Dimensional Properties of Fractional Brownian Motion

Dongsheng Wu and Yimin Xiao *
Department of Statistics and Probability
Michigan State University

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Abstract

Let $B^\alpha = \{B^\alpha(t), t \in \mathbb{R}^N\}$ be an (N,d)-fractional Brownian motion with Hurst index $\alpha \in (0,1)$. By applying the strong local nondeterminism of B^α, we prove certain forms of uniform Hausdorff dimension results for the images of B^α when $N > \alpha d$. Our results extend those of Kaufman [7] for one-dimensional Brownian motion.

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1 Introduction

Let $B^\alpha = \{B^\alpha(t),\ t \in \mathbb{R}^N\}$ be an (N,d)-fractional Brownian motion (fBm) with Hurst index $\alpha \in (0,1)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is, B^α is an N-parameter Gaussian random field with values in $\mathbb{R}^d$; its mean function is zero and its covariance function is given by the following

$$\mathbb{E}\Big(B^\alpha_j(s)B^\alpha_k(t)\Big) = \frac{1}{2}\delta_{jk}\Big(|s|^{2\alpha} + |t|^{2\alpha} - |s-t|^{2\alpha}\Big), \quad s,\, t \in \mathbb{R}^N, \tag{1.1}$$

where $j,\, k = 1, \ldots, d$ and $\delta_{jk} = 1$ if $j = k$ and 0 if $j \neq k$, and where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. Note that the components of B^α are independent and identically distributed.

Fractional Brownian motion has been under extensive investigations in the last decade due to its applications in various areas such as telecommunication networks, hydrology, finance, and so on. Many authors have studied the sample path properties of fractional Brownian motion. See Adler [1], Kahane [5], Monrad and Pitt [10], Pitt [11], Rosen [12], Talagrand [13], Xiao [17] [18], just to mention a few.

It is well known [cf. Kahane ([5], Chapter 18)] that for every Borel set $E \subseteq \mathbb{R}^N$,

$$\dim_{\rm H} X(E) = \min\Big\{d,\ \frac{1}{H}\dim_{\rm H} E\Big\} \quad \text{a.s.}, \tag{1.2}$$

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where $\dim_{\rm H}$ denotes the Hausdorff dimension. We refer to Falconer [3] or Kahane [5] for definition and properties of Hausdorff dimension.

Note that in (1.2) the exceptional null probability event [on which (1.2) does not hold] depends on E. It is natural to ask whether it is possible to find a single null probability event Ω_0 such that for every $\omega \notin \Omega_0$, (1.2) holds for all Borel sets $E \subset \mathbb{R}^N$. Such a result, if exists, is called a uniform Hausdorff dimension result and is applicable even if E is a random set.

The first uniform dimension result was established by Kaufman [6] for the planar Brownian motion. Since then, the problem of establishing uniform Hausdorff dimension results has been studied by several authors for various classes of stochastic processes. See Xiao [19] for a survey on the results for Markov processes and their applications. Monrad and Pitt [10] have proved the following uniform Hausdorff dimension result for the images of B^α: If $N \le \alpha d$, then almost surely

$$\dim_{\rm H} B^\alpha(E) = \frac{1}{\alpha}\dim_{\rm H} E \quad \text{for every Borel set } \ E \subseteq \mathbb{R}^N. \tag{1.3}$$

Of course, the above uniform dimension result can not be true if $N > \alpha d$. This can be easily seen by taking $E = (B^\alpha)^{-1}(0)$ [the zero set of B^α]. Moreover, Monrad and Pitt [10] have shown the following result: If $N > \alpha d$, then almost surely

$$\dim_{\rm H} \big(B^\alpha\big)^{-1}(F) = N - \alpha d + \alpha \dim_{\rm H} F \tag{1.4}$$

for every closed set $F \subseteq \mathbb{R}^d$, where $\big(B^\alpha\big)^{-1}(F) = \{t \in \mathbb{R}^N : B^\alpha(t) \in F\}$ is the inverse image of F.

In this paper, we will prove the following weaker forms of uniform dimension results for B^α when $N > \alpha d$. They are extensions of the results of Kaufman [7] for one-dimensional Brownian motion.

Theorem 1.1 *Suppose $N > \alpha d$. Then with probability 1, for every Borel set $E \subseteq [0,1]^N$,*

$$\dim_{\rm H} B^\alpha(E+t) = \min\Big\{d, \ \frac{1}{\alpha}\dim_{\rm H} E\Big\} \quad \textit{for almost all } \ t \in [0,1]^N. \tag{1.5}$$

Theorem 1.2 *Suppose $N > \alpha d$. Then with probability 1, for every Borel set $E \subseteq [0,1]^N$ with $\dim_{\rm H} E > \alpha d$, we have $\lambda_d\big(B^\alpha(E+t)\big) > 0$ for almost all $t \in [0,1]^N$.*

When $N = d = 1$ and $\alpha = 1/2$, B^α is the ordinary Brownian motion in $\mathbb{R}$. As we mentioned, the above theorems are due to Kaufman [7]. His proofs rely heavily on the independent increment property of Brownian motion as well as the fact that the time-space is one-dimensional, hence can not be carried over to the (N,d)-fractional Brownian motion directly.

Our proofs of Theorems 1.1 and 1.2 are based on Kaufman's arguments and the following property of *strong local nondeterminism* of fBm which was discovered by Pitt [11]: Let B_0^α be an $(N,1)$-fractional Brownian motion with index $\alpha \in (0,1)$. Then, there exists a constant $0 < c_{1,1} < \infty$ such that for all integers $n \ge 1$ and all u, $t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\mathrm{Var}\Big(B_0^\alpha(u)\big|B_0^\alpha(t^1), \ldots, B_0^\alpha(t^n)\Big) \ge c_{1,1} \min_{0 \le k \le n} |u - t^k|^{2\alpha}, \tag{1.6}$$

where $\mathrm{Var}\big(B_0^\alpha(u)\big|B_0^\alpha(t^1), \ldots, B_0^\alpha(t^n)\big)$ denotes the conditional variance of $B_0^\alpha(u)$ given $B_0^\alpha(t^1)$, $\ldots, B_0^\alpha(t^n)$. In the above and in the sequel, $t^0 \equiv 0$.

The strong local nondeterminism has played important rôles in studying various sample path properties of fractional Brownian motion. See Xiao [20] [21] and the references therein for further information. In this paper, we will make use of the following equivalent form of the strong local nondeterminism of B^α.

Lemma 1.3 *The strong local nondeterminism of fBm B^α is equivalent to the following: There exists a constant $0 < c_{1,2} < \infty$ such that for all integers $n \geq 1$ and all u, v, $t^1, \ldots, t^n \in \mathbb{R}^N$,*

$$\begin{aligned} &\mathrm{Var}\Big(B_0^\alpha(u) - B_0^\alpha(v)\big|B_0^\alpha(t^1), \ldots, B_0^\alpha(t^n)\Big) \\ &\geq c_{1,2} \min\Big(\min_{0\leq k\leq n} |u-t^k|^{2\alpha} + \min_{0\leq k\leq n} |v-t^k|^{2\alpha},\ |u-v|^{2\alpha}\Big). \end{aligned} \tag{1.7}$$

Proof Letting $v = 0$ in (1.7), we get (1.6). Hence we only need to prove the implication of (1.6) $\Rightarrow$ (1.7).

We work in the Hilbert space setting and write the conditional variance in (1.7) as the square of the $L^2(\mathbb{P})$-distance between $B_0^\alpha(u) - B_0^\alpha(v)$ and the subspace generated by the random variables $\{B_0^\alpha(t^1), \ldots, B_0^\alpha(t^n)\}$. Hence, by (1.6), there exists a constant $c_{1,3} \in (0, \infty)$ such that

$$\begin{aligned} &\mathrm{Var}\Big(B_0^\alpha(u) - B_0^\alpha(v)\big|B_0^\alpha(t^1), \ldots, B_0^\alpha(t^n)\Big) \\ &= \inf_{a_1,\ldots,a_n\in\mathbb{R}} \mathbb{E}\Big[B_0^\alpha(u) - B_0^\alpha(v) - \sum_{k=1}^n a_k B_0^\alpha(t^k)\Big]^2 \\ &\geq \inf_{a_1,\ldots,a_n,a_v\in\mathbb{R}} \mathbb{E}\Big[B_0^\alpha(u) - a_v B_0^\alpha(v) - \sum_{k=1}^n a_k B_0^\alpha(t^k)\Big]^2 \\ &= \mathrm{Var}\Big(B_0^\alpha(u)\big|B_0^\alpha(v), B_0^\alpha(t^1), \ldots, B_0^\alpha(t^n)\Big) \\ &\geq c_{1,3} \min\Big\{\min_{0\leq k\leq n} |u-t^k|^{2\alpha},\ |u-v|^{2\alpha}\Big\}. \end{aligned} \tag{1.8}$$

By the same token as in the proof of (1.8), there exists a constant $c_{1,4} \in (0, \infty)$ such that

$$\begin{aligned} &\mathrm{Var}\Big(B_0^\alpha(u) - B_0^\alpha(v)\big|B_0^\alpha(t^1), \ldots, B_0^\alpha(t^n)\Big) \\ &\qquad\qquad \geq c_{1,4} \min\Big\{\min_{0\leq k\leq n} |v-t^k|^{2\alpha},\ |u-v|^{2\alpha}\Big\}. \end{aligned} \tag{1.9}$$

Adding up (1.8) and (1.9) yields (1.7). This proves Lemma 1.3. □

The rest of the paper is organized as follows. We give the proofs of Theorem 1.1 and 1.2 in Section 2 and Section 3, respectively. Since the properties of strong local nondeterminism have been established for large classes of Gaussian processes and fields by Xiao [20], our theorems can be further extended. In Section 4, we state some of these extensions and open questions.

Throughout this paper, we use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^m$ respectively, no matter the value of the integer m. We denote the m-dimensional Lebesgue measure by λ_m. Unspecified positive and finite constants will be denoted by c which may have different values from line to line. Specific constants in Section j will be denoted by $c_{j,1}, c_{j,2}$ and so on.

2 Proof of Theorem 1.1

As in Kaufman [7], we define the function H on $\mathbb{R}^d$ such that $H(s) = 1$ if $|s| < 1$ and $H(s) = 0$ otherwise. Define

$$I(x, y, R) = \int_{[0,1]^N} H\big(RB^\alpha(x+t) - RB^\alpha(y+t)\big)\, dt \tag{2.1}$$

provided $R > 0, x, y \in [0,1]^N$. Since for every $s \in \mathbb{R}^d$, $H(Rs)$ is non-increasing in R, we have that for fixed $x, y \in [0,1]^N$, $I(x, y, R)$ is non-increasing in R.

The following lemma is the key for the proof of Theorem 1.1. It will be clear that the strong local nondeterminism of fBm plays an important rôle here.

Lemma 2.1 *For all $x, y \in [0,1]^N$, $R > 1$ and integers $p = 1, 2, \ldots$,*

$$\mathbb{E}\big[(I(x, y, R))^p\big] \le c_{2,1}^p\, (p!)\, R^{-dp}\, |y - x|^{-\alpha dp}. \tag{2.2}$$

Proof Since $B_1^\alpha, \ldots, B_d^\alpha$ are independent copies of B_0^α, the pth moment of $I(x, y, R)$ can be bounded by the following multiple integral:

$$\begin{aligned}
&\mathbb{E}\big[(I(x, y, R))^p\big] \\
&= \int \cdots \int_{[0,1]^{Np}} \mathbb{P}\Big\{ \big|B^\alpha(x + t^j) - B^\alpha(y + t^j)\big| < R^{-1},\ 1 \le j \le p\Big\}\, dt^1 \cdots dt^p \\
&\le \int \cdots \int_{[0,1]^{Np}} \mathbb{P}\Big\{ \big|B_\ell^\alpha(x + t^j) - B_\ell^\alpha(y + t^j)\big| < R^{-1},\ 1 \le j \le p,\ 1 \le \ell \le d\Big\}\, dt^1 \cdots dt^p \\
&= \int \cdots \int_{[0,1]^{Np}} \bigg[\mathbb{P}\Big\{ \big|B_0^\alpha(x + t^j) - B_0^\alpha(y + t^j)\big| < R^{-1},\ 1 \le j \le p\Big\}\bigg]^d dt^1 \cdots dt^p.
\end{aligned} \tag{2.3}$$

Note that

$$\lambda_{Np}\big\{(t^1, \ldots, t^p) \in [0,1]^{Np} :\ t^1, \ldots, t^p \text{ are distinct } \big\} = 1,$$

without loss of generality, we will assume that all the points $t^1, \ldots, t^p$ in (2.3) are distinct. We will estimate the above integral by integrating in the order $dt^p, dt^{p-1}, \ldots, dt^1$.

First let $t^1, \ldots, t^{p-1} \in [0,1]^N$ be fixed and distinct points. We consider the following conditional probability:

$$\begin{aligned}
\mathcal{P}(t^p) \equiv \mathbb{P}\Big\{ &|B_0^\alpha(x + t^p) - B_0^\alpha(y + t^p)| < R^{-1} \Big| \\
&\big|B_0^\alpha(x + t^j) - B_0^\alpha(y + t^j)\big| < R^{-1}, 1 \le j \le p - 1\Big\}.
\end{aligned} \tag{2.4}$$

Since the condition distribution in Gaussian processes are still Gaussian, the above probability can be estimated if the conditional variance of $B_0^\alpha(x + t^p) - B_0^\alpha(y + t^p)$, given $B_0^\alpha(x + t^j) - B_0^\alpha(y + t^j)$ $(1 \le j \le p - 1)$, is bounded from below.

In order to get the desired lower bound for the conditional variance, recall that if $\mathcal{F}$ and $\mathcal{F}'$ are linear subspaces of $L^2(\mathbb{P})$, then for every Gaussian random variable $G \in L^2(\mathbb{P})$,

$$\mathcal{F} \subset \mathcal{F}' \implies \mathrm{Var}(G \,|\, \mathcal{F}) \ge \mathrm{Var}(G \,|\, \mathcal{F}'). \tag{2.5}$$

Moreover, both conditional variances are non-random.

It follows from (2.5) and (1.7) that

$$
\begin{aligned}
&\mathrm{Var}\Big(B_0^\alpha(x+t^p)-B_0^\alpha(y+t^p)\big|B_0^\alpha(x+t^j)-B_0^\alpha(y+t^j), 1\le j\le p-1\Big)\\
&\ge \mathrm{Var}\Big(B_0^\alpha(x+t^p)-B_0^\alpha(y+t^p)\big|B_0^\alpha(x+t^j), B_0^\alpha(y+t^j), 1\le j\le p-1\Big)\\
&\ge c_{2,2}\,\min\Big\{\min_{0\le j\le p-1}\big(|t^p-t^j|^{2\alpha},\ |x+t^p-y-t^j|^{2\alpha}\big)\\
&\qquad\qquad +\min_{0\le j\le p-1}\big(|t^p-t^j|^{2\alpha},\ |y+t^p-x-t^j|^{2\alpha}\big)\,,\ |x-y|^{2\alpha}\Big\}.\\
&\ge c_{2,3}\min\Big\{\min_{0\le j\le p-1}|t^p-t^j-z|^{2\alpha},\ |x-y|^{2\alpha}\Big\},
\end{aligned}
\tag{2.6}
$$

where $z=0,\ x-y$ or $y-x$.

Combining (2.4), (2.6) and Anderson's inequality (see [2]), we derive

$$
\mathcal{P}(t^p)\le c_{2,4}\,R^{-1}\left[\min\Big\{\min_{0\le j\le p-1}|t^p-t^j-z|,\ |x-y|\Big\}\right]^{-\alpha}. \tag{2.7}
$$

Let $\Gamma_p=\big\{t^p\in[0,1]^N:\min_{0\le j\le p-1}|t^p-t^j-z|\le|x-y|\ \text{ for } z=0,\ x-y \text{ or } y-x\big\}$. Note that Γ_p is contained in the union of $3p$ balls $B(t^j+z,\ |x-y|)$ of radius $|x-y|$. Hence we have

$$
\begin{aligned}
&\int_{[0,1]^N}\Big(\mathcal{P}(t^p)\Big)^d dt^p=\left[\int_{\Gamma_p}+\int_{[0,1]^N\setminus\Gamma_p}\right]\Big(\mathcal{P}(t^p)\Big)^d dt^p\\
&\le c_{2,5}\,R^{-d}\left[\sum_z\sum_{j=0}^{p-1}\int_{B(t^j+z,\,|x-y|)}|t^p-t^j-z|^{-\alpha d}\,dt^p+|x-y|^{-\alpha d}\right]\\
&\le c_{2,6}\,p\,R^{-d}|x-y|^{-\alpha d},
\end{aligned}
\tag{2.8}
$$

where $c_{2,6}$ is a finite positive constant. In deriving the last inequality, we have used the fact that $N>\alpha d$ to estimate the integral.

Combining (2.3) and (2.8), we have

$$
\begin{aligned}
&\mathbb{E}\big[(I(x,y,R))^p\big]\le c_{2,6}\,p\,R^{-d}|x-y|^{-\alpha d}\\
&\times\int\cdots\int_{[0,1]^{N(p-1)}}\Big[\mathbb{P}\Big\{\big|B_0^\alpha(x+t^j)-B_0^\alpha(y+t^j)\big|<R^{-1}, 1\le j\le p-1\Big\}\Big)^d dt^1\cdots dt^{p-1}.
\end{aligned}
\tag{2.9}
$$

Continue integrating $dt^{p-1},\dots,dt^1$ in the same way as we did for dt^p, we finally get (2.2) as desired. □

Remark 2.2 For later use in the proof of Theorem 1.2, we remark that the method of the proof of Lemma 2.1 can also be used to prove that

$$
\begin{aligned}
&\int\cdots\int_{[0,1]^{2Np}}\Big[\mathbb{P}\Big\{\big|B_0^\alpha(x+t^j)-B_0^\alpha(y+t^j)\big|<2^{-(1-\varepsilon)n},\ 1\le j\le 2p\Big\}\Big]^{\frac{N}{\alpha}}dt^1\cdots dt^{2p}\\
&\le c_{2,7}^{2p}\,(2p)!\,\Big(2^{-\frac{2(1-\varepsilon)nNp}{\alpha}}n^{2p}+2^{-\frac{2(1-\varepsilon)nNp}{\alpha}}|x-y|^{-2Np}\Big),
\end{aligned}
\tag{2.10}
$$

where $\varepsilon > 0$ is a small positive number whose value will be specified later.

In fact, by taking $R = 2^{(1-\varepsilon)n}$ in (2.7), we obtain

$$\mathcal{P}(t^{2p}) \le c_{2,8} 2^{-(1-\varepsilon)n} \left[\min \left\{ \min_{0\le j\le 2p-1} |t^{2p} - t^j - z|,\ |x-y| \right\} \right]^{-\alpha}. \tag{2.11}$$

Based on (2.11) and the argument in the proof of Lemma 2.1, we follow through (2.8) to get (2.10).

Now, we are ready to prove our Theorem 1.1.

Proof of Theorem 1.1 Since B^α is uniformly Hölder continuous on $[0,1]^N$ of any order smaller than α, we have almost surely

$$\dim_{\rm H} B^\alpha(E+t) \le \min\left\{d,\ \frac{1}{\alpha}\dim_{\rm H} E\right\} \quad \text{for all Borel sets } E \text{ and all } t \in [0,1]^N.$$

Thus, we only need to prove the lower bound in (1.5).

We first show that there exist a constant $c_{2,9}$ and an a.s.-finite random variable $n_0 = n_0(\omega)$ such that almost surely for all $n > n_0(\omega)$,

$$I(x,y,2^n) \le c_{2,9}\, n\, 2^{-nd}\, |x-y|^{-\alpha d}, \qquad \forall x,\, y \in [0,1]^N. \tag{2.12}$$

Let θ be an integer such that $\theta > 2^{1/\alpha}$ and consider the set $Q_n \subseteq [0,1]^N$ defined by

$$Q_n \equiv \left\{ \theta^{-n}\mathbf{k} :\ k_j = 1, 2, \ldots, \theta^n,\ \forall j = 1, \ldots, N \right\}. \tag{2.13}$$

The number of pairs $x,\ y \in Q_n$ is at most θ^{2Nn}. Hence for $u > 1$, Lemma 2.1 implies that

$$\begin{aligned} &\mathbb{P}\left\{ I(x,y,2^n) > u\, n\, 2^{-nd}|x-y|^{-\alpha d} \text{ for some } x, y \in Q_n \cap [0,\,1]^N \right\} \\ &\quad \le \theta^{2Nn}\, c_{2,1}^p\, (p!)\, (u\, n)^{-p}. \end{aligned} \tag{2.14}$$

By choosing $p = n$, $u = c_{2,1}\, \theta^{2N}$, and by Stirling's formula, we know that the probabilities in (2.14) are summable. Therefore, the Borel-Cantelli lemma implies that a.s. for all n large enough,

$$I(x,y,2^n) \le c_{2,10}\, n\, 2^{-nd}|x-y|^{-\alpha d}, \qquad \forall\, x, y \in Q_n \cap [0,\,1]^N. \tag{2.15}$$

Now we are ready to prove (2.12). Note that (2.12) is trivial unless $n\, 2^{-nd} < |x-y|^{\alpha d}$, and we only need to consider this case. For $x,\, y \in [0,1]^N$, we can find $\bar{x}$ and $\bar{y} \in Q_{n-1} \cap [0,\,1]^N$ so that $|x - \bar{x}| \le \sqrt{N}\, \theta^{-n}$ and $|y - \bar{y}| \le \sqrt{N}\, \theta^{-n}$, respectively. By the modulus of continuity of B^α on $[0,1]^N$ (see, e.g., Kahane [5]), we see that $I(x,y,2^n) \le I(\bar{x},\bar{y},2^{n-1})$ for all n large enough. On the other hand, by (2.15) and the assumption $n\, 2^{-nd} < |x-y|^{\alpha d}$, we have

$$I(\bar{x},\bar{y},2^{n-1}) \le c_{2,10}\, (n-1)\, 2^{(1-n)d}\, |\bar{x}-\bar{y}|^{-\alpha d} \le c_{2,9}\, n 2^{-nd}\, |x-y|^{-\alpha d}, \tag{2.16}$$

which proves (2.12).

To prove the the lower bound in (1.5), we fix an $\omega \in \Omega$ such that (2.12) holds. For any Borel set $E \subseteq [0,1]^N$ and all $\gamma \in (0, \dim_{\rm H} E)$, we choose $\eta \in (0, d \wedge \frac{\gamma}{\alpha})$. Then E carries a probability measure μ such that

$$\mu(S) \le c_{2,11}\, ({\rm diam} S)^\gamma \quad \text{for all measurable sets } S \subset [0,1]^N. \tag{2.17}$$

Let ν_t be the image measure of μ under the mapping $x \mapsto B^\alpha(x+t)$ ($x, t \in [0,1]^N$). In order to prove $\dim_{\rm H} B^\alpha(E+t) \geq \eta$, by Frostman's Theorem, we only need to show

$$\int_{\mathbb{R}^{2d}} \frac{\nu_t(du)\nu_t(dv)}{|u-v|^\eta} < \infty. \tag{2.18}$$

Now we follow Kaufman [7], and note that the left-hand side in (2.18) is equal to

$$\begin{aligned}
&\int\int \frac{\mu(dx)\mu(dy)}{\left|B^\alpha(x+t) - B^\alpha(y+t)\right|^\eta} \\
&= \eta \int_0^\infty \int\int H\big(RB^\alpha(x+t) - RB^\alpha(y+t)\big)\, R^{\eta-1}\, \mu(dx)\mu(dy)dR \\
&\leq 1 + \int_1^\infty \int\int H\big(RB^\alpha(x+t) - RB^\alpha(y+t)\big)\, R^{\eta-1}\, \mu(dx)\mu(dy)dR.
\end{aligned} \tag{2.19}$$

To prove that the last integral is finite for almost all $t \in [0,1]^N$, we integrate the above over $[0,1]^N$ and show

$$\int\int\int_1^\infty I(x,y,R)\, R^{\eta-1}\, dR\, \mu(dx)\mu(dy) < \infty. \tag{2.20}$$

We split the above integral over $D = \big\{(x,y) : |x-y| \leq R^{-1/\alpha}\big\}$ and its complement, and denote them by J_1 and J_2, respectively. Since $(\mu \times \mu)(D) \leq c_{2,11}\, R^{-\frac{\gamma}{\alpha}}$ and $I(x,y,R) \leq 1$, we have

$$J_1 \leq c_{2,11} \int_1^\infty R^{-\frac{\gamma}{\alpha}+\eta-1}\, dR < \infty. \tag{2.21}$$

On the other hand, for all $(x,y) \in D^c$, we have $|x-y|^{-\alpha} < R$. By (2.12) and the fact that $I(x,y,R)$ is monotone in R, we have $I(x,y,R) < c(\omega)\, R^{-d} \log R\, |x-y|^{-\alpha d}$. It follows that

$$\begin{aligned}
J_2 &\leq c_{2,12}(\omega) \int\int \frac{1}{|x-y|^{\alpha d}}\, \mu(dx)\mu(dy) \int_{|x-y|^{-\alpha}}^\infty R^{\eta-d-1}\, \log R\, dR \\
&< c_{2,13}(\omega) \int\int \frac{1}{|x-y|^{\alpha\eta}}\, \log(|x-y|^{-\alpha})\mu(dx)\mu(dy) < \infty,
\end{aligned} \tag{2.22}$$

where the last inequality follows from (2.17) and the assumption that $\alpha\eta < \gamma$. Combining (2.21) and (2.22) gives (2.20). This completes the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

Since $\dim_{\rm H} E > \alpha d$, there exists a Borel probability measure μ on E such that

$$\int_E \int_E \frac{\mu(ds)\mu(dt)}{|s-t|^{\alpha d}} < \infty. \tag{3.1}$$

Let ν_t be the image measure of μ as in the proof of Theorem 1.1. It is sufficient to show that

$$\int_{[0,1]^N} \int_{\mathbb{R}^d} |\widehat{\nu}_t(u)|^2\, du\, dt < \infty, \qquad \text{a.s.} \tag{3.2}$$

where

$$\widehat{\nu}_t(u) = \int_{\mathbb{R}^N} e^{i\langle u,\, B^\alpha(x+t)\rangle} \mu(dx)$$

is the Fourier transform of ν_t and exception null probability event does not depend on μ.

We choose and fix a smooth function $\psi \geq 0$ on $\mathbb{R}^d$ such that $\psi(u) = 1$ when $1 \leq |u| \leq 2$ and $\psi(u) = 0$ outside $1/2 < |u| < 5/2$, and satisfying $\psi(b_1u_1, \ldots, b_du_d) = \psi(u)$, where $b_\ell = \pm 1$, $\forall \ell = 1, \ldots, d$ and $u = (u_1, \ldots, u_d)$. Then

$$\begin{aligned}\int_{|u|>1} |\widehat{\nu}_t(u)|^2\, du &\leq \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \psi(2^{-n}u)\, |\widehat{\nu}_t(u)|^2\, du \\ &= \sum_{n=0}^{\infty} 2^n \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \widehat{\psi}\big(2^n B^\alpha(x+t) - 2^n B^\alpha(y+t)\big)\, \mu(dx)\mu(dy).\end{aligned} \tag{3.3}$$

In the above, $\widehat{\psi}$ is the Fourier transform of ψ and the last inequality follows from Fubini's theorem.

Consequently, it suffices to show

$$\sum_{n=0}^{\infty} 2^n \int_{[0,1]^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \widehat{\psi}\big(2^n B^\alpha(x+t) - 2^n B^\alpha(y+t)\big)\, \mu(dx)\mu(dy)\, dt < \infty. \tag{3.4}$$

To this end, we define

$$J(x, y, n) = \int_{[0,1]^N} \widehat{\psi}\big(2^n B^\alpha(x+t) - 2^n B^\alpha(y+t)\big)\, dt.$$

It is clear that $J(x, y, n)$ is bounded in (x, y, n). The following lemma provides a better estimate when $|x-y|$ is relatively large.

Lemma 3.1 *There exist positive and finite constants $c_{3,1}$ and β such that, with probability 1, for all $n \geq n(\omega)$ and $|x-y| \geq c_{3,1}\, 2^{-n/\alpha} n^{1/\alpha}$,*

$$\big|J(x, y, n)\big| \leq (2+\beta)^{-n}\, \big|x-y\big|^{-\alpha d}. \tag{3.5}$$

Proof It suffices to prove that there are positive constants $c_{3,2}$, $c_{3,3}$ and β such that for all integers $n \geq 1$ and $|x-y| \geq c_{3,1}\, 2^{-n/\alpha} n^{1/\alpha}$,

$$\mathbb{E}\big[J(x, y, n)^{2n}\big] \leq c_{3,2}^{2n}\, n^{c_{3,3} n}\, (2+\beta)^{-2n^2}\, |x-y|^{-2n\alpha d}. \tag{3.6}$$

Then (3.5) will follows from a Borel-Cantelli argument as in the proof of Theorem 1.1.

Note that the moment in (3.6) can be written as

$$\begin{aligned}&\mathbb{E}\int_{[0,1]^{2Nn}} \prod_{j=1}^{2n} \widehat{\psi}\big(2^n B^\alpha(x+t^j) - 2^n B^\alpha(y+t^j)\big)\, dt^1 \cdots dt^{2n} \\ &= \mathbb{E}\int_{S_n} \prod_{j=1}^{2n} \widehat{\psi}\big(2^n B^\alpha(x+t^j) - 2^n B^\alpha(y+t^j)\big)\, d\mathbf{t} \\ &\quad + \mathbb{E}\int_{[0,1]^{2Nn}\setminus S_n} \prod_{j=1}^{2n} \widehat{\psi}\big(2^n B^\alpha(x+t^j) - 2^n B^\alpha(y+t^j)\big)\, d\mathbf{t},\end{aligned} \tag{3.7}$$

where $\mathbf{t} = (t^1, \ldots, t^{2n})$, S_n is the set

$$S_n = \bigcup_{k=1}^{2n} \Big\{ \mathbf{t} \in [0,1]^{2Nn} : |t^k - t^j| > r_n \text{ and } |x + t^k - t^j - y| > r_n, \forall j \neq k,\, 0 \leq j \leq 2n \Big\}$$

and $r_n = c_{_{3,1}}\, 2^{-n/\alpha}(n+1)^{1/\alpha}$. The value of $c_{_{3,1}}$ will be determined later.

We consider the above integral over S_n first. By Fubini's theorem, the first term in the right-hand side of (3.7) can be rewritten as

$$\begin{aligned} &\mathbb{E} \int_{S_n} \int_{\mathbb{R}^{2nd}} \prod_{j=1}^{2n} e^{i\langle \xi^j,\ 2^n B^\alpha(x+t^j) - 2^n B^\alpha(y+t^j)\rangle} \psi(\xi^j)\, d\xi d\mathbf{t} \\ &= \int_{S_n} \int_{\mathbb{R}^{2nd}} e^{-\frac{1}{2}\sum_{\ell=1}^d \mathrm{Var}\left(\sum_{j=1}^{2n} \xi_\ell^j [2^n B_0^\alpha(x+t^j) - 2^n B_0^\alpha(y+t^j)]\right)} \prod_{j=1}^{2n} \psi(\xi^j)\, d\xi d\mathbf{t}, \end{aligned} \tag{3.8}$$

where $\xi = (\xi^1, \ldots, \xi^{2n}) \in \mathbb{R}^{2nd}$. Since ψ is supported on the annulus $\big\{\xi \in \mathbb{R}^d :\ |\xi| \in [\frac{1}{2}, \frac{5}{2}]\big\}$, the last integral in $d\xi$ is taken over $\big\{\xi \in \mathbb{R}^d :\ |\xi| \in [\frac{1}{2}, \frac{5}{2}]\big\}^{2n}$.

Note that for every $\mathbf{t} \in S_n$, there is a $k \in \{1, \ldots, 2n\}$ such that $|t^k - t^j| > r_n$ for all $j \neq k$ and $|x + t^k - t^j - y| > r_n$ for all $0 \leq j \leq 2n$. Since $|\xi^k| \in [\frac{1}{2}, \frac{5}{2}]$, there exists $\ell_0 \in \{1, \ldots, d\}$ such that $|\xi_{\ell_0}^k| \geq (2\sqrt{d})^{-1}$. By the same reasoning as we used in proving Lemma 1.3, we derive that

$$\begin{aligned} &\mathrm{Var}\bigg(\sum_{j=1}^{2n} \xi_{\ell_0}^j \big[2^n B_0^\alpha(x+t^j) - 2^n B_0^\alpha(y+t^j)\big] \bigg) \\ &\geq \mathrm{Var}\Big(\xi_{\ell_0}^k \big[2^n B_0^\alpha(x+t^k) - 2^n B_0^\alpha(y+t^k)\big] \big| B_0^\alpha(x+t^j), B_0^\alpha(y+t^j),\, j \neq k \Big) \\ &\geq \frac{1}{4d}\, 2^{2n}\, \mathrm{Var}\Big(B_0^\alpha(x+t^k) \big| B_0^\alpha(x+t^j),\, \forall j \neq k;\, B_0^\alpha(y+t^j),\, \forall j \Big) \\ &\geq c_{_{3,4}}\, 2^{2n} \min_{0 \leq j \leq 2n} \big\{ |t^k - t^j|^{2\alpha}, j \neq k;\, |x + t^k - y - t^j|^{2\alpha} \big\} \\ &\geq c_{_{3,4}}\, 2^{2n}\, r_n^{2\alpha} = c_{_{3,4}}\, c_{_{3,1}}^{2\alpha}\, (n+1)^2. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\mathbb{E} \int_{S_n} \int_{\mathbb{R}^{2nd}} \prod_{j=1}^{2n} e^{i\langle \xi^j,\ 2^n B_0^\alpha(x+t^j) - 2^n B_0^\alpha(y+t^j)\rangle} \psi(\xi^j)\, d\xi d\mathbf{t} \leq e^{-c_{_{3,5}}\, n^2}. \tag{3.10}$$

Note that we can choose the value of $c_{_{3,1}}$ such that $c_{_{3,5}}$ is sufficiently large.

Now, we consider the second integral in (3.7). Let $T_n = [0,1]^{2Np} \backslash S_n$ and we write it as

$$
\begin{aligned}
T_n = \Big\{ & \mathbf{t} \in [0,1]^{2Nn} : \forall k \in \{1, \ldots, 2n\}, \\
& \qquad \exists j_1 \neq k, \ \ j_1 \in \{0, 1, \ldots, 2n\} \ \text{ s.t. } \ \big|t^k - t^{j_1}\big| \leq r_n \\
& \qquad \text{or } \exists j_2 \neq k, \ \ j_2 \in \{0, 1, \ldots, 2n\} \ \text{ s.t. } \ \big|x + t^k - y - t^{j_2}\big| \leq r_n \Big\} \\
= & \bigcap_{k=1}^{2n} \bigg(\Big\{ \mathbf{t} \in [0,1]^{2Nn} : \exists j_1 \neq k \ \text{ s.t. } \min_{j_1 \neq k,\, 0 \leq j_1 \leq 2n} \big|t^k - t^{j_1}\big| \leq r_n \Big\} \\
& \qquad \bigcup \Big\{ \mathbf{t} \in [0,1]^{2Nn} : \exists j_2 \neq k, \ \text{ s.t. } \min_{j_2 \neq k,\, 0 \leq j_2 \leq 2n} \big|x + t^k - y - t^{j_2}\big| \leq r_n \Big\} \bigg).
\end{aligned}
\tag{3.11}
$$

From (3.11), we can see that T_n is a union of at most $(4n)^{2n}$ sets of the form:

$$
A_{\mathbf{j}} = \Big\{ \mathbf{t} \in [0,1]^{2Nn} : \ \big|z + t^k - t^{j_k}\big| \leq r_n, \ \ \forall \ k \in \{1, \ldots, 2n\} \Big\}, \tag{3.12}
$$

where $z = 0$ or $x - y$ and where $\mathbf{j} = (j_k \in \{0, 1, \ldots, 2n\} : 1 \leq k \leq 2n)$ has the property that $j_k \neq k$.

The following lemma is a direct extension of Lemma 3.8 in Khoshnevisan, Wu and Xiao [8], thus its proof will be omitted. We will use it to estimate the Lebesgue measure of T_n.

Lemma 3.2 *For any positive even number m, for any sequence $\{\ell_1, \ldots, \ell_m\} \subset \{0, \ldots, m\}$ satisfying $\ell_j \neq j$, $z^1, \ldots, z^m \in \mathbb{R}^N$, and for any $r > 0$, we have*

$$
\lambda_m \Big\{ \mathbf{s} \in [0,1]^{mN} : \max_{k \in \{1, \ldots, m\}} \big|z^k + s^k - s^{\ell_k}\big| \leq r \Big\} \leq (c_{3,6}\, r)^{mN/2}, \tag{3.13}
$$

where $c_{3,6} > 0$ is a finite constant depending on N only.

We now continue with the proof of Lemma 3.1. It follows from (3.11), (3.12) and Lemma 3.2 that

$$
\lambda_{2Nn}(T_n) \leq c_{3,6}^{Nn} \, (4n)^{2n} \, r_n^{Nn}. \tag{3.14}
$$

We proceed to estimate the integral in (3.7) over T_n. It is bounded above by

$$
\begin{aligned}
& \int_{T_n} \mathbb{E} \bigg[\prod_{j=1}^{2n} \big|\widehat{\psi}(2^n B^\alpha(x + t^j) - 2^n B^\alpha(y + t^j))\big| \bigg] d\mathbf{t} \\
& = \int_{T_n} \mathbb{E} \bigg[\prod_{j=1}^{2n} \big|\widehat{\psi}(2^n B^\alpha(x + t^j) - 2^n B^\alpha(y + t^j))\big| \, \mathbf{1}_{D_n} \bigg] \, d\mathbf{t} \\
& \qquad + \int_{T_n} \mathbb{E} \bigg[\prod_{j=1}^{2n} \big|\widehat{\psi}(2^n B^\alpha(x + t^j) - 2^n B^\alpha(y + t^j))\big| \, \mathbf{1}_{D_n^c} \bigg] \, d\mathbf{t} \\
& \equiv I_1 + I_2,
\end{aligned}
\tag{3.15}
$$

where

$$
D_n = \Big\{ \exists j \in \{1, \ldots, 2n\} \ \text{ s.t. } \ \big|B^\alpha(x + t^j) - B^\alpha(y + t^j)\big| > 2^{-(1-\varepsilon)n} \Big\}.
$$

Since $\widehat{\psi}$ is a rapidly decreasing function, we derive from (3.14) that

$$\begin{aligned} I_1 &\le \lambda_{2Nn}(T_n)\, \widehat{\psi}(2^{\varepsilon n})\, \mathbb{P}(D_n) \\ &\le c_{3,6}^{Nn}\, (4n)^{2n}\, r_n^{Nn}\, e^{-c_{3,7}\, n} \\ &= c_{3,8}^{n}\, n^{n\left(2+\frac{N}{\alpha}\right)}\, 2^{-\frac{Nn^2}{\alpha}}\, e^{-c_{3,7}\, n}. \end{aligned} \tag{3.16}$$

Moreover, we can choose $c_{3,7} > 0$ arbitrarily large, thus I_1 is very small. On the other hand, by the Hölder inequality, I_2 is at most

$$\begin{aligned} &\int_{T_n} \mathbb{P}\Big\{|B^\alpha(x+t^j) - B^\alpha(y+t^j)| \le 2^{-(1-\varepsilon)n}, \forall j = 1, \dots, 2n\Big\}\, d\mathbf{t} \\ &\le \int_{[0,1]^{2Nn}} \mathbf{1}_{T_n}(\mathbf{t}) \left(\mathbb{P}\Big\{|B_0^\alpha(x+t^j) - B_0^\alpha(y+t^j)| \le 2^{-(1-\varepsilon)n}, \forall j = 1, \dots, 2n\Big\}\right)^d d\mathbf{t} \\ &\le \Big(\lambda_{2Nn}\big(T_n\big)\Big)^{(N-\alpha d)/N} \left\{\int_{[0,1]^{2Nn}} \left(\mathbb{P}\Big\{|B_0^\alpha(x+t^j) - B_0^\alpha(y+t^j)| \le 2^{-(1-\varepsilon)n},\, \forall j\Big\}\right)^{N/\alpha} d\mathbf{t}\right\}^{\alpha d/N} \\ &\le c_{3,9}^n\, n^{c_{3,10} n}\, 2^{-n^2\left(\frac{N}{\alpha} + (1-2\varepsilon)d\right)}\, |x-y|^{-2\alpha n d}, \end{aligned} \tag{3.17}$$

where the last inequality follows from (3.14) and (2.10) in Remark 2.2 with $p = n$ and where $c_{3,10} > 0$ is a constant depending on α, d and N only.

Combining (3.7), (3.10) with $c_{3,5}$ large, (3.15), (3.16) and (3.17), we obtain

$$\mathbb{E}\big[J(x,y,n)^{2n}\big] \le c_{3,11}^n\, n^{c_{3,12} n}\, 2^{-n^2\left(\frac{N}{\alpha} + (1-2\varepsilon)d\right)} |x-y|^{-2\alpha n d}. \tag{3.18}$$

We choose and fix $0 < \varepsilon < \frac{N+\alpha d - 2\alpha}{2\alpha d}$. This guarantees that $2^{-\frac{n}{2}\left(\frac{N}{\alpha} + (1-2\varepsilon)d\right)} = (2+\beta)^{-n}$ for some constant $\beta > 0$. Therefore, (3.6) follows from (3.18). This proves Lemma 3.1. □

Finally, we are ready to finish the proof of Theorem 1.2. Thanks to Lemma 3.1, we have

$$\begin{aligned} &2^n \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \big|J(x,y,n)\big|\, \mu(dx)\mu(dy) \\ &\le 2^n (2+\beta)^{-n} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{\alpha d}}\, \mu(dx)\mu(dy) \\ &\le c_{3,13} \left(\frac{2}{2+\beta}\right)^n, \end{aligned} \tag{3.19}$$

which implies (3.4). This finishes the proof of Theorem 1.2.

4 Remarks and Open Questions

The properties of local nondeterminism and/or strong local nondeterminism have been established for large classes of Gaussian processes and fields and they have played important rôles in studying sample path properties of Gaussian processes. We refer to Xiao [20] [21] for further information on various definitions of local nondeterminism and their applications.

We can show that the results similar to Theorems 1.1 and 1.2 hold for a large class of Gaussian random fields considered in Xiao [20]. We leave it to the interested reader to fill in the details.

Another extension of our results is to fractional Brownian sheets. For a given vector $H = (H_1, \ldots, H_N)$ ($0 < H_j < 1$ for $j = 1, \ldots, N$), an (N, d)-dimensional fractional Brownian sheet $W^H = \{W^H(t), t \in \mathbb{R}_+^N\}$ with Hurst index H is an N-parameter centered Gaussian random field in $\mathbb{R}^d$ with covariance function given by

$$\mathbb{E}\Big[W_j^H(s)W_k^H(t)\Big] = \delta_{jk} \prod_{j=1}^{N} \frac{1}{2}\Big(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\Big), \quad s, t \in \mathbb{R}_+^N, \tag{4.1}$$

where $j, k = 1, \ldots, d$ and $\delta_{jk} = 1$ if $j = k$ and 0 if $j \neq k$. When $N > 1$ and $H = \langle \frac{1}{2} \rangle \equiv (\frac{1}{2}, \ldots, \frac{1}{2})$, $W^{\langle \frac{1}{2} \rangle}$ is the (N, d)-Brownian sheet.

It has been long known that fBm is locally nondeterministic, whereas the Brownian sheet and therefore fractional Brownian sheets are not. Hence the arguments in Sections 2 and 3 can not be carried over to W^H or even the Brownian sheet directly. Nevertheless, Khoshnevisan, Wu and Xiao [8] have proved the corresponding weaker forms of uniform dimensional results for the $(N, 1)$-Brownian sheet by using the *sectorial local nondeterminism* of the Brownian sheet proved by Khoshnevisan and Xiao [9].

Thanks to the sectorial local nondeterminism of fractional Brownian sheets established by Wu and Xiao [15], we can modify the methods of Khoshnevisan, Wu and Xiao [8] to extend Theorems 1.1 and 1.2 to a class of fractional Brownian sheets and prove the following theorems:

Theorem 4.1 *Let $W^{\langle \alpha \rangle}$ be an (N, d)-fractional Brownian sheet with Hurst index $H = \langle \alpha \rangle$. Suppose $\alpha d < 1$, then with probability 1, for every Borel set $E \subseteq (0, 1]^N$,*

$$\dim_{\rm H} W^{\langle \alpha \rangle}(E + t) = \min\Big\{d, \ \frac{1}{\alpha} \dim_{\rm H} E\Big\} \quad \textit{for almost all } t \in [0, 1]^N. \tag{4.2}$$

Theorem 4.2 *Let $\alpha d < 1$, then almost surely for every Borel set $E \subseteq (0, 1]^N$ with $\dim_{\rm H} E > \alpha d$, we have $\lambda_d\big(W^{\langle \alpha \rangle}(E + t)\big) > 0$ for almost all $t \in [0, 1]^N$.*

Remark 4.3 Theorems 3.3 and 3.6 of Khoshnevisan Wu and Xiao [8] are special cases of our Theorems 4.1 and 4.2, respectively, by taking $\tau = \frac{1}{2}$. However, contrast to the fractional Brownian motion case, it is an open question whether Theorems 4.1 and 4.2 still hold for $W^{\langle \alpha \rangle}$ when $1 \leq \alpha d < N$.

We end this section with two more open questions. Question 4.4 was raised by Kaufman [7] for Borwnian motion in $\mathbb{R}$. It is still open, and we have reformulated it for the fractional Brownian motion.

It is known (cf. Pitt [11] or Kanahe [5]) that for every Borel set $E \subseteq \mathbb{R}^N$ with $\dim_{\rm H} E > \alpha d$ [this implies $N > \alpha d$], the image set $B^\alpha(E)$ has interior points almost surely. The following question is about a type of uniform version of the above result:

Question 4.4 Suppose $N > \alpha d$. Is it true that, with probability 1, $B^\alpha(E+t)$ has interior points for some $t \in [0,1]^N$ for every Borel set $E \subseteq \mathbb{R}^N$ with $\dim_{\rm H} E > \alpha d$?

Xiao [16] proved the following uniform packing dimension analogue of (1.3) for an (N,d)-fractional Brownian motion B^α: If $N \le \alpha d$, then almost surely

$$\dim_{\rm P} B^\alpha(E) = \frac{1}{\alpha} \dim_{\rm P} E \quad \text{for every Borel set } \; E \subseteq \mathbb{R}^N, \tag{4.3}$$

where $\dim_{\rm P}$ denotes packing dimension, see Falconer [3]. On the other hand, Talagrand and Xiao [14] have shown that when $N > \alpha d$, $\dim_{\rm P} E$ alone is not enough to determine $\dim_{\rm P} B^\alpha(E)$. Xiao [18] has proved that if $N > \alpha d$, then for every Borel set $E \subseteq \mathbb{R}^N$,

$$\dim_{\rm P} B^\alpha(E) = \frac{1}{\alpha}\, {\rm Dim}_{\alpha d} E \qquad \text{a.s.}, \tag{4.4}$$

where ${\rm Dim}_{\alpha d}$ denotes the (αd)-dimensional "packing dimension profile" of E defined by Falconer and Howroyd [4].

In light of (4.4) and Theorem 1.1, we may ask the following natural question:

Question 4.5 Let B^α be the (N,d)-fractional Brownian motion with $N > \alpha d$. Is it true that, a.s. for every Borel set $E \subseteq \mathbb{R}^N$,

$$\dim_{\rm P} B^\alpha(E+t) = \frac{1}{\alpha}\, {\rm Dim}_{\alpha d} E \quad \text{for almost all } t \in [0,1]^N? \tag{4.5}$$

Recently Wu and Xiao [15] have extended the aforementioned result of Pitt [11] on interior points of the image to the fractional Brownian sheet $W^{\langle\alpha\rangle}$ [the result of Wu and Xiao [15] is for general W^H]. Moreover, by using the arguments in Xiao [18] and Wu and Xiao [15], we can easily show that the results (4.3) and (4.4) also hold for the (N,d)-fractional Brownian sheet $W^{\langle\alpha\rangle}$. Therefore, both Questions 4.4 and 4.5 can also be asked for the fractional Brownian sheet $W^{\langle\alpha\rangle}$.

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Dongsheng Wu. Department of Statistics and Probability, A-413 Wells Hall, Michigan State University, East Lansing, MI 48824, U.S.A.
E-mail: `wudongsh@msu.edu`
URL: `http://www.msu.edu/~wudongsh`

Yimin Xiao. Department of Statistics and Probability, A-413 Wells Hall, Michigan State University, East Lansing, MI 48824, U.S.A.
E-mail: `xiao@stt.msu.edu`
URL: `http://www.stt.msu.edu/~xiaoyimi`