\dim G/H \geqq \dim U = n-m$. Thus we have $\dim G/H = n-m$ and the equality $\dim G = \dim H + \dim G/H$ is valid.

Next we consider the case when $\dim G = \infty$. When $\dim H = \infty$, the equality $\dim G = \dim H + \dim G/H$ is trivially true. Hence we consider the case when $\dim H < \infty$. Let $\dim H = m$. In this case we shall prove that dim

5) The author's colleague Dr. Y. Katuta proved this equality for the case when G is a compact group.

$G/H=\infty$, which is not trivial at all.

Let r be an arbitrary positive integer. By Corollary 1.9 there exists an open σ-compact subgroup G_0 of G which is the projective limit of a spectrum $\{G_\alpha, \pi_{\alpha\beta}; \alpha \in A\}$ where G_α are Lie groups. Let $\pi_\alpha: G_0 \to G_\alpha$ be the projections. If

$$\sup\{\dim G_\alpha; \alpha \in A\} = r_1$$

is finite,

$$A_1 = \{\alpha; \dim G_\alpha = r_1\}$$

is equifinal in A. Hence it can easily be seen that the kernel N_α of some π_α is of coverning dimension 0. By Lemma 1.4 there exists a neighborhood U of the identity in G_0 which is the direct product of a Euclidean r_1-cube E and N_α. Since

$$\dim U = \dim E \times N_\alpha \leqq \dim E + \dim N_\alpha = r_1 + 0$$

by Morita [**12**], we have $\dim G = \dim G_0 \leqq r_1$ by Lemma 1.5, which is a contradiction. Hence there exists a compact normal subgroup N of G_0 such that

i) $\dim G_0/N > r+m$,

ii) G_0/N is a Lie group.

Since $H \cap G_0$ is σ-compact and $(H \cap G_0)N$ is closed, we know that $(H \cap G_0)N/N = Q$ is isomorphic to $H \cap G_0/H \cap N$ by Pontrjagin [**16**, G), § 20]. Since every small cell of $H \cap G_0/H \cap N$ can be lifted to $H \cap G_0$ by Montgomery-Zippin [**10**, p. 194], we have

$$\dim Q = \dim H \cap G_0/H \cap N \leqq \dim H \cap G_0 = \dim H = m.$$

Let π be the natural projection of G_0 onto $G_0/N = P$ and let

$$\dim P = p.$$

Let $g_1(t), \cdots, g_p(t)$, $|t| \leqq \delta_1$, be one-parameter subgroups of P which generate a canonical coordinate system of the second kind of P such that $g_{p-q+1}(t), \cdots, g_p(t)$ generate a canonical coordinate system of the second kind of Q, where

$$q = \dim Q.$$

By Montgomery-Zippin [**10**, Theorem 1, p. 192], we can find one-parameter subgroups $g_1^*(t), \cdots, g_p^*(t)$, $|t| \leqq \delta_2$ $(\leqq \delta_1)$, of G_0 such that

i) $\pi(g_i^*(t)) = g_i(t)$ for any t with $|t| \leqq \delta_2$ and $i = 1, \cdots, p$,

ii) $g_{p-q+1}^*(t), \cdots, g_p^*(t)$ are in $H \cap G_0$,

iii) $\pi(LL^{-1}L) \subset \{g_1(t_1) \cdots g_p(t_p); |t_i| \leqq \delta_1\}$, where

$$L = \{g_1^*(t_1) \cdots g_p^*(t_p); |t_i| \leqq \delta_2\},$$

$$\Lambda = \{g_1^*(t_1) \cdots g_{p-q}^*(t_{p-q}); |t_i| \leqq \delta_2\},$$

$$M = \{g_{p-q+1}^*(t_{p-q+1}) \cdots g_p^*(t_p); |t_i| \leqq \delta_2\}.$$

Let $\lambda_1\mu_1$ and $\lambda_2\mu_2$ be two elements of L, where $\lambda_1, \lambda_2 \in \Lambda$ and $\mu_1, \mu_2 \in M$. If $\lambda_1\mu_1$ and $\lambda_2\mu_2$ are elements of the same coset by $H \frown G_0$, then we have $\pi(\lambda_2)^{-1} \cdot \pi(\lambda_1) \in \pi(H \frown G_0) = Q$. Hence we have $\lambda_1 = \lambda_2$.[6] Conversely if $\lambda_1 = \lambda_2 \in \Lambda$ and μ_1, μ_2 be arbitrary elements of M, then $(\lambda_2\mu_2)^{-1} \cdot \lambda_1\mu_1 = \mu_2^{-1}\mu_1 \in M^{-1}M \subset H \frown G_0$. Thus we know that Λ is homeomorphic to $\pi(\Lambda)$ under π. Since $\pi(\Lambda)$ is homeomorphic to a Euclidean $(p-q)$-cube, we have

$$\dim \Lambda = p-q.$$

Similarly we can know that Λ is homeomorphic to $\rho_0(\Lambda)$, where ρ_0 is the natural projection of G_0 onto $G_0/H \frown G_0$. Hence

$$\dim \rho_0(\Lambda) = p-q.$$

Let ρ be the natural projection of G onto G/H; then $G_0/H \frown G_0$ is homeomorphic to $\rho(G_0)$ under the mapping $\rho\rho_0^{-1}$. We have $\dim G/H = \dim \rho(G_0)$ by Lemmas 1.1 and 1.5. Thus we have

$$\dim G/H \geqq \dim \rho(\Lambda) = \dim \rho_0(\Lambda) = p-q > r+m-m = r.$$

Since r was an arbitrary positive integer, we have $\dim G/H = \infty$ and the theorem is completely proved.

THEOREM 2.2. *Let G be a locally compact group with $\dim G = n$ and H a connected compact subgroup of G with $\dim H = m$. Then*

$$\dim G/H = \operatorname{Ind} G/H = \operatorname{ind} G/H = n-m.$$

PROOF. By Lemma 1.6 any point k of G/H has a neighborhood $U(k)$ which is homeomorphic to the direct product of a Euclidean $(n-m)$-cube E and a compact Hausdorff space C with $\dim C = 0$. Hence we have

$$\dim U(k) = n-m$$

as we see in the proof of Theorem 2.1. Since G/H is paracompact by Lemma 1.1, we have

$$\dim G/H = n-m$$

by Lemma 1.5.

Since $n-m = \operatorname{ind} E \leqq \operatorname{ind} U(k) \leqq \operatorname{ind} G/H$, we have

$$\operatorname{ind} G/H \geqq n-m.$$

In general it can easily be seen by an easy induction on $\operatorname{Ind} R$ that $\operatorname{Ind} R \times S \leqq \operatorname{Ind} R$ for compact Hausdorff spaces R and S with $\operatorname{Ind} S = 0$. Hence we have

$$\operatorname{Ind} U(k) \leqq n-m.$$

On the other hand $n-m = \operatorname{Ind} E \leqq \operatorname{Ind} U(k)$. Therefore we have

$$\operatorname{Ind} U(k) = n-m.$$

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6) Cf. Pontrjagin [**16**, A), § 44].

By Lemma 1.9 there exists an open σ-compact subgroup G_0 of G which is the projective limit of Lie groups. Since H is connected, we have $G_0 \supset H$. Since G_0/H is σ-compact, G_0/H is covered by a countable number of compact sets F_i with $\operatorname{Ind} F_i = n-m$, $i = 1, 2, \cdots$.

By Lemma 1.7 G_0/H is the projective limit of $\{K_\alpha, \omega_{\alpha\beta}\}$ where K_α are $(n-m)$-manifolds and $\omega_{\alpha\beta}$ are open continuous and locally topological. Since G_0/H is σ-compact and hence K_α are σ-compact, K_α are separable metric. Hence for any α

$$\dim K_\alpha = \operatorname{Ind} K_\alpha = n-m .$$

Thus all conditions in Lemma 1.11 are satisfied and we conclude that

$$\operatorname{Ind} G_0/H \leqq n-m .$$

Since $n-m = \operatorname{Ind} F_i \leqq \operatorname{Ind} G_0/H$, we have

$$\operatorname{Ind} G_0/H = n-m .$$

Since G/H is, by an analogous argument as in the proof of Corollary 1.10, the sum of mutually disjoint open sets each of which is homeomorphic to G_0/H, we conclude that

$$\operatorname{Ind} G/H = \operatorname{Ind} G_0/H = n-m$$

by an easy induction on $\operatorname{Ind} G_0/H$. Since $\operatorname{ind} G/H \leqq \operatorname{Ind} G/H$, we have also

$$\operatorname{ind} G/H = n-m .$$

Thus the proof is completed.

COROLLARY 2.3. *Let G be a locally compact group with* $\dim G = n$ *which is the projective limit of Lie groups and H a compact subgroup of G with* $\dim H = m$. *Then*

$$\dim G/H = \operatorname{Ind} G/H = \operatorname{ind} G/H = n-m .$$

PROOF. There exists an open subgroup G_0 which is σ-compact. Then $G_1 = G_0 H$ is also an open subgroup which is σ-compact. By an analogous argument to the proof of Theorem 2.2, we have

$$\dim G/H = \operatorname{ind} G/H = \operatorname{Ind} G_1/H = \operatorname{Ind} G/H = n-m ,$$

which proves the corollary.

§ 3. Decomposition theorem.

THEOREM 3.1. *Let G be a locally compact group with* $\dim G = n$. *Then there exist $n+1$ subspaces B_i, $i = 1, \cdots, n+1$, such that for any i B_i is a paracompact space with* $\dim B_i \leqq 0$.

PROOF. Let V be an open symmetric neighborhood of the identity of G such that $\overline{V^2}$ is compact. Let $G_0 = V^\infty$; then G_0 is an open σ-compact subgroup of G. By Lemma 1.3 there exists an open neighborhood of the identity of G which is the direct product of a local Lie group L and a compact group N such that

i) $LN \subset V$,

ii) L is homeomorphic to a Euclidean n-space,

iii) $\dim N = 0$.

Let W be a relatively open neighborhood of the identity in L such that

i) the closure of W in L, say F, is homeomorphic to a Euclidean n-cube,

ii) $FF^{-1}F \subset L$.

Let $x_1, x_2, \cdots$ be a sequence of points of G_0 and $t(1), t(2), \cdots$ be a sequence of positive integers which satisfies the following conditions:

i) $1 \leqq t(1) \leqq t(2) \leqq \cdots$.

ii) $x_i \in \overline{V^m}$ for $i = 1, \cdots, t(m)$, $m = 1, 2, \cdots$.

iii) $x_i \notin \overline{V^m}$ for $i > t(m)$, $m = 1, 2, \cdots$.

iv) $\cup \{x_i WN ; i = 1, \cdots, t(m)\} \supset \overline{V^m}$, $m = 1, 2, \cdots$.

Then $\{x_i WN ; i = 1, 2, \cdots\}$ is a star-finite open covering of G_0. $\{x_i FN ; i = 1, 2, \cdots\}$ is therefore a star-finite closed covering of G.[7] Since F is separable metric, there exist $n+1$ subsets F_i, $i = 1, \cdots, n+1$, of F with $\dim F_i = 0$ for any i (cf. Hurewicz-Wallman [**6**]).

We set

$$H_i = x_1 F_i N \cup \left(\bigcup_{j=2}^{\infty} \left(x_j F_i N - \bigcup_{k<j} x_k FN\right)\right), \qquad i = 1, \cdots, n+1 .$$

It is evident that $G_0 = \bigcup_{i=1}^{n+1} H_i$. Let us prove that every H_i is paracompact. Set for every i

$$H_{i1} = x_1 F_i N ,$$
$$H_{ij} = x_j F_i N - \bigcup_{k<j} x_k FN , \qquad j = 2, 3, \cdots ;$$

then $H_i = \bigcup_{j=1}^{\infty} H_{ij}$ and $\bigcup_{j=1}^{k} H_{ij}$ is relatively closed in H_i for $k = 1, 2, \cdots$. Set

$$J_j = \{k ; x_j FN \cap x_k FN \neq \phi\} , \qquad j = 1, 2, \cdots ;$$

then J_j is a finite set of indices from the star-finiteness of $\{x_i FN ; i = 1, 2, \cdots\}$. It is evident that

$$x_j^{-1} x_k FN \subset FF^{-1}FN \subset LN \quad \text{for any} \quad k \in J_j,\ j = 1, 2, \cdots .$$

Therefore if we set

$$E_{ij} = \cup \{H_{ik} ; k \in J_j\} , \qquad i = 1, \cdots, n+1,\ j = 1, 2, \cdots ,$$

then we have $x_j^{-1} E_{ij} \subset LN$ for $i = 1, \cdots, n+1$, $j = 1, 2, \cdots$.

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7) Cf. the argument in the proof of Remark 1.2.

Write an element of LN as the product of $\lambda \in L$ and $\nu \in N$ and define a mapping π of LN onto L in such a way that

$$\pi(\lambda\nu) = \lambda .$$

Now let us prove the equalities:

$$x_j^{-1}E_{ij} = \pi(x_j^{-1}E_{ij})N, \qquad i=1, \cdots, n+1,\ j=1, 2, \cdots .$$

Evidently $x_j^{-1}E_{ij} \subset \pi(x_j^{-1}E_{ij})N$. To show that $x_j^{-1}E_{ij} \supset \pi(x_j^{-1}E_{ij})N$, let λ be an arbitrary element of $\pi(x_j^{-1}E_{ij})$. Then there exists an index $k \in J_j$ such that $\lambda \in \pi(x_j^{-1}H_{ik})$, and hence $\lambda \in \pi(x_j^{-1}x_kF_iN - \cup\{x_j^{-1}x_{k'}FN; k'<k\})$. Therefore we have

$$\begin{aligned}
\lambda N &\subset \pi(x_j^{-1}x_kF_iN - \cup\{x_j^{-1}x_{k'}FN; k'<k\})N \\
&= \pi(\pi(x_j^{-1}x_kF_i)N - \pi(\cup\{x_j^{-1}x_{k'}F; k'<k\})N)N \\
&= \pi(\pi(x_j^{-1}x_kF_i) - \pi(\cup\{x_j^{-1}x_{k'}F; k'<k\})N)N \\
&= (\pi(x_j^{-1}x_kF_i) - \pi(\cup\{x_j^{-1}x_{k'}F; k'<k\}))N \\
&= \pi(x_j^{-1}x_kF_i)N - \pi(\cup\{x_j^{-1}x_{k'}F; k'<k\})N \\
&= x_j^{-1}x_kF_iN - \cup\{x_j^{-1}x_{k'}FN; k'<k\} \\
&= x_j^{-1}(x_kF_iN - \cup\{x_{k'}FN; k'<k\}) \\
&= x_j^{-1}H_{ik} \subset x_j^{-1}E_{ij}.
\end{aligned}$$

Thus we know that E_{ij} is homeomorphic to the product space of $\pi(x_j^{-1}E_{ij})$ and N. Since $\pi(x_j^{-1}E_{ij})$ is separable metric and N is compact (Hausdorff), the product space $\pi(x_j^{-1}E_{ij}) \times N$ is paracompact by Dieudonné [2]. Therefore we can conclude that E_{ij} is paracompact.

Since

$$\begin{aligned}
H_{ij} \subset x_jFN \cap H_i \subset H_i - \cup\{x_kFN; k \notin J_j\} \\
= (\cup\{H_{ik}; k \in J_j) \cup (\cup\{H_{ik}; k \notin J_j\}) - \cup\{x_kFN; k \notin J_j\} \\
\subset \cup\{H_{ik}; k \in J_j\} = E_{ij},
\end{aligned}$$

the relative closure of H_{ij} in the space H_i, say $\tilde{H}_{ij}$, is contained in $x_jFN \cap H_i$ and hence in E_{ij}. Since $\tilde{H}_{ij}$ is considered as the relative closure of H_{ij} in the space E_{ij}, $\tilde{H}_{ij}$ is paracompact by the paracompactness of E_{ij}. Since $\tilde{H}_{ij} \subset x_jFN$ for $j=1, 2, \cdots$,

$$\{\tilde{H}_{ij}; j=1, 2, \cdots\}$$

is as can easily be seen a locally finite relatively closed covering of H_i. Hence the paracompactness of H_i is established by Morita [**13**].

Next let us prove that $\dim H_i \leqq 0$ for $i=1, \cdots, n+1$. Since F is compact, there exists a sequence of open sets D_r, $r=1, 2, \cdots$, of L such that

$$F = \bigcap_{i=1}^{\infty} D_i .$$

We set

$$H_{ijr}=x_jF_iN-\bigcup\{x_kD_rN\,;k<j\}\,,\qquad i=1,\cdots,n+1\,,\ j=1,2,\cdots,\ r=1,2,\cdots;$$

then H_{ijr} is relatively closed in H_i and contained in H_{ij}. Since $\{\tilde{H}_{ij}\,;j=1,2,\cdots\}$ is locally finite in the space H_i,

$$\{H_{ijr}\,;j=1,2,\cdots\}$$

is also locally finite in the space H_i. Hence

$$K_{ir}=\bigcup_{j=1}^{\infty}H_{ijr}$$

is relatively closed in H_i. Since H_{ijr} is a relatively closed subset of a paracompact space x_jF_iN, we have

$$\dim H_{ijr}\leqq\dim x_jF_iN=\dim F_iN=\dim F_i\times N\leqq\dim F_i+\dim N=0\,.$$

Hence by the sum theorem we have

$$\dim K_{ir}\leqq 0\,.$$

Since it is almost evident that $H_i=\bigcup_{r=1}^{\infty}K_{ir}$, we have

$$\dim H_i\leqq 0\,,\qquad i=1,\cdots,n+1\,,$$

by the sum theorem again.

Let $\{g_\xi G_0\,;\xi\in\Xi\}$ be a collection of all cosets by G_0 such that $g_\xi G_0\cap g_\eta G_0=\phi$ whenever ξ and η are different indices of Ξ. Setting

$$B_i=\bigcup\{g_\xi H_i\,;\xi\in\Xi\}\,,\qquad i=1,\cdots,n+1\,,$$

B_i is evidently a paracompact space with $\dim B_i\leqq 0$ for every i. Thus the theorem is completely proved.

REMARK 3.2. It is to be noted that B_i, $i=1,\cdots,n+1$, constructed above satisfy the following condition: If $I=\{i_1,\cdots,i_j\}$ is any subset of $\{1,\cdots,n+1\}$, then $\bigcup\{B_i\,;i\in I\}$ is a paracompact space with $\dim\bigcup\{B_i\,;i\in I\}\leqq j-1$.

§ 4. Invariance theorem of a domain.

LEMMA 4.1 (Alexandroff-Hopf [1, Theorem IV′, p. 121]). *A compact metric space R with $\dim R=0$ which has no isolated point is homeomorphic to a Cantor discontinuum.*

THEOREM 4.2. *The invariance theorem of a domain does not hold in any locally compact, metric group G with $\dim G<\infty$ which is not locally connected.*

PROOF. Let $\dim G=n$; then by Lemma 1.3 there exists an open neighborhood of the identity of G which is the direct product of a local Lie group L which is homeomorphic to a Euclidean n-space and a compact metric group

N with $\dim N=0$. Since G is not locally connected, N must be infinite. Hence we can consider N as the projective limit of a spectrum $\{N_i, \pi_{ij}: N_i \to N_j; i=1, 2, \cdots\}$ such that

i) N_i is a finite group for every i,

ii) π_{ij} are onto homomorphisms,

iii) for any i the order of the kernel of $\pi_{i+1,i}$ is not less than 3.

Set $M_1=N_1$. By an easy application of the induction we can construct a sequence of finite subsets M_i of N_i, $i=1, 2, \cdots$, such that

$$|\pi_{i+1,i}^{-1}(\mu) \cap M_{i+1}|=2 \quad \text{for any} \quad \mu \in M_i, \ i=1, 2, \cdots.$$

Let M be the projective limit of $\{M_i, \pi_{ij}\}$. Since both N and M are compact metric spaces with $\dim N=\dim M=0$ which have no isolated point, there exists by Lemma 4.1 a homeomorphism φ of N onto M.

Here we notice that M contains no non-empty open set of N. Suppose that a non-empty open set D of N is contained in M; then there exist a point x of D and a positive integer i such that $\pi_i^{-1}\pi_i(x) \subset D$, where π_j is the projection of N onto N_j, $j=1, 2, \cdots$. We have

$$|\pi_{i+1}\pi_i^{-1}\pi_i(x)| \geqq 3.$$

Since $\pi_i(x) \in M_i$ and $\pi_{i+1}\pi_i^{-1}\pi_i(x)=\pi_{i+1,i}^{-1}\pi_i(x)$, we have

$$|\pi_{i+1}\pi_i^{-1}\pi_i(x) \cap M_{i+1}|=2.$$

Since $\pi_{i+1}\pi_i^{-1}\pi_i(x) \subset \pi_{i+1}(M)=M_{i+1}$, we have

$$|\pi_{i+1}\pi_i^{-1}\pi_i(x) \cap M_{i+1}|=|\pi_{i+1}\pi_i^{-1}\pi_i(x)| \geqq 3,$$

which is a contradiction. Thus M contains no non-empty open set of N.

Define a mapping $\psi: LN \to LM$ in such a way that

$$\psi(yx)=y \cdot \varphi(x), \qquad y \in L, \ x \in N.$$

Then ψ is a homeomorphism of LN onto LM. To prove that LM contains no non-empty open set of G, assume the contrary. Then there exist a non-empty open set L_1 of L and a non-empty open set N_1 of N such that $L_1N_1 \subset LM$. Define a mapping $f: LN \to N$ in such a way that

$$f(yx)=x, \qquad y \in L, \ x \in N.$$

Then f is an open continuous mapping. Hence $f(L_1N_1)=N_1$ is open in N and is contained in $f(LM)=M$, which is a contradiction. Thus we know that the invariance theorem of a domain does not hold in G and the proof is completed.

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