

Max-Planck-Institut  
für Mathematik  
in den Naturwissenschaften  
Leipzig

Dirac-Harmonic Maps

by

*Qun Chen, Jürgen Jost, Jiayu Li, and Guofang Wang*

Preprint no.: 83

2004





November 18, 2004

## DIRAC-HARMONIC MAPS

QUN CHEN, JÜRGEN JOST, JIAYU LI, GUOFANG WANG

ABSTRACT. We introduce a functional that couples the nonlinear sigma model with a spinor field:  $L = \int_M [|d\phi|^2 + (\psi, \not{D}\psi)]$ . In two dimensions, it is conformally invariant. The critical points of this functional are called Dirac-harmonic maps. We study some geometric and analytic aspects of such maps, in particular a removable singularity theorem.

### 1. INTRODUCTION

This paper introduces and studies an extension of an established mathematical subject, namely harmonic maps from Riemann surfaces into Riemannian manifolds, that is motivated by a model from quantum field theory, the supersymmetric nonlinear sigma model. What distinguishes harmonic maps from surfaces from those from higher dimensional domains is the feature of conformal invariance. On one hand, this is a global aspect, and it implies that when the domain is the 2-sphere, any such map is automatically conformal; thus, the second order equations that characterize a harmonic map reduce to first order equations. As is typical in the geometric calculus of variations, this leads to important invariants; here, we obtain minimal surfaces in Riemannian manifolds as well as quantum cohomology, and the theory of pseudo-holomorphic curves in symplectic geometry shares the same root. On the other hand, this leads to a non-compact local invariance group and turns the regularity and existence problem into a borderline case of the Palais-Smale condition. In fact, this conformal invariance connects the local regularity theory with global solutions defined on the 2-sphere as we know from the seminal paper of Sacks-Uhlenbeck [14]. The approach of Sacks-Uhlenbeck depended on perturbing the functional to ones that satisfy the Palais-Smale condition and then pass to a limit. This, together with a sophisticated local analysis, then yielded global existence results, with possible obstructions coming from the second homotopy group of the target. (In the absence of those obstructions, the existence had been shown independently by Lemaire [12].) Subsequently, alternative existence schemes were developed by Struwe [15] and Chang [3] (heat flow method) and Jost [9] (local iteration technique). A crucial point of the work of Sacks-Uhlenbeck was the removability of isolated singularities through a detailed blow-up analysis. In the variational context, it is not difficult to reduce the general regularity question to the one of isolated singularities, and therefore, the analysis of Sacks-Uhlenbeck was sufficient for the variational existence scheme. The more general regularity question for weak solutions of the harmonic map equation was solved later by Hélein [7].

Now, in the physics literature, the same model goes under the name non-linear sigma model; when the target is an  $(N - 1)$ -dimensional sphere, a case of special

---

*Key words and phrases.* Dirac-harmonic map, non-linear sigma model, removable singularity.

The research of QC and JYL was partially supported by NSFC. QC also thanks the Max Planck Institute for Mathematics in the Sciences for support and good working conditions during the preparation of this paper.

interest for quantum field theory, it is called more precisely the non-linear  $O(N)$  sigma model. Here, our map  $\phi$  becomes a Bosonic scalar field satisfying a non-linear constraint. Now, this model admits a supersymmetric extension (see [5] for a detailed exposition<sup>1</sup>) where  $\phi$  is coupled to a Fermionic field  $\psi$ . That Fermionic field, and then also the Bosonic field, is Grassmann valued, a property needed for an additional symmetry exchanging the two types of fields, called supersymmetry. One can then try to solve the corresponding equations by expansions in the degree of the Grassmann variables.

Here, we rather let the field  $\psi$  also be real-valued; we then cannot use such expansions anymore, but rather get a coupled set of field equations for the two fields  $\phi$  and  $\psi$ . Mathematically,  $\phi$  as before is a mapping from a Riemann surface into a Riemannian manifold, whereas  $\psi$  is a spinor field with values in the pull-back of the tangent bundle of the target under  $\phi$ . Thus, with respect to coordinate transformations on the target,  $\psi$  transforms as a tangent vector. The important point is that the Lagrangian is set up in such a manner as to still be conformally invariant. We thus obtain an extension of the harmonic map model within the class of conformally invariant variational problems with a rich internal structure. Here, we study the geometric setting of the problem and present a complete variational analysis, that is, we prove the removability of isolated singularities and the regularity of solutions arising from a variational scheme. Our analysis is based on the scheme of Sacks-Uhlenbeck [14], but because of the coupling between the two fields, new difficulties and subtleties arise that make the analysis considerably harder. A regularity result in the sense of Hélein [7] will be presented elsewhere [4]. Having thus laid the analytical foundation, the existence question and the construction of geometric invariants from the solution spaces can be addressed in subsequent work. We should remark that the analysis in our framework is more difficult than in the supersymmetric one because we cannot use the expansions of the physicists. On the other hand, our present model seems to be very natural from the point of view of Riemannian geometry, and it thus falls in a similar category as the Seiberg-Witten or Chern-Simons-Higgs models, and this is our main motivation for its study.

Let us now describe the mathematical framework in more detail: Let  $(M, h)$  be an oriented, compact Riemannian surface and  $P_{SO(2)} \rightarrow \Sigma$  its oriented orthonormal frame bundle. A *Spin*-structure is a lift of the structure group  $SO(2)$  to  $Spin(2)$ , *i.e.* there exists a principal *Spin*-bundle  $P_{Spin(2)} \rightarrow M$  such that there is a bundle map

$$\begin{array}{ccc} P_{Spin(2)} & \longrightarrow & P_{SO(2)} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M. \end{array}$$

Let  $\Sigma^+ M := P_{Spin(2)} \times_{\rho} \mathbb{C}$  be a complex line bundle over  $M$  associated to  $P_{Spin(2)}$  and to the standard representation  $\rho : \mathbb{S}^1 \rightarrow U(1)$ . This is the bundle of positive half-spinors. Its complex conjugate  $\Sigma^- M := \overline{\Sigma^+ M}$  is called the bundle of negative half-spinors. The spinor bundle is  $\Sigma M := \Sigma^+ M \oplus \Sigma^- M$ .

---

<sup>1</sup>For a variant of this model, see [1]

There exists a Clifford multiplication

$$\begin{aligned} TX \times_{\mathbb{C}} \Sigma^+ M &\rightarrow \Sigma^- M \\ TX \times_{\mathbb{C}} \Sigma^- M &\rightarrow \Sigma^+ M \end{aligned}$$

denoted by  $v \otimes \psi \rightarrow v \cdot \psi$ , which satisfies the Clifford relations

$$v \cdot w \cdot \psi + w \cdot v \cdot \psi = -2h(v, w)\psi,$$

for all  $v, w \in TM$  and  $\psi \in \Sigma M$ .

On the spinor bundle  $\Sigma M$  there is a hermitian metric  $\langle \cdot, \cdot \rangle$ . Let  $\nabla$  be the Levi-Civita connection on  $M$  with respect to  $h$ . There is a connection (also denoted by  $\nabla$ ) on  $\Sigma M$  compatible with the hermitian metric.

Let  $\phi$  be a smooth map from  $M$  to another Riemannian manifold  $(N, g)$  of dimension  $n \geq 2$ . Denote  $\phi^{-1}TN$  the pull-back bundle of  $TN$  by  $\phi$  and consider the twisted bundle  $\Sigma M \otimes \phi^{-1}TN$ . On  $\Sigma M \otimes \phi^{-1}TN$  there is a metric induced from the metrics on  $\Sigma M$  and  $\phi^{-1}TN$ . Also we have a natural connection  $\tilde{\nabla}$  on  $\Sigma M \otimes \phi^{-1}TN$  induced from those on  $\Sigma M$  and  $\phi^{-1}TN$ . In local coordinates, the section  $\psi$  of  $\Sigma M \otimes \phi^{-1}TN$  can be expressed by

$$\psi(x) = \sum_{j=1}^n \psi^j(x) \frac{\partial}{\partial y^j}(\phi(x)),$$

where  $\psi^i$  is a spinor and  $\{\frac{\partial}{\partial y^j}\}$  is the natural local basis.  $\tilde{\nabla}$  can be expressed by

$$\tilde{\nabla} \psi = \sum_{i=1}^n \nabla \psi^i(x) \frac{\partial}{\partial y^i}(\phi(x)) + \sum_{i,j,k=1}^n \Gamma_{jk}^i \nabla \phi^j(x) \psi^k(x) \frac{\partial}{\partial y^i}(\phi(x)).$$

It is easy to check that

$$v(\psi_1, \psi_2) = (\tilde{\nabla}_v \psi_1, \psi_2) + (\psi_1, \tilde{\nabla}_v \psi_2),$$

for any vector field  $v$ .

Now we define the *Dirac operator along the map  $\phi$*  by

$$(1.1) \quad \mathcal{D}\psi = \sum_i \not{\partial} \psi^i(x) \frac{\partial}{\partial y^i}(\phi(x)) + \sum_{i,j,k=1}^n \Gamma_{jk}^i \nabla_{e_\alpha} \phi^j(x) e_\alpha \cdot \psi^k(x) \frac{\partial}{\partial y^i}(\phi(x)),$$

where  $e_1, e_2$  is the local orthonormal basis of  $M$  and  $\not{\partial} := \sum_{\alpha=1}^2 e_\alpha \cdot \nabla_{e_\alpha}$  is the usual Dirac operator. The Dirac operator  $\mathcal{D}$  is formally self-adjoint, i.e.,

$$(1.2) \quad \int_M (\psi, \mathcal{D}\xi) = \int_M (\mathcal{D}\psi, \xi),$$

for all  $\psi, \xi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ , the space of smooth section of  $\Sigma M \otimes \phi^{-1}TN$ . Set

$$\mathcal{X} := \{(\phi, \psi) \mid \phi \in C^\infty(M, N) \text{ and } \psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)\}.$$

On  $\mathcal{X}$ , we consider the following functional

$$(1.3) \quad \begin{aligned} L(\phi, \psi) &= \int_M [ |d\phi|^2 + (\psi, \mathcal{D}\psi) ] \sqrt{h} d^2x \\ &= \int_M [ g_{ij}(\phi) h^{\alpha\beta} \frac{\partial \phi^i}{\partial x_\alpha} \frac{\partial \phi^j}{\partial x_\beta} + g_{ij}(\phi) (\psi^i, \mathcal{D}\psi^j) ] \sqrt{h} d^2x, \end{aligned}$$

where  $h := \det(h_{\alpha\beta})$ . The Euler-Lagrange equations of  $L$  are:

$$(1.4) \quad \tau(\phi) = \mathcal{R}(\phi, \psi),$$

$$(1.5) \quad \mathcal{D}\psi = 0,$$

where  $\tau(\phi)$  is the tension field of the map  $\phi$  and  $\mathcal{R}(\phi, \psi) \in \Gamma(\phi^{-1}TM)$  defined by

$$(1.6) \quad \mathcal{R}(\phi, \psi)(x) = \frac{1}{2} \sum R_{lij}^m(\phi(x)) \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \frac{\partial}{\partial y^m}(\phi(x)).$$

Here  $R_{ij}^m$  are the components of the Riemannian curvature tensor of  $h$ . Note that the product  $\langle \cdot, \cdot \rangle$  here is the one of  $\Sigma M$ . Solutions  $(\phi, \psi)$  to (1.4) and (1.5) are called *Dirac-harmonic maps*.

As already mentioned, the functional  $L$  arises from our study of the supersymmetric nonlinear sigma model, the difference being that here  $\psi$  is an ordinary (twisted) spinor. Equations (1.4) and (1.5) couple the harmonic equation and the Dirac equation in a rather natural way.

In this paper, our main aim is to establish some geometric and analytic aspects of solutions to equations (1.4) and (1.5). We first establish some basic properties of the Dirac-harmonic maps and give some examples of nontrivial solutions. We also derive some geometric properties of the Dirac-harmonic maps, the conformal invariance and the existence of a generalized Hopf differential. Then we prove the removability of singularities for the solutions.

This paper is organized as follows: in Section 2, we deduce the Euler-Lagrange equations, and construct nontrivial solutions; in Section 3, we define the energy-momentum tensor of the action  $L$  and construct a holomorphic differential (Proposition 3.2) which plays a role in proving the removable singularity theorem; we also establish the basic Weitzenböck formula (Proposition 3.4) for spinor fields satisfying (1.5); in Section 4 we prove the main result about removable singularities (Theorem 4.7).

## 2. DIRAC-HARMONIC MAPS

In this section, we establish some basic facts for the functional  $L$  and equations (1.4)–(1.5).

**Proposition 2.1.** *The Euler-Lagrange equations for  $L$  are*

$$(2.1) \quad \tau(\phi) = \mathcal{R}(\phi, \psi)$$

$$(2.2) \quad \mathcal{D}\psi = 0,$$

where  $\tau(\phi)$  is the tension field of the map  $\phi$  and  $\mathcal{R}$  is defined by (1.6).

*Proof.* Equation (2.2) is easy to derive. Consider a family of  $\psi_t$  with  $d\psi_t/dt = \eta$  at  $t = 0$  and fix  $\phi$ . Since  $\mathcal{D}$  is formally self-adjoint, we have

$$\begin{aligned} \frac{dL}{dt}|_{t=0} &= \int_M \langle \eta, \mathcal{D}\psi \rangle + \langle \psi, \mathcal{D}\eta \rangle \\ &= 2 \int_M \langle \eta, \mathcal{D}\psi \rangle. \end{aligned}$$

Hence, we get (2.2).

Next, we consider a variation  $\{\phi_t\}$  of  $\phi$  such that  $d\phi_t/dt = \xi$  at  $t = 0$  and fix  $\psi$ . We choose  $\{e_\alpha\}$  as a local orthonormal basis on  $M$  such that  $[e_\alpha, \partial_t] = 0$ ,  $\nabla_{e_\alpha} e_\beta = 0$  at a considered point.

$$(2.3) \quad \frac{dL(\phi_t)}{dt}\Big|_{t=0} = \int_M \frac{\partial}{\partial t} |d\phi_t|^2 \Big|_{t=0} + \int_M \frac{\partial}{\partial t} \langle \psi, \mathcal{D}\psi \rangle \Big|_{t=0} := I + II.$$

It is easy to check that

$$(2.4) \quad I = -2 \int_M \tau^i(\phi) g_{im} \xi^m.$$

See the proof for instance in [16]. Now we compute II. First we compute the variation of  $\mathcal{D}\psi$ . We have

$$\begin{aligned} \frac{d}{dt} \mathcal{D}\psi &= e_\alpha \cdot \nabla_{\partial_t} \nabla_{e_\alpha} \psi \\ &= e_\alpha \cdot \nabla_{e_\alpha} \psi^i \otimes \nabla_{\partial_t} \partial_{y_i} + e_\alpha \cdot \psi^i \otimes \nabla_{\partial_t} \nabla_{e_\alpha} \partial_{y_i} \\ &= e_\alpha \cdot \nabla_{e_\alpha} \psi^i \otimes \nabla_{\partial_t} \partial_{y_i} + e_\alpha \cdot \psi^i \otimes [\nabla_{e_\alpha} \nabla_{\partial_t} \partial_{y_i} + R(\partial_t, e_\alpha) \partial_{y_i}] \\ &= e_\alpha \cdot \nabla_{e_\alpha} (\psi^i \otimes \nabla_{\partial_t} \partial_{y_i}) + e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_t), d\phi(e_\alpha)) \partial_{y_i}. \end{aligned}$$

Hence, we have

$$\begin{aligned} II &= \int_M \langle \xi, \mathcal{D}\psi \rangle + \int_M \langle \psi, \frac{d}{dt} \mathcal{D}\psi \rangle \Big|_{t=0} \\ &= \int_M \langle \psi, \mathcal{D}(\psi^i \otimes \nabla_{\partial_t} \partial_{y_i}) \rangle \Big|_{t=0} + \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_t), d\phi(e_\alpha)) \partial_{y_i} \rangle \Big|_{t=0} \\ &= \int_M \langle \mathcal{D}\psi, \psi^i \otimes \nabla_{\partial_t} \partial_{y_i} \rangle \Big|_{t=0} + \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_t), d\phi(e_\alpha)) \partial_{y_i} \rangle \Big|_{t=0} \\ &= \int_M \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_t), d\phi(e_\alpha)) \partial_{y_i} \rangle \Big|_{t=0} \\ &= \int_M \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(\xi^m \partial_{y_m}, \phi_\alpha^l \partial_{y_l}) \partial_{y_i} \rangle \\ &= \int_M \langle \psi, e_\alpha \cdot \psi^i \otimes \xi^m \phi_\alpha^l R_{iml}^j \partial_{y_j} \rangle \\ &= \int_M \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle R_{mlij} \xi^m, \end{aligned}$$

where we have used (2.2). Consequently, we have

$$\frac{dL(\phi_t)}{dt}\Big|_{t=0} = \int_M [-2g_{mi} \tau^i(\phi) + R_{mlij} (\psi^i, \nabla \phi^l \cdot \psi^j)] \xi^m,$$

and hence (2.1).  $\square$

It is obvious that there are two types of trivial solutions. One is  $(\phi, 0)$ , where  $\phi$  is a harmonic map, and another is  $(y, \psi)$ , where  $y$  is a point in  $N$  viewed as a constant map from  $M \rightarrow N$ , and  $\psi$  is a harmonic spinor, i.e.,  $\mathcal{D}\psi = 0$ .

Let us give a construction of non-trivial solutions. Let  $M = \mathbb{S}^2$  and  $N = \mathbb{S}^2$ . Let  $\Sigma \mathbb{S}^2$  be the spinor bundle over  $\mathbb{S}^2$  with respect to the unique spin structure. For any

map  $\phi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  and a spinor  $\Psi \in \Gamma(\Sigma\mathbb{S}^2)$ , we define a spinor field  $\psi$  along the map  $\phi$  by

$$(2.5) \quad \psi_{\phi, \Psi} = e_\alpha \cdot \Psi \phi_*(e_\alpha),$$

where  $e_\alpha$  ( $\alpha = 1, 2$ ) is a local basis of  $T\mathbb{S}^2$ . It is clear that  $\psi_{\phi, \Psi}$  is well-defined. A spinor  $\Psi \in \Gamma(\Sigma\mathbb{S}^2)$  is called a twistor spinor if

$$\nabla_v \Psi + \frac{1}{2} v \cdot \not\partial \Psi = 0,$$

for any vector field  $v \in \Gamma(T\mathbb{S}^2)$ .

Now we have

**Proposition 2.2.** *Let  $\psi_{\phi, \Psi}$  be defined by (2.5) from a nonconstant map  $\phi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  and a spinor  $\Psi$ . Then  $(\phi, \psi_{\phi, \Psi})$  is a Dirac harmonic map if and only if  $\phi$  is a (possibly branched) conformal map and  $\Psi$  is a twistor spinor.*

*Proof.* We first assume that  $(\phi, \psi_{\phi, \Psi})$  is a Dirac-harmonic map. For the spinor field  $\psi$  constructed from (2.5), we always have

$$\begin{aligned} \langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle &= \langle \nabla \phi^k \cdot \Psi, \nabla \phi^j \cdot \nabla \phi^l \cdot \Psi \rangle \\ &= \phi_\alpha^k \phi_\beta^j \phi_\gamma^l \langle e_\alpha \cdot \Psi, e_\beta \cdot e_\gamma \cdot \Psi \rangle \\ &= 0, \end{aligned}$$

since  $e_\beta \cdot e_\gamma = -e_\gamma \cdot e_\beta$  for  $\beta \neq \gamma$  and  $\phi_\beta \phi_\gamma$  is symmetric. Consequently,

$$\frac{1}{2} R_{jkl}^i (\psi^k, \nabla \phi^j \cdot \psi^l) \equiv 0.$$

Hence, if  $(\phi, \psi_{\phi, \Psi})$  is a Dirac harmonic map, then  $\phi$  is harmonic from equation (1.4). Choosing suitable coordinates such that  $\nabla_{e_\alpha} e_\beta = 0$  and  $x \in M$ , by (1.5) at  $x$  we have

$$\begin{aligned} 0 &= \not\partial \psi_{\phi, \Psi} \\ &= e_\beta \cdot \tilde{\nabla}_{e_\beta} (e_\alpha \cdot \Psi \phi_*(e_\alpha)) \\ &= e_\beta \cdot e_\alpha \cdot \{ \nabla_{e_\beta} \Psi \phi_*(e_\alpha) + \Psi \nabla_{e_\beta} \phi_*(e_\alpha) \} \\ &= -\{ \nabla_{e_\alpha} \Psi \phi_*(e_\alpha) + \Psi \tau(\phi) \} \\ &\quad + e_1 \cdot e_2 \cdot \{ \nabla_{e_1} \Psi \phi_*(e_2) - \nabla_{e_2} \Psi \phi_*(e_1) + \Psi [\nabla_{e_1} \phi_*(e_2) - \nabla_{e_1} \phi_*(e_2)] \} \\ &= -\nabla_{e_\alpha} \Psi \phi_*(e_\alpha) - e_1 \cdot e_2 \cdot \{ \nabla_{e_2} \Psi \phi_*(e_1) - \nabla_{e_1} \Psi \phi_*(e_2) \} \end{aligned}$$

Since  $\phi$  is conformal (and non-constant), the above equation is equivalent to

$$(2.6) \quad e_1 \cdot \nabla_{e_1} \Psi = e_2 \cdot \nabla_{e_2} \Psi,$$

which is equivalent to the condition that  $\Psi$  is a twistor spinor.

Conversely, from the above proof, it is easy to see that if  $\phi$  is a conformal map and  $\Psi$  is a twistor spinor, then  $(\phi, \psi_{\phi, \Psi})$  is a Dirac-harmonic map.  $\square$

**Remark 2.3.** The twistor spinors form a linear space of complex dimension 2, whereas the conformal maps form a nonlinear space whose dimension depends on their degree; for degree 1, we have dimension 3.



## 3. GEOMETRIC ASPECTS OF DIRAC-HARMONIC MAPS

In this section, we will study geometric properties of Dirac-harmonic maps. First

**Lemma 3.1.** *The functional  $L(\phi, \psi)$  is conformally invariant. Namely, for any conformal diffeomorphism  $f : M \rightarrow M$ , set*

$$\tilde{\phi} = \phi \circ f \quad \text{and} \quad \tilde{\psi} = \lambda^{-1/2} \psi \circ f.$$

Then  $L(\phi, \psi) = L(\tilde{\phi}, \tilde{\psi})$ . Here  $\lambda$  is the conformal factor of the conformal map  $f$ .

*Proof.* The proof is standard. In fact, the following terms are invariant under a conformal transformation

$$\int |\nabla \phi|^2 d\text{vol}(g), \int \langle \psi, \mathcal{D}\psi \rangle d\text{vol}(g) \quad \text{and} \quad \int |\psi|^4 d\text{vol}(g).$$

Here we only check the conformality of  $\int \langle \psi, \mathcal{D}\psi \rangle d\text{vol}(g)$ . Let  $\tilde{g} = f^*g$  and  $\tilde{\mathcal{D}}$  the usual Dirac operator with respect to the new metric  $\tilde{g}$ . By the conformality of  $f$ , we have  $\tilde{g} = (\lambda^*)g$ . We identify the new and old spin bundles as in [8]. Recall that the relation between two Dirac operators  $\mathcal{D}$  and  $\tilde{\mathcal{D}}$  is (cf. [8])

$$(3.1) \quad \tilde{\mathcal{D}}\tilde{\psi} = \lambda^{-\frac{3}{2}} \mathcal{D}\psi,$$

from which one can deduce that

$$(3.2) \quad \tilde{\mathcal{D}}\tilde{\psi} = \lambda^{-\frac{3}{2}} \mathcal{D}\psi,$$

and hence the conformal invariance of  $\int \langle \psi, \mathcal{D}\psi \rangle d\text{vol}(g)$ .  $\square$

For a two dimensional harmonic map, there is an important holomorphic quadratic differential, the Hopf differential. For a Dirac-harmonic map, we also have an analogue.

Let  $(\phi, \psi)$  be a Dirac-harmonic map. On a small domain  $\Omega$  of  $M$ , choose  $z = x + iy$  a local isothermal parameter  $z = x + iy$  with  $g = ds^2 = \rho|dz|^2$ . Define

$$(3.3) \quad T(z)dz^2 = \{(|\phi_x|^2 - |\phi_y|^2 - 2i\langle \phi_x, \phi_y \rangle) + (\langle \psi, \partial_x \cdot \tilde{\nabla}_{\partial_x} \psi \rangle - i\langle \psi, \partial_x \cdot \tilde{\nabla}_{\partial_y} \psi \rangle)\} dz^2.$$

Here  $\partial_x = \frac{\partial}{\partial x}$  and  $\partial_y = \frac{\partial}{\partial y}$ .

**Proposition 3.2.** *The quadratic differential  $Tdz^2$  is holomorphic.*

One can prove this proposition by a direct computation, or as a consequence of a conservation law as follows.

Define a two-tensor by

$$(3.4) \quad T_{\alpha\beta} := (2\langle \phi_\alpha, \phi_\beta \rangle - \delta_{\alpha\beta} \langle \phi_\gamma, \phi_\gamma \rangle) + \langle \psi, e_\alpha \cdot \tilde{\nabla}_{e_\beta} \psi \rangle,$$

where  $\phi_\alpha := \phi_*(e_\alpha)$ . Here,  $\{e_\alpha\}$  is a local orthonormal basis on  $M$  and  $\{\theta^\alpha\}$  a coframe dual to  $\{e_\alpha\}$ . The tensor  $T_{\alpha\beta}\theta^\alpha \otimes \theta^\beta$  is called the energy-momentum tensor. Using the equation  $\mathcal{D}\psi = 0$  one can verify that  $T$  is symmetric:

$$T_{\alpha\beta} = T_{\beta\alpha}.$$

**Proposition 3.3.** *Let  $(\phi, \psi)$  be a smooth solution of (1.4) (1.5) on  $M$ , then the energy-momentum tensor is conserved, i.e.,*

$$(3.5) \quad \sum_{\alpha} \nabla_{e_{\alpha}} T_{\alpha\beta} = 0.$$

*Proof.*

$$\begin{aligned} \nabla_{e_{\alpha}} T_{\alpha\beta} &= \nabla_{e_{\alpha}} (2\langle \phi_{\alpha}, \phi_{\beta} \rangle - \delta_{\alpha\beta} \langle \phi_{\gamma}, \phi_{\gamma} \rangle) + \nabla_{e_{\alpha}} \langle \psi, e_{\alpha} \cdot \tilde{\nabla}_{e_{\beta}} \psi \rangle \\ &:= I + II. \end{aligned}$$

We choose a local orthonormal basis  $\{e_{\alpha}\}$  on  $M$  such that  $\nabla_{e_{\alpha}} e_{\beta} = 0$  at a considered point. We compute

$$\begin{aligned} I &= 2\langle \nabla_{e_{\alpha}} \phi_{*}(e_{\alpha}), \phi_{*}(e_{\beta}) \rangle + 2\langle \phi_{*}(e_{\alpha}), \nabla_{e_{\alpha}} \phi_{*}(e_{\beta}) \rangle \\ &\quad - 2\delta_{\alpha\beta} \langle \phi_{*}(e_{\gamma}), \nabla_{e_{\alpha}} \phi_{*}(e_{\gamma}) \rangle \\ &= 2\langle \tau(\phi), \phi_{\beta} \rangle + 2\langle \phi_{\alpha}, \nabla_{e_{\beta}} \phi_{*}(e_{\alpha}) \rangle - 2\langle \phi_{\gamma}, \nabla_{e_{\beta}} \phi_{*}(e_{\gamma}) \rangle \\ &= 2\langle \tau(\phi), \phi_{\beta} \rangle, \end{aligned}$$

and

$$\begin{aligned} II &= \langle \psi_{\alpha}, e_{\alpha} \cdot \psi_{\beta} \rangle + \langle \psi, e_{\alpha} \cdot \tilde{\nabla}_{e_{\alpha}} \tilde{\nabla}_{e_{\beta}} \psi \rangle \\ &= -\langle e_{\alpha} \cdot \psi_{\alpha}, \psi_{\beta} \rangle + \langle \psi, \mathcal{D}\psi_{\beta} \rangle \\ &= \langle \psi, \mathcal{D}\psi_{\beta} \rangle. \end{aligned}$$

Therefore, we have

$$(3.6) \quad \nabla_{e_{\alpha}} T_{\alpha\beta} = 2\langle \tau(\phi), \phi_{\beta} \rangle + \langle \psi, \mathcal{D}\psi_{\beta} \rangle.$$

Now

$$\begin{aligned} 2\langle \tau(\phi), \phi_{\beta} \rangle &= 2\langle \frac{1}{2} R_{lij}^m \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \partial_{y^m}, \phi_{\beta}^p \partial_{y^p} \rangle \\ &= g_{mp} R_{lij}^m \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \phi_{\beta}^p \\ (3.7) \quad &= R_{mlij} \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \phi_{\beta}^m. \end{aligned}$$

We compute  $\mathcal{D}\psi_{\beta} = e_{\alpha} \cdot \tilde{\nabla}_{e_{\alpha}} \tilde{\nabla}_{e_{\beta}} \psi$ . By a direct computation, we have:

$$\tilde{\nabla}_{e_{\alpha}} \tilde{\nabla}_{e_{\beta}} \psi - \tilde{\nabla}_{e_{\beta}} \tilde{\nabla}_{e_{\alpha}} \psi = R^{\Sigma M}(e_{\alpha}, e_{\beta}) \psi^i \otimes \partial_{y^i} + R_{lij}^m \phi_{\alpha}^i \phi_{\beta}^j \psi^l \otimes \partial_{y^m},$$

where  $R^{\Sigma M}$  is the curvature operator of the connection  $\nabla$  on the spinor bundle  $\Sigma M$ . For this curvature operator, we have (cf. [10])

$$(3.8) \quad e_{\alpha} \cdot R^{\Sigma M}(e_{\alpha}, X) \psi^i = \frac{1}{2} Ric(X) \cdot \psi^i, \quad \forall X \in \Gamma(M).$$

It follows that

$$\begin{aligned} \langle \psi, e_{\alpha} \cdot R^{\Sigma M}(e_{\alpha}, e_{\beta}) \psi^i \otimes \partial_{y^i} \rangle &= \langle \psi^j \otimes \partial_{y^j}, e_{\alpha} \cdot R^{\Sigma M}(e_{\alpha}, e_{\beta}) \psi^i \otimes \partial_{y^i} \rangle \\ &= g_{ij} \langle \psi^j, e_{\alpha} \cdot R^{\Sigma M}(e_{\alpha}, e_{\beta}) \psi^i \rangle \\ &= \frac{1}{2} g_{ij} \langle \psi^j, Ric(e_{\beta}) \cdot \psi^i \rangle \\ &= 0. \end{aligned}$$

Therefore

$$\begin{aligned}
\langle \psi, \mathcal{D}\psi_\beta \rangle &= \langle \psi, e_\alpha \cdot \tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\beta} \psi \rangle \\
&= \langle \psi, \tilde{\nabla}_{e_\beta} (e_\alpha \cdot \tilde{\nabla}_{e_\alpha} \psi) \rangle + R_{ij}^m \phi_\beta^j \langle \nabla \phi^i \cdot \psi^l \otimes \partial_{y^m}, \psi^p \otimes \partial_{y^p} \rangle \\
&= R_{ij}^m \phi_\beta^j \langle \psi^p, \nabla \phi^i \cdot \psi^l \rangle g_{mp} \\
&= -R_{mlj} \langle \psi^i, \nabla \phi^l \cdot \psi^j \rangle \phi_\beta^m.
\end{aligned}$$

From (3.6),(3.7) and (3.9) we conclude that  $T_{\alpha\beta}$  is conserved.  $\square$

*Proof of Proposition 3.2.* The proof follows directly from Proposition 3.3.  $\square$

We now consider (1.5), i.e.,

$$\mathcal{D}\psi = (\mathcal{D}\psi^i + \Gamma_{jk}^i(\phi) \partial_\alpha \phi^j e_\alpha \cdot \psi^k) \otimes \partial_{y^i} = 0,$$

for  $\phi : M^m \rightarrow N^n$ ,  $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ . Here  $M^n$  is an  $n$  dimensional spin manifold with  $n \geq 2$ . Clearly, one can also discuss (1.4)-(1.5) for higher dimensional spin manifolds. We call such a  $\psi$  a harmonic spinor field along the map  $\phi$ . Note that here we do not assume  $(\phi, \psi)$  to be a solution of (1.4).

For a harmonic spinor field along a map  $\phi$ , we have the following Weitzenböck formula.

**Proposition 3.4.** *Let  $M^m$  and  $N^n$  be Riemannian manifolds,  $\phi : M \rightarrow N$ , and  $\psi = \psi^i \otimes \frac{\partial}{\partial y^i}(\phi) \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ . Then*

$$(3.9) \quad \mathcal{D}^2\psi = -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{4}R\psi + \frac{1}{2}R_{kpj}^i \phi_\alpha^p \phi_\beta^j (e_\alpha \cdot e_\beta \cdot \psi^k) \otimes \frac{\partial}{\partial y^i}.$$

If  $\phi$  and  $\psi$  satisfy (1.5), then

$$(3.10) \quad \frac{1}{2}\Delta|\psi|^2 = |\tilde{\nabla}\psi|^2 + \frac{1}{4}R|\psi|^2 - \frac{1}{2}R_{ijkl}(\nabla\phi^k \cdot \psi^i, \nabla\phi^l \cdot \psi^j),$$

where  $R$  is the scalar curvature of  $M$ ,  $\tilde{\nabla}$  denotes the connection on  $\Sigma M \otimes \phi^{-1}TN$  and  $|\psi|^2 := g_{ij}(\phi)(\psi^i, \psi^j)$ .

*Proof.* One can apply a general Weitzenböck formula (see for example [11] or [10]) to prove the proposition. For the convenience of the reader, we present a proof here. Choose an orthonormal basis  $\{e_\alpha | \alpha = 1, 2, \dots, m\}$  on  $M$  such that  $\nabla_{e_\beta} e_\alpha = 0$  at a considered point. Noting that  $e_\alpha^2 = -1$ ,  $e_\alpha e_\beta + e_\beta e_\alpha = 0$  and  $\alpha \neq \beta$ , we have

$$(3.11) \quad \begin{aligned} \mathcal{D}^2\psi &= e_\beta \cdot e_\alpha \cdot \tilde{\nabla}_{e_\beta} \tilde{\nabla}_{e_\alpha} \psi \\ &= -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \sum_{\alpha < \beta} e_\alpha \cdot e_\beta \cdot \tilde{R}(e_\alpha, e_\beta)\psi \end{aligned}$$

$$(3.12) \quad = -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{2} \sum_{\alpha, \beta} e_\alpha \cdot e_\beta \cdot \tilde{R}(e_\alpha, e_\beta)\psi$$

where  $\tilde{R}(\cdot, \cdot)$  is the curvature operator on  $\Sigma M \otimes \phi^{-1}TN$ , namely,

$$\tilde{R}(e_\alpha, e_\beta)\psi = \tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\beta} \psi - \tilde{\nabla}_{e_\beta} \tilde{\nabla}_{e_\alpha} \psi - \tilde{\nabla}_{[e_\alpha, e_\beta]}\psi.$$

Since

$$\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\beta} \psi = (\nabla_{e_\alpha} \nabla_{e_\beta} \psi^i + \Gamma_{jk,p}^i \phi_\alpha^p \phi_\beta^j \psi^k) \otimes \frac{\partial}{\partial y^i}$$

and

$$\tilde{\nabla}_{e_\beta} \tilde{\nabla}_{e_\alpha} \psi = (\nabla_{e_\beta} \nabla_{e_\alpha} \psi^i + \Gamma_{jk,p}^i \phi_\beta^p \phi_\alpha^j \psi^k) \otimes \frac{\partial}{\partial y^i},$$

we have

$$\begin{aligned} \tilde{R}(e_\alpha, e_\beta) \psi &= R^{\Sigma M}(e_\alpha, e_\beta) \psi^i \otimes \frac{\partial}{\partial y^i} + (\Gamma_{jk,p}^i - \Gamma_{kp,j}^i) \phi_\alpha^p \phi_\beta^j \psi^k \otimes \frac{\partial}{\partial y^i} \\ &= R^{\Sigma M}(e_\alpha, e_\beta) \psi + R_{kpj}^i \phi_\alpha^p \phi_\beta^j \psi^k \otimes \frac{\partial}{\partial y^i}. \end{aligned}$$

Putting this formula into (3.11) we have

$$\begin{aligned} \mathcal{D}^2 \psi &= -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{2} e_\alpha \cdot e_\beta \cdot R^{\Sigma M}(e_\alpha, e_\beta) \psi \\ &\quad + \frac{1}{2} R_{kpj}^i \phi_\alpha^p \phi_\beta^j (e_\alpha \cdot e_\beta \cdot \psi^k) \otimes \frac{\partial}{\partial y^i}, \end{aligned} \quad (3.13)$$

where  $R^{\Sigma M}(\cdot, \cdot)$  is the curvature operator on  $\Sigma M$ . It is known (cf. [10]) that

$$e_\alpha \cdot e_\beta \cdot R^{\Sigma M}(e_\alpha, e_\beta) \psi^i = \frac{1}{2} R \psi^i.$$

Thus, we obtain

$$\mathcal{D}^2 \psi = -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{4} R \psi + \frac{1}{2} R_{kpj}^i \phi_\alpha^p \phi_\beta^j (e_\alpha \cdot e_\beta \cdot \psi^k) \otimes \frac{\partial}{\partial y^i}, \quad (3.14)$$

from which

$$\langle \mathcal{D}^2 \psi, \psi \rangle = -\langle \tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi, \psi \rangle + \frac{1}{4} R |\psi|^2 + \frac{1}{2} R_{ikpj} \phi_\alpha^p \phi_\beta^j (e_\alpha \cdot e_\beta \cdot \psi^k, \psi^i).$$

Therefore,

$$\begin{aligned} \frac{1}{2} \Delta |\psi|^2 &= \langle \tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi, \psi \rangle + \langle \tilde{\nabla}_{e_\alpha} \psi, \tilde{\nabla}_{e_\alpha} \psi \rangle \\ &= |\tilde{\nabla} \psi|^2 - \langle \mathcal{D}^2 \psi, \psi \rangle + \frac{1}{4} R |\psi|^2 - \frac{1}{2} R_{ijkl} (e_\alpha \cdot \psi^i, e_\beta \cdot \psi^j) \phi_\alpha^k \phi_\beta^l. \end{aligned}$$

□

Let  $(N', g')$  be another Riemannian manifold and  $f : N \rightarrow N'$  a smooth map. For any  $(\phi, \psi) \in \mathcal{X}$  we set

$$\phi' = f \circ \phi \quad \text{and} \quad \psi' = f_* \psi.$$

It is clear that  $\psi'$  is a spinor along the map  $\phi'$ . Let  $A$  be the second fundamental form of  $f$ , i.e.,  $A(X, Y) = (\nabla_X df)(Y)$  for any  $X, Y \in \Gamma(TN)$ . The tension fields of  $\phi$  and  $\phi'$  have the following relation

$$\tau(\phi') = \sum_{\alpha=1}^2 A(d\phi(e_\alpha), d\phi(e_\alpha)) + df(\tau(\phi)). \quad (3.15)$$

It is also easy to check that the Dirac operators  $\mathcal{D}$  and  $\mathcal{D}'$  corresponding to  $\phi$  and  $\phi'$  respectively are related by the following formula

$$\mathcal{D}' \psi' = f_*(\mathcal{D} \psi) + A(d\phi(e_\alpha), e_\alpha \cdot \psi). \quad (3.16)$$

Here in local coordinates

$$A(d\phi(e_\alpha), e_\alpha \cdot \psi) = \phi_\alpha^i A\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) e_\alpha \cdot \psi^j,$$

where  $\{\frac{\partial}{\partial y^i}\}$  is a local basis of  $N$ . Furthermore, if  $f : N \rightarrow N'$  is an isometric immersion, then  $A(\cdot, \cdot)$  is the second fundamental form of the submanifold  $N$  in  $N'$ , and

$$\nabla'_X \xi = -P(\xi; X) + \nabla_X^\perp \xi, \quad \nabla'_X Y = \nabla_X Y + A(X, Y)$$

$\forall X, Y \in \Gamma(TN)$ ,  $\xi \in \Gamma(T^\perp N)$ , where  $P(\xi; \cdot)$  denotes the shape operator. Note that  $\langle P(\xi; X), Y \rangle = \langle A(X, Y), \xi \rangle$ . We can rewrite equations (1.4)-(1.5) in terms of  $A$  and the geometric data of the ambient space  $N'$ . By the equation of Gauss, one has

$$(3.17) \quad \mathcal{R}(\phi, \psi) = P(A(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi) + \frac{1}{2}R'(e_\alpha \cdot \psi, \psi)d\phi(e_\alpha).$$

Therefore, by using (3.15) and (3.16), and identifying  $\phi$  with  $\phi'$  and  $\psi$  with  $\psi'$ , we can rewrite (1.4) and (1.5) as follows:

$$(3.18) \quad \tau(\phi) = A(d\phi(e_\alpha), d\phi(e_\alpha)) + \frac{1}{2}R'(e_\alpha \cdot \psi, \psi)d\phi(e_\alpha) + P(A(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi),$$

$$(3.19) \quad \mathcal{D}'\psi = A(d\phi(e_\alpha), e_\alpha \cdot \psi).$$

In particular, if  $N' = \mathbb{R}^K$ , then these become

$$(3.20) \quad -\Delta\phi = A(d\phi, d\phi) + P(A(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)$$

$$(3.21) \quad \mathcal{D}\psi = A(d\phi(e_\alpha), e_\alpha \cdot \psi).$$

#### 4. ANALYTIC ASPECTS: REMOVABLE SINGULARITIES

Embed  $(N, h)$  into some  $\mathbb{R}^K$  isometrically and denote by  $A(\cdot, \cdot)$  the second fundamental form as in the previous section. Any map  $\phi$  from  $(M, g)$  to  $(N, h)$  can be seen as a map  $\phi$  from  $(M, g)$  to  $\mathbb{R}^K$  with  $\phi(x) \in N$ . And any spinor field  $\psi$  along the map  $\phi$  can be seen as a  $K$ -tuple of (usual) spinors  $(\psi^1, \psi^2, \dots, \psi^K)$  satisfying the condition that for any normal vector  $\nu = \sum_{i=1}^K \nu_i E_i$  of  $N$  at  $\phi(x)$ , we have

$$\sum_i \nu_i \psi^i = 0,$$

where  $\{E_i, i = 1, 2, \dots, K\}$  is the standard basis of  $\mathbb{R}^K$ . In this section, we always view  $(\phi, \psi)$  in this way. By the discussion in the previous section, such a pair  $(\phi, \psi)$  is a Dirac-harmonic map if and only if  $(\phi, \psi)$  satisfies (3.20) and (3.21).

In this section, we will prove the removable singularity theorem for Dirac-harmonic maps with “finite energy” defined below.

**Definition 4.1.** Let  $U$  be a domain on  $M$ . The energy of  $(\phi, \psi)$  on  $U$  is:

$$(4.0) \quad E(\phi, \psi, U) := \int_U (|d\phi|^2 + |\psi|^4).$$

Note that the energy is conformally invariant. It is crucial for our results.

Before we consider the analytic aspects of Dirac-harmonic maps, let us note that on a surface the (usual) Dirac operator  $\mathcal{D}$  can be seen as the Cauchy-Riemann operator. Consider  $\mathbb{R}^2$  with the Euclidean metric  $dx^2 + dy^2$ . Let  $e_1 = \frac{\partial}{\partial x}$  and  $e_2 = \frac{\partial}{\partial y}$  be

the standard orthonormal frame. A spinor field is simply a map  $\Psi : \mathbb{R}^2 \rightarrow \Delta_2 = \mathbb{C}^2$ , and  $e_1$  and  $e_2$  acting on spinor fields can be identified by multiplication with matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If  $\Psi := \begin{pmatrix} f \\ g \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{C}^2$  is a spinor field, then the Dirac operator is

$$\not\partial\Psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial x} \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial y} \end{pmatrix} = 2 \begin{pmatrix} \frac{\partial g}{\partial \bar{z}} \\ -\frac{\partial f}{\partial z} \end{pmatrix},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Therefore, the elliptic estimates developed for (anti-) holomorphic functions can be used to study the Dirac equation.

**Proposition 4.2** *Let  $(N^n, h)$  be a compact Riemannian manifold and  $(M^2, g)$  a surface with a fixed spin structure. Then there is a small constant  $\varepsilon > 0$  such that if  $(\phi, \psi)$  is a smooth Dirac-harmonic map satisfying*

$$(4.1) \quad \int_M (|d\phi|^2 + |\psi|^4) < \varepsilon,$$

then  $\phi$  is constant and consequently  $\psi$  is a usual harmonic spinor.

*Proof.* In view of (3.20), we have

$$|\Delta\phi| \leq C(|d\phi|^2 + |d\phi||\psi|^2),$$

where  $C > 0$  is a constant depending only on  $N$ . Hence we have

$$(4.2) \quad \begin{aligned} \|\Delta\phi\|_{L^{\frac{4}{3}}} &\leq C(\|\nabla\phi\|_{L^{\frac{4}{3}}}^2 + \|\psi\|_{L^4}^2 \|\nabla\phi\|_{L^4}) \\ &\leq C(\|\nabla\phi\|_{L^2} + \|\psi\|_{L^4}^2) \|\nabla\phi\|_{L^{1, \frac{4}{3}}} \\ &\leq \varepsilon C \|\nabla\phi\|_{L^{1, \frac{4}{3}}} \end{aligned}$$

Therefore, if  $\varepsilon$  is small enough, we can show that  $\phi \equiv \text{const}$ , and hence  $\psi$  is a harmonic spinor.  $\square$

Now we consider the local behavior of Dirac-harmonic maps. Since they are conformally invariant, in the sequel we may assume  $M$  to be the unit disk  $D$  with trivial spin structure.

**Theorem 4.3.** *There is a small constant  $\varepsilon > 0$  such that if  $(\phi, \psi)$  is a Dirac harmonic map satisfying*

$$(4.3) \quad \int_D (|d\phi|^2 + |\psi|^4) < \varepsilon,$$

then

$$(4.4) \quad \|\phi\|_{C^k(D_{\frac{1}{2}})} + \|\psi\|_{C^k(D_{\frac{1}{2}})} \leq C(\|\nabla\phi\|_{L^2(D)} + \|\psi\|_{L^4(D)}),$$

where  $C > 0$  is a constant depending only on  $k$  and the geometry of  $N$ .

*Proof.* In the sequel, we denote  $\|\cdot\|_{L^{k,p}(D)}$  by  $|\cdot|_{D,k,p}$ . If there is no confusion, we may drop the subscript  $D$ . In the proof,  $C$  is a constant, varying from line to line and

$$D_{\frac{1}{2}} \subset D^2 \subset D^1 \subset D.$$

We devide the proof into several steps.

**Step 1.** There is an  $\varepsilon > 0$  such that

$$(4.5) \quad |d\phi|_{D^1,0,4} \leq C(D^1)(|d\phi|_{0,2} + |\psi|_{0,4}^2), \quad \forall D^1 \subset D,$$

where  $C(D^1) > 0$  is a constant depending only on  $D^1$ .

Choose a cut-off function  $\eta : 0 \leq \eta \leq 1$ , with  $\eta|_{D^1} \equiv 1$  and  $\text{Supp}\eta \subset D$ . By (3.20) we have

$$\begin{aligned} |\Delta(\eta\phi)| &\leq C(|\phi| + |d\phi|) + |A|_{\infty}|d\phi|(|d(\eta\phi)| + |\phi d\eta|) + |\eta\alpha| \\ &\leq |A|_{\infty}|d\phi||d(\eta\phi)| + C(|\phi| + |d\phi|) + |\eta\alpha|, \end{aligned}$$

where we denote  $\alpha := P(A(d\phi(e_{\alpha}), e_{\alpha} \cdot \psi); \psi)$ . Thus, for any  $p > 1$ ,

$$(4.6) \quad |\Delta(\eta\phi)|_{0,p} \leq |A|_{\infty}(|d\phi||d(\eta\phi)|)_{0,p} + C|\phi|_{1,p} + |\eta\alpha|_{0,p}.$$

Let  $p = \frac{4}{3}$ , and without loss of generality we assume  $\int_D \phi = 0$  so that  $|\phi|_{1,p} \leq C'|d\phi|_{0,p}$ , then

$$|A|_{\infty}(|d\phi||d(\eta\phi)|)_{0,\frac{4}{3}} \leq |A|_{\infty}|\eta\phi|_{1,4}|d\phi|_{0,2},$$

from this and (4.6) we have

$$|\eta\phi|_{2,\frac{4}{3}} \leq C(|A|_{\infty}|\eta\phi|_{1,4}|d\phi|_{0,2} + |d\phi|_{0,\frac{4}{3}} + |\eta\alpha|_{0,\frac{4}{3}}).$$

By the Sobolev inequality,  $|\eta\phi|_{1,4} \leq C'|\eta\phi|_{2,\frac{4}{3}}$ , so,

$$(4.7) \quad (C'^{-1} - C|A|_{\infty}|d\phi|_{0,2})|\eta\phi|_{1,4} \leq C(|d\phi|_{0,\frac{4}{3}} + |\eta\alpha|_{0,\frac{4}{3}}).$$

Moreover,

$$\begin{aligned} |\eta\alpha|_{0,\frac{4}{3}} &\leq C_N(|\psi|^2|\eta d\phi|)_{0,\frac{4}{3}} \\ &= C_N(|\psi|^2|d(\eta\phi) - \phi d\eta|)_{0,\frac{4}{3}} \\ &\leq C|\psi|_{0,4}^2|\eta\phi|_{1,4} + C|\psi|_{0,4}^2. \end{aligned}$$

Putting this into (4.7) and choosing  $\varepsilon$  small, we get

$$|\eta\phi|_{1,4} \leq C(|d\phi|_{0,\frac{4}{3}} + \sqrt{\varepsilon}|\eta\phi|_{1,4} + |\psi|_{0,4}^2),$$

which yields

$$|\eta\phi|_{1,4} \leq C(|d\phi|_{0,\frac{4}{3}} + |\psi|_{0,4}^2) < 2\sqrt{\varepsilon}C.$$

**Step 2.** If  $\varepsilon > 0$  is small enough, then

$$(4.8) \quad |\psi|_{D^2,0,q} \leq C(D^2)|\psi|_{D,0,4}, \quad \forall q > 1, \quad D^2 \subset D^1 \subset D,$$

where  $C(D^2) > 0$  is a constant depending only on  $D^2$ .

Choose a cut-off function  $\eta : 0 \leq \eta \leq 1$ , with  $\eta|_{D^2} \equiv 1$  and  $\text{Supp}\eta \subset D^1$ . Let  $\xi = \eta\psi$  have compact support in  $D^1$ . By the well-known Lichnerowitz' formula, we have

$$\not\partial^2 \xi = -\Delta \xi + \frac{1}{4}R\xi = -\Delta \xi,$$

because the scalar curvature  $R \equiv 0$  on  $D$ . Integrating this yields

$$\begin{aligned}
|\nabla\xi|_{D^{1,0,2}} &= |\partial\xi|_{D^{1,0,2}}^2 \\
&= |\partial(\eta\psi)|_{D^{1,0,2}}^2 = |\nabla\eta \cdot \psi + \eta\partial\psi|_{D^{1,0,2}}^2 \\
&\leq C(|\psi|_{D^{1,0,2}}^2 + |\eta\partial\psi|^2) \\
&\leq C(|\psi|_{D^{1,0,2}}^2 + C|d\phi|^2|\eta\psi|_{D^{1,0,2}}^2) \\
&\leq C(|\psi|_{D^{1,0,2}} + |d\phi|_{D^{1,0,4}}^2|\psi|_{D^{1,0,4}}^2) \\
&\leq C|\psi|_{D^{1,0,4}}(1 + |d\phi|_{D^{1,0,4}}) \\
&\leq C'|\psi|_{D^{1,0,4}}.
\end{aligned}$$

Hence, Step 2 is proved.

**Step 3.** If  $\varepsilon$  is small enough, we have

$$(4.9) \quad |d\phi|_{D^{2,0,4}} \leq C(D^2)|d\phi|_{D,0,2}, \quad \forall D^2 \subset D^1 \subset D,$$

where  $C(D^2) > 0$  is a constant depending only on  $D^2$ . This follows from Steps 1 and 2. For higher order estimates, it is rather standard. See for example [4].  $\square$

From this theorem we know that a sequence of Dirac-harmonic maps with small energy has a convergent subsequence. However, if the energy is large, then a blow-up may occur. In this case, the concentration of energy may happen. Namely,  $|\nabla\phi_k(x_k)|^2 \rightarrow \infty$  as  $k \rightarrow \infty$ , for a sequence  $x_k \rightarrow x_0$  as  $k \rightarrow \infty$ . After a suitable rescaling, we get a ‘‘bubble’’, an entire solution of (3.20)–(3.21) with finite energy. By the conformal invariance of the Dirac-harmonic map, such an entire solution can be viewed as a Dirac-harmonic map from  $\mathbb{S}^2 \setminus \{p\} \rightarrow N$  with finite energy. In this section, we prove that such a singularity can be removed as in many conformal problems. Hence, at the end we obtain a Dirac-harmonic map from  $\mathbb{S}^2 \rightarrow N$ . Using the Theorem 4.3, we can describe the behavior of solutions  $(\phi, \psi)$  near a singular point as follows:

**Corollary 4.4.** *There is an  $\varepsilon > 0$  small enough such that if  $(\phi, \psi)$  is a smooth solution of (1.4)–(1.5) on  $D \setminus \{0\}$  with energy  $E(\phi, \psi, D) < \varepsilon$ , then for any  $x \in D_{\frac{1}{2}}$*

$$(4.10) \quad |d\phi(x)||x| \leq C\left(\int_{D(2|x)} |d\phi|^2\right)^{\frac{1}{2}},$$

$$(4.11) \quad |\psi(x)||x|^{\frac{1}{2}} + |\nabla\psi(x)||x|^{\frac{3}{2}} \leq C\left(\int_{D(2|x)} |\psi|^4\right)^{\frac{1}{4}},$$

*Proof.* Fix any  $x_0 \in D \setminus \{0\}$ , define  $\tilde{\phi}$  and  $\tilde{\psi}$  by

$$\tilde{\phi}(x) := \phi(x_0 + |x_0|x) \quad \text{and} \quad \tilde{\psi}(x) := |x_0|^{\frac{1}{2}}\psi(x_0 + |x_0|x).$$

It is clear that  $(\tilde{\phi}, \tilde{\psi})$  is a  $C^\infty$  solution of (1.4)–(1.5) on  $D$  and  $E(\tilde{\phi}, \tilde{\psi}, D) < \varepsilon$ . Applying Theorem 4.3, we have

$$|d\tilde{\phi}|_{L^\infty(D_{\frac{1}{2}})} \leq C|d\tilde{\phi}|_{D,0,2}$$

and

$$|\tilde{\psi}|_{C^1(D_{\frac{1}{2}})} \leq C|d\tilde{\psi}|_{D,0,4}.$$



Scaling back, we prove the Corollary.  $\square$

**Lemma 4.5.** *Let  $(\phi, \psi)$  be a  $C^\infty$  solution of (1.4)-(1.5) on  $D \setminus \{0\}$  satisfying  $E(\phi, \psi, D) < \varepsilon$ , then,*

$$(4.12) \quad \begin{aligned} \int_0^{2\pi} \frac{1}{r^2} |\phi_\theta|^2 d\theta &= \int_0^{2\pi} |\phi_r|^2 d\theta + \int_0^{2\pi} \langle \psi, \partial_r \cdot \psi_r \rangle d\theta \\ &= \int_0^{2\pi} |\phi_r|^2 d\theta - \int_0^{2\pi} \frac{1}{r^2} \langle \psi, \partial_\theta \cdot \psi_\theta \rangle d\theta, \end{aligned}$$

where  $(r, \theta)$  are the polar coordinates in  $D$  centered at 0,  $\psi_r := \tilde{\nabla}_{\partial_r} \psi$ .

*Proof.* From Proposition 3.2,

$$\begin{aligned} T &= [|\phi_x|^2 - |\phi_y|^2 - 2i\langle \phi_x, \phi_y \rangle] + [\langle \psi, \partial_x \cdot \tilde{\nabla}_{\partial_x} \psi \rangle - i\langle \psi, \partial_x \cdot \tilde{\nabla}_{\partial_y} \psi \rangle] \\ &=: (A - iB) + (A' - iB'), \quad z = x + iy \in D \end{aligned}$$

is holomorphic on  $D \setminus \{0\}$ . By Corollary 4.4, we know that

$$|A - iB| \leq 2(|\phi_x|^2 + |\phi_y|^2) \leq C|z|^{-2}.$$

Noting that  $\tilde{\nabla}_{e_\alpha} \psi = (\nabla_{e_\alpha} \psi^i + \Gamma_{jk}^i \phi_\alpha^j \psi^k) \otimes \partial_{y^i}$ , we have

$$|\tilde{\nabla} \psi| \leq C(|\nabla \psi| + |d\phi||\psi|).$$

By Corollary 4.4 again, we have

$$|A' - iB'| \leq 2|\psi||\tilde{\nabla} \psi| \leq C(|\psi||\nabla \psi| + |d\phi||\psi|^2) \leq C|z|^{-2}.$$

Therefore  $|T(z)| \leq C|z|^{-2}$ . Furthermore,

$$\int_D |A' - iB'| \leq 2 \int_D |\psi||\tilde{\nabla} \psi| < \infty$$

and  $\int_D |A - iB| \leq 2 \int_D |d\phi|^2 < \infty$ . Thus,  $\int_D |T(z)| < \infty$ , which implies that  $T(z)$  has a pole at  $z = 0$  of order at most one. Hence,  $zT(z)$  is holomorphic on  $D$  and

$$(4.13) \quad 0 = \text{Im} \left[ \int_{|z|=r} zT(z) dz \right] = \int_0^{2\pi} \text{Re}[z^2 T(z)] d\theta.$$

It is easy to compute that

$$(4.14) \quad \begin{aligned} \text{Re}[z^2 T(z)] &= r^2[(A \cos 2\theta + B \sin 2\theta) + (A' \cos 2\theta + B' \sin 2\theta)] \\ &= r^2|\phi_r|^2 - |\phi_\theta|^2 - \langle \psi, \partial_\theta \cdot \psi_\theta \rangle. \end{aligned}$$

Now the desired equalities follow.  $\square$

**Remark 4.6.** Integrating (4.12) yields:

$$(4.15) \quad \int_D |\phi_r|^2 - \int_D \frac{1}{r^2} |\phi_\theta|^2 = - \int_D \langle \psi, \partial_r \cdot \psi_r \rangle := I.$$

From (4.12) and

$$\int_{|z|=r} |\phi_r|^2 + \int_{|z|=r} \frac{1}{r^2} |\phi_\theta|^2 = \int_{|z|=r} |d\phi|^2 := E_r(\phi)$$

we have

$$(4.16) \quad \int_{|z|=r} |\phi_r|^2 = \frac{1}{2}E_r + \frac{1}{2}I_r,$$

and

$$(4.17) \quad \int_{|z|=r} \frac{1}{r^2} |\phi_\theta|^2 = \frac{1}{2}E_r - \frac{1}{2}I_r,$$

where  $I_r := - \int_{|z|=r} (\psi, \partial_r \cdot \psi_r)$ .

Now we can state the following

**Theorem 4.7 (Removable singularity theorem)** *Let  $(\phi, \psi)$  be a solution of (1.4) and (1.5) which is  $C^\infty$  on  $U \setminus \{p\}$  for some  $p \in U \subset M$ . If  $(\phi, \psi)$  has finite energy, then  $(\phi, \psi)$  extends to a  $C^\infty$  solution on  $U$ .*

*Proof.* By rescaling, we may assume that

$$(4.18) \quad \int_{D(2)} (|d\phi|^2 + |\psi|^4) < \varepsilon.$$

Choose a function  $q(r)$  on  $D$  which is piecewise linear in  $\log r$  with

$$q(2^{-m}) = \frac{1}{2\pi} \int_0^{2\pi} \phi(2^{-m}, \theta) d\theta,$$

then we have (cf. [14])

$$(4.19) \quad \int_D |dq - d\phi|^2 = \int_{r=1} (q - \phi)\phi_r - \int_D (q - \phi)\Delta(q - \phi)$$

with

$$(4.20) \quad |q - \phi|_\infty := |q - \phi|_{C^0(D)} < 2^3 \sqrt{\varepsilon}.$$

Using (3.21), we have  $\Delta(q - \phi) = -\Delta\phi = -A(\phi)(d\phi, d\phi) - \alpha$ , so,

$$(4.21) \quad \begin{aligned} \left| \int_D (q - \phi)\Delta(q - \phi) \right| &\leq |q - \phi|_\infty |A|_\infty \int_D |d\phi|^2 + |q - \phi|_\infty \int_D |\alpha| \\ &< \delta \int_D |d\phi|^2 + C\sqrt{\varepsilon} \int_D |\psi|^2 |d\phi|, \end{aligned}$$

where  $2^3 \sqrt{\varepsilon} |A|_\infty < \delta$  for some constant  $\delta > 0$  small, and  $C > 0$  is a constant. As for the first term on the RHS of (4.19),

$$(4.22) \quad \begin{aligned} \int_{r=1} (q - \phi)\phi_r &\leq \left( \int_{r=1} |q - \phi|^2 \right)^{\frac{1}{2}} \left( \int_{r=1} |\phi_r|^2 \right)^{\frac{1}{2}} \\ &\leq \left( \int_{r=1} |\phi_\theta|^2 \right)^{\frac{1}{2}} \left( \int_{r=1} |\phi_r|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand, by (4.16),

$$(4.23) \quad \int_D |dq - d\phi|^2 \geq \int_D \frac{1}{r^2} |\phi_\theta|^2 = \frac{1}{2}E(\phi) - \frac{1}{2}I,$$

where  $I := -\int_D \langle \psi, \partial_r \cdot \psi_r \rangle$ . Inserting (4.21), (4.22) and (4.23) into (4.19), we get

$$(4.24) \quad \begin{aligned} \frac{1}{2} \int_D |d\phi|^2 - \frac{1}{2} I &\leq \left( \int_{r=1} |\phi_\theta|^2 \right)^{\frac{1}{2}} \left( \int_{r=1} |\phi_r|^2 \right)^{\frac{1}{2}} \\ &+ \delta \int_D |d\phi|^2 + C\sqrt{\varepsilon} \int_D |\psi|^2 |d\phi|. \end{aligned}$$

By (4.16) and (4.17), we have

$$\begin{aligned} \left( \int_{r=1} |\phi_\theta|^2 \right)^{\frac{1}{2}} \left( \int_{r=1} |\phi_r|^2 \right)^{\frac{1}{2}} &= \left( \frac{1}{2} E_1 - \frac{1}{2} I_1 \right)^{\frac{1}{2}} \left( \frac{1}{2} E_1 + \frac{1}{2} I_1 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} E_1 = \frac{1}{2} \int_{r=1} |d\phi|^2. \end{aligned}$$

It follows

$$(4.25) \quad (1 - 2\delta) \int_D |d\phi|^2 \leq \int_{r=1} |d\phi|^2 + [2C\sqrt{\varepsilon} \int_D |\psi|^2 |d\phi| - \int_D \langle \psi, \partial_r \cdot \psi_r \rangle].$$

By a scaling argument, this yields

$$(4.26) \quad \begin{aligned} (1 - 2\delta) \int_{D_r} |d\phi|^2 &\leq r \int_{\partial D_r} |d\phi|^2 + [2C\sqrt{\varepsilon} \int_{D_r} |\psi|^2 |d\phi| - \int_{D_r} \langle \psi, \partial_r \cdot \psi_r \rangle] \\ &\leq r \int_{\partial D_r} |d\phi|^2 + C\sqrt{\varepsilon} \int_{D_r} |\psi|^4 + C\sqrt{\varepsilon} \int_{D_r} |d\phi|^2 + \int_{D_r} |\psi| |\nabla \psi| \\ &\leq r \int_{\partial D_r} |d\phi|^2 + C\sqrt{\varepsilon} \int_{D_r} |d\phi|^2 + C \int_{D_r} |\psi|^4 + C \int_{D_r} |\nabla \psi|^{\frac{4}{3}}. \end{aligned}$$

Now we need the following lemma. It is the elliptic estimate with boundary. For completeness we give a proof here.

**Lemma 4.8.** *Let  $u$  be a complex function satisfying*

$$(4.27) \quad \begin{cases} \bar{\partial}u = f, & \text{in } D, \\ u|_{\partial D} = \varphi, \end{cases}$$

with  $\varphi \in L^{1,p}(\partial D)$  and  $f \in L^p(D)$  for some  $p > 1$ , where  $D$  is the unit disc on  $\mathbb{R}^2$  centered at the origin, then the following estimate holds

$$(4.28) \quad |u|_{D,1,p} \leq C(|f|_{D,0,p} + |\varphi|_{\partial D,1,p}).$$

If instead  $u$  satisfies

$$(4.29) \quad \begin{cases} \partial u = f, & \text{in } D, \\ u|_{\partial D} = \varphi, \end{cases}$$

then the same estimate holds.

*Proof.* We first consider the following boundary value problem:

$$(4.30) \quad \begin{cases} \bar{\partial}w_1 = 0, \\ \operatorname{Re} w_1|_{\partial D} = \varphi, \quad \int_{\partial D} \operatorname{Im} w_1 = 2\pi c_0, \end{cases}$$

where  $c_0 := \frac{1}{2\pi} \int_{\partial D} \operatorname{Im} \varphi$ . It is clear that (cf. [2], Theorem 38):

$$(4.31) \quad |w_1|_{C^\alpha(D)} + |\nabla w_1|_{L^p(D)} \leq C(|\varphi|_{\partial D,1,p} + c_0),$$

where  $C > 0$  is a constant.

Next, let  $w_2 := u - w_1$ , then it satisfies

$$(4.32) \quad \begin{cases} \bar{\partial} w_2 = f & \text{in } D, \\ \text{Re} w_2|_{\partial D} = 0, \quad \int_{\partial D} \text{Im} w_2 = 0. \end{cases}$$

The elliptic estimates yield that

$$(4.33) \quad |w_2|_{L^{1,p}(D)} \leq C|f|_{L^p(D)}.$$

In fact, by the Schwarz-Poisson formula (see Theorem 21 in [2]), one has

$$(4.34) \quad w_2 = -\frac{1}{\pi} \int_{|\zeta| < 1} \left( \frac{f}{\zeta - z} + \frac{\bar{z}\bar{f}}{1 - z\bar{\zeta}} \right) d\xi d\eta.$$

From this and the boundedness of the Riesz transformation, we obtain (4.33). Combining (4.31) and (4.33) then gives the desired estimate (4.28). The proof is similar if  $u$  satisfies (4.29).  $\square$

Now we return to the proof of Theorem 4.7. Recall (3.22)

$$(4.35) \quad \bar{\partial} \psi = A(d\phi(e_\alpha), e_\alpha \cdot \psi) \quad \text{in } D \setminus \{0\}.$$

We choose a cut-off function  $\eta_\varepsilon \in C_0^\infty(D_{2\varepsilon})$  such that  $\eta_\varepsilon = 1$  in  $D_\varepsilon(0)$  and  $|d\eta_\varepsilon| < C/\varepsilon$ . Then we have

$$\bar{\partial}((1 - \eta_\varepsilon)\psi) = (1 - \eta_\varepsilon)A(d\phi(e_\alpha), e_\alpha \cdot \psi) - \nabla \eta_\varepsilon \cdot \psi.$$

From Lemma 4.8, we have

$$(4.36) \quad |(1 - \eta_\varepsilon)\psi|_{D,1,\frac{4}{3}} \leq C|d\phi|_{D,0,2}|\psi|_{D,0,4} + C|\psi|_{\partial D,1,\frac{4}{3}} + C|\nabla \eta_\varepsilon \cdot \psi|_{D,0,\frac{4}{3}}.$$

Letting  $\varepsilon \rightarrow 0$ , using

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\frac{4}{3}}} \int_{D_{2\varepsilon}} |\psi|^{\frac{4}{3}} = 0,$$

the smallness of  $|d\phi|_{D,0,2}$  and the Sobolev embedding theorem, we obtain

$$\left( \int_D |\psi|^4 \right)^{\frac{1}{4}} \leq C \left( \int_{\partial D} |\nabla \psi|^{\frac{4}{3}} \right)^{\frac{3}{4}} + C \left( \int_{\partial D} |\psi|^4 \right)^{\frac{1}{4}}.$$

By rescaling, we have for any  $0 \leq r \leq 1$

$$\begin{aligned} \left( \int_{D_r} |\psi|^4 \right)^{\frac{1}{4}} &\leq C(r \int_{\partial D_r} |\nabla \psi|^{\frac{4}{3}})^{\frac{3}{4}} + C(r \int_{\partial D_r} |\psi|^4)^{\frac{1}{4}} \\ &\leq C(r \int_{\partial D_r} |\nabla \psi|^{\frac{4}{3}})^{\frac{1}{4}} + C(r \int_{\partial D_r} |\psi|^4)^{\frac{1}{4}}. \end{aligned}$$

Thus,

$$(4.37) \quad \int_{D_r} |\psi|^4 \leq Cr \int_{\partial D_r} |\nabla \psi|^{\frac{4}{3}} + Cr \int_{\partial D_r} |\psi|^4.$$

Let  $\bar{\psi} := \frac{1}{2\pi} \int_{\partial D} \psi$ . Note that

$$\bar{\partial}(\psi - \bar{\psi}) = A(d\phi(e_\alpha), e_\alpha \cdot (\psi - \bar{\psi})) + A(d\phi(e_\alpha), e_\alpha \cdot \bar{\psi}) \quad \text{in } D \setminus \{0\}.$$

By an argument similar to the one used in obtaining (4.36) and using the Poincaré inequality, we have

$$(4.38) \quad \begin{aligned} |\psi - \bar{\psi}|_{D,1,\frac{4}{3}} &\leq C|d\phi|_{D,0,2}|\psi - \bar{\psi}|_{D,1,\frac{4}{3}} + C(|d\phi||\bar{\psi}|)_{D,0,\frac{4}{3}} \\ &\quad + C|\nabla \psi|_{\partial D,0,\frac{4}{3}}. \end{aligned}$$

Again, by the smallness of  $|d\phi|_{D,0,2}$  we obtain

$$\begin{aligned} \left(\int_D |\nabla\psi|^{\frac{4}{3}}\right)^{\frac{3}{4}} &\leq C\left(\int_{\partial D} |\nabla\psi|^{\frac{4}{3}}\right)^{\frac{3}{4}} + C|\bar{\psi}|\left(\int_D |d\phi|^2\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{\partial D} |\nabla\psi|^{\frac{4}{3}}\right)^{\frac{3}{4}} + C\left(\int_{\partial D} |\psi|^4\right)^{\frac{1}{4}}\left(\int_D |d\phi|^2\right)^{\frac{1}{2}}. \end{aligned}$$

So we have

$$\begin{aligned} \int_D |\nabla\psi|^{\frac{4}{3}} &\leq C\int_{\partial D} |\nabla\psi|^{\frac{4}{3}} + C\left(\int_{\partial D} |\psi|^4\right)^{\frac{1}{3}}\left(\int_D |d\phi|^2\right)^{\frac{2}{3}} \\ &\leq C\int_{\partial D} |\nabla\psi|^{\frac{4}{3}} + \varepsilon_1\int_D |d\phi|^2 + \frac{C}{\varepsilon_1}\int_D |\psi|^4, \end{aligned}$$

where  $\varepsilon_1 > 0$  is a small constant. Hence, for  $0 \leq r \leq 1$ ,

$$(4.39) \quad \int_{D_r} |\nabla\psi|^{\frac{4}{3}} \leq Cr\int_{\partial D_r} |\nabla\psi|^{\frac{4}{3}} + \varepsilon_1\int_{D_r} |d\phi|^2 + \frac{Cr}{\varepsilon_1}\int_{\partial D_r} |\psi|^4.$$

Putting (4.26), (4.37) and (4.39) together, we have for any  $0 \leq r \leq 1$  and some constant  $C > 0$

$$(4.40) \quad \int_{D_r} |d\phi|^2 + \int_{D_r} |\psi|^4 + \int_{D_r} |\nabla\psi|^{\frac{4}{3}} \leq Cr\left(\int_{\partial D_r} |d\phi|^2 + \int_{\partial D_r} |\psi|^4 + \int_{\partial D_r} |\nabla\psi|^{\frac{4}{3}}\right).$$

Denote  $F(r) := \int_{D_r} |d\phi|^2 + \int_{D_r} |\psi|^4 + \int_{D_r} |\nabla\psi|^{\frac{4}{3}}$ . (4.40) implies that

$$(4.41) \quad F(r) \leq CrF'(r).$$

Integrating this inequality yields

$$(4.42) \quad F(r) \leq F(1)r^{\frac{1}{C}}.$$

From this we can easily conclude that there are some  $\beta > 1$  and  $2 > q > \frac{4}{3}$  such that

$$(4.43) \quad \phi \in L^{1,2\beta}(D), \quad \psi \in L^{1,q}(D).$$

Now let us consider  $\psi$ . Recall that  $\partial\psi = A(d\phi(e_\alpha), e_\alpha \cdot \psi) := a$ , and note that

$$\begin{aligned} \int_{D_{\frac{1}{2}}} |a|^{q_1} &\leq C\int_{D_{\frac{1}{2}}} (|d\phi||\psi|)^{q_1} \\ &\leq C\left(\int_{D_{\frac{1}{2}}} |d\phi|^{q_1 p'}\right)^{\frac{1}{p'}}\left(\int_{D_{\frac{1}{2}}} |\psi|^{q_1 q'}\right)^{\frac{1}{q'}}, \end{aligned}$$

where  $q_1 > 0, p'$  and  $q'$  are constants chosen as follows

$$q_1 = \frac{2\beta q}{2\beta - (\beta - 1)q}, \quad p' = \frac{2q}{2q - (2 - q)q_1}, \quad q' = \frac{2q}{(2 - q)q_1}.$$

Since  $\beta > 1$ , we have  $q_1 > q$  and

$$(4.44) \quad \int_{D_{\frac{1}{2}}} |a|^{q_1} \leq C\left(\int_{D_{\frac{1}{2}}} |d\phi|^{2\beta}\right)^{\frac{1}{p'}}\left(\int_{D_{\frac{1}{2}}} |\psi|^{q^*}\right)^{\frac{1}{q'}} < \infty,$$

where  $q^* := \frac{2q}{2-q}$ . We note that  $\psi \in L^{q_1}$  because of  $q_1 < q^*$ .

From the regularity theory of Cauchy-Riemann operators we have

$$(4.45) \quad \psi \in L_1^{q_1} \quad \text{with} \quad q_1 = \frac{2\beta q}{2\beta - (\beta - 1)q} > q.$$

From above, we see that  $\phi \in L^{1,2\beta}$  and  $\psi \in L^{1,q}(4/3 < q < 2)$  imply  $\phi \in L^{1,2\beta}$  and  $\psi \in L^{1,q_1}$ . By iteration,  $\phi \in L^{1,2\beta}$  and  $\psi \in L^{1,q_n}$  imply  $\phi \in L^{1,2\beta}$  and  $\psi \in L^{1,q_{n+1}}$  with

$$q_{n+1} = \frac{2\beta q_n}{2\beta - (\beta - 1)q_n}.$$

Since

$$\frac{q_{n+1}}{q_n} = \frac{2\beta}{2\beta - (\beta - 1)q_n} > \frac{3\beta}{\beta + 2} > 1,$$

there exists  $q_n$  such that  $q_n > 2$ . Therefore, we have that  $\phi \in L^{1,p_0}(D)$  and  $\psi \in L^{1,q_0}(D)$  for some  $p_0 > 2$  and  $q_0 > 2$ . We can then conclude that  $(\phi, \psi)$  is smooth on  $D$  through the standard bootstrap method. We omit the details here. This completes the proof of the theorem.  $\square$

## REFERENCES

- [1] O.Alvarez, I.M.Singer, Beyond elliptic genus, *Nucl. Phys.* B633(2002), 309-344
- [2] H.G.W.Begehr, Complex analytic methods for partial differential equations, *World Scientific*, 1994.
- [3] K.C.Chang, Heat flow and boundary value problem for harmonic maps, *Anal.Nonlinéaire* 6(1989), 363-396
- [4] Q.Chen, J.Jost, J.Y.Li, G.F.Wang, Regularity and energy identities for Dirac-harmonic maps, *Math.Zeitschr.*, to appear
- [5] P.Deligne et al.(eds.), Quantum fields and strings: A course for mathematicians, Vol.1, *AMS & Inst.Adv.Study* 1999
- [6] *dto*, Vol.2
- [7] F.Hélein, Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne, *CRAS Paris* 312(1991), 591-596
- [8] N.Hitchin, Harmonic spinors, *Adv. Math.* 14(1974),1-55.
- [9] J.Jost, Two-dimensional geometric variational problems, *Wiley-Interscience*, 1991
- [10] J.Jost, Riemannian Geometry and geometric analysis, *3rd edition*, *Springer-Verlag*, 2002
- [11] H.Lawson and M.L.Michelsohn, Spin geometry, *Princeton University Press*, 1989
- [12] L.Lemaire, Applications harmoniques de surfaces riemanniennes, *J.Diff.Geom.* 13(1978), 51-78
- [13] T.H.Parker, Gauge theories on four dimensional Riemannian manifolds, *Comm. Math. Phys.* 85(1982), 563-602.
- [14] J.Sacks and K.Uhlenbeck, The existence of minimal immersions of 2-spheres, *Ann. Math.* 113(1981), No.1, 1-24.
- [15] M.Struwe, On the evolution of harmonic mappings, *Comment.Math.Helv.* 60(1985), 558-581
- [16] Y.L.Xin, Geometry of harmonic maps, *Birkhäuser*, 1996.

SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL CHINA NORMAL UNIVERSITY, WUHAN 430079, CHINA

*E-mail address:* qunchen@mail.ccnu.edu.cn

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22-26, D-04103 LEIPZIG, GERMANY

*E-mail address:* jjost@mis.mpg.de

PARTNER GROUP OF MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSTITUTE OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING 100080, P. R. OF CHINA

*E-mail address:* lijia@mail.amss.ac.cn

MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22-26, D-04103 LEIPZIG, GERMANY

*E-mail address:* gwang@mis.mpg.de