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Dirac Equation in Spatially Homogeneous Cosmological Models

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Abstract

The propagation of spin -1/2 particles in the background of spatially homogeneous cosmological models has been analysed. By restricting the direction of propagation of the spinors, it is shown that Dirac equation reduces to a decoupled system of differential equations depending only on the time variable. A new exact solution is presented in the Joseph cosmological models.

Key-words: Dirac Equation; Cosmology.

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I Introduction

The Dirac equation is an outstanding achievement whose consequences the physicists are very far to exhaust due to the complexity of the calculations involved. The establishment of this equation was the starting point for many research problems, one of which is the study of spin 1/2 particles in the presence of gravitational fields. Brill and Wheeler[1] studied the interaction of neutrinos with the gravitational field of a spherically symmetrical body. They developed the mathematical formalism to study spinors in curved space-time and opened the road for many other authors.

The quantum mechanics of particles in curved spacetimes has been analysed in many articles and some important results are the creation of particles due to the expansion of spacetime[2] and the Hawking radiation of a black hole[3]. These developments go in the direction of a unified theory of quantum and gravitational fields. In this context, the knowledge of exact solutions of the Dirac equation in curved space-time is demanded.

The most studied cosmological backgrounds are the Robertson-Walker ones, due to their simplicity and their accuracy in the fitting of the astronomical data of the recent eras of cosmological evolution. We can refer to Parker[4], Isham and Nelson[5], Ford[6], Andretsch and Schäfer[7], Kovalyov and Légaré[8], Barut and Duru[9] and Villalba and Percoco[10].

Recently a great effort has been done to solve the massless Dirac equation by applying the method of separation of variables to a general diagonal metric[11,12]. This study could generalize the analysis of the propagation of spin 1/2 particles in curved space-time, since most of the background used so far depends on just one space-time variable.

There is an important class of spacetimes – spatially homogeneous models – that is characterized by having a three dimensional hypersurface of type space passing in every point of the space-time[13,14]. They have been classified from the geometrical point of view as the Bianchi cosmological models, and they are the simplest generalization of the Robertson-Walker models. The Bianchi models are spatially homogeneous but they admit anisotropies. The formalism of differential forms allows one to treat these models in a unified way. Some authors have used this formalism to study the Maxwell equations in the background of Bianchi models[15]. In the present article we use this formalism to analyse in an unified way the Dirac equations in the background of the spatially homogeneous cosmological models.

The Dirac equation has already been studied in some anisotropic Bianchi models. The

Bianchi type I model is the most used one, since the flatness of the space hypersurface simplifies the analysis[9,16,17]. There are also exact solutions of the Dirac equation in Bianchi type VI_0 models[18].

Henneaux[19] has studied the problem of finding the most general spinor field with the same symmetry of the gravitational field. In the present article we study the separation of variable obeying the Ansatz (15) in the background of the diagonal Bianchi models. Our purpose is to reduce Dirac equation to a decoupled system of differential equations depending only on the time variable. So, we restrict the direction of propagation of the spinor. We show that for Bianchi type II models, there are two directions of propagation allowed, for the Bianchi types III to VII there is only one particular direction of propagation and for the Bianchi types VIII and IX it is not possible to separate the variables using the Ansatz (15).

We present a new exact solution to the massless Dirac equation in the Joseph cosmological models[20], which are Bianchi type V vacuum space-times.

In section II, we stablish the Dirac equation in the background of the diagonal Bianchi models and separate the variables to obtain a decoupled purely temporal system of differential equations. In section III, we give two applications of the formalism developed here.

II The Dirac Equation

The spatially homogeneous cosmological models have a homogeneous three-dimensional hypersurface passing in every point of the space-time. If we choose the direction of the axis of the timelike parameter to be orthogonal to these three-dimensional surfaces, we have a synchronous coordinate system and the line element is given by the following form[13]:

$$ds^2 = dt^2 - g_{ij}(t)w^i w^j \tag{1}$$

where g_{ij} is a symmetric three-dimensional matrix depending only on t coordinate. The one-forms $\{w^i, i = 1 \cdots 3\}$ form a basis for the three-dimensional dual space. These forms are invariant under a three-dimensional transitive Lie group which defines a Lie algebra with the structure constants C^i_{jk} given by

$$dw^i = -\frac{1}{2}C^i_{jk} w^j \wedge w^k \ . \tag{2}$$

The enumeration for all distinct groups in the context of spatially homogeneous cosmological models has been done by Taub[14] following the original work of Bianchi[21] in a pure mathematical context. There are nine types that are splitted into two main classes, namely: class A when $\Sigma_k C^k_{ik} = 0$ and class B when $\Sigma_k C^k_{ik} \neq 0$.

There are alternative conventions for the choice of C^i_{jk}. The choice of different conventions should not alter the physical results, but can simplify or complicate the intermediate calculations. From our experience, we follow the convention of Luminet[22] and Eardley[23]. The structure constants of all Bianchi types are listed in the appendix.

In general, the three-dimensional metric g_{ij} does not need to be diagonal. In the present article we restrict ourselves to analyse the models with diagonal metric. In this case the line element (1) reduces to:

$$ds^2 = dt^2 - R_1^2(w^1)^2 - R_2^2(w^2)^2 - R_3^2(w^3)^2 \tag{3}$$

where the metric components R_1, R_2 and R_3 depend on t only. We use a tetrad basis of one forms $\{\sigma^2, a = 0 \cdots 3\}$ given by

$$\begin{aligned} &\sigma^0 = dt \\ &\sigma^i = R_i w^i \quad \text{(no sum).} \end{aligned} \tag{4}$$

The Dirac equation in curved space-time is[24]

$$(\gamma^a H_a{}^\mu \partial_\mu - \gamma^a \Gamma_a + im)\Psi = 0 \tag{5}$$

where the $H_a{}^\mu$ establish the connection between tetrad indices (Latin indices) and world indices (Greek indices). The components $H_a{}^\mu$ are obtained from

$$\sigma^a = H^a{}_\mu dx^\mu \tag{6}$$

where $H^a{}_\mu$ and $H_a{}^\mu$ are inverse components. The quantities Γ_a are given by

$$\Gamma_a = -\frac{1}{4}\Gamma_{bca}\gamma^b\gamma^c \tag{7}$$

for not charged spinors. γ^a are the Dirac matrices of Minkowski space-time and Γ_{abc} are the Ricci rotation coefficients defined by

$$d\sigma^a = \Gamma^a{}_{bc}\sigma^b \wedge \sigma^c \tag{8}$$

and are antisymmetric in the first two indices $\Gamma_{abc} = -\Gamma_{bac}$. Using eqs. (2), (4) and (8) we can obtain the components of the Ricci coefficients:

$$\Gamma_{0ii} = \frac{\dot{R}_i}{R_i} \tag{9}$$

and

$$\Gamma_{ijk} = -\frac{1}{2}\left(\frac{R_i}{R_j R_k}\eta_{ip}C^p_{jk} + \frac{R_j}{R_i R_k}\eta_{jp}C^p_{ki} + \frac{R_k}{R_i R_j}\eta_{kp}C^p_{ji}\right) \tag{10}$$

where $\eta_{ij} = diag(-1,-1,-1)$. Using eq. (7) we can find Γ_i

$$\Gamma_i = -\frac{1}{2}\Sigma^3_{i=1}\frac{\dot{R}_i}{R_i}\gamma^0\gamma^i - \frac{1}{4}\Gamma_{jki}\gamma^j\gamma^k. \tag{11}$$

We have already discussed the choice of a convention for the structure constants. Once this convention has been stablished, it is possible to have alternative choices for the coordinate system. Different from the choice of C^i_{jk}, the coordinate system does not alter the form of the spinor. The basis w^i can be expressed as

$$w^{(i)} = h^{(i)}{}_j dx^j \tag{12}$$

and the w^i's are listed in appendix for all Bianchi type models with the respective dual basis ξ_i. We use parenthised indices to distinguish tetrad indices from coordinate indices when necessary. The components $h^{(i)}_j$ are time-independent but they are related to the time-dependent components $H^{(i)}{}_j$ by eqs. (4) and (6)

$$H^{(i)}{}_j = R_i h^{(i)}{}_j \quad \text{(no sum)} \tag{13}$$

Using eqs. (11) and (13), Dirac equation (5) reduces to:

$$\left(\partial_t + \frac{\gamma^0\gamma^j h_{(j)}{}^\ell}{R_j}\partial_\ell + \frac{1}{2}\sum_i \frac{\dot{R}_i}{R_i} + im\gamma^0 + \frac{1}{4}\Gamma_{kji}\gamma^0\gamma^i\gamma^k\gamma^j\right)\Psi = 0. \tag{14}$$

Eq. (14) is the Dirac equation for the diagonal Bianchi models. Our main interest here is to try a separation of variables of the form

$$\Psi = \frac{e^{i\vec{k}.\vec{x}}}{(2\pi)^{3/2}\sqrt{R_1 R_2 R_3}} f(\vec{k},t) \tag{15}$$

to obtain a differential equation depending only on the time coordinate. Note that the dependence on space coordinates of eq. (14) is restricted to the second term (aside the spinor itself), since the inverse components ${h_{(j)}}^k$ is the unique term that can depend on space coordinates. Using the Ansatz (15), eq. (14) reduces to

$$\left(\partial_t + \frac{ik_\ell {h_{(j)}}^\ell \alpha^j}{R_j} + im\gamma^0 + \frac{1}{4}\Gamma_{kji}\gamma^0\gamma^i\gamma^k\gamma^j \right) f(\vec{k},t) = 0. \tag{16}$$

The components ${h_{(j)}}^\ell$ can be read as the coefficients of the ∂_ℓ terms of the vector basis given in appendix. By inspection, we can see that Bianchi type VIII and IX models do not admit a separation of variables of the form (15), since the second term of eq. (16) can be independent of the space coordinates for no choice of $\vec{k}$. On the other hand, for Bianchi types III to VII we must impose $\vec{k} = (k_1, 0, 0)$, for eq. (16) be purely temporal. For Bianchi type II, we must impose $\vec{k} = (k_1, 0, k_3)$ and for Bianchi type I, $\vec{k}$ can be arbitrary since the ${h_{(j)}}^\ell$ vanish. The last result was expected, since Bianchi I models admit three independent Killing vectors of the form ∂_i. Therefore, a spinor of the form (15) can propagate in all directions in a Bianchi I model. In the other models the spinor is restricted to propagate in two directions (Bianchi II), or in one direction (Bianchi III to VII).

In order to continue in a unified way, we impose $\vec{k} = (k_1, 0, 0)$. The Bianchi models of type II to VII admit this kind of momentum. For these models, the second term of eq. (16) reduces to $i\varepsilon k_1 \alpha^3/R_3$ where $\varepsilon = 1$ for Bianchi II and $\varepsilon = -1$ for Bianchi III to VII. Using eq. (10), use can put the fourth term of eq. (16) in the form $iA(t)\gamma^5 + B(t)\alpha^3$ where $A(t)$ and $B(t)$ are given by:

$$\text{Bianchi II}: A(t) = \frac{R_1}{4R_2R_3}, \qquad B(t) = 0 \tag{17}$$

$$\text{Bianchi IV}: A(t) = \frac{R_1}{4R_2R_3}, \qquad B(t) = -\frac{1}{R_3} \tag{18}$$

$$\text{Bianchi V}: A(t) = 0, \qquad B(t) = -\frac{1}{R_3} \tag{19}$$

$$\text{Bianchi VI}_h: A(t) = \frac{1}{4}\left(\frac{R_1}{R_2R_3} - \frac{R_2}{R_1R_3}\right), \quad B(t) = -\frac{\sqrt{-h}}{R_3} \tag{20}$$

$$\text{Bianchi VII}_h: A(t) = \frac{1}{4}\left(\frac{R_1}{R_2R_3} + \frac{R_2}{R_1R_3}\right), \quad B(t) = -\frac{\sqrt{h}}{R_3}. \tag{21}$$

The expressions for Bianchi type III and VI_0 can be obtained from VI_h using $h = -1$ and $h = 0$ respectively, and the type VII_0 from VII_h using $h = 0$. Now we can put eq. (16) in the form

$$\left(\partial_t + im\gamma^0 + \left(B + \frac{i\varepsilon k_1}{R_3}\right)\alpha^3 + iA\gamma^5 \right) f(\vec{k},t) = 0 \tag{22}$$

which is equivalent to

$$\left(\begin{array}{cc} \partial_t + im & iA + \left(B + \frac{i\varepsilon k_1}{R_3}\right)\sigma^3 \\ iA + \left(B + \frac{i\varepsilon k_1}{R_3}\right)\sigma^3 & \partial_t - im \end{array} \right) \left(\begin{array}{c} f_I \\ f_{II} \end{array} \right) = 0. \tag{23}$$

This equation decouples in two 2×2 systems:

$$\left(\begin{array}{cc} \partial_t + im & iA + B + \frac{i\varepsilon k_1}{R_3} \\ iA + B + \frac{i\varepsilon k_1}{R_3} & \partial_t - im \end{array} \right) \left(\begin{array}{c} f_I^+ \\ f_{II}^+ \end{array} \right) = 0 \tag{24}$$

and

$$\left(\begin{array}{cc} \partial_t + im & iA - B - \frac{i\varepsilon k_1}{R_3} \\ iA - B - \frac{i\varepsilon k_1}{R_3} & \partial_t - im \end{array} \right) \left(\begin{array}{c} f_I^- \\ f_{II}^- \end{array} \right) = 0. \tag{25}$$

Let us call

$$C^{\pm} = iA \pm \left(B + \frac{i\varepsilon k_1}{R_3}\right) . \tag{26}$$

Eqs. (24) and (25) yield the following equations:

$$\left(\partial_t^2 - \frac{\dot{C}^{\pm}}{C^{\pm}}(\partial_t + im) + C^{\pm 2} + m^2\right) f_I^{\pm} = 0 \tag{27}$$

and

$$f_{II}^{+} = -\frac{1}{C^{\pm}}(\partial_t + im) f_I^{\pm} . \tag{28}$$

Eq. (27) can be put in the form:

$$\left(\partial_t^2 + \frac{1}{2}\frac{\ddot{C}}{C} - \frac{3}{4}\frac{\dot{C}^2}{C^2} - C^2 - im\frac{\dot{C}}{C} + m^2\right)\left(\frac{f_I}{\sqrt{C}}\right) = 0 \tag{29}$$

where we have taken out the $\pm$ sign. The problem of finding the solutions for the Dirac equation in a spatially homogeneous models satisfying the Ansatz (15) now reduces to find the solutions for an equation of the form (29). This differential equation is rather complicated in the general case even when m is zero (neutrinos). The function $C(t)$ must obey eq. (26), where $A(t)$ and $B(t)$ are given by eqs. (17)-(21). Most of the known exact solutions of Einstein Equations[25] yield a differential equation (eq. (29)) that has no known exact solution. In the following section we give two applications where eq. (29) can be integrated.

III Two Examples: Bianchi VI_0 and V

As an application of results obtained so far, let us consider the line element

$$ds^2 = dt^2 - \frac{t^2 dx^2}{(\mu-\nu)^2} - \frac{dy^2}{t^{2(\mu+\nu)}e^{2x}} - \frac{e^{2x}dz^2}{t^{2(\mu+\nu)}} \tag{30}$$

studied in ref. 18, where solutions for the Dirac equation have been presented both for neutrinos and massive spinors using the coordinate system of eq. (30). With the coordinate transformation

$$\begin{aligned} x &= x^1 \\ y &= \frac{e^{x^1}}{\sqrt{2}}(x^2 + x^3) \\ z &= \frac{e^{-x^1}}{\sqrt{2}}(x^2 - x^3) \end{aligned} \tag{31}$$

the line element (30) can be put in the form

$$ds^2 = dt^2 - \frac{t^2(dx^1)^2}{(\mu-\nu)^2} - \frac{(x^2dx^1 + dx^3)^2}{t^{2(\mu+\nu)}} - \frac{(x^3dx^1 + dx^2)^2}{t^{2(\mu+\nu)}} \tag{32}$$

Chosing

$$\begin{aligned} w^1 &= x^3dx^1 + dx^2 \\ w^2 &= x^2dx^1 + dx^3 \\ w^3 &= -dx^1 \end{aligned} \tag{33}$$

we can see that the line element (32) can be put in the form (3) with $R_1 = R_2 = 1/t^{(\mu+\nu)}$ and $R_3 = t/(\mu-\nu)$. The structure constants of this model are $C^1_{23} = C^2_{13} = 1$, therefore it is of type Bianchi VI_0 (see the appendix). The function $C(t)$, given by eqs. (26) and (20), is

$$C(t) = \frac{ik_1(\mu-\nu)}{t} \ . \tag{34}$$

Substituting this value in eq. (29), we obtain

$$\left(\partial_t + \frac{\frac{1}{4} + k_1^2(\mu-\nu)^2}{t^2} + \frac{im}{t} + m^2\right)(\sqrt{t}f_I) = 0 \ . \tag{35}$$

It can be verified that eq. (35) is equal to eq. (20) of ref. 18. Therefore, using the formalism developed here, we obtained the same Dirac equation by a suitable choice of the basis w^i. The explicit form of the spinor will differ for these coordinate systems, since in one case it will be given in coordinates (x, y, z) and in other case in (x^1, x^2, x^3). But the temporal part will be the same. Eqs. (35) has been solved in ref. 18.

Let us consider a second application. By inspection in the eqs. (17) to (21) we can see that Bianchi type V furnishes the simplest Dirac equation since $A(t) = 0$ and $B(t)$ has the same form of the term $i\varepsilon k_1/R_3$ of eq. (22). The vacuum Bianchi V models have been studied by Joseph[20]. The line element[1] can be put in the form[25]

$$ds^2 = (sinh\ 2a\tau)\left[d\tau^2 - (tanh\ a\tau)^{\sqrt{3}}(w^1)^2 - (tanh\ a\tau)^{-\sqrt{3}}(w^2)^2 - \left(\frac{w^3}{a}\right)^2\right] \tag{36}$$

where w^1, w^2 and w^3 are listed in appendix. Since this line element has a coefficient in $d\tau^2$ term, it is interesting to introduce the coefficient $R_0(t)^2$ in the general line element (3) to obtain the correspondent Dirac equation of the form (29). The equations that change are eqs. (4), (9), (11), (14), (16), (22) and the subsequent ones. Eq. (16) changes to

$$\left(\partial_t + \frac{ik_\ell R_0 h_{(j)}{}^\ell \alpha^j}{R_j} + imR_0\gamma^0 + \frac{R_0}{4}\Gamma_{kji}\gamma^0\gamma^i\gamma^k\gamma^j\right) f(\vec{k}, t) = 0 \tag{37}$$

and eq. (29) to

$$\left(\partial_t^2 + \frac{1}{2}\frac{\ddot{C}}{C} - \frac{3}{4}\frac{\dot{C}^2}{C^2} - C^2 - \frac{imR_0\dot{C}}{C} + m^2R_0^2 + im\dot{R}_0\right)\left(\frac{f_I}{\sqrt{C}}\right) = 0 \tag{38}$$

where C is given by

$$C^{\pm} = iR_0A \pm R_0\left(B + \frac{i\varepsilon k_1}{R_3}\right) . \tag{39}$$

For the line element (36) we have

$$\left(\partial_t^2 - (ik_1a + a)^2 + m^2 \sinh\ 2a\tau + \frac{iam\cosh 2a\tau}{\sqrt{\sinh 2a\tau}}\right)(f_I) = 0 \tag{40}$$

and

$$f_{II} = \frac{1}{a + ik_1a}(\partial_t + im\sinh 2a\tau)f_I$$

For neutrinos, the solutions are

$$f_I = c_1\ e^{(a+ik_1a)\tau} + c_2e^{-(a+ik_1a)\tau} \tag{41}$$

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[1]In ref. 25 the constant a of the last term of eq. 36 is missing.

where c_1 and c_2 are the integration constants. The four independent solutions of the form (15) are

$$\Psi_I^{\pm} = \frac{e^{i(k_1x+ak_1\tau)+a\tau}}{(2\pi)^{3/2}(\sinh 2a\tau)^{3/4}} \begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}$$

$$\Psi_{II}^{\pm} = \frac{e^{i(k_1x-ak_1\tau)-a\tau}}{(2\pi)^{3/2}(\sinh 2a\tau)^{3/4}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \pm 1 \end{pmatrix}$$

This spinor cannot be normalized due to the term $e^{\pm a\tau}$. This fact indicates that this solution is non stable. A similar situation happens with a scalar field ϕ that has a Lagrangian with a negative mass squared $-\mu^2$. In this case the scalar field has a term $e^{\pm\sqrt{\mu^2-k^2}t}$. The usual interpretation is that the solution $\phi = 0$ is not stable and the spontaneous symmetry breaking occurs after a time of order μ^{-1} (see ref. 26).

Conclusion

We have analysed the Dirac equation in the background of the spatially homogeneous cosmological models. We have used the method of separation of variables through the Ansatz (15). The dynamics of the spatially homogeneous models is caracterized by functions depending only on the time variables. For this reason we have chosen a spinor of the form of eq. (15). With this choice it is possible to reduce the Dirac equation to a system of decoupled differential equations, depending only on the time variable.

We have showed that Bianchi type VIII and IX models do not admit this kind of separation of variables, since too much components of the Killing vectors depend on space coordinates. In this case the spatial dependence of the spinor cannot be $e^{i\vec{k}\vec{x}}$. For the other Bianchi models, it is possible to have solutions of the form of eq. (15), if we impose restrictions on the direction of propagation of the spinor. For Bianchi types III to VII, the spinor is allowed to propagate in only one direction. The direction is determined by the spatial dependence of the space-like Killing vectors. In the present article the direction is $\vec{k} = (k_1, 0, 0)$, since the first component of all space-like Killing vectors does not depend on the spatial variables. For the Bianchi type II model, there are two directions allowed and for Bianchi type I, there are no restrictions on the direction of propagation.

We have made two applications. In the first one, the Dirac equation was analysed in the background of a Bianchi type VI_0 model that has been widely studied in the literature. In the second one, the background used was the vacuum Bianchi type V model studied by Joseph[20]. In this case, the spinor found cannot be normalized. This probably indicates that the Dirac equation yields non stable solutions when the spinor is in the presence of a external gravitational field.

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Appendix

The structure constants (${C^i}_{jk}$), the vector basis (ξ_i) and the invariant 1-form basis[22] (w^i) of the Bianchi models.

The structure constants (${C^i}_{jk} = -{C^i}_{kj}$):

CLASS A($C^{\ell}_{i\ell} = 0$)

Bianchi I: $C^i_{jk} = 0$

Bianchi II: $C^1_{23} = 1$

Bianchi VI$_0$: $C^1_{23} = C^2_{13} = 1$

Bianchi VII$_0$: $C^1_{23} = C^2_{31} = 1$

Bianchi VIII: $C^1_{23} = C^2_{31} = C^3_{21} = 1$

Bianchi IX: $C^1_{23} = C^2_{31} = C^3_{12} = 1$

CLASS B($C^{\ell}_{i\ell} \neq 0$)

Bianchi III: Bianchi VI$_{-1}$

Bianchi IV: $C^1_{31} = C^1_{23} = C^1_{32} = 1$

Bianchi V: $C^1_{31} = C^2_{32} = 1$

Bianchi VI$_h$: $C^1_{23} = C^2_{13} = 1, C^1_{31} = C^2_{32} = \sqrt{-h}, h < 0$

Bianchi VII$_h$: $C^1_{23} = C^2_{31} = 1, C^1_{31} = C^2_{32} = \sqrt{h}, h > 0$

The vector and 1-form basis:

Bianchi Type I :

$\xi_1 = \partial_1 \qquad w^1 = dx^1$

$\xi_2 = \partial_2 \qquad w^2 = dx^2$

$\xi_3 = \partial_3 \qquad w^3 = dx^3$

Bianchi Type II :

$\xi_1 = \partial_2 \qquad w^1 = -x^3 dx^1 + dx^2$

$\xi_2 = \partial_3 \qquad w^2 = dx^3$

$\xi_3 = \partial_1 + x^3\partial_2 \qquad w^3 = dx^1$

Bianchi Type IV :

$\xi_1 = \partial_2 \qquad w^1 = (x^3 - x^2)dx^1 + dx^2$

$\xi_2 = \partial_3 \qquad w^2 = -x^3dx^1 + dx^3$

$\xi_3 = -\partial_1 + (x^3 - x^2)\partial_2 - x^3\partial_3 \qquad w^3 = -dx^1$

Bianchi Type V :

$\xi_1 = \partial_2 \qquad w^1 = -x^2dx^1 + dx^2$

$\xi_2 = \partial_3 \qquad w^2 = -x^3dx^1 + dx^3$

$\xi_3 = -\partial_1 - x^2\partial_2 - x^3\partial_3 \qquad w^3 = -dx^1$

Bianchi Type VI$_h$:

$\xi_1 = \partial_2 \qquad w^1 = (x^3 - ax^2)dx^1 + dx^2$

$\xi_2 = \partial_3 \qquad w^2 = (x^2 - ax^3)dx^1 + dx^3$

$\xi_3 = -\partial_1 + (x^3 - ax^2)\partial_2 + (x^2 - ax^3)\partial_3 \qquad w^3 = -dx^1$

Bianchi Type VII$_h$:

$\xi_1 = \partial_2 \qquad w^1 = (x^3 - ax^2)dx^1 + dx^2$

$\xi_2 = \partial_3 \qquad w^2 = -(x^2 + ax^3)dx^1 + dx^3$

$\xi_1 = -\partial_1 + (x^3 - ax^2)\partial_2 - (x^2 + ax^3)\partial_3 \qquad w^3 = -dx^1$

Bianchi Type VIII :

$\xi_1 = \frac{1}{2}e^{-x^2}\partial_1 - \frac{1}{2}(e^{x^3} + (x^2)^2e^{-x^3})\partial_2 - x^2e^{-x^3}\partial_3 \qquad w^1 = (e^{x^3} - (x^2)^2e^{-x^3})dx^1 - e^{-x^3}dx^2$

$\xi_2 = \partial_3 \qquad w^2 = 2x^2dx^1 + dx^3$

$\xi_3 = -\frac{1}{2}e^{-x^3}\partial_1 - \frac{1}{2}(e^{x^3} - (x^2)^2e^{-x^3})\partial_2 + x^2e^{-x^3}\partial_3 \qquad w^3 = -(e^{x^3} + (x^2)^2e^{-x^3})dx^1 - e^{x^3}dx^2$

Bianchi Type IX :

$$\xi_1 = \partial_2 \qquad w^1 = dx^2 + \cos\ x^1 dx^3$$

$$\xi_2 = \cos\ x^2\partial_1 - cotan\ x^1 \sin\ x^2\partial_2 + \frac{\sin\ x^2}{\sin\ x^1}\partial_3 \quad w^2 = \cos\ x^2\ dx^1 + \sin\ x^2\ \sin\ x^1\ dx^3$$

$$\xi_3 = -\sin\ x^2\partial_1 - cotan\ x^1\ cos\ x^2\partial_2 + \frac{\cos\ x^2}{\sin\ x^1}\partial_3 \quad w^3 = -\sin\ x^2\ dx^1 + \cos\ x^2\ \sin\ x^1\ dx^3$$

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