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Dirac field on Newtonian space-time (*)

by

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ABSTRACT. — Lévy-Leblond's Galilei invariant theory of a spin-1/2-wave equation is generalized to a general curved Newtonian space-time. This is achieved by finding the general quadratic Lagrangian that leads to first order homogeneous equations in the four component spinor field and is invariant under the previously introduced general covariance group of Newtonian gravity. In addition to the field equations expressions for the matter current vector and the stress-energy tensor of the Dirac field are derived. It turns out that this derivation is less straightforward than in the relativistic case. The (gauge invariant) local energy density, for example, retains an arbitrary parameter reminiscent of the free constant in the potential in classical mechanics.

RÉSUMÉ. — On généralise l'équation d'onde de Lévy-Leblond pour une particule de spin 1/2 invariante par le groupe de Galilée au cas d'un espace-temps newtonien courbe. On construit un lagrangien invariant par le groupe de covariance générale de la gravitation newtonienne. Cela conduit

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à des équations linéaires homogènes du premier ordre pour un champ de spineurs à quatre composantes. En outre, on déduit les expressions du courant de matière et du tenseur énergie-impulsion du champ de Dirac. Le calcul de ces expressions est moins élémentaire qu'en relativité générale. La densité d'énergie locale, par exemple, fait apparaître un paramètre libre qui rappelle la constante arbitraire figurant dans le potentiel de la mécanique classique.

1. INTRODUCTION

We continue in this paper the programme [*8, 9*] of studying classical fields theories in Newtonian space-time by constructing the theory of a Dirac particle, i. e. a 4-component spinor field coupled to the Newtonian gravitational field via the minimal interaction principle applied to the Lagrangian.

This work should provide some additional insight into the role played by general covariance arguments in relativistic classical field theories. It can also be motivated by Kuchar's [*14*] suggestion that to learn about relativistic quantum gravity (and quantum field theory on curved space-time) one might for comparison purposes look at the nonrelativistic case in a similar covariant formulation.

This theory also underlies the COW-experiment [*6, 20*] since it leads within the nonrelativistic domain to a nearly unique wave equation for a spinning particle in a gravitational field in an arbitrarily rotating or accelerated reference frame, an equation which can then be approximated by WKB methods. In [*20*], for example, a slight doubt is expressed whether the Hamiltonian on which the theory of the experiment is based is the correct one. In our approach the interaction between the spinor field and the gravitational field follows from well established physical principles, namely the general covariance under coordinate transformations and, in this case, observer field and gauge changes. We find indeed an interaction term between the spinor field and the gravitational field in the case of rotating reference frames. But this analysis is not pursued in this paper where we only present the general fourdimensional formalism.

The plan is as follows. We first develop in section 2 the formalism for 4-component spinor fields on a general Galileian space-time manifold with a compatible linear connection. This is best done when an explicit action of the spin Galilei group on $\mathbb{C}^4$ is used to define spinor bundles associated to the bundle of spin Galilei frames. In this form the construction is a simple adaptation of the well known one for the Lorentz case.

The approach via Galilei Clifford algebra representations [*2-4*] might here provide a wider perspective but has perhaps not yet been developed far enough to be used as a suitable starting point for our purpose. We therefore defer a discussion of this alternative point of view to section 5 where it can be combined with an analysis of the role played by the Bargmann group (= 11-dimensional nontrivial central extension of the inhomogeneous Galilei group) in this context.

In section 3 we review the Galilei invariant wave equation found by Lévy-Leblond [17] as a « square root » of the free Schrödinger equation. We write this equation in 4-component spinor form and observe that it derives from a Lagrangian density $\mathscr{L}$ quadratic in the spinor fields and linear in their derivatives.

The general structure of the Lagrangian which is needed to yield first order linear homogeneous field equations we wish to preserve. But otherwise we allow $\mathscr{L}$ to arbitrarily depend on the Galilei space-time structure and the given Newtonian connection [15]. Requiring $\mathscr{L}$ to be invariant under the full automorphism group of Newtonian gravity [9], i. e. under diffeomorphisms, changes of the observer field and the gauge field which determine the connection, and local spin frame rotations then turns out to fix its form up to three nontrivial constants. They can be set to zero in order to preserve the connection with the Schrödinger equation.

After an algebraic-geometric discussion in section 5 we derive the field equations of the spinor field on the curved Newtonian space-time which also imply a gravitational Schrödinger-Pauli equation. Finally, in sections 7 and 8, we apply the general theory of [8, 9] to derive systematically the matter current and the stress-energy tensor of the spinor field treated here as purely classical.

This method is basically the Newtonian analogue of defining the relativistic stress-energy tensor by variational derivatives of the Lagrangian with respect to local Lorentz frames. It would produce invariantly defined expressions (i. e. independent of the local spin frame and the Galilei connection gauge fields A and u) were it not for the built-in constraint of a symmetric connection. This forces the timelike 1-form $\theta^0 = \psi$ to be closed, and as a consequence there is some arbitrariness in the definition of energy density and energy flow. By requiring these quantities to be gauge invariant the arbitrariness can be reduced to two constants. This phenomenon should probably be understood better still, also in a more general context.

All expressions in this paper are 4-dimensionally covariant. For many applications and even just to recognize familiar aspects of the theory a $(3+1)$-dimensional formulation is preferable. It is also particularly useful for a post-Newtonian approximation of the relativistic theory. But since the techniques needed (cf. [16]) are somewhat different, this discussion will be published elsewhere.

The basic definitions of Galileian manifolds and Newtonian structures are not repeated here. The reader is referred to references [*15*, *7*, *8*, *9*]. Capital Latin indices run from 0 to 3 and are raised and lowered by means of the Euclidean metric. Lower case Latin indices run from 0 to 3 and label frame vectors and frame components, Greek indices refer to arbitrary space-time coordinates. Every space-time tensor or spinor field is given a unique physical dimension, general coordinates being dimensionless. For example, for the basic spinor field Φ the physical dimension is $[\Phi] = L^{-3/2}$. This also applies to the individual vectors of the Galilei frame $\{ e_a \}$ and dual coframe $\{ \theta^a \}$, e. g. $[e_0] = T^{-1}$, $[\theta^A] = L$. Frame components of tensors may thus have different dimensions.

2. GALILEI SPIN GROUP AND SPINOR CONNECTION

The universal covering group of the homogeneous Galilei group $G_h = SO(3) \underline{\times} \mathbb{R}^3$ is $\tilde{G}_h = SU(2) \underline{\times} \mathbb{R}^3$ where $\mathbb{R}^3$ is the abelian group of boosts and $\underline{\times}$ denotes the semidirect product with respect to the standard action of SO (3) on $\mathbb{R}^3$ and of SU (2) via its projection onto SO (3). Explicitly we describe SU (2) by unimodular quaternions,

$$SU(2) = \{ a = (a_0, a) \in \mathbb{H} / | a |^2 = a_0{}^2 + a^2 = 1 \},$$

the group SO (3) by orthogonal matrices of determinant 1 and the homomorphism by $\lambda : SU(2) \to SO(3) : a \to R$ where

$$R^A{}_B = (a_0{}^2 - a^2)\delta^A_B + 2a^A a_B - 2a_0 \varepsilon^A{}_{BC} a^C .$$

The group G_h will be parameterized by the matrix [1]

$$\Lambda = (\Lambda^a{}_b) = \begin{pmatrix} 1 & 0 \\ w^{-1}b & R \end{pmatrix}, \qquad b \in \mathbb{R}^3, \qquad R \in SO(3) \tag{2.1}$$

which corresponds to its left action on the space-time coframes $\{ \theta^a \}$. A convenient representation of $\tilde{G}_h$ in $\mathbb{C}^4$ is given by

$$\sigma : \tilde{G}_h \to GL(4, \mathbb{C}) : (a, b) \to \Sigma$$

where

$$\Sigma := \begin{pmatrix} \alpha & 0 \\ -\frac{1}{2}\beta\alpha & \alpha \end{pmatrix}, \qquad \alpha := a_0 1 - ia \cdot \sigma, \qquad \beta = w^{-1} b \cdot \sigma \tag{2.2}$$

and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. From the well known relation

$$\alpha^+ \sigma^A \alpha = R^A{}_B \sigma^B$$

---

(1) We introduce here a « universal » constant w with $[w] = LT^{-1}$ in order to have consistent physical dimensions later for all geometric objects on space-time.

(+ denoting the Hermitian conjugate) follows then that

$$\Sigma^{-1}\gamma^a\Sigma = \Lambda^a{}_b\gamma^b \tag{2.3}$$

where

$$\gamma^0 := w^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \gamma^A := \begin{pmatrix} \sigma^A & 0 \\ 0 & -\sigma^A \end{pmatrix}. \tag{2.4}$$

The matrices $\{ \gamma^a \}$ satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\gamma^{ab}1 \tag{2.5}$$

($(\gamma^{ab}) = \mathrm{diag}\,(0, 1, 1, 1)$) and generate a nonuniversal Galilei Clifford algebra [*3, 4*] a basis of which is $\{ 1, \gamma^a, I, J^A, K^A \}$ where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J^A = \begin{pmatrix} \sigma^A & 0 \\ 0 & \sigma^A \end{pmatrix}, \quad K^A = \begin{pmatrix} 0 & 0 \\ w^{-1}\sigma^A & 0 \end{pmatrix}. \tag{2.6}$$

A basis for the whole matrix algebra $\mathbb{C}^{4\times 4}$ will also include

$$\overset{*}{\gamma}{}^0 := w\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \overset{*}{\gamma}{}^A := w\begin{pmatrix} 0 & \sigma^A \\ 0 & 0 \end{pmatrix}. \tag{2.7}$$

The group $\tilde{G}_h$ operates on these matrices as follows:

$$\begin{aligned}
\Sigma^{-1}I\Sigma &= I + b_B R^B{}_C K^C, \\
\Sigma^{-1}J^A\Sigma &= R^A{}_B J^B - i\varepsilon^A{}_{BC} b^B R^C{}_D K^D, \\
\Sigma^{-1}K^A\Sigma &= R^A{}_B K^B, \\
\Sigma^{-1}\overset{*}{\gamma}{}^0\Sigma &= \overset{*}{\gamma}{}^0 - \frac{1}{2} b_B R^B{}_C \gamma^C - \frac{1}{4} b^2 \gamma^0, \\
\Sigma^{-1}\overset{*}{\gamma}{}^A\Sigma &= R^A{}_B\overset{*}{\gamma}{}^B - \frac{1}{2} b^A I - \frac{i}{2}\varepsilon^A{}_{BC} R^C{}_D J^D - \frac{1}{2} b^A b_B R^B{}_C K^C + \frac{1}{4} b^2 R^A{}_B K^B.
\end{aligned}$$

Let now $\mathscr{G}_h(M, \pi, G_h)$ be the bundle of Galilei frames over the space-time manifold M that is given a Galilei structure (γ, ψ) [*15*]. Then there exists a Galilei spin frame bundle $\tilde{\mathscr{G}}_h(M, \tilde{\pi}, \tilde{G}_h)$ and a two-to-one principal bundle homomorphism $\tilde{\lambda} : \tilde{\mathscr{G}}_h \to \mathscr{G}_h$ covering the identity of M under the same topological conditions ([2]) as are needed for the existence of a spin frame bundle over a Lorentz manifold (parallelizability) [*11*]. If such a space-time is given a Galilei connection Γ there exists a unique connection $\tilde{\Gamma}$ on $\tilde{\mathscr{G}}_h$ that projects onto Γ [*12*]. A contravariant Dirac spinor field Φ (which will be written as a complex (4×1)-matrix) is then a section of a vector bundle E associated with $\tilde{\mathscr{G}}_h$ via the representation σ of $\tilde{G}_h$ in $\mathbb{C}^4$. In a local chart its covariant derivative is given by

$$\nabla_\alpha\Phi := \partial_\alpha\Phi + \Gamma_\alpha\Phi. \tag{2.9}$$

([2]) Since both SL(2, C) and $\tilde{G}_h$ have topology $S^3 \times \mathbb{R}^3$ and hence have the same homotopy group sequences.

The complex (4 × 4)-matrix Γ_α can be written in the form

$$\Gamma_\alpha = \frac{1}{4} \underset{\alpha}{\omega}{}^{A}{}_{B}\gamma_A\gamma^B + \frac{1}{2} \underset{\alpha}{\omega}{}^{A}{}_{0}\gamma_A\gamma^0 \tag{2.10}$$

and the $\underset{\alpha}{\omega}{}^{a}{}_{b}$ are the components of the pullback of the connection form on $\mathcal{G}_h$ to M via the section $s : x \mapsto e_a^\lambda(x)$

$$s^*(\omega^a{}_b) = \underset{\alpha}{\omega}{}^{a}{}_{b} dx^\alpha = \theta^a_\lambda \nabla_\alpha e^\lambda_b dx^\alpha$$

in terms of the Galilei frame field. (Compare, for example, [*18*, *5*] for the same construction in the Lorentz case.).

The covariant derivative (2.9) extends to the algebra of tensor-spinor fields over M. For a covariant spinor field Ψ we have

$$\nabla_\alpha \Psi = \partial_\alpha \Psi - \Psi \Gamma_\alpha$$

so that ∇_α agrees on complex scalars with the partial derivative and that by virtue of (2.10)

$$\nabla_\alpha \gamma^\beta := \partial_\alpha \gamma^\beta + \Gamma^\beta_{\alpha\gamma}\gamma^\lambda + \Gamma_\alpha \gamma^\beta - \gamma^\beta \Gamma_\alpha = 0 \tag{2.11}$$

where $\gamma^\alpha := e^\alpha_a \gamma^a$.

The tensor-spinors γ^α satisfy

$$\gamma^{(\alpha}\gamma^{\beta)} = \gamma^{\alpha\beta} 1 , \tag{2.12}$$

where we have introduced the tensor field $\gamma^{\alpha\beta} = e^\alpha_A \delta^{AB} e^\beta_B$ which together with $\psi_\alpha = \theta^0_\alpha$ characterizes the Galilei structure. But, even if the torsion vanishes, the spinor connection components Γ_α can not be expressed in terms of the γ^α's alone as in the Lorentz case just as symmetric Galilei connection coefficients are not determined by $\gamma^{\alpha\beta}$ and ψ_α.

Under local spin frame changes we have

$$\Phi \to \hat{\Phi} = \Sigma^{-1}\Phi, \qquad e \to \hat{e} = e\Lambda$$

where $\Sigma = \sigma(g)$, $\Lambda = \lambda(g)$ for $g \in \tilde{G}_h$. It follows from (2.2) and (2.10) that then

$$\nabla_\alpha \Phi \to \Sigma^{-1} \nabla_\alpha \Phi . \tag{2.13}$$

There is no Galileian analogue of a charge conjugation operation on 4-spinors. But we can, following Lichnerowicz [*18*] (see also Kosmann [*13*]), define a *Dirac conjugation* as an involutive antilinear antihomomorphism of the algebra of spinor-tensors that is homogeneous of degree 0, commutes with contractions, maps (1, 0)-spinors into (0, 1)-spinors, is the complex conjugate on tensors, and maps the fundamental spinor-tensor γ^α into $-\gamma^\alpha$.

Then this map is given on (1, 0)-spinors Φ in the form

$$\text{Dirac conjugation}: \Phi \to \tilde{\Phi} := \Phi^+ D \tag{2.14}$$

where the matrix D (in our representation) is uniquely determined up to a real factor which we fix equal to w,

$$D = iw\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.15}$$

For (1, 1)-spinors (in matrix notation) we have

$$\tilde{C} = D^{-1}C^{+}D. \tag{2.16}$$

It follows that under Dirac conjugation the matrices 1 and J^A are invariant while γ^α, I, K^A and $\overset{*}{\gamma}{}^\alpha$ change the sign.

Therefore

$$\tilde{\Gamma}_\alpha = -\Gamma_\alpha \tag{2.17}$$

whence

$$(\nabla_\alpha \Phi)^\sim = \nabla_\alpha \tilde{\Phi}. \tag{2.18}$$

Finally

$$(\Sigma^{-1}\Phi)^\sim = \tilde{\Phi}\Sigma \qquad \text{for any} \qquad \Sigma \in \tilde{G}_h.$$

3. LÉVY-LEBLOND'S GALILEI INVARIANT SPIN 1/2 WAVE EQUATION

Lévy-Leblond [17] applied Dirac's original construction to the Schrödinger equation and produced a first order set of equations on two pairs of two-component spinors which can be written (in a Cartesian coordinate system with $x^0 = t$) (3) in the form

$$E_L[\Phi] := \frac{\hbar}{i}\gamma^a \partial_a \Phi + 2mM_0\Phi = 0. \tag{3.1}$$

This equation is invariant under the inhomogeneous Galilei group acting on space-time,

$$x^a \to \hat{x}^a = \Lambda^a{}_r x^r + c^a$$

if the spinor field Φ transforms at the same time up to a unimodular factor under the representation σ of $\tilde{G}_h$, namely $\Phi \to \hat{\Phi}$ with

$$\hat{\Phi}(\Lambda x + c) = e^{if(x)}\Sigma\Phi(x), \qquad f(x) = m/(2\hbar)(b \times Rx + \frac{1}{2}b^2 t + \kappa). \tag{3.2}$$

Explicitly, this means that

$$\hat{E}_L[\hat{\Phi}](\hat{x}) = e^{if(x)}\Sigma E_L[\Phi](x) \tag{3.3}$$

as follows directly from (3.2) with the help of (2.3) and (2.8).

(3) Cartesian coordinates are given the obvious physical dimensions.

Lévy-Leblond's operator E_L is not the most general one that satisfies (3.3) when Φ transforms as in (3.2) for some real scalar f. Suppose we look for the most general linear first order operator with constant coefficients that has this property, i. e. we put

$$E[\Phi] = C^a\partial_a\Phi + M\Phi$$

and require that (3.3) hold for any spin Galilei transformation for some function f which may depend on x^a as well as the transformation. A longer calculation gives then that

$$C^a = \frac{\hbar}{i}(1 - k\gamma^0)\gamma^a, \qquad M = 2m\overset{*}{\gamma}{}^0 + il1 + a\gamma^0 + km\mathrm{I},$$

$$f = \frac{m}{\hbar}\left(b \cdot \mathrm{R}x + \frac{1}{2}b^2t + \kappa\right)$$

where now $\hbar$, k, l, m are arbitrary complex constants (except that $\hbar/m$ must be real) and κ can be interpreted as the 11th parameter of the Bargmann group. Then

$$E^2[\Phi] = 2mE_S[\Phi] + 2ilE[\Phi] + (2am + k^2m^2 + l^2)\Phi \tag{3.4}$$

where

$$E_S[\Phi] := -\frac{\hbar^2}{2m}\Delta\Phi + \frac{\hbar}{i}\partial_t\Phi = 0 \tag{3.5}$$

is the free Schrödinger equation. Depending on whether one wants only solutions of $E[\Phi] = 0$ to satisfy the free Schrödinger equation or E^2 itself to be proportional to E_S several restrictions on the free constant parameters can be obtained. One can also make a global basis change in the spinor space to transform k to zero and make $\hbar$ and m real.

We will only consider the case $a = k = l = 0$, $\hbar$ and m real with their usual physical interpretations as Planck's constant and particle rest mass, respectively.

It is easy to verify that a Lagrangian density that leads to the equation $E_L[\Phi] = 0$ is $\tilde{\mathscr{L}} = v\mathscr{L}$ where $v d^4x = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$ is the Galileian volume element and

$$\mathscr{L} = \frac{\hbar}{2}(\tilde{\Phi}\gamma^a\partial_a\Phi - \partial_a\tilde{\Phi}\gamma^a\Phi) + 2im\tilde{\Phi}\overset{*}{\gamma}{}^0\Phi .$$

4. INVARIANT LAGRANGIAN ON CURVED SPACE

We now wish to couple this spinor field minimally to gravitation by constructing a Lagrangian on a curved Newtonian manifold that is invariant under the Newtonian automorphism group $\mathscr{A}$. i. e. under *i*) diffeomorphisms of space-time,

ii) gauge transformations and transformations of the observer field:

$$
\begin{aligned}
u^\alpha &\to u^\alpha + \gamma^{\alpha\lambda} W_\lambda , \\
A_\alpha &\to A_\alpha + \partial_\alpha f + W_\alpha - \psi_\alpha (W_\lambda u^\lambda + 1/2 W_\lambda W^\lambda) , \\
\Phi &\to e^{i(m/\hbar)f}\Phi
\end{aligned}
\tag{4.1}
$$

for an arbitrary scalar field f and 1-form W,

iii) local frame changes:

$$e \to \hat{e} = e\Lambda, \qquad \Phi \to \hat{\Phi} = \Sigma^{-1}\Phi$$

where $\Lambda = \lambda(g)$, $\Sigma = \sigma(g)$ for any $g \in \tilde{G}_h$.

Here the timelike unit vector field u^α and the 1-form A_α determine the Newtonian connection $\Gamma^\alpha_{\beta\gamma}$ together with $(\gamma^{\alpha\beta}, \psi_\sigma)$ such that

$$\nabla_\alpha \gamma^{\beta\gamma} = \nabla_\alpha \psi_\beta = 0 \tag{4.2}$$

and

$$\overset{u}{\gamma}_{\lambda[\alpha} \nabla_{\beta]} u^\lambda + \partial_{[\alpha} A_{\beta]} = 0 . \tag{4.3}$$

where $\overset{u}{\gamma}_{\alpha\lambda}$ is defined by $\overset{u}{\gamma}_{\alpha\lambda} u^\lambda = 0$ and $\overset{u}{\gamma}_{\alpha\lambda}\gamma^{\lambda\beta} = \delta^\beta_\alpha - u^\beta \psi_\alpha$. (Cf. [*7*] or [*9*].)

The field equations should still be linear homogeneous first order equations in Φ so that we will only consider Lagrangians quadratic in Φ and linear in $\partial_\alpha \Phi$, i. e., without loss of generality, of the general form

$$\mathscr{L} = \frac{\hbar}{2}(\tilde{\Phi} C^\alpha \nabla_\alpha \Phi - \nabla_\alpha \tilde{\Phi} N^\alpha \Phi) + i\tilde{\Phi} M \Phi \tag{4.4}$$

We consider the Galilei structure defined by the frame field $\{ e^\alpha_a \}$,

$$\gamma^{\alpha\beta} = e^\alpha_A \delta^{AB} e^\alpha_B , \qquad \psi_\alpha = \theta^0_\alpha , \tag{4.5}$$

the Newtonian connection by (4.2) and (4.3) and regard the (eight real) components of Φ, e^α_a and the e-frame components u^A and A_b of the observer field u and the vector potential A as independent dynamical variables on which the coefficients $C^a = \theta^a_\lambda C^\lambda$, N^a and M may depend.

The Lagrangian $\mathscr{L}$ must be real-valued which implies that

$$N^a = -\tilde{C}^a \qquad \text{and} \qquad M = -\tilde{M} . \tag{4.6}$$

Diffeomorphism invariance requires that neither C^a nor M may depend explicitly on the space-time coordinates. Invariance under frame changes requires in view of (2.13) that

$$\Lambda^a{}_b C^b(\hat{u}, \hat{A}) = \Sigma^{-1} C^a(u, A)\Sigma \tag{4.7}$$

and

$$M(\hat{u}, \hat{A}) = \Sigma^{-1} M(u, A)\Sigma . \tag{4.8}$$

Assuming that C^a and M depend differentiably on their arguments and using the frame component transformation laws

$$\hat{u}_A = (u_B - b_B)R^B_A, \qquad \hat{A}_0 = A_0 + b^K A_K, \qquad \hat{A}_B = A_K R^K{}_B$$

we differentiate (4.6) and (4.7) with respect to the group parameters a^A and b_B at the identity of $\tilde{G}_h$. This gives

$$\mathscr{L}_A[M] := \varepsilon_A{}^K{}_L\left(u_K \frac{\partial M}{\partial u_L} + A_K \frac{\partial M}{\partial A_L}\right) = -\frac{i}{2}(J_A M - MJ_A), \quad (4.9)$$

$$\mathscr{K}_A[M] := -\frac{\partial M}{\partial u_A} + A_A \frac{\partial M}{\partial A_0} = \frac{1}{2}(K_A M - MK_A), \quad (4.10)$$

$$\mathscr{L}_A[C^b] + \varepsilon_A{}^B{}_K C^K \delta^b_B = -\frac{i}{2}(J_A C^b - C^b J_A), \quad (4.11)$$

$$\mathscr{K}_A[C^b] + \delta^b_A C^0 = \frac{1}{2}(K_A C^b - C^b K_A). \quad (4.12)$$

Changes of the observer field and gauge transformations amount to

$$u_A \to \check{u}_A = u_A + W_A,$$
$$A_A \to \check{A}_A = A_A + W_A + e^\alpha_A \partial_\alpha f, \quad A_0 \to \check{A}_0 = A_0 - W_K u^K - \frac{1}{2}W^2 + e^\alpha_0 \partial_\alpha f$$

and leave $\mathscr{L}$ invariant provided $C^a(\check{u}, \check{A}) = C^a(u, A)$ and

$$M(\check{u}, \check{A}) = M(u, A) - \frac{1}{2}m(C^a + N^a)e^\alpha_a \partial_\alpha f \quad (4.13)$$

for arbitrary W_A and f.

Clearly C^a must not depend on u and A. Therefore (4.11) and (4.12) yield $C^a = c(1 - k\gamma^0)\gamma^a$ whence, by (4.6), $N^a = \bar{c}(1 - \bar{k}\gamma^0)\gamma^a$. Differentiating (4.13) with respect to $f_a = e^\alpha_a \partial_\alpha f$ and u_A, respectively, yields

$$\frac{\partial M}{\partial A_b} = -\frac{1}{2}m(C^b + N^b)$$

and

$$\frac{\partial M}{\partial u_A} = -\frac{1}{2}m[C^A + N^A - u^A(C^0 + N^0)]$$

which can be integrated. The result is

$$M = -\frac{1}{2}m(\overset{u}{\gamma}_\alpha + A_\alpha)(C^\alpha + N^\alpha) + M_0 \quad (4.14)$$

where M_0 is a constant matrix and we have introduced [*15*, *7*]

$$\overset{u}{\gamma}_\alpha := \overset{u}{\gamma}_{\alpha\lambda} e^\lambda_0 - \frac{1}{2}\overset{u}{\gamma}_{\lambda\mu} e^\lambda_0 e^\mu_0 \psi_\alpha. \quad (4.15)$$

The matrix M of equation (4.14) now satisfies indeed (4.13), and (4.6) yields only

$$M_0 = - \tilde{M}_0 . \tag{4.16}$$

Equation (4.7) is now also identically satisfied while (4.8) and (4.16) give

$$M_0 = 2m\overset{*}{\gamma}{}^0 + il1 + a\gamma^0 + kmI$$

where we have dropped an overall real factor. The parameters a, l, k and m may be chosen real as it turns out that their imaginary parts would only contribute to divergence terms.

Summarizing we have the

THEOREM 1. — A Lagrangian for a four-component spinor field Φ on a Newtonian manifold that is invariant under the Newtonian automorphism group and leads to first order linear homogeneous field equations for Φ can be chosen to be of the form

$$\mathscr{L} = \frac{\hbar}{2}(\tilde{\Phi}C^\alpha\nabla_\alpha\Phi - \nabla_\alpha\tilde{\Phi}C^\alpha\Phi) + i\Phi M\Phi \tag{4.17}$$

with $C^\alpha = (1 - k\gamma^0)\gamma^\alpha$ and

$$M = 2m\overset{*}{\gamma}{}^0 + il1 + a\gamma^0 + kmI - m(\overset{u}{\gamma}_\alpha + A_\alpha)C^\alpha \tag{4.18}$$

where $\hbar$, m, a, k, l are real constants.

There are thus the same number of free parameters as for the Galilei invariant wave equation in section 3. We will from now on concentrate on the physically interesting case and set the parameters a, k, l equal to zero.

5. BARGMANN SPINORS

Let us now return to the definition of a theory of spinors on a Newtonian manifold in order to elucidate the geometric nature of the spinors appearing in equation (4.17). In addition to the fundamental Galileian spinor tensor γ^a (2.4), an additional spinor, $\overset{*}{\gamma}{}^0$ is needed in (4.18) to formulate consistently the Levy-Leblond equation. The purpose of this section is to geometrically motivate a new spinor theory that incorporates all previously introduced objects. We start by taking a non relativistic limit ($c^{-2} \to 0$) of a Lorentzian spinor theory over a 4-dimensional general relativistic space time.

Let (M, g) be a Lorentzian manifold with signature $(- + + +)$. Suppose that there exists a spin structure on M, given by a (global) section of the bundle of Lorentz frames.

The Gram matrix of a frame $\{ e_a \}$ $(a=0, 1, 2, 3)$ is given by

$$g_{ab} = - c^2 \delta_a^0 \delta_b^0 + \delta_a^A \delta_b^B \delta_{AB} \tag{5.1}$$

and accordingly

$$g^{ab} = - c^{-2} \delta_0^a \delta_0^b + \delta_A^a \delta_B^b \delta^{AB} . \tag{5.2}$$

There exists a smooth assignment $M \ni x \mapsto \Gamma_x$ of $L(E_x)$-valued covectors of T_xM such that, if $\Gamma^a := \Gamma(\theta^a)$, then

$$\Gamma^{(a}\Gamma^{b)} = g^{ab} . \tag{5.3}$$

In order to take a non relativistic limit, we leave the frames unchanged and expand the spinor-tensors Γ^a and $\Gamma_a := g_{ab}\Gamma^b$ according to

$$\Gamma^a = \gamma^a + c^{-2}\bar{\gamma}^a, \qquad \Gamma_a = c^2 \overset{+}{\gamma}_a + \gamma_a \tag{5.4}$$

where the c-dependence in (5.4) reflects that of (5.1) and (5.2). From (5.1-4) we obtain

$$\begin{aligned} &\overset{+}{\gamma}_a = \bar{\gamma}^A = 0, \qquad \gamma_A = \gamma^A \qquad (A = 1, 2, 3), \\ &\gamma^0 = - \overset{+}{\gamma}_0 =: \gamma, \qquad \gamma_0 = - \bar{\gamma}^0 =: \overset{*}{\gamma} \end{aligned} \tag{5.5}$$

and furthermore

$$\begin{aligned} &\gamma^{(A}\gamma^{B)} = \delta^{AB}, \qquad \gamma\gamma^A + \gamma^A\gamma = \gamma^A\overset{*}{\gamma} + \overset{*}{\gamma}\gamma^A = 0 , \\ &\gamma\overset{*}{\gamma} + \overset{*}{\gamma}\gamma = 1, \qquad \gamma^2 = \overset{*}{\gamma}{}^2 = 0 . \end{aligned} \tag{5.6}$$

Introducing the Galilei structure $\gamma^{ab} = \lim_{c^{-2}\to 0} (g^{ab})$ and chosing ψ_a as a square root of $\psi_a\psi_b = \lim_{c^{-2}\to 0} (- c^{-2} g_{ab})$ we define the Galilei spinors in view of (5.4) as

$$\gamma^a = \lim_{c^{-2}\to 0} \Gamma^a$$

Now there is one additional spinor left, namely $\overset{*}{\gamma}$ (see (5.5)) and the results (5.6) are summarized as follows

$$\gamma^{(a}\gamma^{b)} = \gamma^{ab}, \qquad \gamma^a\overset{*}{\gamma} + \overset{*}{\gamma}\gamma^a = \delta_0^a, \qquad \overset{*}{\gamma}{}^2 = 0 \tag{5.7}$$

or, putting $\gamma^\alpha := \gamma^a e_a^\alpha$,

$$\gamma^{(\alpha}\gamma^{\beta)} = \gamma^{\alpha\beta}, \qquad \gamma^\alpha\overset{*}{\gamma} + \overset{*}{\gamma}\gamma^\alpha = e_0^\alpha, \qquad \overset{*}{\gamma}{}^2 = 0 . \tag{5.8}$$

So we really end up with five matrices $\gamma^a, \overset{*}{\gamma}$ that satisfy the anticommutation relations (5.7). A purely Galileian spinor theory (cf. [*2*, *3*]) would only involve the first of equations (5.7) or (5.8). We will see that we have in fact introduced a Bargmann theory ([4]) of spinors associated with a gauge

([4]) After Bargmann's [*1*] original discovery of the extended Galilei group.

representative of our Newtonian structure (cf. section 4 and [9] for more details).

If we choose for the Γ^a the representation

$$\Gamma^0 = w^{-1}\begin{pmatrix} 0 & -c^{-2}w^2 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma^A = \begin{pmatrix} \sigma^A & 0 \\ 0 & -\sigma^A \end{pmatrix}$$

we have by (5.4) in the limit the expressions (2.4) and

$$\dot{\gamma} = w\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \dot{\gamma}^0 \tag{5.9}$$

Let us now prove that the additional spinor $\overset{*}{\gamma}$ is uniquely determined by the Galilei structure $(\psi_\alpha, \gamma^{\alpha\beta})$, γ^α and a timelike unit vector. Put for the moment $\overset{*}{\gamma} =: \overset{*}{\gamma}{}^0$ and look for all solutions $\overset{u}{\gamma}$ of (5.8) where e_0^α is to be replaced by a new unit vector u^α. The solution is unique and reads

$$\overset{u}{\gamma} = \overset{*}{\gamma}{}^0 - 1/2\overset{u}{\gamma}_\alpha \gamma^\alpha. \tag{5.10}$$

Note that the transformation law (5.10) reflects the change of observer (unit vector) in a specific Bargmann-like manner [7].

We stress that the five Bargmann spinors generate a *universal Clifford algebra* while Galilei spinors do not. See [4]. We will thus write the fundamental Clifford anticommutation relations associated with a given Bargmann structure $(\gamma^{\alpha\beta}, \psi_\alpha, u^\alpha, A_\alpha)$ over a Newtonian space time as

$$\gamma^{(\alpha}\gamma^{\beta)} = \gamma^{\alpha\beta}, \tag{5.11}$$

$$\gamma^\alpha \overset{u}{\gamma} + \overset{u}{\gamma}\gamma^\alpha = u^\alpha, \tag{5.12}$$

$$\overset{u}{\gamma}{}^2 = 0, \tag{5.13}$$

Let us prove that these γ-matrices serve to represent the universal covering $\tilde{B}_n$ of the homogeneous Bargmann group $B_n (= (SO(3) \underline{\times} \mathbb{R}^3) \times \mathbb{R})$ which can be given as the groups of (6×6)-matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ b & R & 0 & 0 \\ b^2/2 & b^T R & 1 & \lambda \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.14}$$

where $R \in SO(3)$, $b \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$.

We define the following unitary representation of $\tilde{B}_h = \tilde{G}_h \times \mathbb{R}$ on C^4

$$\rho_m(\Sigma, \lambda) := e^{im\lambda/\hbar}\Sigma \tag{5.15}$$

where Σ is given by (2.2) and the constant m has the dimension of a mass.

Then, if we let B_n act on the affine subspace $\mathbb{R}^5$ of $\mathbb{R}^6$ of those columns $(x^0, x^A, x^4, 1)^T$, the image $\hat{\phi}$ of a 1-form $\phi = \phi_j dx^j = \phi_0 dx^0 + \phi_A dx^A + \phi_4 dx^4$ under the typical element (5.14) is

$$\begin{aligned} \hat{\phi}_A &= \phi_K (R^{-1})^K{}_A - \phi_4 b_A \\ \hat{\phi}_0 &= \phi_0 - \phi_A (R^{-1})^A{}_K b^K + \phi_4 b^2/2 \\ \hat{\phi}_4 &= \phi_4 \end{aligned} \qquad (5.16)$$

If $\gamma^4 := -2\overset{*}{\gamma}{}^0$, we find (using (2.8)) that the representation (5.15) is such that

$$\rho_m \gamma^j \phi_j \rho_m^{-1} = \gamma^j \hat{\phi}_j \qquad (j = 0, \ldots, 4) \qquad (5.17)$$

Note that the Lie algebra of the Bargmann spin group $\tilde{B}_h$ is spanned by

$$E^{AB} = 1/4\gamma^A\gamma^B \quad (A < B), \qquad E^A = 1/2\gamma^A\gamma^0, \qquad E^* = 1\,. \quad (5.18)$$

These considerations already lead to a modification of the definition of non relativistic spinors and of the corresponding notion of covariant derivative. The principal spin bundle to consider is actually an $(\mathbb{R}, +)$-extension of the bundle of Galilei spin frames, namely a principal $\tilde{B}_h$-bundle over space time endowed with a Bargmann connection $(\omega^a_b, \omega^* = dx^4 - A_\alpha dx^\alpha)$, see [*7*, *9*]. The spinor bundle is nothing but the $\mathbb{C}$-bundle associated with the representation (5.15) and the Galilei covariant derivative (2.9, 10) of a section now takes the new form

$$D_\alpha \Phi = \nabla_\alpha \Phi - (im/\hbar) A_\alpha \Phi\,. \qquad (5.19)$$

The section Φ represents a spin 1/2 particle of mass *m*. Note that definition (5.19) is actually invariant under the full automorphism group $\mathscr{A}$ (4.1). This remark justifies the intrinsic Newtonian character of the local definition (5.19). Again, from the global point of view, the homotopy group sequences of $\tilde{B}_h$ are the same as those of $\tilde{G}_h$.

The covariant derivative D is extended to the whole algebra of spinor-tensor fields according to the usual rules: it is a derivation, commutes with contractions and the Dirac (or complex) conjugation, and agrees with ∇ on tensor fields. Then, for any (1, 1)-spinor field K, DK $= \nabla$K, so that, in particular,

$$D_\alpha \gamma^\beta = \nabla_\alpha \gamma^\beta = 0\,. \qquad (5.20)$$

We will also need $D_\alpha \overset{u}{\gamma} = \nabla_\alpha \overset{u}{\gamma}$ which is best obtained by differentiating (5.12) and becomes

$$D_\alpha \overset{u}{\gamma} = (1/2)\nabla_\alpha u^\lambda \overset{u}{\gamma}_{\lambda\mu}\gamma^\mu\,. \qquad (5.21)$$

Observe also that the Dirac conjugate

$$(\overset{u}{\gamma})^\sim = -\overset{u}{\gamma}\,. \qquad (5.22)$$

6. LÉVY-LEBLOND'S EQUATION ON A CURVED NEWTONIAN SPACE TIME

It is now easy to give a covariant formulation of the Lévy-Leblond equation on a Newtonian space time.

The general gauge invariant Lagrangian (4.17) becomes for $a=k=l=0$

$$\mathscr{L} = (\hbar/2)(\tilde{\Phi}\gamma^\alpha D_\alpha\Phi - D_\alpha\tilde{\Phi}\gamma^\alpha\Phi) + 2mi\tilde{\phi}\gamma\Phi \tag{6.1}$$

since the term $-mA_\alpha\gamma^\alpha$ in the matrix M of equation (4.18) may be viewed as part of the covariant derivative D. (cf. (4.15) and (5.10)). The variation of the action integral with respect to (the eight real components of) Φ then gives

$$E_L[\Phi] := (\hbar/i)\gamma^\alpha D_\alpha\Phi + 2m\overset{u}{\gamma}\tilde{\Phi} = 0, \tag{6.2}$$

and its Dirac conjugate

$$(\hbar/i)D_\alpha\tilde{\Phi}\gamma^\alpha - 2m\tilde{\Phi}\overset{u}{\gamma} = 0. \tag{6.3}$$

Equation (6.2) is the generalized Lévy-Leblond equation on a Newtonian manifold. Despite its local formulation this field equation is invariant under the general gauge group $\mathscr{A}$ of (4.1) since the Lagrangian $\mathscr{L}$ is. This can, of course, be verified directly.

Let us show that « squaring » the Lévy-Leblond equation (6.2) results in the *Schrödinger equation* (5) on a curved Newtonian space time [*9*]. We have

$$E_LE_L[\Phi] = -\hbar^2\gamma^\alpha D_\alpha\gamma^\beta D_\beta\Phi - \hbar^2\gamma^\alpha\gamma^\beta D_\alpha D_\beta\Phi + 2m(\hbar/i)(\gamma^\alpha D_\alpha\overset{u}{\gamma}\Phi + (\gamma^\alpha\overset{u}{\gamma} + \overset{u}{\gamma}\gamma^\alpha)D_\alpha\Phi) + 4m^2\overset{u}{\gamma}{}^2\Phi.$$

Now using

$$D_{[\alpha}D_{\beta]}\Phi = \nabla_{[\alpha}\nabla_{\beta]}\Phi - (im/\hbar)\nabla_{[\alpha}A_{\beta]}\Phi,$$

(4.3), (5.11-13) and (5.20, 21), we find

$$E_L \circ E_L[\Phi] = -\hbar^2 D^\alpha D_\alpha\Phi - \hbar^2\gamma^\alpha\gamma^\beta\nabla_{[\alpha}\nabla_{\beta]}\Phi - i\hbar m\nabla_\alpha u^\alpha\Phi - 2i\hbar m u^\alpha D_\alpha\Phi.$$

It follows from (2.10) that $\nabla_{[\alpha}\nabla_{\beta]}\Phi = \frac{1}{2}\mathscr{R}_{\alpha\beta}\Phi$ with

$$\mathscr{R}_{ab} = \frac{1}{2}R^K{}_{0ab}\gamma_K\gamma^0 + \frac{1}{4}R^K{}_{Lab}\gamma_K\gamma^L. \tag{6.4}$$

Using the anticommutation relations of the γ^a's as well as the symmetries

(5) Or rather a gravitational Schrödinger-Pauli equation.

of the curvature tensor of the Newtonian connection [15] one derives from this that

$$\gamma^{\alpha}\gamma^{\beta}\nabla_{[\alpha}\nabla_{\beta]}\Phi = -(\mathrm{R}/4)\Phi \tag{6.5}$$

where $\mathrm{R} := \mathrm{R}_{\alpha\beta}\gamma^{\alpha\beta}$ is the scalar curvature. We thus have finally

$$\begin{aligned}\mathrm{E_L} \circ \mathrm{E_L}[\Phi] &= \hbar^2/(2m)\mathrm{D}^{\alpha}\mathrm{D}_{\alpha}\Phi + i\hbar u^{\alpha}\mathrm{D}_{\alpha}\Phi. \\ &+ i\hbar 2\nabla_{\alpha}u^{\alpha}\Phi - \hbar^2/(2m)\mathrm{R}\Phi = 0\end{aligned} \tag{6.7}$$

which is identical in form to the *Schrödinger* equation [14, 9], modulo an additional term involving the scalar curvature R. If the Newtonian field equations, $\mathrm{R}_{\alpha\beta} = 4\pi\rho\mathrm{G}\psi_{\alpha}\psi_{\beta}$, are to be satisfied, that extra term vanishes with R.

7. MATTER CURRENT AND STRESS-ENERGY TENSOR: GENERAL CONSIDERATIONS

In classical field theories on general relativistic space-times the most satisfactory stress-energy tensor (which has to be symmetric to serve as a source of Einstein's equations) is obtained as a variational derivative of the matter Lagrangian with respect to the space-time metric. It is often also possible and convenient, in particular for spinor fields, to vary instead with respect to a tetrad field $\{ e^{\lambda}_{a} \}$ that defines the metric. The stress-energy tensor $\mathrm{T}^{\alpha\beta}$ of, for example, a Dirac field ψ is then not by definition symmetric, but instead its symmetry depends on the fact that the Lagrangian density considered as a functional of ψ and e^{λ}_{a} does not change when the frame field and ψ are subject to local Lorentz transformations.

We find here that this situation still largely prevails in the Newtonian theory, except that some arbitrariness in the definition of energy density and energy flow is introduced through constraints imposed by forcing the connection to be symmetric.

We first adapt proposition 2 of [9] to the present case.

PROPOSITION 1. — If the functional

$$\delta \mathrm{S}_m = \int_{\mathrm{M}} d^4x v(\mathrm{P}^{a}{}_{\lambda}\delta e^{\lambda}_{a} + \mathrm{J}^{\alpha}\delta\mathrm{A}_{\alpha} + \mathrm{K}_{\mathrm{A}}\delta u^{\mathrm{A}}) \tag{7.1}$$

is invariant under diffeomorphisms, local observer field (u, A) transformations and local Galilei frame transformations then

i) $$\nabla_{\alpha}\mathrm{J}^{\alpha} = 0, \tag{7.2}$$

ii) $$\mathrm{K_A} = \rho u_{\mathrm{A}} - \mathrm{J_A}, \tag{7.3}$$

iii) $$\mathrm{P}^{0}{}_{\mathrm{A}} = \rho u_{\mathrm{A}} - \mathrm{J_A}, \qquad \mathrm{P}^{r}{}_{[\mathrm{A}}\delta_{\mathrm{B}]r} = u_{[\mathrm{A}}\mathrm{J}_{\mathrm{B}]}, \tag{7.4}$$

or, equivalently,

$$T^{\alpha}{}_{\beta} := P^{\alpha}{}_{\beta} + J^{\alpha}(\overset{u}{\gamma}_{\beta\lambda}v^{\lambda} - \frac{1}{2}\overset{u}{\gamma}_{\lambda\mu}v^{\lambda}v^{\mu}\psi_{\beta}) \tag{7.5}$$

satisfies

$$T^{[\alpha}{}_{\lambda}\gamma^{\beta]\lambda} = 0\,, \tag{7.6}$$

iv)

$$\nabla_{\lambda}T^{\lambda}{}_{\alpha} = \rho\overset{v}{\gamma}_{\alpha\lambda}v^{\mu}\nabla_{\mu}v^{\lambda}\,. \tag{7.7}$$

Here we have introduced again the frame components $P^{a}{}_{b} := P^{a}{}_{\lambda}e^{\lambda}{}_{b}$, etc., the matter density $\rho := J^{\lambda}\psi_{\lambda}$, and the 4-velocity of matter $v^{\alpha} := (1/\rho)J^{\alpha}$.

Proof. — *i*) Let first $A_{\alpha} \to A_{\alpha} + \partial_{\alpha}f$ and keep u^{α} and e^{α}_{a} constant. Then $\delta A_{\alpha} = \partial_{\alpha}\chi$, $\delta e^{\alpha}_{a} = 0$, $\delta u^{\alpha} = 0$. Equation (7.2) now follows by Green's theorem.

ii) Under infinitesimal observer boosts we have $\delta u^{A} = W^{A}$,

$$\delta A_{\alpha} = W_{A}(\theta^{A}_{\alpha} - u^{A}\psi_{\alpha}),$$

$\delta e^{\alpha}_{a} = 0$ for arbitrary W^{A}. This leads to (7.3).

iii) For infinitesimal Galilei boosts, $\delta e^{\alpha}_{0} = e^{\alpha}_{K}\beta^{K}$, $\delta u^{A} = -\beta^{A}$ and infinitesimal rotations, $\delta e^{\alpha}_{A} = \varepsilon_{AB}{}^{C}\alpha^{B}e^{\alpha}_{C}$, $\delta u^{A} = -\varepsilon^{A}{}_{BC}u^{B}\alpha^{C}$ for arbitrary α^{A} and β^{A}, respectively, lead to (7.4).

iv) Substituting $\delta e^{\alpha}_{a} = \pounds_{\xi}e^{\alpha}_{a}$, $\delta A_{\alpha} = \pounds_{\xi}A_{\alpha}$ and $K_{A} = \pounds_{\xi}u_{A}$ into (7.1) for an arbitrary (asymptotically vanishing) vector field ξ on space-time (where $\pounds_{\xi}$ denotes the Lie derivative) leads by Green's theorem to

$$0 = \nabla_{\lambda}P^{\lambda}{}_{\alpha} + P^{r}{}_{\lambda}\nabla_{\alpha}e^{\lambda}_{r} + J^{\lambda}\nabla_{\alpha}A_{\lambda} - \nabla_{\lambda}A_{\alpha}J^{\lambda} - A_{\alpha}\nabla_{\lambda}J^{\lambda} + \nabla_{\alpha}u^{A}K_{A}\,.$$

With the help of (7.2)-(7.4) and (4.3) this can be written in the form

$$\nabla_{\lambda}P^{\lambda}{}_{\alpha} = J^{\lambda}\nabla_{\lambda}u^{\mu}\overset{u}{\gamma}_{\mu\alpha}\,. \tag{7.8}$$

Finally, equations (7.4) and (7.8) are easily seen to be equivalent to (7.6) and (7.7), respectively.

Suppose now we make a finite local Galilei frame and observer boost transformation of the variables in the action integral. This means for the variations that

$$\delta e^{\lambda}_{a} \to \delta\hat{e}^{\lambda}_{a} = \delta e^{\lambda}_{r}\Lambda^{r}{}_{a}, \qquad \delta u^{A} \to \delta\hat{u}^{A} = (R^{-1})^{A}{}_{B}\delta u^{B}\,,$$

$$\delta A_{\alpha} \to \delta\hat{A}_{\alpha} = \delta A_{\alpha} - \psi_{\alpha}W_{A}du^{A} - W_{A}\left(\theta^{A}_{\lambda} - u^{A}\psi_{\lambda} - \frac{1}{2}W^{A}\psi_{\lambda}\right)\theta^{r}_{\alpha}\delta e^{\lambda}_{r}\,.$$

But then

$$\delta\hat{S}_{m} = \int_{M} d^{4}xv(\hat{P}^{a}{}_{\lambda}\delta\hat{e}^{\lambda}_{a} + J^{\alpha}\delta\hat{A}_{\alpha} + \hat{K}_{A}\delta\hat{u}^{A})$$

Comparing the coefficients of δe^λ_a, δu^A and δA_α gives

$$J^\alpha = \hat{J}^\alpha, \qquad K_A = \hat{K}_B(R^{-1})^B{}_A - \rho W_A,$$

$$P^a{}_\lambda = \Lambda^a{}_r \hat{P}^r{}_\lambda - W_K\left(\theta^K{}_\lambda - u^K\psi_\lambda - \frac{1}{2}W^K\psi_\lambda\right)J^a$$

from which it follows that $T^\alpha{}_\beta = \hat{T}^\alpha{}_\beta$. □

It would thus seem that J^α and $T^\alpha{}_\beta$ defined by (7.1) are gauge invariant quantities as we also stated in proposition 2 of [*9*]. This is not quite correct however, since we tacitly assumed that $P^a{}_\lambda$, J^α and K_A were uniquely defined by (7.1) which would only be true if the variations δe^λ_a, δA_α and δu^A were completely arbitrary. Since the Newtonian structure is defined in terms of e^λ_a, A_α and u^α these variations can, in fact, be chosen arbitrarily at each space-time point, but forcing the connection to be symmetric imposes the constraint $d\psi = 0$ [*15*]. We must therefore add a Lagrange multiplier term $\Delta\mathcal{L} = Q^{\alpha\beta}\nabla_\alpha\psi_\beta$ (with $Q^{\alpha\beta} = Q^{[\alpha\beta]}$) to the Lagrangian $\mathcal{L}$.

Variation of this term with respect to the gravitational variables $(e^\alpha_a, A_\alpha, u^\alpha)$ gives

$$\delta\int d^4x\upsilon\Delta\mathcal{L} = -\int d^4x\upsilon Q^{\alpha\beta}\nabla_\alpha(\theta^r_\beta\delta e^\lambda{}_r\psi_\lambda) = \int d^4x\upsilon\nabla_\alpha Q^{\alpha\beta}\theta^r{}_\beta\psi_\lambda\delta e^\lambda_r\,.$$

Since there are no restrictions on $Q^{\alpha\beta}$ we conclude that the quantity $P^\alpha{}_\beta$ (and similarly $T^\alpha{}_\beta$) is only determined up to an additive term of the form $\nabla_\lambda Q^{\alpha\lambda}\psi_\beta$. The energy density and heat flow part of the stress-energy tensor $T^\alpha{}_\beta$ derived according to proposition 1 from an invariant matter Lagrangian is therefore not necessarily gauge invariant, but only invariant up to a divergence term, namely,

$$\hat{T}^\alpha{}_\beta = T^\alpha{}_\beta + \nabla_\lambda(\hat{Q}^{\alpha\lambda} - Q^{\alpha\lambda})\psi_\beta\,.$$

Summarizing we have

PROPOSITION 2. — Under the assumptions of proposition 1 the matter current vector J^α and the stress tensor $T^{\alpha\beta} := T^\alpha{}_\lambda\gamma^{\beta\lambda}$ are invariantly defined, i. e. independent of the Galilei frame e^α_a and the gauge fields u^α and A_α. The energy current is only determined up to an additive term of the form $\nabla_\lambda Q^{\lambda\alpha}$ for an arbitrary skew symmetric tensor field $Q^{\alpha\beta}$.

This result is somewhat analogous to the well known fact that the energy in non relativistic particle mechanics is always only definable up to a constant. (See, for example, [*19*].) It also explains our inability in [*9*] to obtain a unique correspondence between the stress-energy tensor of the Klein-Gordon field and the stress-energy tensor of the Schrödinger field on Newtonian space-time.

In specific cases, including the present one of the Dirac field, the arbitrariness of $T^\alpha{}_\beta$ can be reduced considerably be *requiring* that $T^\alpha{}_\beta$ be gauge invariant. How to do this in the general case does not seem obvious at present.

8. MATTER CURRENT AND STRESS-ENERGY TENSOR OF THE NEWTONIAN DIRAC FIELD

It remains to carry out explicitly the variation of

$$S = \int d^4x v \mathscr{L} \tag{8.1}$$

with $\mathscr{L}$ given by (6.1) with respect to e^λ_a, u^λ and A_α. This is straight forward though very laborious. The work can be organized best if done in frame components. Let

$$E^a{}_b := \theta^a_\lambda \delta e^\lambda{}_b, \tag{8.2}$$

$$\mathscr{E}_a{}^b{}_c := e^\lambda_a \nabla_\lambda \delta e^\mu_c \theta^b_\mu, \tag{8.3}$$

$$\underset{a}{\omega}{}^b{}_c := e^\lambda_a \underset{\lambda}{\omega}{}^b{}_c = e^\lambda_a \nabla_\lambda e^\mu_c \theta^b_\mu. \tag{8.4}$$

The frame components $P^a{}_b$ of $P^a{}_\lambda$ are then obtained as coefficients of $E^b{}_a$ in the integral expression for δS. Terms involving $\mathscr{E}_a{}^b{}_c$ will be reduced by means of the formula

$$\int d^4x v \mathscr{E}_a{}^b{}_c \tilde{\Phi} Q \Phi = \int d^4x v [- \nabla_a(\tilde{\Phi} Q \Phi) E^b{}_c + \tilde{\Phi} Q \Phi (\underset{a}{\omega}{}^b{}_r E^r{}_c - \underset{r}{\omega}{}^r{}_a E^b{}_c)] \tag{8.5}$$

which follows from Stokes' theorem for any (1, 1)-spinor field Q.

We use the following easily verified formulae

$$\delta v = - v E^r{}_r, \tag{8.6}$$

$$\delta \gamma^\alpha = e^\alpha_r E^r{}_s \gamma^s, \tag{8.7}$$

$$\delta \overset{u}{\gamma} = 1/2(\gamma_A - u_A \gamma^0) \delta u^A, \tag{8.8}$$

$$\delta(D_\alpha \Phi) = D_\alpha \delta \Phi + (\delta \Gamma_\alpha - im/\hbar \delta A_\alpha) \Phi, \tag{8.9}$$

$$\delta(D_\alpha \tilde{\Phi}) = D_\alpha \delta \tilde{\Phi} - \tilde{\Phi}(\delta \Gamma_\alpha - im/\hbar \delta A_\alpha), \tag{8.10}$$

It then follows quickly that

$$P^a{}_b = (\hbar/2)(\tilde{\Phi} \gamma^a D_b \Phi - D_b \tilde{\Phi} \gamma^a \Phi) + \mathscr{P}^a{}_b, \tag{8.11}$$

$$J^\alpha = - im \tilde{\Phi} \gamma^\alpha \Phi + \mathscr{J}^\alpha \tag{8.12}$$

$$K_A = im \tilde{\Phi}(\gamma_A - u_A \gamma^0) \Phi + \mathscr{K}_A \tag{8.13}$$

where we used that $\mathscr{L} = 0$ when the field equations hold, and where

$$\int d^4x v (\mathscr{P}^a{}_b E^b{}_a + \mathscr{J}^\alpha \delta A_\alpha + \mathscr{K}_A \delta u^A) = (\hbar/2) \int d^4x v \tilde{\Phi}(\gamma^\alpha \delta \Gamma_\alpha + \delta \Gamma_\alpha \gamma^\alpha) \Phi. \tag{8.14}$$

Now, using (2.10), (8.2-4) as well as the definitions (2.6) ,we have

$$\gamma^\alpha\delta\Gamma_\alpha+\delta\Gamma_\alpha\gamma^\alpha = i[1/2\mathscr{E}_{0AB}+\mathscr{E}_{AB0}+(1/2\underset{0}{\omega}{}^K{}_A-\underset{A}{\omega}{}^K{}_0)E_{BK}-1/2\delta\Gamma_{0AB}]\varepsilon^{AB}{}_C K^C + i/2(E_{ABC}+\underset{A}{\omega}{}^K{}_B E_{CK})\varepsilon^{ABC} I \quad (8.15)$$

where $\delta\Gamma_a{}^b{}_c := e^\alpha_a\theta^b_\beta e^\gamma_c\delta\Gamma^\beta_{\alpha\gamma}$. But $\Gamma^\alpha_{\beta\gamma}$ is determined by (4.2) and (4.3) and its variation is thus obtained by solving the variations of these two equations. We obtain for the only needed frame components

$$\delta\Gamma_{O[AB]}=\mathscr{E}_{[AB]O}-\mathscr{E}_{[A}{}^K{}_{B]}u_K+u_K u^K{}_{/[A}E^O{}_{B]}-u_{K/[A}E^K{}_{B]}-\omega_{[A}{}^K{}_O E_{B]K} - \underset{[A}{\omega}{}_{B]K}u^L E^K{}_L+\partial_{[A}\delta u_{B]}+\underset{[A}{\omega}{}_{B]K}\delta u^K-e^\lambda_A e^\mu_B\partial_{[\lambda}\delta A_{\mu]} \quad (8.16)$$

where $u^a{}_{/b} := \theta^a_\alpha\nabla_\beta u^\alpha e^\beta_b$.

The expressions (8.15) and (8.16) must be substituted into (8.14). It is easy to see that

$$\mathscr{I}^\alpha = -(i\hbar/4)\nabla_\lambda(e^\lambda_K e^\alpha_L \varepsilon^{KL}{}_M\tilde{\Phi}K^M\Phi) = -(\hbar/4)\nabla_\lambda(\tilde{\Phi}\Sigma^{\lambda\alpha}\Phi) \quad (8.17)$$

where we have defined

$$\Sigma^{\alpha\beta\gamma} := \gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]} \quad (8.18)$$

and

$$\Sigma^{\alpha\beta} := \psi_\lambda\Sigma^{\lambda\alpha\beta}. \quad (8.19)$$

To obtain $\mathscr{P}^\alpha{}_\beta$ Stokes' theorem in the form (8.5) must be used. This is a long calculation whose details are not particularly interesting. The result can be brought into the form

$$\mathscr{P}^\alpha{}_\beta = (\hbar/4)\nabla_\lambda(\tilde{\Phi}\Sigma^{\lambda\alpha\mu}\Phi)\overset{u}{\gamma}_{\mu\beta}+\mathscr{P}^\alpha\psi_\beta \quad (8.20)$$

where

$$\mathscr{P}^\alpha=(\hbar/4)\tilde{\Phi}\Sigma^{\alpha\lambda\mu}\Phi\nabla_\lambda u^\nu\overset{u}{\gamma}_{\mu\nu}-(\hbar/4)\nabla_\lambda(\tilde{\Phi}\Sigma^{\lambda\alpha\mu}\Phi\overset{u}{\gamma}_{\mu\nu}e^\nu_0). \quad (8.21)$$

The second term of (8.21) is manifestly not invariant under local Galilei frame transformations. But then it is a divergence and therefore arbitrary anyhow. The first term is not a divergence and is needed for $P^\alpha{}_\beta$ to satisfy (7.8).

According to propositions 1 and 2 the obtained $P^\alpha{}_\beta$ has the correct symmetries and divergence, but not necessarily the correct transformation property under frame and observer changes. Indeed, we have under local Galilei frame transformations $e^\lambda_a \to \hat{e}^\lambda_a=e^\lambda_r\Lambda^r{}_a$, $\hat{u}^\alpha=u^\alpha$, $\hat{A}_\alpha=A_\alpha$,

$$\hat{P}^\alpha{}_\beta = P^\alpha{}_\beta-(\hbar/4)\nabla_\lambda(\tilde{\Phi}\Sigma^{\lambda\alpha\mu}\Phi\overset{u}{\gamma}_{\mu\nu}e^\nu_K b^K\psi_\beta) \quad (8.22)$$

(where b^K are the boost parameters in $\Lambda^a{}_b$ according to (2.1)), while under local observer boosts, $\check{u}^\alpha=u^\alpha+e^\alpha_k W^k$, $\check{A}_\alpha=A_\alpha+W_K\left(\theta^K_\alpha-u^K\psi_\alpha-\frac{1}{2}W^K\psi_\alpha\right)$, $\check{e}^\alpha_a = e^\alpha_a$, we have

$$\check{P}^\alpha{}_\beta=P^\alpha{}_\beta+J^\alpha(\check{A}_\beta-A_\beta)-(\hbar/4)\nabla_\lambda[\tilde{\Phi}\Sigma^{\lambda\alpha}\Phi W_K(u^K+1/2W^K)\psi_\beta] \quad (8.23)$$

It is the divergence terms in (8.22) and (8.23) which destroy the gauge invariance of $T^{\alpha}{}_{\beta}$. We will therefore look for a skew-symmetric tensor field $Q^{\alpha\beta}$ such that

$$\overline{T}^{\alpha}{}_{\beta} = T^{\alpha}{}_{\beta} + \nabla_{\lambda} Q^{\lambda\alpha} \psi_{\beta} \tag{8.24}$$

will be gauge invariant.

In order to preserve the general structure of the functional $T^{\alpha}{}_{\beta}(\Phi)$ we will assume that the frame components of $Q^{\alpha\beta}$ have the form

$$Q^{ab} = \tilde{\Phi} X^{ab} \Phi \tag{8.25}$$

where X^{ab} is a (1, 1)-spinor that may depend on u^{A} only. Equations (8.22) and (8.23) then tell us that we must have

$$\Lambda^{a}{}_{r} \Sigma X^{rs}(\hat{u}) \Sigma^{-1} \Lambda^{b}{}_{s} = X^{ab}(u) + (\hbar/4) \Sigma^{abr} \overset{u}{\gamma}_{rK} b^{K} \tag{8.26}$$

and

$$X^{ab}(u) = X^{ab}(u) + (\hbar/4) \Sigma^{ab} W_{K}(u^{K} + 1/2 W^{K}). \tag{8.27}$$

To determine what kind of matrix X^{ab} will have to be one can now proceed in exactly the same fashion as in section 4. If we also use the fact that $Q^{\alpha\beta}$ should be real the result is unique up to two real constants, q and r, namely

$$X^{\alpha\beta} = (\hbar/4)(2S^{\alpha\beta} + \Sigma^{\alpha\beta\lambda} \overset{u}{\gamma}_{\lambda\mu} e^{\mu}_{0} + q\Sigma^{\alpha\beta} + 2ir\gamma^{[\alpha}\gamma^{\beta]}) \tag{8.28}$$

where

$$S^{\alpha\beta} = (1/6)(\gamma^{\alpha}\gamma^{\beta}\overset{u}{\gamma} + \gamma^{\beta}\overset{u}{\gamma}\gamma^{\alpha} + \overset{u}{\gamma}\gamma^{\alpha}\gamma^{\beta} - \gamma^{\beta}\gamma^{\alpha}\overset{u}{\gamma} - \gamma^{\alpha}\overset{u}{\gamma}\gamma^{\beta} - \overset{u}{\gamma}\gamma^{\alpha}\gamma^{\beta}). \tag{8.29}$$

(Observe that $S^{\alpha\beta}$ appears to be another component of a « Bargmann-analogue » of the Galileian spin tensor-spinor $\Sigma^{\alpha\beta\gamma}$.)

Substituting (8.29) and (8.28) into (8.25), (8.24) and also using the field equations we have finally

$$\overline{P}^{\alpha}{}_{\beta} = (\hbar/2)(\tilde{\Phi}\gamma^{\alpha} D_{\beta}\Phi - D_{\beta}\tilde{\Phi}\gamma^{\alpha}\Phi) + (\hbar/4)\nabla_{\lambda}(\tilde{\Phi}\Sigma^{\lambda\alpha\mu}\Phi)\overset{u}{\gamma}_{\mu\beta} + \overline{\mathscr{P}}^{\alpha}\psi_{\beta} \tag{8.30}$$

where

$$\begin{aligned} \overline{\mathscr{P}}^{\alpha} = (\hbar/4) \{ & \tilde{\Phi}\Sigma^{\alpha\lambda\mu}\Phi\nabla_{\lambda} u^{\nu} \overset{u}{\gamma}_{\mu\nu} + \nabla_{\lambda}[\tilde{\Phi}(2S^{\lambda\alpha} + q\Sigma^{\lambda\alpha})\Phi] \\ & + 2ir(\tilde{\Phi}D^{\alpha}\Phi - D^{\alpha}\tilde{\Phi}\Phi) - (4m/\hbar) r u^{\alpha}\tilde{\Phi}\Phi \} \end{aligned} \tag{8.31}$$

and $\overline{T}^{\alpha}{}_{\beta}$ is given by (7.5) in terms of $\overline{P}^{\alpha}{}_{\beta}$ and

$$J^{\alpha} = - im\tilde{\Phi}\gamma^{\alpha}\Phi - (\hbar/4)\nabla_{\lambda}(\tilde{\Phi}\Sigma^{\lambda\alpha}\Phi). \tag{8.32}$$

This stress-energy tensor $\overline{T}^{\alpha}{}_{\beta}$ and matter current J^{α} are thus independent of the choice of the Galilei frame e^{λ}_{a} and the fields u^{α}, A_{α} and depend only on the spinor field Φ and the Newtonian structure $(\gamma^{\alpha\beta}, \psi_{a}, \Gamma^{\alpha}_{\beta\gamma})$ of space-time.

The structure of $T^{\alpha}{}_{\beta}$ and even of J^{α} is not particularly transparent in this fourdimensional formalism. It will be analyzed elsewhere in a (3+1)-dimensional splitting of space-time and a two-component spinor form. The space-time approach, however, is useful to systematically derive these quantities from first principles and gives more insight into the relation between physical fields and space-time structure. We have also observed in section 5 and again, perhaps somewhat surprisingly, in the appearance of the spinor-tensor $S^{\alpha\beta}$ in (8.28) that perhaps a global 5-dimensional approach fully based on the Bargmann group would be more appropriate. This question also deserves to be studied in more detail.

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REFERENCES

[1] V. BARGMANN, *Ann. Math.*, t. **59**, 1952, p. 1-46.
[2] J. A. BROOKE, *J. Math. Phys.*, t. **19**, 1978, p. 952-959.
[3] J. A. BROOKE, *J. Math. Phys.*, t. **21**, 1980, p. 617-621.
[4] J. A. BROOKE, *Ph. D. thesis*, University of Alberta, 1980.
[5] Y. CHOQUET-BRUHAT, C. DEWITT-MORETTE and M. DILLARD-BLEICK, *Analysis, manifolds and physics* (revised edition), North-Holland, Amsterdam, 1982.
[6] R. COLLELA, A. W. OVERHAUSER and S. A. WERNER, *Phys. Rev. Lett.*, t. **34**, 1975, p. 1472-1474.
[7] C. DUVAL and H. P. KÜNZLE, *C. R. Acad. Sc., Paris*, t. **285 A**, 1977, p. 813-816.
[8] C. DUVAL and H. P. KÜNZLE, *Rep. Math. Phys.*, t. **13**, 1978, p. 351-368.
[9] C. DUVAL and H. P. KÜNZLE, *Gen. Rel. Grav.*, t. **16**, 1984, p. 333-347.
[10] J. EHLERS, p. 65-84 in J. Nitsch et al. (eds.), *Grundlagenprobleme der modernen Physik*, Bibliographisches Institut, Mannheim, 1981.
[11] R. GEROCH, *J. Math. Phys.*, t. **11**, 1968, p. 1739-1744.
[12] S. KOBAYASHI and K. NOMIZU, *Foundations of differential geometry*, Vol. 1, Wiley, New York, 1963.
[13] Y. KOS MANN, *Ann. Math. Pura Appl.*, t. **91**, 1972, p. 317-395.
[14] K. KUCHAŘ, *Phys. Rev.*, t. **D 22**, 1980, p. 1285-1299.
[15] H. P. KÜNZLE, *Ann. Inst. H. Poincaré*, t. **27**, 1972, p. 337-362.
[16] H. P. KÜNZLE and J. M. NESTER, *J. Math. Phys.*, t. **25**, 1984, p. 1009-1018.
[17] J. M. LÉVY-LEBLOND, *Commun. Math. Phys.*, t. **6**, 1967, p. 286-311.
[18] A. LICHNEROWICZ, *Bull. Soc. Math., France*, t. **92**, 1964, p. 11-100.
[19] J. M. SOURIAU, *Structure des systèmes dynamiques*, Dunod, Paris, 1970.
[20] J. L. STAUDEMANN et al., *Phys. Rev.*, t. **A 21**, 1980, p. 1419-1438.

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