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DIRAC STRUCTURES AND BOUNDARY CONTROL SYSTEMS ASSOCIATED WITH SKEW-SYMMETRIC DIFFERENTIAL OPERATORS

Y. Le Gorrec[1], H. Zwart[2], and B. Maschke[1]

Abstract. Associated with a skew-symmetric linear operator on the spatial domain $[a, b]$ we define a Dirac structure which includes the port variables on the boundary of this spatial domain. This Dirac structure is a subspace of a Hilbert space. Naturally, associated to this Dirac structure is infinite dimensional system. We parameterize the boundary port variables for which the C_0-semigroup associated to this system is contractive or unitary. Furthermore, this parameterization is used to split the boundary port variables into inputs and outputs. Similarly, we define a linear port controlled Hamiltonian system associated with the previously defined Dirac structure and a symmetric positive operator defining the energy of the system. We illustrate this theory on the example of the Timoshenko Beam.

Port Hamiltonian systems, strongly continuous semigroup, boundary control systems, Dirac structures

1. Introduction

Port Hamiltonian systems have been introduced in the finite dimensional case as an analytical frame for the modeling and control of open physical systems [18, 15, 13, 24]. The key concepts are the definition of pairs of power conjugated power variables and the geometric structure defined on them. This geometric structure is called the Dirac structure [4, 7]. These Dirac structures define as well the internal geometric structure of the physical system as the structure of their interaction with the environment [13, 25]. It reflects the (discrete) topology and the geometry of the physical system under consideration such as the port connection graph, constraints, or inter-domain coupling [18, 16, 6]. Furthermore, it is a geometric structure which allows to define *implicit* Hamiltonian systems and Hamiltonian systems with *port variables* [24, 6, 23]. This formalism revealed to be also a very useful frame for the design of stabilizing control laws by shaping the closed-loop Hamiltonian function, the dissipation, and the Dirac structures [1, 2, 22].

An extension of port Hamiltonian systems to infinite-dimensional systems has been recently proposed for distributed parameter systems with energy flow at their boundary [17, 26]. The state space is a vector space of differential forms defined on the spatial domain and the port variables are defined on the boundary of the spatial domain. The port Hamiltonian system is defined with respect to a so-called *Stokes-Dirac structure*, which in turn is uniquely defined by the exterior derivatives and the order of the differential forms. The Stokes-Dirac structure represents the canonical inter-domain coupling in physical systems (the paradigm is the harmonic oscillator [14]). Finally, the Stokes-Dirac structures have been extended in order to encompass fluid dynamics and beam models [26].

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In this paper we propose a definition of Dirac structures and port Hamiltonian systems associated with linear skew-symmetric differential operators, restricting ourselves to one-dimensional spatial domains. It extends the definition of Stokes-Dirac structures where the operator has differential degree one to higher-dimensional degrees. We use an alternative definition of a Dirac structure on Hilbert spaces as proposed in [20] and [11]. In [11] Dirac structures on Hilbert spaces have also been used for the study of their composition (interconnection) and the definition of scattering representations.

The organization of the paper is the following. In Section 2 we recall the definition of Dirac structures on Hilbert spaces. In Section 3 we define Dirac structures associated with skew-symmetric linear differential operators and its conjugated port variables on the boundary of the spatial domain. In Section 4 we associate to our Dirac structure a family of boundary control systems. The input of this boundary control system are chosen to lie in a subspace of the boundary port variables. The semigroup associated to this system is a contraction semigroup. By choosing the output to lie in the complementary of the "input subspace" we get a power balance system. The above contraction give the parameterization of all the systems. In Section 5 we give the definition of a port Hamiltonian system associated with a skew-symmetric differential operator and some Hamiltonian defined by a symmetric linear operator.

2. Dirac structures defined on Hilbert spaces

In this section we recall the definition of Dirac structures defined on Hilbert spaces proposed by Parsian and Shafei Deh Abad in [20] and by G. Golo and coauthors in [11, 8]. We shall follow the definitions and notations of [11, 8] for the purpose of analyzing and treating the composition of Dirac structures in the frame of port-based modeling and control.

Let us first define the space of *bond variables* which is constituted of pairs of conjugated variables endowed with a pairing corresponding for models of physical systems to an associated instantaneous power [12, 3]. Let the *space of flow variables,* denoted by $\mathcal{F}$ and the *space of effort variables,* denoted by $\mathcal{E}$ be real Hilbert spaces endowed with the inner products $\langle . \,, \,. \rangle_{\mathcal{F}}$ and $\langle . \,, \,. \rangle_{\mathcal{E}}$, respectively. Assume moreover that $\mathcal{F}$ and $\mathcal{E}$ are isometrically isomorphic, that is there exists an isometry: $r_{\mathcal{F},\mathcal{E}} : F \longrightarrow \mathcal{E}$. Denote furthermore its inverse by $r_{\mathcal{E},\mathcal{F}}$. Define now the *space of bond variables* as the Hilbert space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ endowed with the natural inner product

$$\langle b^1, b^2 \rangle = \langle f^1, f^2 \rangle_{\mathcal{F}} + \langle e^1, e^2 \rangle_{\mathcal{E}}, \quad b^1 = \left(f^1, e^1\right), b^2 = \left(f^2, e^2\right) \in \mathcal{B}.$$

In order to define a Dirac structure, let us moreover endow the bond space $\mathcal{B}$ with a *canonical symmetrical pairing*, i.e., a bilinear form defined as follows:

$$\langle b^1, b^2 \rangle_+ = \langle f^1, r_{\mathcal{E},\mathcal{F}} e^2 \rangle_{\mathcal{F}} + \langle e^1, r_{\mathcal{F},\mathcal{E}} f^2 \rangle_{\mathcal{E}}, \; b^1 = \left(f^1, e^1\right), b^2 = \left(f^2, e^2\right) \in \mathcal{B}. \tag{2.1}$$

Now we may define a Dirac structure on the bond space $\mathcal{B}$ using this canonical pairing. Denote by $\mathcal{D}^{\perp}$ the orthogonal subspace to $\mathcal{D}$ with respect to the symmetrical pairing (2.1):

$$\mathcal{D}^{\perp} = \left\{ b \in \mathcal{B} | \left\langle b, b' \right\rangle_+ = 0 \text{ for all } b' \in \mathcal{D} \right\}. \tag{2.2}$$

Definition 2.1. A Dirac structure $\mathcal{D}$ on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is a subspace of $\mathcal{B}$ which is maximally isotropic with respect to the canonical symmetrical pairing (2.1), i.e.,

$$\mathcal{D}^{\perp} = \mathcal{D}. \tag{2.3}$$

■

One may find different examples of such Dirac structures as well as some properties concerning their representations and their composition in [11, chapter 5]. We shall now give a canonical example of a Dirac structure in the context of port based modeling of physical systems. Therefore we consider the example a lossless vibrating string. Firstly, we recall the port based model structure [26, 14] which gives rise to the definition of a Stokes-Dirac structure on Hilbert spaces of functions with a one-dimensional domain [11]. Secondly, we recall the formulation of the evolution equation as a port Hamiltonian systems as an introduction to the objectives on the paper.

Example 2.2. Consider an elastic string defined on the one-dimensional spatial domain $Z = [a, b] \subset \mathbb{R}$ and subject to boundary conditions which allow some energy flow. Let us denote by $u(t, z)$ the displacement of the string at time t and position z. We first recall the variables and functions defining the elasto-dynamic energy of the string. The elastic potential energy is a function of the *strain*, the energy variable defined by:

$$\epsilon(t, z) = \frac{\partial u}{\partial z}(t, z). \tag{2.4}$$

The associated co-energy variable is the *stress* given by:

$$\sigma(t, z) = T(z)\, \epsilon(t, z) \tag{2.5}$$

with T denoting the elasticity modulus. Hence the potential energy is the quadratic function

$$U(\epsilon(t, \cdot)) = \frac{1}{2} \int_a^b T(z)\, \epsilon(t, z)^2 dz. \tag{2.6}$$

The kinetic energy K is a function of the kinetic *momentum*, $p(t, z)$, and defined by the quadratic function

$$K(p(t, \cdot)) = \frac{1}{2} \int_a^b \frac{p^2(t, z)}{\mu(z)}\, dz. \tag{2.7}$$

The associated co-energy variable is the *velocity* given by

$$v(t, z) = \frac{1}{\mu(z)} p(t, z), \tag{2.8}$$

where μ denotes the mass density.

The dynamical model of the vibrating string is then obtained by coupling the elastic energy physical domain and the kinetic domain through the following relations. Consider the time variation of the energy variables, called flow variables:

$$\frac{\partial}{\partial t} \begin{pmatrix} p \\ \epsilon \end{pmatrix} = \begin{pmatrix} f_K \\ f_U \end{pmatrix}. \tag{2.9}$$

The canonical inter-domain coupling between the elastic-potential and the kinetic energies relates the flow variables and the co-energy variables by the canonical differential operator [14]:

$$\begin{pmatrix} f_K \\ f_U \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} v \\ \sigma \end{pmatrix}. \tag{2.10}$$

Finally, the interaction of the vibrating string through its boundary is expressed by the definition of the boundary port variables, i.e., the velocity and stress at the boundaries of the string

$$\begin{pmatrix} w_K \\ w_U \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} v|_{a,b} \\ \sigma|_{a,b} \end{pmatrix}. \tag{2.11}$$

The canonical inter-domain coupling equation (2.10) and the boundary coupling equation (2.11) actually define a Dirac structure [11, chap. 5], called *Stokes-Dirac structure* in the following way. Consider the Hilbert spaces of flow variables $\mathcal{F} = L^2([a,b],\mathbb{R}) \times L^2([a,b],\mathbb{R}) \times \mathbb{R}^2 \ni (f_K, f_U, w_K)$ and of effort variables $\mathcal{E} = L^2([a,b],\mathbb{R}) \times L^2([a,b],\mathbb{R}) \times \mathbb{R}^2 \ni (v, \sigma, w_U)$. Furthermore, endow the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with following pairing:

$$\begin{aligned} \langle (f_K^1, f_U^1, v^1, \sigma^1, w_K^1, w_U^1)\,,\ (f_K^2, f_U^2, v^2, \sigma^2, w_K^2, w_U^2) \rangle_+ \\ = \int_a^b f_K^1\, v^2\, dz + \int_a^b f_K^2\, v^1\, dz + \\ \int_a^b f_U^1\, \sigma^2\, dz + \int_a^b f_U^2\, \sigma^1\, dz + {w_K^1}^T\, \Sigma w_U^2 + {w_K^2}^T\, \Sigma w_U^1, \end{aligned}$$

where $\Sigma = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$.

This pairing on the bond space corresponds to the general definition given in equation (2.1) where the flow is a product space $\mathcal{F} = \mathcal{F}_{(a,b)} \times \mathcal{F}_\partial$ as well the effort vector space: $\mathcal{E} = \mathcal{E}_{(a,b)} \times \mathcal{E}_\partial$. The subspace of flow variables defined on the domain $[a,b]$ $\mathcal{F}_{(a,b)} = L^2([a,b],\mathbb{R}) \times L^2([a,b],\mathbb{R}) \ni (f_K, f_U)$ and the conjugated subspace of variables $\mathcal{E}_{(a,b)} = L^2([a,b],\mathbb{R}) \times L^2([a,b],\mathbb{R}) \ni (v,\sigma)$ are equal and the isometry $r_{\mathcal{F}_{(a,b)},\mathcal{E}_{(a,b)}}$ is the identity. On the contrary for the pairing on the boundary port variables, the matrix Σ actually corresponds to the definition of an isometry $r_{\mathcal{F}_\partial,\mathcal{E}_\partial}$ between the boundary port spaces $\mathcal{F}_\partial = \mathbb{R}^2 \ni w_K$ and $\mathcal{E}_\partial = \mathbb{R}^2 \ni w_U$ endowed with the canonical Euclidean metric.

It has been shown in [11, 8], that the equations (2.10) and (2.11) define a Dirac structure, the Stokes-Dirac structure on $\mathcal{B}$ associated with the differential operator given in equation (2.10). We shall denote this Dirac structure by $\mathcal{D}_1$.

The system defined by the equations (2.10), (2.9), (2.5) and (2.8), may be rewritten as follows

$$\frac{\partial}{\partial t} \begin{pmatrix} p \\ \epsilon \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \delta_p \mathcal{H} \\ \delta_\epsilon \mathcal{H} \end{pmatrix}, \tag{2.12}$$

where $\mathcal{H}$ denotes the Hamiltonian function corresponding to the total energy of the system and $\delta_p \mathcal{H}(x) = v$, $\delta_\epsilon \mathcal{H}(x) = \sigma$ denote the *variational derivatives* [19] of $\mathcal{H}$ with respect to the momentum p, and the strain ϵ, respectively. This system is indeed a Hamiltonian system [19] if the differential operator in equation (2.12) is skew-symmetric, i.e., if the boundary variables are such that there is *no energy flow at the boundary* of the system:

$${w_K^1}^T\, \Sigma w_U^2 + {w_K^2}^T\, \Sigma w_U^1 = 0. \tag{2.13}$$

If there is some energy flow at the boundary, then the evolution equation (2.12) may be completed using the *port boundary variables* defined in equation (2.11), i.e., the velocity and the strain at the boundary

$$\begin{pmatrix} w_K \\ w_U \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \delta_p \mathcal{H}\,|_{a,b} \\ \delta_\epsilon \mathcal{H}\,|_{a,b} \end{pmatrix}. \tag{2.14}$$

The system composed of (2.12) and (2.14) defines a port Hamiltonian system with respect to the Stokes-Dirac structure and generated by the Hamiltonian $\mathcal{H}$ [26, 11, 8] and it may be written in the following implicit way:

$$\left(\frac{\partial p}{\partial t}, \frac{\partial \epsilon}{\partial t}, w_K, \delta_p \mathcal{H}, \delta_\epsilon \mathcal{H}, w_U\right) \in \mathcal{D}_1. \tag{2.15}$$

■

This canonical example has shown that the Stokes-Dirac structure $\mathcal{D}_1$, associated with the canonical inter-domain coupling, is derived from a skew-symmetric differential operator of order one. In the Section 3, we consider a generalization of this differential operator by considering skew-symmetric operators of any order and derive Dirac structures on Hilbert spaces from them. In Example 2.2 we have also seen how the dynamics is defined with respect to the canonical Dirac structure $\mathcal{D}_1$ and by a Hamiltonian function defining the energy of the system. In Section 4 we consider energy functions which are equal to the norm of the Hilbert space. Hence there the co-energy variables and the state-variables are identical. We show how to parameterize the contractive semigroups associated with the Dirac structures defined in Section 3. In Section 5 finally, we distinguish between the state and the coenergy variables by introducing more general Hamiltonian functions and define port Hamiltonian systems associated with skew-symmetric differential operators of any order.

3. Dirac structure associated with a skew-symmetric operator

In this section we extend the definition of Stokes-Dirac structures to skew-symmetric differential operators of any order. Therefore we first recall how one may extend Stokes theorem to such operators and how Stokes theorem induces a symmetric pairing on the boundary variables. Secondly, we define boundary port variables as linear combination of the boundary variables associated with the differential operator. Using these boundary port variables, we define a bond space and a Dirac structure associated with the differential operator.

Consider the differential operator $\mathcal{J}$of order N

$$\mathcal{J}e = \sum_{i=0}^{N} P(i) \frac{d^i e}{dz^i}(z) \qquad z \in [a,b]\,, \tag{3.1}$$

where $e \in C^\infty((a,b);\mathbb{R}^n)$ and $P(i)$, $i = 0, \dots, N$, is a $n \times n$ real matrix. The formal adjoint $\mathcal{J}^\star$ of $\mathcal{J}$ is given by

$$\mathcal{J}^\star e = \sum_{i=0}^{N} P(i)^T (-1)^i \frac{d^i e}{dz^i}(z) \qquad z \in [a,b]\,.$$

Now assume that $\mathcal{J}$ is skew symmetric, i.e., $\mathcal{J} = -\mathcal{J}^*$. From the above expression of $\mathcal{J}^*$ we see that this is equivalent to

$$P(i) = P(i)^T (-1)^{i+1}. \tag{3.2}$$

Using this property, we show that the bilinear symmetric pairing of e and $\mathcal{J}e$ only depends on the boundary values. Thus if the boundary values are zero, then $\langle e_1, \mathcal{J}e_2\rangle + \langle e_2, \mathcal{J}e_1\rangle = 0$, which is corresponding to the fact that $\mathcal{J}$ is (formally) skew symmetric.

Theorem 3.1. *Let $\mathcal{J}$ be a skew symmetric operator defined by (3.1), and let $H^N((a,b);\mathbb{R}^n)$ denote the Sobolev space of N times differentiable functions on the interval (a,b). Then for* *any two functions $e_i \in H^N((a,b);\mathbb{R}^n)$, $i \in \{1,2\}$ we have that*

$$\int_a^b e_1^T(z)(\mathcal{J}e_2)(z) + e_2^T(z)(\mathcal{J}e_1)(z)dz = \tag{3.3}$$
$$\left[\begin{pmatrix} e_1^T(z), & \cdots & \frac{d^{N-1}e_1^T}{dz^{N-1}}(z) \end{pmatrix} Q \begin{pmatrix} e_2(z) \\ \vdots \\ \frac{d^{N-1}e_2}{dz^{N-1}}(z) \end{pmatrix}\right]_a^b,$$

where

$$Q = (Q_{ij}) \qquad i,j = 1,\cdots,N$$

with

$$Q_{ij} = \begin{cases} 0 & i+j > N, \\ P(k)(-1)^{j-1} & i+j-1 = k. \end{cases} \tag{3.4}$$

Furthermore, Q is a symmetric matrix.

Proof. Step 1: By the definition of $\mathcal{J}$ it is easy to see that

$$\begin{aligned} \int_a^b e_1^T(z)\left(\mathcal{J}e_2\right)(z) &+ e_2^T(z)\left(\mathcal{J}e_1\right)(z)dz \\ &= \sum_{i=0}^N \int_a^b e_1^T(z)P(i)\frac{d^i e_2}{dz^i}(z) + e_2^T(z)P(i)\frac{d^i e_1}{dz^i}(z)dz \\ &= \sum_{i=1}^N \int_a^b e_1^T(z)P(i)\frac{d^i e_2}{dz^i}(z) + e_2^T(z)P(i)\frac{d^i e_1}{dz^i}(z)dz, \end{aligned} \tag{3.5}$$

where we have used that $P(0)$ is skew-adjoint. From (3.5) we see that we can restrict our proof to the particular operator $\mathcal{J}e = P(i)\dfrac{d^i e}{dz^i}$ with $i \geq 1$.

Step 2: We consider first the case of even differentiation, *i.e.*, we assume that $i = 2\ell$, and thus $\mathcal{J}e = P(i)\dfrac{d^i e}{dz^i}$, $P(i) = -P(i)^T$.

Using integration by parts, we find

$$\begin{aligned} \int_a^b e_1^T(z)P(i)\frac{d^i e_2}{dz^i}(z) &+ e_2^T P(i)\frac{d^i e_1}{dz^i}(z)dz \\ &= \left[e_1^T(z)P(i)\frac{d^{i-1}e_2}{dz^{i-1}}(z) + e_2^T(z)P(i)\frac{d^{i-1}e_1}{dz^{i-1}}(z)\right]_a^b \\ &\quad - \int_a^b \frac{de_1^T}{dz}(z)P(i)\frac{d^{i-1}e_2}{dz^{i-1}}(z) + \frac{de_2}{dz}(z)^T P(i)\frac{d^{i-1}e_1}{dz^{i-1}}(z)dz. \end{aligned}$$

Using iteratively the same kind of integration by parts, we obtain

$$\begin{aligned}\int_a^b e_1^T(z)P(i)\frac{d^i e_2}{dz^i}(z) + e_2^T(z)P(i)\frac{d^i e_1}{dz^i}(z)dz\\ = \sum_{j=0}^{\ell-1}\Bigg[(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_2}{dz^{i-1-j}}(z) + \\ (-1)^j\frac{d^j e_2}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_1}{dz^{i-1-j}}(z)\Bigg]_a^b + \\ (-1)^\ell\int_a^b \frac{d^\ell e_1}{dz^\ell}(z)^T P(i)\frac{d^\ell e_2}{dz^\ell}(z) + \frac{d^\ell e_2}{dz^\ell}(z)^T P(i)\frac{d^\ell e_1}{dz^\ell}(z)dz.\end{aligned}$$

Using now the fact that $P(i)^T = -P(i)$ and $x^T y = y^T x$, we find that

$$\begin{aligned}\int_a^b e_1^T(z)P(i)\frac{d^i e_2}{dz^i}(z) + e_2^T(z)P(i)\frac{d^i e_1}{dz^i}(z)dz\\ = \Bigg[\sum_{j=0}^{\ell-1}(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}}{dz^{i-i-j}}e_2(z) + \\ \sum_{j=0}^{\ell-1}(-1)^{j+1}\frac{d^{i-1-j}}{dz^{i-1-j}}e_1(z)^T P(i)\frac{d^j e_2}{dz^j}(z)\Bigg]_a^b + \\ (-1)^\ell\int_a^b \frac{d^\ell e_1}{dz^\ell}(z)^T P(i)\frac{d^\ell e_2}{dz^\ell}(z) - \frac{d^\ell e_1}{dz^\ell}(z)^T P(i)\frac{d^\ell e_2}{dz^\ell}(z)dz.\end{aligned}$$

Calling $i-1-j=k$ in the second sum:

$$\begin{aligned}\int_a^b e_1^T(z)P(i)\frac{d^i e_2}{dz^i}(z) + e_2^T(z)P(i)\frac{d^i e_1}{dz^i}(z)dz\\ = \Bigg[\sum_{j=0}^{\ell-1}(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_2}{dz^{i-i-j}}(z) + \\ \sum_{k=i-1}^{\ell}(-1)^{i-k}\frac{d^k e_1}{dz^k}(z)^T P(i)\frac{d^{i-1-k}e_2}{dz^{i-1-k}}(z)\Bigg]_a^b + 0\\ = \Bigg[\sum_{j=0}^{\ell-1}(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_2}{dz^{i-1-j}}(z)\\ + \sum_{j=\ell}^{i-1}(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_2}{dz^{i-1-j}}(z)\Bigg]_a^b\end{aligned}$$

which shows (3.3).

Step 3: Consider now the uneven case, i.e., we assume that $i = 2\ell + 1$ and $\mathcal{J}e = P(i)\frac{d^i e}{dz^i}$. As in Step 2, we find

$$\begin{aligned}
&\int_a^b e_1^T(z)P(i)\frac{d^i e_2}{dz^i}(z) + e_2^T(z)P(i)\frac{d^i e_1}{dz^i}(z)dz \\
&= \sum_{j=0}^{\ell-1}\left[(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_2}{dz^{i-1-j}}(z) + (-1)^j\frac{d^j e_2}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_1}{dz^{i-1-j}}(z)\right]_a^b + \\
&\quad (-1)^\ell\int_a^b \frac{d^\ell e_1}{dz^\ell}(z)^T P(i)\frac{d^{\ell+1}e_2}{dz^{\ell+1}}(z) + \frac{d^\ell e_2}{dz^\ell}(z)^T P(i)\frac{d^{\ell+1}e_1}{dz^{\ell+1}}(z)dz.
\end{aligned}$$

Using the fact that $p^T q = q^T p$ and $P(i)^T = P(i)$ we find

$$\begin{aligned}
&\int_a^b e_1^T(z)P(i)\frac{d^i e_2}{dz^i}(z) + e_2^T P(i)\frac{d^i e_1}{dz^i}(z)dz \\
&= \left[\sum_{j=0}^{\ell-1}(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_2}{dz^{i-1-j}}(z) + (-1)^j\frac{d^{i-1-j}e_1}{dz^{i-1-j}}(z)^T P(i)\frac{d^j e_2}{dz^j}(z)\right]_a^b + \\
&\quad (-1)^\ell\int_a^b \frac{d^\ell e_1}{dz^\ell}(z)^T P(i)\frac{d^{\ell+1}e_2}{dz^{\ell+1}}(z) + \frac{d^{\ell+1}e_1}{dz^{\ell+1}}(z)^T P(i)\frac{d^\ell e_2}{dz^\ell}(z)dz \\
&= \left[\sum_{j=0}^{\ell-1}(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}e_2}{dz^{i-1-j}}(z) + \right. \\
&\quad \left.\sum_{\kappa=\ell+1}^{i-1}(-1)^{i-1-\kappa}\frac{d^\kappa e_1}{dz^\kappa}(z)^T P(i)\frac{d^{i-1-\kappa}e_2}{dz^{i-1-\kappa}}(z)\right]_a^b + (-1)^\ell\left[\frac{d^\ell e_1(z)^T}{dz^\ell}P(i)\frac{d^\ell e_2}{dz^\ell}(z)\right]_a^b \\
&= \left[\sum_{j=0}^{i-1}(-1)^j\frac{d^j e_1}{dz^j}(z)^T P(i)\frac{d^{i-1-j}}{dz^{i-1-j}}e_2(z)\right]_a^b.
\end{aligned}$$

Step 4: Combining Step 1–3 and using equation (3.5), we see that we have proved the result. The symmetry follows directly from (3.4) and (3.2). ■

The above theorem shows that any skew symmetric differential operator $\mathcal{J}$ gives rise to a symmetric bilinear product on the space of boundary conditions $e(a), \cdots, \frac{d^{N-1}e}{dz^{N-1}}(a), e(b), \cdots, \frac{d^{N-1}e}{dz^{N-1}}(b)$. The coefficients of this symmetric product, captured in the matrix Q, are uniquely defined by the coefficients of the skew-symmetric differential operator $\mathcal{J}$. In the sequel, we shall define port boundary variables and a bond space in such a way that Stokes' theorem applied to the differential operator may be expressed using the canonical symmetric pairing defined in equation (2.1). Therefore, let us focus, in a first step, on the properties of Q and define the matrix R_{ext} which is used for defining the port variables. First of all, note that Q has the following form

$$Q = \begin{pmatrix} P(1) & P(2) & P(3) & \cdots & P(N-1) & P(N) \\ -P(2) & -P(3) & -P(4) & \cdots & -P(N) & 0 \\ P(3) & P(4) & \iddots & \iddots & 0 & 0 \\ -P(4) & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \iddots & & & & \vdots \\ (-1)^{N-1}P(N) & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \tag{3.6}$$

From the form of Q, the proof of the following lemma is immediately.

Lemma 3.2. *The matrix Q introduced in Theorem 3.1 is symmetric and*

$$\ker Q = \{0\}$$

if and only if $\ker P(N) = \{0\}$.

From now on we assume that Q is non-singular.

Definition 3.3. The matrix Q_{ext} in $\mathbb{R}^{2nN \times 2nN}$ associated with the differential operator $\mathcal{J}$ is defined by:

$$Q_{\text{ext}} = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}. \tag{3.7}$$

Lemma 3.4. *The matrix R_{ext} defined as*

$$R_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{pmatrix} Q & -Q \\ I & I \end{pmatrix} \tag{3.8}$$

is invertible, and satisfies

$$\begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} = R_{\text{ext}}^T \Sigma R_{\text{ext}}, \tag{3.9}$$

where

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3.10}$$

All possible matrices R which satisfies (3.9) are given by the formula

$$R = U R_{\text{ext}},$$

with U satisfying $U^T \Sigma U = \Sigma$.

Proof. We have that

$$\frac{1}{\sqrt{2}} \begin{pmatrix} Q & I \\ -Q & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} Q & -Q \\ I & I \end{pmatrix} \frac{1}{\sqrt{2}} = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}.$$

Thus using the fact that Q is symmetric $R_{\text{ext}} := \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} Q & -Q \\ I & I \end{smallmatrix}\right)$ satisfies (3.9). Since Q is invertible, the invertibility of R_{ext} follows from equation (3.9).

Let W be another solution of (3.9). Hence

$$W^T \Sigma W = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} = R_{\text{ext}}^T \Sigma R_{\text{ext}}.$$

This can be written in the equivalent form

$$R_{\mathrm{ext}}^{-T} W^T \Sigma W R_{\mathrm{ext}}^{-1} = \Sigma.$$

Calling $W R_{\mathrm{ext}}^{-1} = U$, we have that $U^T \Sigma U = \Sigma$ and $W = U R_{\mathrm{ext}}$, which proves the assertion. ■

The crucial step in order to define Dirac structures associated with the operator $\mathcal{J}$, is to define the boundary port variables as the following linear combination of the boundary conditions.

Definition 3.5. The boundary port variables associated with the differential operator $\mathcal{J}$ are the vectors $e_\partial, f_\partial \in \mathbb{R}^{nN}$, defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\mathrm{ext}} \begin{pmatrix} e(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ e(a) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(a) \end{pmatrix}, \tag{3.11}$$

where R_{ext} is defined by (3.9). ■

Consider the effort and flow space $\mathcal{E} = \mathcal{F} = L^2((a,b);\mathbb{R}^n) \times \mathbb{R}^{nN}$ with their natural inner product. We define the bond space $\mathcal{B}$ as $\mathcal{F} \times \mathcal{E}$ with the canonical symmetrical pairing

$$\begin{aligned} &\left\langle \left(f^1, f_\partial^1, e^1, e_\partial^1\right), \left(f^2, f_\partial^2, e^2, e_\partial^2\right) \right\rangle_+ = \\ &\qquad \langle e^1, f^2 \rangle_{L^2} + \langle e^2, f^1 \rangle_{L^2} - \langle e_\partial^1, f_\partial^2 \rangle - \langle e_\partial^2, f_\partial^1 \rangle, \end{aligned} \tag{3.12}$$

where

$$\left(f^i, f_\partial^i, e^i, e_\partial^i\right) \in \mathcal{B} \quad i = \{1,2\}.$$

Let us stress that this pairing on the bond space corresponds to the general definition given in equation (2.1), where the pairing on the bond space is defined modulo an isometry $r_{\mathcal{F},\mathcal{E}}$. The space of flow variables is the product space $\mathcal{F} = L^2((a,b);\mathbb{R}^n) \times \mathbb{R}^N$. Thus every flow-element is a pair, with the top element a function, and the bottom element is a part of the (boundary) port variable. The same description holds for the space of effort variables. The spaces $\mathcal{F}$ and $\mathcal{E}$ are equal and the natural isometry would be the identity. However, we choose

$$r_{\mathcal{F},\mathcal{E}} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

It is easy to see that this is an isometry, which is equal to its own inverse. Furthermore, with this choice equation (2.1) equals (3.12).

On the bond space $\mathcal{B}$ with the symmetrical pairing (3.12) we define the Dirac structure, $\mathcal{D}_{\mathcal{J}}$, associated with the linear skew symmetric operator $\mathcal{J}$.

This Dirac structure is nothing else than the expression of Stokes' theorem (recalled in Theorem 3.1) with respect to the port variables defined in Definition 3.5.

Theorem 3.6. *Let $H^N((a,b);\mathbb{R}^n)$ denote the Sobolev space of N times differentiable functions on the interval (a,b). The subspace $\mathcal{D}_{\mathcal{J}}$ of $\mathcal{B}$ defined as*

$$\mathcal{D}_{\mathcal{J}} = \left\{ \begin{pmatrix} f \\ f_\partial \\ e \\ e_\partial \end{pmatrix} \middle| \, e \in H^N((a,b);\mathbb{R}^n), \mathcal{J}e = f, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ e(a) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(a) \end{pmatrix} \right\} \tag{3.13}$$

is a Dirac structure.

Proof. The Dirac structure is defined by the fact that $\mathcal{D}_{\mathcal{J}} = \mathcal{D}_{\mathcal{J}}^{\perp}$.

Step 1: We show first that the power of every element of $\mathcal{D}_{\mathcal{J}}$ is zero. From equation (3.12) we have that

$$\begin{aligned}
&\langle (f, f_\partial, e, e_\partial,), (f, f_\partial, e, e_\partial) \rangle_+ \\
&\quad = \langle e, \mathcal{J}e \rangle_{L^2} + \langle e, \mathcal{J}e \rangle_{L^2} - e_\partial^T f_\partial - e_\partial^T f_\partial \\
&\quad = \left[\begin{pmatrix} e^T(z), & \cdots, & \frac{d^{N-1}e^T}{dz^{N-1}}(z) \end{pmatrix} Q \begin{pmatrix} e(z) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(z) \end{pmatrix} \right]_a^b - 2e_\partial^T f_\partial \\
&\quad = \begin{pmatrix} e^T(b), & \cdots, & \frac{d^{N-1}e^T}{dz^{N-1}}(a) \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} \begin{pmatrix} e(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(a) \end{pmatrix} - 2e_\partial^T f_\partial \\
&\quad = \begin{pmatrix} f_\partial^T, & e_\partial^T \end{pmatrix} \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} - 2e_\partial^T f_\partial = 0,
\end{aligned}$$

where we have used Theorem 3.1 and (3.9).

Step 2: Let $(\phi, \phi_\partial, \varepsilon, \varepsilon_\partial) \in \mathcal{D}_{\mathcal{J}}^{\perp}$. Choose $e \in H^N((a,b);\mathbb{R}^n)$ with compact support strictly included in (a,b). Thus $\frac{d^k e}{dz^k}$, $k \in \{0, \cdots, N-1\}$ are zero in a and b. Then it is easy to see that $(\mathcal{J}e, 0, e, 0) \in \mathcal{D}_{\mathcal{J}}$. Using equation (3.12) we have

$$0 = \langle e, \phi \rangle + \langle \varepsilon, f \rangle = \langle e, \phi \rangle + \langle \varepsilon, \mathcal{J}e \rangle$$

for all such e. This implies that $\varepsilon \in H^N((a,b);\mathbb{R}^n)$ and $\mathcal{J}\varepsilon = \phi$.

Step 3: Let $(\phi, \phi_\partial, \varepsilon, \varepsilon_\partial) \in \mathcal{D}_{\mathcal{J}}^\perp$ and let $(f, f_\partial, e, e_\partial) \in \mathcal{D}_{\mathcal{J}}$. From Step 2 and equation (3.12) we obtain

$$\begin{aligned}
0 =& \langle e, \mathcal{J}\varepsilon\rangle + \langle \varepsilon, \mathcal{J}e\rangle - e_\partial^T \phi_\partial - \varepsilon_\partial^T f_\partial \\
=& \left[\begin{pmatrix} e^T(z), & \cdots, & \frac{d^{N-1}e^T(z)}{dz^{N-1}}(z) \end{pmatrix} Q \begin{pmatrix} \varepsilon(z) \\ \vdots \\ \frac{d^{N-1}\varepsilon(z)}{dz^{N-1}}(z) \end{pmatrix} \right]_a^b - e_\partial^T \phi_\partial - \varepsilon_\partial^T f_\partial \\
=& \begin{pmatrix} f_\partial^T, e_\partial^T \end{pmatrix} \Sigma R_{\text{ext}} \begin{pmatrix} \varepsilon(b) \\ \vdots \\ \frac{d^{N-1}\varepsilon}{dz^{N-1}}(a) \end{pmatrix} - e_\partial^T \phi_\partial - \varepsilon_\partial^T f_\partial \\
=& \begin{pmatrix} e_\partial^T, f_\partial^T \end{pmatrix} \left[R_{\text{ext}} \begin{pmatrix} \varepsilon(b) \\ \vdots \\ \frac{d^{N-1}\varepsilon}{dz^{N-1}}(a) \end{pmatrix} - \begin{pmatrix} \phi_\partial \\ \varepsilon_\partial \end{pmatrix} \right].
\end{aligned}$$

By the proper choice of e, we can let the vectors e_∂ and f_∂ have arbitrary values. Thus the above equality has to hold for all $e_\partial \in \mathbb{R}^{nN}$ and $f_\partial \in \mathbb{R}^{nN}$. Consequently, we have that

$$R_{\text{ext}} \begin{pmatrix} \varepsilon(b) \\ \vdots \\ \frac{d^{N-1}\varepsilon}{dz^{N-1}}(a) \end{pmatrix} = \begin{pmatrix} \phi_\partial \\ \varepsilon_\partial \end{pmatrix}.$$

Concluding we have that $\mathcal{D}_{\mathcal{J}} = \mathcal{D}_{\mathcal{J}}^\perp$, and so $\mathcal{D}_{\mathcal{J}}$ is a Dirac structure. ■

4. Contraction semigroups, boundary control systems and their parameterization

In the previous section we have associated to the skew symmetric operator $\mathcal{J}$ a Dirac structure $\mathcal{D}_{\mathcal{J}}$. In this section, we shall define dynamic systems with inputs, states, and outputs with respect to this Dirac structure. These systems will be boundary control systems in the sense of semigroup theory [5], which implies that the controls and observations act on the boundary of the spatial domain. With respect to this Dirac structure it is possible to define many systems. However, we only consider those systems for which the energy does not grow, when the input is zero. This implies that the associated semigroup is contractive. We parameterize all these systems by nN-dimensional linear subspaces of the port variables. As a consequence of this parameterization, we identify those systems for which the associated semigroup is unitary.

We begin by showing that $\mathcal{J}$ is the infinitesimal generator of a contraction semigroup for appropriate choices of the boundary conditions.

4.1. **Contraction semigroups associated to $\mathcal{D}_{\mathcal{J}}$.** We begin by studying the differential operator $\mathcal{J}$ for different boundary conditions. In the first lemma we identify the adjoint of this operator. As can be expected from the skew symmetry of $\mathcal{J}$, the adjoint of this operator will be $-\mathcal{J}$, but the boundary conditions will in general differ from the original operator. We use this adjoint in order to characterize all boundary conditions for which $\mathcal{J}$ is the infinitesimal generator of a contraction or of a unitary semigroup.

Lemma 4.1. *Let K be a full rank matrix of size $2nN \times k$. Define the operator A and its domain $D(A)$, as*

$$Ae = \mathcal{J}e \tag{4.1}$$

and

$$\begin{aligned} D(A) \quad = \quad & \big\{e \in L^2((a,b),\mathbb{R}^n) \mid \textit{the port variable associated to } e, \\ & \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \textit{is in } \operatorname{ran} K \textit{and there exists} \\ & \textit{an } f \in L^2((a,b);\mathbb{R}^n) \textit{ such that } (f, f_\partial, e, e_\partial) \in \mathcal{D}_\mathcal{J}\big\}. \end{aligned} \tag{4.2}$$

Then the adjoint of A equals $-\mathcal{J}$ with domain

$$\begin{aligned} D(A^\star) \quad = \quad & \big\{e \in L^2((a,b),\mathbb{R}^n) \mid \textit{the port variable associated to } e, \\ & \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \textit{is in } \ker K^T\Sigma \textit{ and there exists} \\ & \textit{an } f \in L^2((a,b);\mathbb{R}^n) \textit{ such that } (f, f_\partial, e, e_\partial) \in \mathcal{D}_\mathcal{J}\big\}. \end{aligned} \tag{4.3}$$

Proof. By the definition of A^* and its domain, we have that an element $e_1 \in L^2((a,b);\mathbb{R}^n)$ is in the domain of A^* if and only if there exists an $\tilde{e}_1$ such that for every $e_2 \in D(A)$ we have that

$$\langle e_1, Ae_2 \rangle = \langle \tilde{e}_1, e_2 \rangle. \tag{4.4}$$

If this holds, then A^*e_1 is by definition equal to $\tilde{e}_1$.

Now compute for $e_2 \in D(A)$

$$\langle e_1, Ae_2 \rangle = \int_a^b e_1(z)^T (\mathcal{J}e_2)(z)dz. \tag{4.5}$$

We have that every function in $H^N((a,b);\mathbb{R}^n)$ which is zero at the boundaries is in the domain of A. Because of this and (4.4), we get that every $e_1 \in D(A^\star)$ must be an element of $H^N((a,b);\mathbb{R}^n)$. Using Theorem 3.1, Definition 3.11, and equation (3.9), we can write (4.5) as

$$\begin{aligned} \langle e_1, Ae_2 \rangle &= \int_a^b e_1(z)^T (\mathcal{J}e_2)(z)dz + \int_a^b e_2^T(z)(\mathcal{J}e_1)(z)dz - \int_a^b e_2^T(z)(\mathcal{J}e_1)(z)dz \\ &= \begin{pmatrix} f_{\partial,1}^T, & e_{\partial,1}^T \end{pmatrix} \Sigma \begin{pmatrix} f_{\partial,2} \\ e_{\partial,2} \end{pmatrix} - \int_a^b e_2^T(z)(\mathcal{J}e_1)(z)dz, \end{aligned}$$

where $\left(\begin{smallmatrix} f_{\partial,1} \\ e_{\partial,1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} f_{\partial,2} \\ e_{\partial,2} \end{smallmatrix}\right)$ denote the port boundary variables associated with e_1 and e_2, respectively. Since $\left(\begin{smallmatrix} f_{\partial,2} \\ e_{\partial,2} \end{smallmatrix}\right)$ lies in the range of K, we have that $\left(\begin{smallmatrix} f_{\partial,2} \\ e_{\partial,2} \end{smallmatrix}\right) = K\ell$ for some $\ell \in \mathbb{R}^k$. Hence

$$\langle e_1, Ae_2 \rangle = \begin{pmatrix} f_{\partial,1}^T, & e_{\partial,1}^T \end{pmatrix} \Sigma K\ell + \int_a^b e_2^T(z)(-\mathcal{J}e_1)(z)dz.$$

Using the defining condition (4.4), and the fact that the above equality must hold for all $\ell \in \mathbb{R}^k$, we conclude that

$$\left(\begin{smallmatrix} f_{\partial,1} \\ e_{\partial,1} \end{smallmatrix}\right) \in \ker\left(K^T\Sigma\right) \qquad \text{and} \qquad A^\star e_1 = -\mathcal{J}e_1,$$

which proves the assertion. ∎

Using the above result, we now derive necessary conditions on the boundary port variables such that A is the infinitesimal generator of a contraction or a unitary C_0-semigroup.

Theorem 4.2. *Let W be a full rank matrix of size $k \times 2nN$. Define the operator J_W and its domain, $D(J_W)$, as*

$$J_W e = \mathcal{J} e, \tag{4.6}$$

$$D(J_W) = \Big\{ e \in L^2((a,b),\mathbb{R}^n) \mid \text{the port variable associated to } e, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \text{ is in } \ker W \text{ and there exists an } f \in L^2((a,b);\mathbb{R}^n) \text{ such that } (f, f_\partial, e, e_\partial) \in \mathcal{D}_{\mathcal{J}} \Big\}. \tag{4.7}$$

The operator J_W is the infinitesimal generator of a contraction semigroup only if W has rank nN and satisfies $W\Sigma W^T \geq 0$.

If J_W is the infinitesimal generator of a unitary group, then W has rank nN and satisfies $W\Sigma W^T = 0$.

Proof. The proof consists of several steps. In the first two steps we show that the rank of W must be nN, in the third step we show that the relation $W^T\Sigma W \geq 0$ must hold, and in the last step we prove the unitary case.

Note that a densely defined operator is the infinitesimal generator of a contraction semigroup if and only if $\langle J_W e, e\rangle \leq 0$ on $D(J_W)$ and $\langle J_W^* e, e\rangle \leq 0$ on $D(J_W^*)$, see [5, Corollary 2.2.3].

Step 1: We begin by investigating the inequality $\langle J_W e, e\rangle \leq 0$ on $D(J_W)$. Let e be an element in the domain of J_W, then by the definition of this domain and Theorem 3.6 there holds

$$\langle (J_W e, f_\partial, e, e_\partial), (J_W e, f_\partial, e, e_\partial)\rangle_+ = 0.$$

This is equivalent to:

$$\langle e, J_W e\rangle_{L^2} + \langle J_W e, e\rangle_{L^2} = \begin{pmatrix} f_\partial^T & e_\partial^T \end{pmatrix} \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}. \tag{4.8}$$

We know that $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right)$ lies within the kernel of W, and that this is the only restriction on the port variables. Combining this with $\langle J_W e, e\rangle \leq 0$, we have that the subspace $\ker W$ satisfies

$$(\ker W)^T \Sigma (\ker W) \leq 0. \tag{4.9}$$

Since Σ has nN negative eigenvalues, this implies that $\dim(\ker W) \leq nN$. Since W has rank k, we have that $2nN - k \leq nN$. Or equivalently,

$$k \geq nN. \tag{4.10}$$

Step 2: Here we investigate the inequality $\langle J_W^* e, e\rangle \leq 0$ on $D(J_W^*)$, which is similar to the investigation in Step 1. It is easy to see that there exists a full rank matrix K of size $2nN \times (2nN - k)$ such that $\ker W = \operatorname{ran} K$. So Lemma 4.1 gives us the expression for J_W^* and its domain. By this lemma we also find that $(-J_W^* e, f_\partial, e, e_\partial) \in \mathcal{D}_{\mathcal{J}}$. Thus

$$\langle (-J_W^* e, f_\partial, e, e_\partial), (-J_W^* e, f_\partial, e, e_\partial)\rangle_+ = 0.$$

This is equivalent to:

$$\langle e, J_W^* e\rangle_{L^2} + \langle J_W^* e, e\rangle_{L^2} = -\begin{pmatrix} f_\partial^T & e_\partial^T \end{pmatrix} \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}.$$

From the domain we know that $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right)$ lies within the kernel of $K^T\Sigma$. There are no other conditions on these port variables, and so, like in Step 1, we have that

$$(\ker(K^T\Sigma))^T\Sigma(\ker(K^T\Sigma)) \geq 0. \tag{4.11}$$

Since Σ has nN positive eigenvalues, this implies that $\dim(\ker(K^T\Sigma)) \leq nN$. Since K has rank $2nN-k$, we obtain that $k \leq nN$. Combining this with (4.10), gives that k equals nN.

Step 3: In the previous step we have shown that $k = nN$. Among other this implies that the dimension of the kernel W is nN. Now we can write K, see Step 2., as

$$K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}.$$

Since the kernel of W equals the image of K, we find by (4.9) that

$$K_2^T K_1 + K_1^T K_2 \leq 0.$$

Since $\ker W = \operatorname{ran} K$, we have that W can be written as $W = S(-K_2, K_1)$, for some invertible S. Combining this with the above inequality, proves $W\Sigma W^T \geq 0$.

Step 4: It is basically copying the above step, and using the fact that J_W generates a unitary group if and only if $\langle J_W e, e\rangle = 0$ on $D(J_W)$ and $\langle J_W^* e, e\rangle = 0$ on $D(J_W^*)$. Hence the two inequalities in Step 3. become equalities. ■

In conclusion, the kernel of full rank matrices W of size $nN \times 2nN$ satisfying

$$W\Sigma W^T \geq 0, \tag{4.12}$$

with $\Sigma = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, give necessary conditions for the differential operators J_W to be infinitesimal generators of contractive semigroups and are associated with the Dirac structure $\mathcal{D}_{\mathcal{J}}$. Next we show a converse proposition. Thus the above theorem can be formulated as an if and only if result, or equivalently, gives a complete characterization of all subspaces of boundary port variables which define the domain of the generator of a contraction semigroup associated to the Dirac structure $\mathcal{D}_{\mathcal{J}}$.

Before stating this result, let us define a parameterization of the matrices W of size $nN \times 2nN$ satisfying (4.12) (see the Appendix A for the proof):

$$W = S\begin{pmatrix} I+V & I-V \end{pmatrix}, \tag{4.13}$$

with S is an invertible matrix, and V satisfying $VV^T \leq I$. Furthermore, these matrices satisfy

$$\ker W = \operatorname{ran}\begin{pmatrix} I-V \\ -I-V \end{pmatrix}. \tag{4.14}$$

In the following, it is shown that if the port variables are restricted to the kernel of W, this defines the domain of a contraction semigroup associated with the operator $\mathcal{J}$.

Theorem 4.3. *Let W be a matrix of size $nN \times 2nN$ satisfying (4.12). Define the operator J_W and its domain, $D(J_W)$, as*

$$J_W e = \mathcal{J}e, \tag{4.15}$$

and

$$D(J_W) = \Big\{e \in L^2((a,b),\mathbb{R}^n) \mid \textit{the port variable associated to } e, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \textit{ is in } \ker W \textit{ and there exists an } f \in L^2((a,b);\mathbb{R}^n) \textit{ such that } (f, f_\partial, e, e_\partial) \in \mathcal{D}_{\mathcal{J}}\Big\}. \tag{4.16}$$

Then on $L^2((a,b);\mathbb{R}^n)$ *we have that*

(1) *For all* $e \in D(J_W)$ *there holds* $\langle e, J_W e\rangle \leq 0$*;*
(2) *The adjoint of* J_W *equals* $-\mathcal{J}$ *with domain*

$$D(J_W^\star) = \Big\{e \in L^2((a,b),\mathbb{R}^n) \mid \textit{the port variable associated to } e, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \textit{ is in } \ker\left(-I - V^T,\ I - V^T\right) \textit{ and there exists an } f \in L^2((a,b);\mathbb{R}^n) \textit{ such that } (f, f_\partial, e, e_\partial) \in \mathcal{D}_{\mathcal{J}}\Big\}; \tag{4.17}$$

(3) *For all* $e \in D(J_W^*)$ *there holds* $\langle e, J_W^* e\rangle \leq 0$*;*
(4) *The operator* J_W *generates a contraction group* $T(t)$ *on* $L^2((a,b);\mathbb{R}^n)$*, i.e.,* $\|T(t)e\| \leq \|e\|$ *for all* $t \geq 0$ *and* $e \in L^2((a,b);\mathbb{R}^n)$*.*

Proof. 1: Let e be an element in the domain of J_W, then by the definition of this domain and Theorem 3.6 there holds

$$\langle (J_W e, f_\partial, e, e_\partial), (J_W e, f_\partial, e, e_\partial)\rangle_+ = 0.$$

This is equivalent to:

$$\langle e, J_W e\rangle_{L^2} + \langle J_W e, e\rangle_{L^2} = \begin{pmatrix} f_\partial^T & e_\partial^T \end{pmatrix} \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right)$ lies within the kernel of W, we know by Lemma A.2 that there exists a vector $\ell \in \mathbb{R}^{nN}$ such that

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} I - V \\ -I - V \end{pmatrix} \ell.$$

Thus

$$\langle e, J_W e\rangle_{L^2} = \frac{1}{2}\ell^T \begin{pmatrix} I - V^T & -I - V^T \end{pmatrix} \Sigma \begin{pmatrix} I - V \\ -I - V \end{pmatrix} \ell = \ell^T[-I + V^T V]\ell \leq 0,$$

where we have used that VV^T, and thus also $V^T V$, is less than or equal to the identity.
2: By equation (4.14) we have that J_W equals the operator A of Lemma 4.1 for $K = \left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$. Thus from this lemma we obtain that $J_W^* = -\mathcal{J}$ on the domain

$$D(J_W^\star) = \Big\{e \in L^2((a,b),\mathbb{R}^n) \mid \text{the port variable associated to } e, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \text{ is in } \ker\left(\begin{pmatrix} I - V^T & -I - V^T \end{pmatrix}\Sigma\right) \text{ and there exists an } f \in L^2((a,b);\mathbb{R}^n) \text{ such that } (f, f_\partial, e, e_\partial) \in \mathcal{D}_{\mathcal{J}}\Big\};$$

Since $\begin{pmatrix} I - V^T & -I - V^T \end{pmatrix}\Sigma = \begin{pmatrix} -I - V^T & I - V^T \end{pmatrix}$, we have proved the result.
3: This is similar to the proof of part 1. By part 2. we find that $(-J_W^* e, f_\partial, e, e_\partial) \in \mathcal{D}_{\mathcal{J}}$. Thus

$$\langle (-J_W^* e, f_\partial, e, e_\partial), (-J_W^* e, f_\partial, e, e_\partial)\rangle_+ = 0.$$

This is equivalent to:

$$\langle e, J_W^* e\rangle_{L^2} + \langle J_W^* e, e\rangle_{L^2} = -\begin{pmatrix} f_\partial^T & e_\partial^T \end{pmatrix} \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right)$ lies within the kernel of $\begin{pmatrix} -I - V^T & I - V^T \end{pmatrix}$, we have by a similar argument as used in Lemma A.2 that there exists a vector $\ell \in \mathbb{R}^{nN}$ such that

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} I - V^T \\ I + V^T \end{pmatrix} \ell.$$

Thus

$$\langle e, J_W^* e\rangle_{L^2} = -\frac{1}{2}\ell^T \begin{pmatrix} I - V & I + V \end{pmatrix} \Sigma \begin{pmatrix} I - V^T \\ I + V^T \end{pmatrix} \ell = \ell^T[-I + VV^T]\ell \leq 0,$$

where we have used that VV^T is less than or equal to the identity.

4: It is easily seen that $D(J_W)$ is dense in $L^2((a,b);\mathbb{R}^{nN})$. Using part 1. and 3., and Corollary 2.2.3 of [5] we conclude the assertion. ■

An important particular case of the Theorem 4.3 consist in the case when J_W generates a unitary semigroup. The next corollary expresses this case explicitely. It is a direct consequence of the Theorem 4.3 and the Lemmas A.1 and A.2.

Corollary 4.4. *Let W be a matrix of size $nN \times 2nN$ with full rank and satisfying $W\Sigma W^T = 0$. Define the operator J_W and its domain, $D(J_W)$, as*

$$J_W e = \mathcal{J} e, \tag{4.18}$$

$$D(J_W) = \Big\{ e \in L^2((a,b),\mathbb{R}^n) \mid \textit{the port variable associated to } e, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \textit{ is in } \ker W \textit{ and there exists an } f \in L^2((a,b);\mathbb{R}^n) \textit{ such that } (f, f_\partial, e, e_\partial) \in \mathcal{D}_{\mathcal{J}} \Big\}. \tag{4.19}$$

Then on $L^2((a,b);\mathbb{R}^n)$ the operator J_W is the infinitesimal generator of a unitary group, i.e., $\|T(t)e\| = \|e\|$ for all $t \in \mathbb{R}$ and for all $e \in L^2((a,b);\mathbb{R}^n)$.

4.2. **Boundary control system and port conjugated output.** In the previous subsection we have derived the family of contraction semigroups from the Dirac structure $\mathcal{D}_{\mathcal{J}}$ associated with a skew-symmetric differential operator $\mathcal{J}$. More precisely, we have parameterized these semigroups by a family of subspaces of the port boundary variables, defined as the kernel of a class of matrices W (matrices of size $nN \times 2nN$ satisfying (4.12)). In the following theorem, we show that the image of the matrices W can be chosen as an input space of a boundary control system, and derive the definition of a conjugated output. For more information on boundary control systems, we refer to Section 3.3 of [5]. Note that the term input and output are used to make the relation with infinite-dimensional systems theory. It does not necessarily mean that the input is completely free, i.e., can be chosen arbitrarily in $L^2_{\text{loc}}((0,\infty);\mathbb{R}^{nN})$, nor does it implies that for every initial condition in $L^2(a,b);\mathbb{R}^n)$ the output is well-defined.

Theorem 4.5. *For the differential operator $\mathcal{J}$ and the associated Dirac structure $\mathcal{D}_{\mathcal{J}}$, see Theorem 3.6, we consider the dynamical system:*

$$(\dot{x}(t), f_\partial(t), x(t), e_\partial(t)) \in \mathcal{D}_{\mathcal{J}}, \qquad t \geq 0, \tag{4.20}$$

where $(f_\partial(t), e_\partial(t))$ are the boundary port variable associated to $x(t)$, see Definition 3.5.

Let W be a full rank matrix of size $nN \times 2nN$ satisfying (4.12), and define $\mathcal{B} : H^N((a,b),\mathbb{R}^n) \to \mathbb{R}^{nN}$ as

$$\mathcal{B}x(t) := W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \tag{4.21}$$

Then the system (4.20) with the input defined as

$$u(t) = \mathcal{B}x(t) \tag{4.22}$$

is a boundary control system.

Furthermore, if we define the linear mapping $\mathcal{C} : H^N((a,b),\mathbb{R}^n) \to \mathbb{R}^{nN}$ as, see (4.13),

$$\mathcal{C}x(t) := S_2 \begin{pmatrix} I - V^T & -I - V^T \end{pmatrix} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{4.23}$$

with S_2 invertible and the output as

$$y(t) = \mathcal{C}x(t), \tag{4.24}$$

then for $u \in C^2((0,\infty);\mathbb{R}^{nN})$ and $x(0) - Bu(0) \in D(J_W)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{4.25}$$

where

$$P_W = \frac{1}{4}\begin{pmatrix} S^{-T}(P_1^2 - P_1VV^TP_1)S^{-1} & -2S^{-T}P_1VP_2S_2^{-1} \\ -2S_2^{-T}P_2V^TP_1S^{-1} & S_2^{-T}(-P_2^2 + P_2V^TVP_2)S_2^{-1} \end{pmatrix}, \tag{4.26}$$

$$P_1 = (I + VV^T)^{-1}, \qquad P_2 = (I + V^TV)^{-1}. \tag{4.27}$$

Remark 4.6. The system defined by (4.20)–(4.22) may be equivalently written in the more usual form of a boundary control system

$$\begin{aligned} \dot{x}(t) &= \mathcal{J}x(t), \\ \mathcal{B}x(t) &= u(t). \end{aligned} \tag{4.28}$$

Proof. of Theorem 4.5

In Step 1. and 2. we show that we have a boundary control system. In step 3. and 4. we prove (4.25). For a boundary control system we have to show that for zero inputs, the system is a C_0-semigroup, and furthermore that there exists a bounded operator B mapping into the domain of $\mathcal{B}$ and such that $\mathcal{B}Bu = u$ for all $u \in \mathbb{R}^{nN}$.

Step 1: As mentioned above, we have to show that J_W defined as

$$J_W x = \mathcal{J}x$$

on

$$D(J_W) = D(\mathcal{J}) \cap \ker \mathcal{B}$$

is an infinitesimal generator. This follows directly from Theorem 4.3.

Step 2: We have to find a bounded linear operator B such that $Bu \in D(\mathcal{B}) = H^N((a,b);\mathbb{R}^{nN})$ and $\mathcal{B}Bu = u$ for all $u \in \mathbb{R}^{nN}$.

Let $\{u^1, \cdots, u^{nN}\}$ be the standard basis of the input space $\mathbb{R}^{nN}$, *i.e.*, $u^i = (\delta_{ij})^T_{j=1,\cdots,nN}$. Since R_{ext} is invertible, and since W has rank nN there exists for every u^i, a $v^i \in \mathbb{R}^{2nN}$ such that

$$WR_{\text{ext}}v^i = u^i. \tag{4.29}$$

Let v^i_k denote the k'th block of v^i, $k = 1, \cdots, 2N$. Using the functions $f_{r,j}$ and $f_{l,j}$ introduced in Lemma A.3, we define the i'th column of B as

$$B_i = \sum_{k=1}^{N} v^i_k f_{r,k-1}(z) + \sum_{k=1}^{N} v^i_{k+N} f_{l,k-1}(z).$$

It is straightforward that B is a bounded operator mapping into the domain of $\mathcal{J}$. Furthermore, by Definition 3.5 we have that

$$\mathcal{B}B_i = WR_{\text{ext}} \begin{pmatrix} B_i(b) \\ \vdots \\ \frac{d^{N-1}B_i(b)}{dz^{N-1}} \\ B_i(a) \\ \vdots \\ \frac{d^{N-1}B_i(a)}{dz^{N-1}} \end{pmatrix}.$$

Now by definition

$$\frac{d^p B_i}{dz^p}(z) = \sum_{k=1}^{N} v^i_k f^{(p)}_{r,k-1}(z) + \sum_{k=1}^{N} v^i_{k+N} f^{(p)}_{l,k-1}(z).$$

From equations (A.3) and (A.4) of Lemma A.3 we have that

$$\frac{d^p B_i}{dz^p}(b) = v^i_{p+1} \quad \text{and} \quad \frac{d^p B_i}{dz^p}(a) = v^i_{p+N+1}$$

and so B satisfies $\mathcal{B}Bu = WR_{\text{ext}} \begin{pmatrix} v^i_1 \\ \vdots \\ v^i_{2N} \end{pmatrix} = u^i.$

Step 3: Under the conditions stated in the theorem, we know by Theorem 3.3.3 of [5] that there exists a classical solution of (4.20)–(4.22). Hence, in particular, $x(t) \in H^N((a,b),\mathbb{R}^n)$ holds pointwise in t, $x(t)$ is differentiable as a function of t, and $\dot{x}(t) = \mathcal{J}x(t)$. Using this, we obtain

$$\begin{aligned} \frac{d}{dt}\|x(t)\|^2 &= \frac{d}{dt}\langle x(t), x(t)\rangle \\ &= \langle \dot{x}(t), x(t)\rangle + \langle x(t), \dot{x}(t)\rangle \\ &= \langle \mathcal{J}x(t), x(t)\rangle + \langle x(t), \mathcal{J}x(t)\rangle \\ &= \begin{pmatrix} f^T_\partial(t) & e^T_\partial(t) \end{pmatrix} \Sigma \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \end{aligned} \tag{4.30}$$

On the other hand, we have that

$$\begin{aligned} \begin{pmatrix} u \\ y \end{pmatrix} &= \begin{pmatrix} & W & \\ S_2(I-V)^T, & & -S_2(I+V)^T \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\ &= \begin{pmatrix} S(I+V) & S(I-V) \\ S_2(I-V)^T & -S_2(I+V)^T \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}. \end{aligned} \tag{4.31}$$

We study next the above two by two block matrix

$$\begin{pmatrix} S(I+V) & S(I-V) \\ S_2(I-V^T) & -S_2(I+V^T) \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} I+V & I-V \\ I-V^T & -I-V^T \end{pmatrix}$$
$$= \begin{pmatrix} S & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} I & V \\ -V^T & I \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

So the matrix $\begin{pmatrix} S(I+V) & S(I-V) \\ S_2(I-V^T) & -S_2(I+V^T) \end{pmatrix}$ is invertible, and its inverse is given by

$$\begin{pmatrix} S(I+V) & S(I-V) \\ S_2(I-V^T) & -S_2(I+V^T) \end{pmatrix}^{-1}$$
$$= \frac{1}{2} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} P_1 & -VP_2 \\ V^TP_1 & P_2 \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix},$$

where

$$P_1 = (I+VV^T)^{-1}, \qquad P_2 = (I+V^TV)^{-1}. \tag{4.32}$$

Using now (4.31) and the above inverse, we can rewrite equation (4.30)

$$\begin{aligned}
&\begin{pmatrix} f_\partial^T & e_\partial^T \end{pmatrix} \Sigma \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} u^T & y^T \end{pmatrix} \begin{pmatrix} S^{-T} & 0 \\ 0 & S_2^{-T} \end{pmatrix} \begin{pmatrix} P_1 & P_1V \\ -P_2V^T & P_2 \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \Sigma \cdot \\
&\quad \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} P_1 & -VP_2 \\ V^TP_1 & P_2 \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} u^T & y^T \end{pmatrix} \begin{pmatrix} S^{-T} & 0 \\ 0 & S_2^{-T} \end{pmatrix} \begin{pmatrix} P_1 & P_1V \\ -P_2V^T & P_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \cdot \\
&\quad \begin{pmatrix} P_1 & -VP_2 \\ V^TP_1 & P_2 \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} u^T & y^T \end{pmatrix} \begin{pmatrix} S^{-T} & 0 \\ 0 & S_2^{-T} \end{pmatrix} \cdot \\
&\quad \begin{pmatrix} P_1^2 - P_1VV^TP_1 & -2P_1VP_2 \\ -2P_2V^TP_1 & -P_2^2 + P_2V^TVP_2 \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} u^T & y^T \end{pmatrix} \\
&\quad \begin{pmatrix} S^{-T}(P_1^2 - P_1VV^TP_1)S^{-1} & -2S^{-T}P_1VP_2S_2^{-1} \\ -2S_2^{-T}P_2V^TP_1S^{-1} & S_2^{-T}(-P_2^2 + P_2V^TVP_2)S_2^{-1} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}.
\end{aligned}$$

Combining the above equality with (4.30), we see that (4.25) holds. ∎

Now we consider two particular cases which are canonical in the following sense. The first one corresponds to the definition of a contraction semigroup with the balance equation (4.25) canonical to scattering variables whereas the second one corresponds to the definition of a unitary semigroup with the canonical balance equation corresponding to a lossless system.

We begin by considering the case when the boundary control system is generated by the matrix $W = \frac{1}{2}(I, I)$.

Corollary 4.7. *Consider the system defined as*

$$\dot{x}(t) = \mathcal{J}x(t), \tag{4.33}$$

$$u(t) = \frac{1}{2}\left(f_\partial(t) + e_\partial(t)\right) \tag{4.34}$$

$$y(t) = \frac{1}{2}\left(f_\partial(t) - e_\partial(t)\right), \tag{4.35}$$

where $(f_\partial(t), e_\partial(t))$ are the boundary port variable associated to $x(t)$, see Definition 3.5.

The above system is a boundary control system, with the associated semigroup a contraction. Furthermore, for $u \in C^2((0,\infty);\mathbb{R}^{nN})$ and $x(0) - Bu(0)$ in the domain of the generator we have that

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2. \tag{4.36}$$

Proof. We take in Theorem 4.5 to be

$$W = \frac{1}{2}\begin{pmatrix} I & I \end{pmatrix}$$

and thus $V = 0$ and $S = \frac{1}{2}I$. Furthermore, if we take $S_2 = S$, then the system of Theorem 4.5 becomes the system as defined above. In particular, from Theorem 4.5, it is a boundary control system, and the associated semigroup is a contraction, see also Theorem 4.3. Furthermore, from equation (4.25) we have that

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix},$$

with $P_W = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. This is the same as (4.36). ■

Let now consider another special case from Theorem 4.5, for which $W = \frac{1}{2}(2I, 0)$ and the semigroup is unitary.

Corollary 4.8. *Consider the system defined by*

$$\dot{x}(t) = \mathcal{J}x(t), \tag{4.37}$$

$$u(t) = f_\partial(t), \tag{4.38}$$

$$y(t) = -e_\partial(t), \tag{4.39}$$

where $(f_\partial(t), e_\partial(t))$ are the boundary port variable associated to $x(t)$, see Definition 3.5.

The above system is a boundary control system, with the associated semigroup unitary. Furthermore, for $u \in C^2((0,\infty);\mathbb{R}^{nN})$ and $x(0) - Bu(0)$ in the domain of the generator we have that

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t). \tag{4.40}$$

Proof. We take in Theorem 4.5

$$W = \frac{1}{2}\begin{pmatrix} 2I & 0 \end{pmatrix}$$

and thus $S = \frac{1}{2}I$, and $V = I$. Furthermore, if we take $S_2 = -S$, then the system of Theorem 4.5 becomes the system as defined above. In particular, from Theorem 4.5, it is a boundary control system, and the associated semigroup is a unitary group, see also Theorem 4.3. Finally, using the V, S, and S_2 and equations (4.26), (4.27), it is easy to see that (4.40) holds. ■

However, it is important to note that the definition of the output given in the Theorem 4.5 is such that the balance equation (4.25) get the canonical form (4.40) for *all* choice of subspaces of boundary port variables leading to a *unitary* semigroup. This is stated in the following theorem.

Theorem 4.9. *Let S and V be $nN \times nN$ matrices, with S invertible and V unitary. Associate to these matrices the following system*

$$\dot{x}(t) = \mathcal{J}x(t), \tag{4.41}$$

$$u(t) = S\begin{pmatrix} I+V & I-V \end{pmatrix}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{4.42}$$

$$y(t) = \frac{1}{4}S^{-T}\begin{pmatrix} I-V & I+V \end{pmatrix}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}, \tag{4.43}$$

where $(f_\partial(t), e_\partial(t))$ are the boundary port variable associated to $x(t)$, see Definition 3.5.

The above system is a boundary control system, with the associated semigroup being unitary. Furthermore, for $u \in C^2((0,\infty);\mathbb{R}^{nN})$ and $x(0) - Bu(0)$ in the domain of the generator the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t). \tag{4.44}$$

Proof. If we take in Theorem 4.5 to be

$$W = S\begin{pmatrix} I+V & I-V \end{pmatrix},$$

then the system of Theorem 4.5 becomes the system as defined above. In particular, from Theorem 4.5 it is a boundary control system, and the associated semigroup is a unitary group, see also Theorem 4.3. Choosing, S_2 to be equal to $-\frac{1}{4}S^{-T}V$, it is easy to see that (4.23) equals (4.35). So from Theorem 4.5 we obtain for u and $x(0)$ satisfying the given smoothness conditions that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|x(t)\|^2 = & \begin{pmatrix} u(t)^T & y^T(t) \end{pmatrix} \cdot \\
& \tfrac{1}{4}\begin{pmatrix} 0 & -2S^{-T}\tfrac{1}{2}V\tfrac{1}{2}(-4V^TS^T) \\ -2(-4SV)\tfrac{1}{2}V^T\tfrac{1}{2}S^{-1} & 0 \end{pmatrix}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \\
= & \, u(t)^T y(t).
\end{aligned}$$

■

5. Port Hamiltonian system

In this section we define port Hamiltonian systems associated with (constant) skew-symmetric matrix operators. These systems are defined in terms network based modeling [3, 18, 24] which is based on the definition of two objects: the interconnection structure defined by a Dirac structure and the Hamiltonian function representing the total energy of the system. Firstly, using the definition given in Section 3 of the Dirac structure associated with a skew-symmetric operator, we define a port Hamiltonian system with boundary port variables. Secondly, using the results of Section 4, we formulate these port Hamiltonian systems as boundary control systems. In Subsection 5.2 we treat extensively the example of the Timoshenko beam.

5.1. **Linear port Hamiltonian systems with boundary port variables.** We now extend the definition of linear port Hamiltonian systems as defined for finite-dimensional state spaces [24] to infinite dimensional state spaces. The interconnection structure is defined by a Dirac structure associated with skew-symmetric differential operator, see also Theorem 3.6. The Hamiltonian function, generating this port Hamiltonian system, is defined by a coercive operator relating the state variable to the effort variable.

In the introductory example of the Section 2, the skew-symmetric operator was the 2×2 matrix differential operator of differential order 1, corresponding to the canonical inter-domain coupling, and the Dirac structure was the Stokes-Dirac structure. The symmetric operator was defined by the elasticity modulus and the mass distribution defining the elasto-dynamic energy of the string.

Definition 5.1. Consider the domain $Z = (a, b) \subset \mathbb{R}$. Let the space of flow variables $\mathcal{F}_Z$ be equal to $L^2((a, b); \mathbb{R}^n)$ and let the space of effort variables $\mathcal{E}_Z$ be equal to $\mathcal{F}_Z$. Consider a $n \times n$ matrix skew-symmetric differential operator of differential order N denoted by $\mathcal{J}$ defined by (3.1) and (3.2). Define the bond space $\mathcal{B} = \mathcal{F}_Z \times \mathbb{R}^{nN} \times \mathcal{E}_Z \times \mathbb{R}^{nN}$ and the Dirac structure $\mathcal{D}_{\mathcal{J}}$ associated with the skew-symmetric differential operator $\mathcal{J}$ as defined in Theorem 3.6. Let $\mathcal{L}$ be a coercive operator on $\mathcal{E}_Z$. The port Hamiltonian system with boundary port variables associated with $\mathcal{J}$ and generated by $\mathcal{L}$ is defined by:

$$(\dot{x}(t), f_\partial(t), \mathcal{L}x(t), e_\partial(t)) \in \mathcal{D}_{\mathcal{J}}, \qquad t \geq 0, \tag{5.1}$$

where $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right)$ is the boundary port associated with $e := \mathcal{L}x$, see Definition 3.5.

Remark 5.2. It may be noted that the Definition 5.1 corresponds to the abstract system $\dot{x}(t) = A\, x(t)$ defined by the differential operator:

$$A = \mathcal{J}\mathcal{L} \tag{5.2}$$

which need not to be skew-symmetric nor with constant coefficients.

It is also worth to explicit the Hamiltonian function, representing the energy of the system, and which has not been explicitly used in the definition of the port Hamiltonian system. Consider the following Hamiltonian:

$$H(x) = \frac{1}{2}\langle x,\, \mathcal{L}x\rangle, \tag{5.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural inner product on the space $\mathcal{E}_Z$.

Noting that $\frac{dH(x(t))}{dt} = \langle \dot{x}(t),\, \mathcal{L}x(t)\rangle$ and by definition of the Dirac structure, one obtains the following energy balance equation

$$\frac{dH(x(t))}{dt} = \frac{1}{2}\left(f_\partial^T(t), e_\partial^T(t)\right)\Sigma \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}.$$

This expresses that the variation of the energy of the boundary port Hamiltonian system is equal to the flow of energy at the boundary of the system's domain.

This also motivates to take the state space equals to those x for which the Hamiltonian is finite. Since $\mathcal{L}$ is coercive on $\mathcal{E}_Z = L^2((a, b); \mathbb{R}^n)$, we see that the state space $\mathcal{X}$ is $L^2((a, b); \mathbb{R}^n)$ with the new inner product

$$\langle x_1, x_2\rangle_{\mathcal{X}} = \langle x_1, \mathcal{L}x_2\rangle_{L^2((a,b);\mathbb{R}^n)}. \tag{5.4}$$

In the previous definition we have defined linear port Hamiltonian systems with boundary port variables using the definition of Dirac structure for which the port variables are not split into input and output variables. However, we have seen in Section 4 that using a specific subspaces of the port variables, one may define input and output variables as belonging to complementary subspaces of the boundary port variables. Moreover, by choosing in an appropriate way these subspaces, one may define a boundary control system with its associated semigroup being a contraction. In the sequel we reformulate the boundary port Hamiltonian system of Definition 5.1 as a boundary control system. We use the parameterization of the input and output variables and the contractive semigroups associated with the Dirac structure $\mathcal{D}_{\mathcal{J}}$ given in Section 4. However, there is an essential difference with these boundary control systems as now the effort variables need no longer be identical to the state variables. The state variable have become the image of the effort variables through the coercive operator $\mathcal{L}^{-1}$.

Theorem 5.3. *The port Hamiltonian system of the Definition 5.1 may be formulated as a boundary control system on the state space $\mathcal{X}$:*

$$(\dot{x}(t), f_\partial(t), \mathcal{L}x(t), e_\partial(t)) \in \mathcal{D}_{\mathcal{J}}, \qquad t \geq 0 \tag{5.5}$$

with the input variables defined by choosing some full rank matrix W of size $nN \times 2nN$ satisfying (4.12) and the map

$$\mathcal{B}x(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = u(t) \tag{5.6}$$

on the domain

$$D(\mathcal{B}) = D(\mathcal{J}). \tag{5.7}$$

Furthermore, define the port conjugated output

$$y(t) = S_2 \begin{pmatrix} I - V^T & -I - V^T \end{pmatrix} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$

with S, V defined in (4.13) and S_2 invertible. Then for $u \in C^2((0,\infty); \mathbb{R}^{nN})$ and $x(0) - Bu(0) \in D(A_W)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}H(x(t)) = \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}. \tag{5.8}$$

where P_W is defined in the equations (4.26), (4.27).

The proof is a straightforward extension of the proof of the Theorem 4.3 using the following lemma.

Lemma 5.4. *The differential operator $A_W = \mathcal{J}\mathcal{L}$ with domain $D(A_W) = \{x \in \mathcal{X} \mid \mathcal{L}x \in D(J_W)\}$, see (4.16), generates a contraction semigroup on $\mathcal{X}$.*

Proof. We know that

$$\langle e, \mathcal{J}e \rangle_{L_2} = (e_\partial^T, f_\partial^T)\Sigma \begin{pmatrix} e_\partial \\ f_\partial \end{pmatrix},$$

where (e_∂, f_∂) is the boundary port variable associated to e. Take now $e = \mathcal{L}x$, with $\mathcal{L}$ coercive, then

$$\langle e, \mathcal{J}e \rangle_{L_2} = \langle \mathcal{L}x, \mathcal{J}\mathcal{L}x \rangle_{L_2} = \langle x, \mathcal{L}\mathcal{J}\mathcal{L}x \rangle_{L_2} = \langle x, \mathcal{J}\mathcal{L}x \rangle_{\mathcal{X}}.$$

Now for our choice of domain we have that $A_W = J_W\mathcal{L} = \mathcal{J}\mathcal{L}$. Furthermore, we know for $e = \mathcal{L}x \in D(J_W)$ that $(e_\partial^T, f_\partial^T)\Sigma \begin{pmatrix} e_\partial \\ f_\partial \end{pmatrix} \leq 0$. Thus we conclude that

$$\langle x, A_W x \rangle_{\mathcal{X}} \leq 0.$$

This together with a similar argument for the adjoint gives that A_W generates a contraction semigroup on $\mathcal{X}$. ■

5.2. **Example: The Timoshenko's beam model.** Timoshenko's beam model describes the infinitesimal planar deformations of a flexible beam reduced to its neutral fiber with some particular geometrical assumptions. We briefly recall the Hamiltonian formulation as proposed by Golo et al. [10]. Note that this corresponds to taking the Legendre transform of the usual Lagrangian formulation. Consider the spatial domain $Z = [a, b]$. Denote the angular displacement by q_θ, the transversal displacement of the beam by q_y, and the conjugated momenta by p_θ and p_y. The elastic potential energy density is given by $\mathcal{U}(q) = \frac{1}{2}\int_Z F^T\, qdz$, where the strain wrench (torque and force) is $F = K\, q$. Let $K = \operatorname{diag}(c_\theta, c_y)$ denotes the positive definite compliance matrix which depends on the elasticity properties of the material and its geometry. The kinetic energy is given by $\mathcal{K}(p) = \frac{1}{2}\int_Z v^T\, pdz$, where the co-energy variable is the velocity, $v = M^{-1}\, p$. M denotes the positive definite inertia matrix which is given as $M = \operatorname{diag}(\iota, \mu)$ with ι the momentum of inertia of the beam per unit length and μ the mass per unit length. It is immediate that $F = \delta_q \mathcal{U}(q)$ and $v = \delta_p \mathcal{K}(q)$, where δ denotes the variational derivative [19].

Choose the state vector x as

$$x = \begin{pmatrix} q_\theta \\ q_y \\ p_\theta \\ p_y \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}.$$

The Timoshenko beam model may be expressed as the following Hamiltonian evolution equations [11, 10]

$$\frac{\partial x}{\partial t} = \mathcal{J} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial q} \\ \frac{\partial \mathcal{H}}{\partial p} \end{pmatrix}, \tag{5.9}$$

where $\mathcal{H}(q, p) = \mathcal{U}(q) + \mathcal{K}(p)$ is the total elasto-dynamic energy of the beam and the skew-symmetric differential operator $\mathcal{J}$ is

$$\mathcal{J} = \begin{pmatrix} 0_2 & \begin{pmatrix} \frac{\partial}{\partial z} & 0 \\ -1 & \frac{\partial}{\partial z} \end{pmatrix} \\ \begin{pmatrix} \frac{\partial}{\partial z} & 1 \\ 0 & \frac{\partial}{\partial z} \end{pmatrix} & 0_2 \end{pmatrix}. \tag{5.10}$$

We now derive the port Hamiltonian formulation of this system. The time variation of the energy variables are defined as flow variables

$$\frac{\partial}{\partial t} \begin{pmatrix} q \\ p \end{pmatrix} := \begin{pmatrix} f_q \\ f_p \end{pmatrix}.$$

The variational derivative of the total energy $\delta_x \mathcal{H}$, defines the effort variables

$$\begin{pmatrix} e_q \\ e_p \end{pmatrix} := \mathcal{L} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{5.11}$$

Note that

$$\mathcal{L} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial q} \\ \frac{\partial \mathcal{H}}{\partial p} \end{pmatrix}.$$

More precisely,

$$e_q = \begin{pmatrix} c_\theta & 0 \\ 0 & c_y \end{pmatrix} \begin{pmatrix} q_\theta \\ q_y \end{pmatrix} = \begin{pmatrix} T \\ F_y \end{pmatrix} \tag{5.12}$$

is the vector composed of the torque and the force, and

$$e_p = \begin{pmatrix} \iota^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_y \end{pmatrix} = \begin{pmatrix} \omega \\ v_y \end{pmatrix} \tag{5.13}$$

is the vector composed of the angular and longitudinal velocities.

Hence, according to the evolution equation (5.9), the flow variables are related to the co-energy variables by the skew symmetric differential operator $\mathcal{J}$ defined in (3.1)

$$\begin{pmatrix} f_q \\ f_p \end{pmatrix} = \mathcal{J} \begin{pmatrix} e_q \\ e_p \end{pmatrix}.$$

This differential operator may be written:

$$\mathcal{J} = P(0) + P(1)\frac{\partial}{\partial z},$$

where

$$P(0) = \begin{pmatrix} 0_2 & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0_2 \end{pmatrix}, \quad P(1) = \begin{pmatrix} 0_2 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0_2 \end{pmatrix}.$$

The symmetric matrix Q corresponding to the bilinear term on the boundary variables in Theorem 3.1 and given in equation (3.6) reduces to $Q = P(1)$. The matrix R_{ext} defining the boundary port variables equals, see (3.8)

$$R_{\text{ext}} = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.14}$$

According to Definition 3.5 the port variables are

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L} \end{pmatrix} \begin{pmatrix} q(b) \\ p(b) \\ q(a) \\ p(a) \end{pmatrix}.$$

Considering the relations (5.11), (5.12), (5.13)

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} \omega(b) - \omega(a) \\ v_y(b) - v_y(a) \\ T(b) - T(a) \\ F_y(b) - F_y(a) \\ T(b) + T(a) \\ F_y(b) + F_y(a) \\ \omega(b) + \omega(a) \\ v_y(b) + v_y(a) \end{pmatrix}.$$

The associated Dirac structure is given by

$$\mathcal{D}_\mathcal{J} = \left\{ \begin{pmatrix} f_q \\ f_p \\ f_\partial \\ e_q \\ e_p \\ e_\partial \end{pmatrix} \middle| \begin{pmatrix} e_q \\ e_p \end{pmatrix} \in H^1((a,b);\mathbb{R}^4), \mathcal{J}\begin{pmatrix} e_q \\ e_p \end{pmatrix} = \begin{pmatrix} f_q \\ f_p \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} \omega(b) - \omega(a) \\ v_y(b) - v_y(a) \\ T(b) - T(a) \\ F_y(b) - F_y(a) \\ T(b) + T(a) \\ F_y(b) + F_y(a) \\ \omega(b) + \omega(a) \\ v_y(b) + v_y(a) \end{pmatrix} \right\}.$$

We now illustrate the derivation of boundary control systems from the port Hamiltonian system using two different choices of the matrix W defining them according to the Theorem 5.3. The first choice corresponds to boundary control system is associated with a unitary semigroup and the other choice where it is associated with a contractive semigroup.

For the unitary case let us choose the matrix matrix W given in equation (4.13) with the invertible matrix S and matrix V satisfying $VV^T = I$ chosen as follows:

$$S = \frac{1}{2\sqrt{2}} \begin{pmatrix} -I_2 & I_2 \\ I_2 & I_2 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

This choice corresponds to define the inputs:

$$\begin{aligned} u =& S \begin{pmatrix} I_4 + V & I_4 - V \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\ =& \frac{1}{\sqrt{2}} \begin{pmatrix} -I_2 & 0 & 0 & I_2 \\ 0 & I_2 & I_2 & 0 \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \omega(a) \\ v_y(a) \\ T(b) \\ F_y(b) \end{pmatrix}. \end{aligned}$$

The unitary semigroup associated to the boundary control $u = 0$ corresponds to the following boundary conditions

$$\omega(a,t) = v_y(a,t) = M(b,t) = F_y(b,t) = 0,$$

which are the so-called *clamped-free* boundary conditions. According to Theorem 4.5 the output conjugated to this input is:

$$y = \frac{1}{4} S^{-T} \begin{pmatrix} I_4 - V & I_4 + V \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} -T(a) \\ -F_y(a) \\ \omega(b) \\ v_y(b) \end{pmatrix}.$$

For the contractive case let us choose the matrix matrix W given in equation (4.13) with the invertible matrix S and matrix V satisfying $VV^T \leq I$ chosen as follows:

$$S = \frac{\sqrt{2}}{4} \begin{pmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Theorem 5.3 the inputs are:

$$u = \frac{\sqrt{2}}{4} \begin{pmatrix} I_2 & I_2 & I_2 & I_2 \\ -I_2 & I_2 & -I_2 & I_2 \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \omega(b) + T(b) \\ v_y(b) + F_y(b) \\ \omega(a) - T(a) \\ v_y(a) - F_y(a) \end{pmatrix}.$$

For S_2 we choose S and so the outputs are:

$$y = \frac{\sqrt{2}}{4} \begin{pmatrix} I_2 & I_2 & -I_2 & -I_2 \\ -I_2 & I_2 & I_2 & -I_2 \end{pmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \omega(a) + T(a) \\ v_y(a) + F_y(a) \\ \omega(b) - T(b) \\ v_y(b) - F_y(b) \end{pmatrix}.$$

In this case the boundary inputs and outputs correspond to the *scattering variables* and $\frac{1}{2}\|x(t)\|^2_{\mathcal{X}} = \|u(t)\|^2 - \|y(t)\|^2$.

6. Conclusion

In this paper we have defined infinite dimensional linear port Hamiltonian systems associated to skew-symmetric differential operators. Furthermore, we have related them to boundary control systems.

Therefore we have, in a first instance, defined a Dirac structure on a Hilbert space associated with skew-symmetric differential operators with constant coefficients. Using Stokes' theorem, we have defined port boundary variables as the image of the boundary values under a linear map, which is derived from the differential operator. Then we have shown that the differential operator together with the boundary port variables defines a Dirac structure on a vector space (the space of bond variables) endowed with a canonical symmetric pairing.

In a second instance, we have shown that one may derive from the Dirac structure infinitesimal generators of contraction semigroups. These infinitesimal generators are obtained by restricting the domain of the skew-symmetric operator to subspaces for which the boundary port variables belong to the kernel of a certain family of linear maps. Conversely, this family of maps gives a parameterization of all contraction semigroups which are associated with the skew-symmetric operator.

In a third instance we have derived a formulation of the port Hamiltonian system as boundary control systems associated with the class of contraction semigroups obtained from the Dirac structure. We have defined outputs conjugated to the inputs of the boundary control systems in such a way that the system satisfies a power balance equation, in a similar way as dissipative systems [21].

In a forth instance, we have used these results in order to define infinite-dimensional port Hamiltonian systems. These systems are defined with respect to the Dirac structure associated with a skew-symmetric differential operator and a coercive operator defining the Hamiltonian functional, i.e., total energy of the system. Again from such a port Hamiltonian system one may derive a class of boundary control system associated with contraction semigroups. This is illustrated by the example of the Timoshenko's beam.

Future work will concern the relation of the proposed linear port Hamiltonian system with the formulation of dissipative systems in terms of systems nodes, as conservative and well-posed systems [9]. This works opens also the way for the generalization to infinite dimensional systems of the synthesis of stabilizing controllers using the immersion and Hamiltonian reduction proposed in [2, 22].

Appendix A. Technical lemma's

Lemma A.1. *Let W be a $nN \times 2nN$ matrix and let $\Sigma = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. Then W has rank nN and $W\Sigma W^T \geq 0$ if and only if there exist a matrix $V \in \mathbb{R}^{nN\times nN}$ and an invertible matrix $S \in \mathbb{R}^{nN\times nN}$ such that*

$$W = S \begin{pmatrix} I+V & I-V \end{pmatrix} \tag{A.1}$$

with $VV^T \leq I$.

Furthermore, $W\Sigma W^T = 0$ if and only if V is unitary.

Proof. If W is of the form (A.1), then we find

$$W\Sigma W^T = S \begin{pmatrix} I+V & I-V \end{pmatrix} \Sigma \begin{pmatrix} I+V^T \\ I-V^T \end{pmatrix} S^T = S[2I - 2VV^T]S^T,$$

which is non-negative, since $VV^T \leq I$.

Now we prove that if W is of full rank and is such that $W\Sigma W^T \geq 0$, then (A.1) holds. Writing W as $W = \begin{pmatrix} W_1 & W_2 \end{pmatrix}$, we that $W\Sigma W^T \geq 0$ is equivalent to $W_1W_2^T + W_2W_1^T \geq 0$. Hence

$$(W_1 + W_2)(W_1 + W_2)^T \geq (W_1 - W_2)(W_1 - W_2)^T \geq 0. \tag{A.2}$$

If $x \in \ker((W_1 + W_2)^T)$, then the above inequality implies that $x \in \ker((W_1 - W_2)^T)$. Thus $x \in \ker(W_1^T) \cap \ker(W_2^T)$. Since W has full rank, this implies that $x = 0$. Hence $W_1 + W_2$ is invertible.

Using (A.2) once more, we see that

$$(W_1 + W_2)^{-1}(W_1 - W_2)(W_1 - W_2)^T(W_1 + W_2)^{-T} \leq I$$

and thus $V := (W_1 + W_2)^{-1}(W_1 - W_2)$ satisfies $VV^T \leq I$. Summarizing, we have

$$\begin{aligned} \begin{pmatrix} W_1 & W_2 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} W_1 + W_2 + W_1 - W_2 & W_1 + W_2 - W_1 + W_2 \end{pmatrix} \\ &= \frac{1}{2}(W_1 + W_2)\begin{pmatrix} I+V & I-V \end{pmatrix}. \end{aligned}$$

Defining $S := \frac{1}{2}(W_1 + W_2)$, we have shown the representation (A.1).

If instead of inequality, we have equality for W, then it is easy to show that we have equality in the equation for V as well. Thus V is unitary. ■

Lemma A.2. *Suppose that the $nN \times 2nN$ matrix W can be written in the format of equation (A.1), i.e., $W = S(I+V,\ I-V)$ with S and V square matrices, and S invertible. Then the kernel of W equals the range of $\left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$.*

If V is unitary, then the kernel of W equals the range of ΣW^T.

Proof. Let $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ be in the range of $\left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$. By the equality (A.1), we have that

$$\begin{aligned} W\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= S\begin{pmatrix} I+V & I-V \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= S\begin{pmatrix} I+V & I-V \end{pmatrix}\begin{pmatrix} I-V \\ -I-V \end{pmatrix} l = 0. \end{aligned}$$

Hence we see that the range of $\left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$ lies in the kernel of W. It is easy to show that W has rank nN, and so the kernel of W has dimension nN. Thus if we can show that the $2nN \times nN$ matrix $\left(\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right)$ has full rank, then we have proved the first assertion. If this matrix would not have full rank, then there should be a non-trivial element in its kernel. It is easy to see that the kernel consists of zero only, and so we have proved the the first part of the lemma.

Suppose now that V is unitary, then

$$\begin{pmatrix} I-V \\ -I-V \end{pmatrix} = \begin{pmatrix} -I+V^T \\ -I-V^T \end{pmatrix} V = -\Sigma W^T S^{-T} V.$$

Since the range of ΣW^T equals the range of $-\Sigma W^T S^{-T} V$, we have proved the second assertion. ∎

Lemma A.3. *Given the interval $[a,b]$ and a positive number $N \in \mathbb{N}$. There exist polynomials $f_{lj}(z), f_{rj}(z), j = 0, \cdots, N-1$ such that*

$$\begin{aligned} &\frac{d^k f_{l,j}}{dz^k}(a) = \delta_{kj}; \quad k = 0, \cdots, N-1 \\ &\frac{d^k f_{l,j}}{dz^k}(b) = 0; \qquad k = 0, \cdots, N-1 \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} &\frac{d^k f_{r,j}}{dz^k}(a) = 0; \qquad k = 0, \cdots, N-1 \\ &\frac{d^k f_{r,j}}{dz^k}(b) = \delta_{kj}; \quad k = 0, \cdots, N-1. \end{aligned} \tag{A.4}$$

Proof. Since the construction of $f_{r,j}$ is very similar to that of $f_{l,j}$, we only show how is constructed $f_{l,j}$. These functions are constructed using backward induction. It is easily seen that

$$f_{l,N-1}(z) := \frac{1}{(N-1)!}(z-a)^{N-1}(z-b)^N \frac{1}{(a-b)^N}$$

satisfies the condition (A.3). Suppose next that we have constructed the functions $f_{l,j}(z)$ for $j = j_0+1, \cdots, N-1$. We next construct $f_{l,j_0}(z)$. Define $\tilde{f}_{l,j_0}(z)$ as

$$\tilde{f}_{l,j_0}(z) = \frac{1}{j_0!}(z-a)^{j_0}(z-b)^N \frac{1}{(a-b)^N}.$$

It is easy to see that

$$\frac{d^k \tilde{f}_{l,j_0}}{dz^k}(a) = \delta_{kj_0}; \qquad k = 0, \cdots, j_0$$

and

$$\frac{d^k \tilde{f}_{l,j_0}}{dz^k}(b) = 0; \qquad k = 0, \cdots, N-1.$$

If we define the function $f_{l,j_0}(z)$ as

$$f_{l,j_0}(z) = \tilde{f}_{l,j_0}(z) - \sum_{i=j_0+1}^{N-1} \frac{d^i f_{l,j_0}}{dz^i}(a) f_{l,i}(z),$$

then it is straightforward to see that it satisfies (A.3). ■

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