DIRAC'S COSMOLOGY AND THE GENERAL THEORY OF RELATIVITY

C. Gilbert

(Received 1956 November 9)

Summary

An explanation is given, according to the principles of the general theory of relativity, of some results previously obtained by Dirac. It is assumed that there is a different unit of distance for electromagnetic phenomena from the unit of distance used for describing gravitational phenomena. The ratio of these units varies with the time. The former unit gives the usual "cosmic" distance and a Newtonian law of gravitation which is independent of the time, the latter unit leads to a Newtonian law of gravitation for which the gravitational power of matter varies inversely as the epoch.

Formulae are derived which enable the age of the universe ($\bar{t}$), Hubble's constant (H) and the mean density of matter in the universe (ω), to be calculated from the values of the constants of atomic theory and the present observed value of the gravitational "constant". It is found that $\bar{t}=4{\cdot}1\times10^9$ years, $H=160$ km/sec/megaparsec, and $\omega=4{\cdot}8\times10^{-29}$ gm/cm^3.

1. *Introduction.*—Nearly twenty years ago Dirac gave "a new basis for cosmology" (**1**), in which he used some numerical coincidences which occur between large dimensionless numbers formed from the constants of Nature, to show that the gravitational "constant" is in fact not constant, but a quantity γ which varies inversely as the epoch. Dirac gave a relativistic treatment of the theory, but unfortunately the method he used is open to criticism (**2**), and consequently his results have not been generally accepted. It is, in fact, generally believed that a theory of gravitation with variable γ is beyond the scope of the general theory of relativity. In this paper I show that this is not the case and I obtain results similar to those given by Dirac by use of the principles of the general theory of relativity.

The usual derivation of the Einstein constant κ depends on obtaining an approximation to Poisson's equation for weak static fields, without considering the relation of these fields to the field generated by the rest of the matter in the universe. This has led to the assumption having been made that the spatial coordinates and the time occurring in Newtonian theory are the same as those for which the velocity of light *in vacuo* is constant. It will be shown that this is not necessarily the case, and, in order to bring this point out clearly, we derive the value of κ in a different manner, which makes use of equations for the relativistic models of the universe, which form the basis of "Newtonian Cosmology" (**3**, **4**). We then show how Newton's law of gravitation can be derived from the relativistic equations of motion, for both local and large-scale gravitational phenomena, in two forms. In one form the usual "cosmic" coordinates are used and the gravitational power of matter is constant, in the other form a different unit of distance is used and the gravitational power γ varies inversely as the epoch.

Classical electrodynamics is independent of Newtonian mechanics. It is not therefore necessary for the spatial coordinates used in forming Maxwell's equations *in vacuo* to be the same as those occurring in Newton's law of gravitation. We shall assume that the former coordinates correspond to *cosmic* distance coordinates and the latter coordinates are those which give rise to the value of γ which varies inversely as the epoch. We also assume that the measurements of the coordinates and other quantities occurring in the description of the Universe are based on atomic units of mass, length, and time. The rather tentative nature of the approach, which comes from making these assumptions, is made more rigorous in Section 10, where it is shown that Dirac's equation expresses the relation between the units of the electrical and gravitational fields in a local gauge-system.

We consider the Einstein–de Sitter model of the universe (**5**), and we show that the present epoch may be calculated for this model, from the present values of the physical constants. The properties of the model may then be fully determined without making assumptions about the values of any arbitrary constants.

P. Jordan (**6, 7**) and E. A. Milne (**8**), have described models for which the physical constants varied with the time, but the present approach is different from theirs, which used different mathematical techniques. The cosmology of Jordan is based on Dirac's results, and uses the technique of projective relativity, whereas the rather different results of Milne's theory were obtained by the methods of kinematic relativity.

2. *The atomic units of measurement.*—We assume that the metric of the expanding universe has the form

$$ds^2 = c^2\,dt^2 - R^2(dx_1^2 + dx_2^2 + dx_3^2), \tag{1}$$

where R is a function of t only, c is the velocity of light and the 3-spaces $t=$ constant are conformal to Euclidean space in which radial distance $r=(x_1^2+x_2^2+x_3^2)^{1/2}$.

We assume that the matter of the universe exerts negligible pressure and that the cosmological constant occurring in the field equations is zero. We then find from the field equations that the mean density of matter ρ_{00} is given by

$$\kappa\rho_{00} = 3\dot{R}^2/R^2c^2, \tag{2}$$

where κ is Einstein's constant, and

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = 0. \tag{3}$$

From (3) we find

$$\dot{R}^2 = K/R, \tag{4}$$

where K is a constant of integration. A further integration gives

$$R = (\tfrac{3}{2}K^{1/2}t + A)^{2/3}, \tag{5}$$

where A is a constant. We can without loss of generality take $A=0$ and

$$K = \tfrac{4}{9}C^2. \tag{6}$$

Equation (5) then gives

$$R = (Ct)^{2/3}. \tag{7}$$

The system of coordinates (t, x_1, x_2, x_3) we call *basic* coordinates. These coordinates are usually said to be co-moving because the coordinate distance between any two particles of matter is constant. We assume that although the matter in the universe may be treated mathematically as a continuous distribution, it does in fact consist of protons and electrons of masses m_p and m_e gm and that associated with these particles there are fundamental intervals of time ϵ secs and distance δ cm. The basic coordinates are assumed to be measured in terms of these fundamental constants as units of time and length. We also assume that the average charge density in the neighbourhood of any event is zero, and that t at any event is measurable from the local properties of matter, and therefore has the nature of an absolute time and gives a meaning to simultaneity of events independently of light-signals between events.

3. *The value of Einstein's constant.*—The equation governing the radial motion of matter in the universe can be put in Newtonian form when the distance coordinates are

$$X_\alpha = R x_\alpha \quad (\alpha = 1, 2, 3). \tag{8}$$

Writing

$$\rho = (X_1^2 + X_2^2 + X_3^2)^{1/2} = (Ct)^{2/3} r, \tag{9}$$

and $v = d\rho/dt$, we find that

$$v = \frac{2}{3}\frac{\rho}{t}. \tag{10}$$

The equation giving the radial motion of matter is then

$$\frac{dv}{dt} = -\frac{2}{9}\frac{\rho}{t^2}. \tag{11}$$

Writing $M(\rho)$ for the mass contained within a sphere of radius ρ, we find from (2), (4), (6) and (7)

$$M(\rho) = \frac{16\pi}{9\kappa c^2}\frac{\rho^3}{t^2}, \tag{12}$$

$$\frac{dv}{dt} = -\frac{\Gamma M(\rho)}{\rho^2}, \tag{13}$$

where Γ is a constant satisfying

$$\kappa = \frac{8\pi\Gamma}{c^2}. \tag{14}$$

An observer who uses coordinates (t, X_1, X_2, X_3) and uses Euclidean space will describe the system as subject to a Newtonian law of gravitation, with gravitational constant Γ, provided he assumes that matter at distances greater than ρ gives no resultant gravitational force on the matter inside a sphere of radius ρ. This result, which was obtained by Milne (3), is used in the present instance to obtain the relation (14) defining κ.

4. *The local gravitational field.*—The coordinates (t, X_1, X_2, X_3) are usually called *cosmic* coordinates. It has been shown (9) that when *cosmic* spatial coordinates are used the gravitational field of a particle of mass m in the universe is of the static Schwarzschild form, having the line element

$$ds^2 = c^2 e^\alpha d\tau^2 - e^{-\alpha} d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{15}$$

where

$$e^\alpha = 1 - \frac{2\Gamma m}{c^2 \rho}, \tag{16}$$

and (ρ, θ, ϕ) are the spherical polar coordinates corresponding to cosmic coordinates (X_1, X_2, X_3) for Euclidean space.

The condition that the field (15) shall join continuously onto the field (1) at the 3-space $r=a$ has been shown to be (9)

$$m=\frac{Ka^3}{2\Gamma}. \tag{17}$$

We assume that Γ is chosen to make $a=1$ when $m=1$. From (6) and (17) we then find

$$\Gamma=\frac{2}{9}C^2. \tag{18}$$

The time τ in (15) in general differs from cosmic time, but for speeds very much less than the speed of light, and when $\Gamma m/c^2\rho \ll 1$, $d\tau$ is approximately equal to ds/c, which is the cosmic time measured by an observer at himself. It is also known that for speeds very much less than the speed of light the equations of motion for a free particle in the field (15) have the Newtonian form when ds/c plays the part of Newtonian time. *Cosmic* time is therefore approximately the same as Newtonian time for local gravitational phenomena.

5. *The velocity of light.*—The equations of the general theory of relativity governing both the large scale motions of matter in the universe and the orbital motions about a condensation of matter, have been shown to agree with the equations of motion derived from Newtonian theory, when *cosmic* coordinates are used. We now show that the relativistic equations giving the propagation of light *in vacuo* are in agreement with Maxwell's equations provided that *cosmic* time and *cosmic* spatial coordinates are used in the latter equations.

Consider a spherical wave emitted from $r=0$ at time $t=t_0$. Since the light paths are the null geodesics of (1), we find that in basic coordinates the wave front at time t is given by

$$r=3cC^{-2/3}(t^{1/3}-t_0^{1/3}). \tag{19}$$

From (7) and (19) the wave front in *cosmic* coordinates is

$$\rho=3c(t-t^{2/3}t_0^{1/3}). \tag{20}$$

From (20) we find that the velocity of light in cosmic coordinates tends to the limiting value $3c$ as t tends to infinity. When $t=\infty$, the coordinate density of matter is zero, and therefore when *cosmic* coordinates are used the velocity of light *in vacuo* is $3c$.

The constant C in (7) was arbitrary. We now assume that it is a dimensionless number formed from the constants of atomic theory. The cosmic coordinates are then expressed in terms of constants of atomic theory. Since it is always possible to form a quantity having the dimensions of a length from one having the dimensions of time by multiplication by the velocity of light, it is possible to choose one of the quantities ϵ and δ so that the velocity of light in cosmic coordinates is unity. Hence we can take

$$c=\tfrac{1}{3}. \tag{21}$$

6. *The variation of the gravitational power of matter with time.*—We now show that the relativistic equations can be put into the Newtonian form when spatial coordinates differing from *cosmic* coordinates are used. In these coordinates γ is found to vary with the time. Let these coordinates ζ_α $(\alpha=1, 2, 3)$ be defined by

$$\zeta_\alpha=\lambda x_\alpha, \tag{22}$$

48

where λ is a slowly varying function of t, and write $\zeta=(\zeta_1^2+\zeta_2^2+\zeta_3^2)^{\frac{1}{2}}$, $\bar{v}=d\zeta/dt$. We then find that the radial motion of the matter in the universe is given by

$$\frac{d\bar{v}}{dt}=\frac{\ddot{\lambda}}{\lambda}\zeta. \tag{23}$$

Writing $M(\zeta)$ for the mass of the homogeneous distribution contained within a sphere of radius ζ we find from (12) and (22) that

$$M(\zeta)=\frac{\zeta^3}{\lambda^3} \tag{24}$$

and

$$\frac{d\bar{v}}{dt}=-\frac{\gamma M(\zeta)}{\zeta^2}, \tag{25}$$

where

$$\gamma=-\lambda^2\ddot{\lambda}. \tag{26}$$

Since λ varies slowly with the time we find for the local field (15) that a free particle will move under a Newtonian law of gravitation, approximately, with

$$\gamma=\frac{\lambda^3}{R^3}\Gamma=\frac{2}{9}\frac{\lambda^3}{t^2}. \tag{27}$$

Equating the values of γ from (26) and (27) we find that λ must satisfy the equation

$$\frac{\ddot{\lambda}}{\lambda}=-\frac{2}{9t^2}. \tag{28}$$

Integration of (28) gives

$$\lambda=C_1t^{2/3}+C_2t^{1/3}, \tag{29}$$

where C_1 and C_2 are arbitrary constants.

7. The case $C_1=C^{2/3}$, $C_2=0$ gives the *cosmic* system of coordinates, which have already been discussed. We now discuss the case $C_1=0$, $C_2=1$ giving coordinates $(t, \zeta_1, \zeta_2, \zeta_3)$, which we shall call *local* coordinates.

We have from (22)

$$\zeta_\alpha=t^{1/3}x_\alpha, \tag{30}$$

and from (27)

$$\gamma=\frac{2}{9}\cdot\frac{1}{t}. \tag{31}$$

This shows that in *local* coordinates the gravitational power of matter varies inversely as the epoch, in agreement with a result obtained by Dirac (1).

8. We assume that a terrestial observer uses *cosmic* coordinates for the description of electromagnetic phenomena and *local* coordinates for the description of gravitional phenomena. Also instead of using the atomic units of length mass and time he uses the c.g.s. system of measurement in which the atomic units have the values δ, m_p and ϵ respectively.

Writing $\bar{c}$, $\bar{t}$ and $\bar{\gamma}$ for the velocity of light, the epoch and the gravitational power of matter in the c.g.s. system we have

$$\bar{c}=\frac{\delta c}{\epsilon}=\frac{\delta}{3\epsilon}, \tag{32}$$

$$\bar{t}=\epsilon t, \tag{33}$$

$$\bar{\gamma}=\frac{\delta^3}{m_p\epsilon^2}\gamma, \tag{34}$$

From (31), (32), (33) and (34) we find

$$\bar{\gamma}=\frac{6\epsilon^2\bar{c}^3}{m_p\bar{t}}. \tag{35}$$

The quantity ϵ has to be chosen from amongst the quantities having the dimensions of time which can be formed from the constants of atomic theory. We take $\epsilon=e^2/m_e\bar{c}^3$, where e, m_e are the charge and mass of an electron. This was the unit of time taken by Dirac and we find that it leads to numerical results which are in good agreement with observation. Substituting this value for ϵ in (35) we find

$$t=\frac{\bar{t}}{\epsilon}=\frac{6e^2}{m_p m_e\bar{\gamma}}. \tag{36}$$

9. *Comparison with observation.*—From the present value of the gravitational "constant" and the atomic constants, we can calculate the value of $\bar{t}$ from (36). We call this value of $\bar{t}$ the "age of the universe". We find that

$$\textit{the age of the universe } \bar{t}=4\cdot1\times10^9 \text{ years.} \tag{37}$$

It can be shown that *cosmic* distance corresponds closely to "luminosity distance" (2) and must therefore be used in making estimates of Hubble's constant and the mean density of matter in the universe.

From (10) we find that Hubble's constant is given by

$$H=\frac{2}{3\bar{t}}. \tag{38}$$

Substituting for $\bar{t}$ from (36) we find that

$$\textit{Hubble's constant } H=160 \text{ km/sec/megaparsec,} \tag{39}$$

corresponding to an age calculated from its reciprocal of 6×10^9 years.

The mean density of matter in the universe, ω, is the same as the proper density of matter, when cosmic distance is used. Therefore

$$\omega=\frac{1}{6\pi\Gamma t^2} \text{ protons/unit cosmic volume} \tag{40}$$

$$=\frac{1}{6\pi\bar{\gamma}\bar{t}^2}\text{ gm/cm}^3. \tag{41}$$

This gives

$$\textit{the mean density of matter } \omega=4\cdot8\times10^{-29} \text{ gm/cm}^3. \tag{42}$$

10. *The local gauge-system.*—The following generalization may be regarded as the essentially new concept of the work which has been done. *The measurements of all quantities occurring in field theory shall be based on units of mass, length and time, derived from the values of the atomic "constants" measured in a local frame with arbitrarily chosen units.* In order to discuss this generalization with regard to the Einstein–de Sitter universe we first make a transformation $t=C^2T^3$, where T is a new basic time coordinate. We find from (1), (7) and (21),

$$ds^2=(CT)^4(dT^2-dx_1^2-dx_2^2-dx_3^2). \tag{43}$$

Let the proper interval in a local frame with arbitrarily chosen units be given by

$$ds'^2=\bar{c}^2\,dT'^2-dx_1'^2-dx_2'^2-dx_3'^2, \tag{44}$$

and let the measured values of the atomic constants be $\bar{c}$, e, m_e, m_p. If we choose new units of mass, length and time m_p, $e^2/m_e\bar{c}^2$ and $e^2/m_e\bar{c}^3$ the measured values of the atomic constants are 1, $\pm(m_e/m_p)^{1/2}$, m_e/m_p, 1, respectively, and in basic coordinates (44) becomes

$$ds_0^2 = dT^2 - dx_1^2 - dx_2^2 - dx_3^2 \tag{45}$$

where $ds' = (e^2/m_e\bar{c}^2)\, ds_0$. The atomic constants are therefore absolute constants for basic coordinates provided m_e/m_p is an absolute constant, and their values will not be affected by transformations of gauge. The electric field of a stationary charge will thus transform differently from the gravitational field of a particle under a gauge transformation. From the point of view of field theory Dirac's equation determines the gauge-system in which the unit of the electrical field is defined in terms of the unit of the gravitational field. In the preceding work measurements of distances in this gauge-system have been made in *local* coordinates, and the corresponding interval of proper time in the local gauge is $ds_1 = ds/C^{4/3}T$. It is possible that this local gauge-system may be determined by a condition for the existence of free charges of the type proposed by Dirac (**10**).

In the uncharged state of the universe which we have been discussing, gauge transformations are merely transformations to conformal space-times which do not alter the field equations. The structure of the universe is then the same in the different gauge-systems, but its description is different. It is from the description of the local gravitational field that we have been able to determine the local gauge-system without a full knowledge of the field theory.

The generalization which has been arrived at is derived from the necessity for field theory to use a system of measurement in which the atomic constants are absolute constants. The suggestion of M. Born (**11, 12**) that field theory should "introduce an absolute length $(e^2/m_e\bar{c}^2)$ right from the beginning" is probably closely related, but was made for different reasons.

Department of Mathematics,
King's College,
Newcastle-upon-Tyne:
1956 *November*.

References

(1) P. A. M. Dirac, *Proc. Roy. Soc.* A, **165**, 199, 1938.
(2) H. Bondi, *Cosmology*, 162, 108, C.U.P., 1950.
(3) E. A. Milne, *Quart. Journ. of Math.*, **5**, 64, 1934.
(4) W. H. McCrea and E. A. Milne, *Quart. Journ. of Math.*, **5**, 73, 1934.
(5) A. Einstein and W. de Sitter, *Proc. Nat. Acad. of Sciences*, **18**, 213, 1932.
(6) P. Jordan, *Die Herkunft der Sterne*, Stuttgart, 1947.
(7) P. Jordan, *Nature*, **164**, 637, 1949.
(8) E. A. Milne, *Kinematic Relativity*, Oxford, 1948.
(9) C. Gilbert, *M.N.*, **116**, 678, 1956.
(10) P. A. M. Dirac, *Proc. Roy. Soc.* A, **209**, 291, 1951.
(11) M. Born, *Nature*, **169**, 1105, 1952.
(12) M. Born, *Rev. of Mod. Phys.*, **21**, 463, 1949.