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DIRECT AND INDIRECT METHODS TO OPTIMIZE THE MUSCULAR FORCE RESPONSE TO A PULSE TRAIN OF ELECTRICAL STIMULATION

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Abstract. Recent force-fatigue mathematical models in biomechanics [7] allow to predict the muscular force response to functional electrical stimulation (FES) and leads to the optimal control problem of maximizing the force. The stimulations are Dirac pulses and the control parameters are the pulses amplitudes and times of application, the number of pulses is physically limited and the model leads to a sampled data control problem. The aim of this article is to present and compare two methods. The first method is a direct optimization scheme where a further refined numerical discretization is applied on the dynamics. The second method is an indirect scheme: first-order Pontryagin type necessary conditions are derived and used to compute the optimal sampling times.

Résumé. En biomécanique les modèles mathématiques de force-fatigue de la réponse musculaire aux impulsions électriques [7] permettent de prédire et de contrôler la réponse à un train de stimulations électriques et donc de maximiser la force produite à la fin du train. Mathématiquement les stimulations sont des impulsions de Dirac dont on peut moduler les temps d'applications et les amplitudes, le nombre d'impulsions étant physiquement limité sur le train et le modèle conduit à un problème où le contrôle est de dimension fini. L'objectif de ce travail est de présenter deux méthodes d'optimisation en vue de les comparer. La première méthode est dite directe et l'on utilise une discrétisation numérique de la dynamique pour transformer le problème d'optimisation en un problème en dimension finie. La seconde méthode dite indirecte utilise le principe du maximum de Pontriaguine dans le contexte où le contrôle est de dimension finie et on établit des conditions nécessaires d'optimalité qui peuvent être implémentées numériquement pour calculer les temps d'impulsions optimaux.

Introduction

Optimized force response to FES is an important problem for muscular reeducation and in case of paralysis. An historical model is known as the Hill model [8] and more refined models are taking into account the muscular fatigue ([3], [7], [9]). See [12] for a comparison of the models. In this article, we shall consider the Ding et al. force-fatigue model. The physical control amounts to apply on $[0,T]$ a finite sequence of Dirac pulses at times $0 = t_1 < t_2 < \ldots < t_n < T$, with amplitudes $\eta_i \in [0,1]$, $i = 1,\ldots,n$ and in the model they are integrated using a linear dynamics to produce the so-called E_s input (see Fig.1), which drives the force response. This fits in the

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sampled-data control frame since between each interpulse $t_i - t_{i-1}$, $i = 1, \dots, n$ the control is constant. Our aim is to optimize a cost function to maximize the force response. The control parameters are the interpulse $t_i - t_{i-1}$, $i = 2, \dots, n$ and the Dirac amplitudes η_i, $i = 1, \dots, n$. There is a minimal interpulse due to the digital control but the dynamics can be discretized over a refined interval and this led to a so-called direct optimization problem which can be handled with an optimization routine. Another method is an indirect optimal control scheme. The optimal problem fits in the frame of sampled-data control problem and some Pontryagin type necessary optimality conditions were obtained in [6] and they were refined to be applied to our specific problem [2]. This leads to numerical methods to compute the optimal control in which we can distinguish between control by interpulses and control by amplitudes. The objective of this article is to present and to give preliminary numerical simulations.

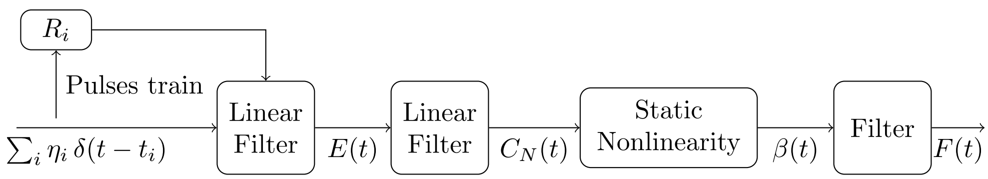

Figure 1. Ding et al. model description from the input (pulses train) to the output (force F).

The article is organized as follows. In section 1, the Ding et al. model is presented and the sampled data optimal control problem is introduced. Using a numerical discretization method, the direct optimization scheme is presented. In section 2, we give the Pontryagin type necessary optimality condition for sampled-data optimal control problem, derived for our category of problem and coming from [2]. Both sections contain numeric simulations.

1. The mathematical model and the direct scheme

1.1. **Force-fatigue model**

Let us briefly recall the Ding et al. force-fatigue model [7]. The control associated to electrical pulses is described by

$$v(t) = \sum_{i=1}^{n} \eta_i \delta(t - t_i) \tag{1}$$

applied at times $t_1 = 0 < t_2 < ... < t_n < T$ with train $T :=$ *train duration* (fixed) and $\eta_i \in [0, 1]$ are the amplitudes of each pulse. The FES signal denoted by $E_s(t)$ is the solution of the linear dynamics

$$\dot{E}_s(t) + \frac{E_s(t)}{\tau_c} = \frac{1}{\tau_c} \sum_{i=1}^{n} \eta_i \delta(t - t_i) R_i \tag{2}$$

with $E_s(0) = 0$ and R_i is the parameter describing the *phenomenon of tetania* defined by

$$R_i = \begin{cases} 1 \text{ for } i = 1 \\ 1 + (R_0 - 1)\exp(-\frac{t_i - t_{i-1}}{\tau_c}) \text{ for } i > 1, \end{cases} \tag{3}$$

and τ_c is a parameter. In the model, the input $E_s(t)$ is the control to drive the force response using the dynamics

$$\begin{cases} \frac{\mathrm{d}C_N(t)}{\mathrm{d}t} + \frac{C_N(t)}{\tau_c} = E_s(t) \\ \frac{\mathrm{d}F(t)}{\mathrm{d}t} = -\gamma(t)F(t) + A\beta(t) \end{cases} \tag{4}$$

where C_N is the concentration of Ca^{2+}, F is the force, A, K_m, τ_1, τ_2 are parameters and

$$\beta(t) = \frac{C_N(t)}{K_m + C_N(t)}, \qquad \gamma(t) = \frac{F(t)}{\tau_1 + \tau_2, \beta(t)} \tag{5}$$

are the Hill functions. The triplet (A, K_m, τ_1) contain the fatigue variables following the linear dynamics:

$$\begin{cases} \frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{A - A_{\text{rest}}}{\tau_{fat}} + \alpha_A F \\ \frac{\mathrm{d}K_m}{\mathrm{d}t} = -\frac{K_m - K_{m,\text{rest}}}{\tau_{fat}} + \alpha_{K_m} F \\ \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = -\frac{\tau_1 - \tau_{1,\text{rest}}}{\tau_{fat}} + \alpha_{\tau_1} F. \end{cases} \tag{6}$$

with $(A_{\text{rest}},\ K_{m,\text{rest}}, \tau_{1,\text{rest}})$ being the equilibrium, while τ_{fat}, α_A, α_{K_m} and α_{τ_1} are parameters.
The force-fatigue model can be written shortly as:

$$\dot{x}(t) = g(x(t)) + B(t) \sum_{i=1}^{n} \eta_i H(t - t_i) G(t_{i-1}, t_i), \tag{7}$$

with $x = (C_N, F, A, K_m, \tau_1)$, H being the Heaviside function defined by: $H(x) := 1$ if $x > 0$ and 0 if $x < 0$, and

$$\begin{cases} G(t_{i-1}, t_i) = (R_0 - 1) e^{\frac{t_{i-1}}{\tau_c}} + e^{\frac{t_i}{\tau_c}} \\ B(t) = (b(t), 0, \ldots, 0)^T,\ b(t) = \frac{1}{\tau_c} e^{\frac{-t}{\tau_c}}. \end{cases} \tag{8}$$

We use the convention $t_0 = -\infty$, $t_{n+1} = T$ and $g(x)$ is defined by Eq. (4) and Eq. (6). Note that once the sampling times are fixed, the control is constant on each $[t_{i-1},\ t_i]$ and we get a sampled control system. The optimal sampled-data control problem that we want to study is the Mayer problem of the form: $\min_{u(\cdot)} \varphi(x(T))$ where $\varphi : \mathbb{R}^5 \mapsto \mathbb{R}$ is the cost function and the minimum is taken over the set of controls $u := (\eta_1,\ \eta_2,\ \cdots,\ , \eta_n,\ t_2,\ \cdots,\ t_n) \in \mathbb{R}^{2n-1}$ with the constraints $\eta_i \in [0, 1]$, $0 = t_1 < t_2 < \cdots < t_n < T = t_{n+1}$ and the interpulse constraints $t_{i+1} - t_i \geq I_m$, $i = 1, \ldots, n$.

1.2. **Direct method**

Direct methods can solve efficiently optimal control problems in the permanent control case and for the sampled-data control case. In the permanent control case, the optimal control problem is transformed into a non linear finite dimensional optimization problem and this is done by a discretization in time of the state and control variables in the dynamics. For the sampled-data control problems, the dynamics is discretized in time.

1.2.1. *Direct method using Maltab*

We present the general frame (stimulation time and amplitude as control variables). To control the force and/or the fatigue levels, three cases are considered ($t_1 = 0$ and $T = t_{n+1}$ being fixed):

- Stimulation times t_i as control variables (with fixed amplitudes η_i):

$$u_t = [t_2\ \ldots\ t_n]^\intercal. \tag{9}$$

- Stimulation amplitudes ($\eta_i = \eta(t_i)$) as control variables (with fixed stimulation times t_i):

$$u_\eta = [\eta_1\ \eta_2\ \ldots\ \eta_n]^\intercal, \tag{10}$$

  with $\eta_1 = \eta(t_1)$ and $t_1 = 0$.
- Stimulations times and amplitudes as control variables: $u_{(t,\eta)} = \begin{bmatrix} u_t \\ u_\eta \end{bmatrix}$.

In the third case (stimulations times and amplitudes as control variables), and from (7) and (8), we define ζ_n:

$$\begin{cases} \zeta_1 = u_\eta(1)G(t_0, t_1), t_0 = -\infty, \\ \quad t \in [(t_1 = 0), (t_2 = u_t(1))[\\ \vdots \\ \zeta_l = \zeta_{l-1} + u_\eta(l)G(u_t(l-2), u_t(l-1)), \\ \quad t \in [(t_l = u_t(l-1)), (t_{l+1} = u_t(l))[\\ \vdots \\ \zeta_n = \zeta_{n-1} + u_\eta(n)G(u_t(n-2), u_t(n-1)), \\ \quad t \in [(t_n = u_t(n-1)), (t_{n+1} = T)[. \end{cases} \tag{11}$$

Note that ζ_i is constant for $t \in [t_i, t_{i+1}[$.

Eq. (7) is of the form

$$\dot{x}(t) = f(x(t), E_s(t)), \tag{12}$$

which, using an Euler discretizing scheme on each interval $[t_{i-1}, t_i],\ i = 2, \dots, n+1$, becomes

$$x_{k+1} = x_k + \mathrm{d}t_i\, f(x_k, \zeta_k),\ k = 0, \dots, K_i, \tag{13}$$

where $\mathrm{d}t_i$ is the discretizing step $\mathrm{d}t_i = (t_i - t_{i-1})/K_i$ and $K_i \in \mathbb{N}^*$ (see Fig. (2)).

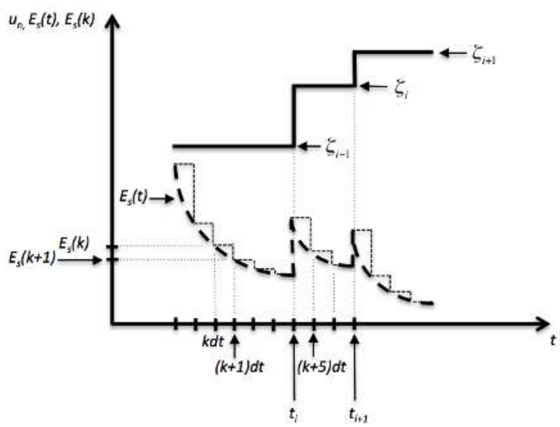

FIGURE 2. Evolution of $E_s(t)$, $E_s(k)$ and ζ_i (case: $\mathrm{d}t_i = \mathrm{d}t$)

To maximize the force level, we minimize the cost function $\varphi(x(T)) := -F(T)$ subject to the state constraints (13) and the control constraints $t_1 < (t_2 = u_t(1)) < \dots < (t_N = u_t(n-1)) < T$ and $t_i - t_{i-1} \geq I_m,\ I_m > 0$, where we have fixed the amplitudes $\eta_i,\ i = 1, \dots, n+1$ to 1.

We present the numerical results concerning the final force optimization problem in Table 1.

1.2.2. *Direct method using Bocop*

We use the Bocop software [4] to implement a direct approach, where the NLP is solved using a primal-dual interior point algorithm and the derivatives required for the optimization are computed by automatic differentiation with `CppAD`.

We give in Fig.4 the results with Bocop for $n = 8$, $I_m = 10$ms and $T = 400$ms. Matlab toolbox gives **315.6N** as a maximum force versus Bocop result **339.3N**, enlightening the existence of several local minima for the non-convex final force maximization problem.

$F(T)$ (kN)	t_i optimal vector (ms)	Final state vector $x(T)$	Optimization method	Simulation time (s)
0.303	[0 32.8834 88.6003 139.0000 183.8992 265.6509 337.7939 370.0000 400]	[0.4931 303.2002 2.9743 0.1046 52.7804]	Interior-point	34.964578
0.315	[0 45.0000 90.0000 135.0000 180.0000 268.7500 313.7500 341.2500 400]	[0.2295 315.6880 2.9738 0.1047 52.8037]	Active-set	8.451400
0.315	[0 27.6470 46.4705 135.8821 181.1762 270.5878 315.8818 343.52 400]	[0.2476 315.6504 2.9739 0.1047 52.7986]	SQP	7.157023

Table 1. Optimization methods (Matlab toolbox) comparison in the case of maximizing $F(T)$: initial stimulation times [0 50 100 150 200 300 350 380 400](ms) and initial state vector $x(0) = [C_N(0)\ F(0)\ A(0)\ \tau_1(0)\ K_m(0)]^T = [0\ 0\ 3.0090\ 0.1030\ 50.9570]^T$ $(\eta_i = 1)$. The execution time are given for a standard computer (processor: 4 Intel@Core^{TM} i5 CPU @ 2.4Ghz).

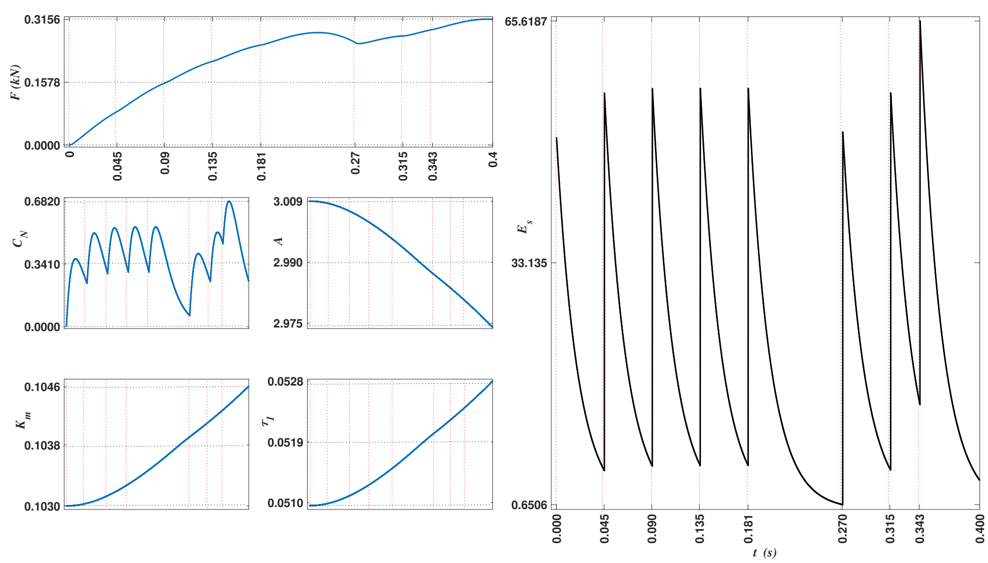

Figure 3. Force maximization result using Interior point methods (see Table 1). Time evolutions of state and FES signal E_s obtained by Matlab.

2. Pontryagin maximum principle with sampled-data control and necessary optimality conditions, indirect method

In this section we propose a first brief recap on the Pontryagin maximum principle for optimal control problems and we compare two situations: the permanent control case versus the sampled-data control case. We consider in this section a very simple framework (smooth dynamics and Mayer cost, fixed initial condition and no final state constraint) and we give some recalls on the main techniques (related to the classical calculus of variations) leading to each version of the Pontryagin maximum principle. Let $T > 0$ and let d, $m \in \mathbb{N}^*$ be fixed positive integers. Let us consider a general nonlinear control system of the form

$$\dot{x}(t) = f(t, x(t), u(t)), \quad \text{a.e. } t \in [0, T], \tag{14}$$

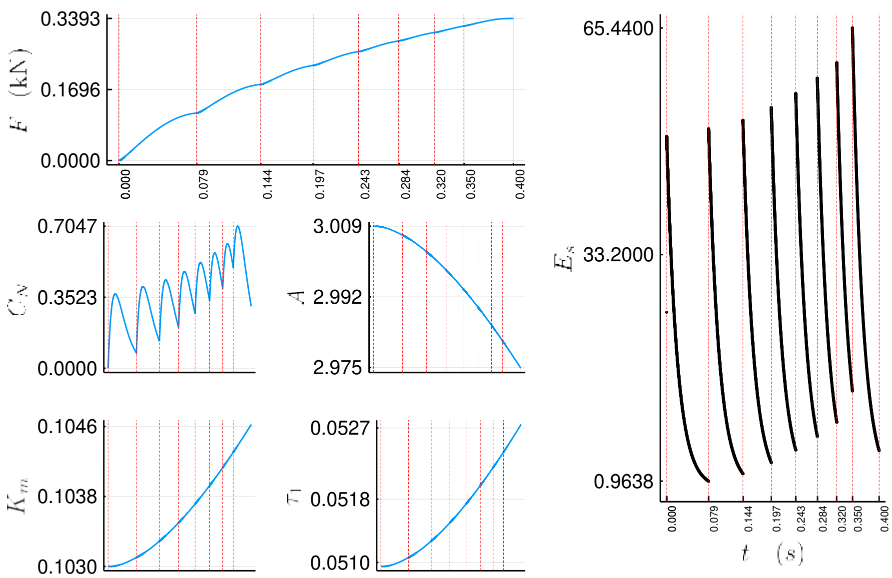

FIGURE 4. *(Direct method)* Case $n = 8$, $T = 400$ms and $I_m = 10$ms (see Table 2). Time evolutions of state and FES signal E_s obtained by Bocop. The primal-dual interior point method converges in less than a hundred iterations and it corresponds to a total execution time of 15s on a standard computer (processor: 4 Intel@Core^{TM} i5 CPU @ 2.4Ghz).

where $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is of class $\mathcal{C}^1$, together with the fixed initial condition $x(0) = x_0 \in \mathbb{R}^d$ and the control constraint $u(t) \in \Omega \subset \mathbb{R}^m$. We focus in this section on the Mayer optimal control problem given by

$$\min_{u \in \mathcal{U}} \varphi(x(T)), \tag{15}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is of class $\mathcal{C}^1$ and where $\mathcal{U}$ stands for the set of admissible controls (see details in each subsection below). Recall that the *Hamiltonian function* $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ associated to Problem (15) is defined by $H(t,x,p,u) = \langle p, f(t,x,u) \rangle$.

2.1. **Permanent control case**

If the set $\mathcal{U}$ of admissible controls in Problem (15) is the set of all bounded measurable functions $u : [0,T] \to \Omega$, then there is no restriction on the modification of the value of the control and thus it can occur at any time in $[0,T]$. In such case, Problem (15) is said to be with *permanent control*. This situation corresponds to the very well-known framework deeply studied and developed in the literature (see, e.g., [10] and references therein). From now our aim is to briefly recall the derivation of the Pontryagin maximum principle for Problem (15) in the case of permanent control. Let x^* be a reference optimal curve associated to the control u^*. Take a L^1-perturbation (or needle-variation) defined by $u_\varepsilon(t) = v \in \Omega$ on $[s, s+\varepsilon)$, where s is a Lebesgue time of the function $t \mapsto f(t, x^*(t), u^*(t))$, and $u_\varepsilon(t) = u^*(t)$ elsewhere. The corresponding variation vector satisfies the linear equation

$$\dot{w}(t) = \frac{\partial f}{\partial x}(t, x^*(t), u^*(t)) \times w(t), \quad \text{a.e. } t \in [s,T],$$

with the initial condition $w(s) = f(s, x^*(s), v) - f(s, x^*(s), u^*(s))$. From optimality, it holds that $\frac{\varphi(x_\varepsilon(T))-\varphi(x^*(T))}{\varepsilon} \geq 0$, where x_ε denotes the response to u_ε. Taking the limit $\varepsilon \to 0^+$, one gets

$$\langle \nabla\varphi(x^*(T)), w(T)\rangle \geq 0. \tag{16}$$

Write the co-state equation as

$$\dot{p}(t) = -\frac{\partial f}{\partial x}(t, x^*(t), u^*(t))^\intercal \times p(t)$$

with the final condition $p(T) = -\nabla\varphi(x^*(T))$. Using in (16) the equalities $w(T) = \Phi(T,s) \times w(s)$ and $p(s) = \Phi(T,s)^\intercal \times p(T)$, where $\Phi(\cdot,\cdot)$ stands for the state-transition matrix associated to the matrix function $t \mapsto \frac{\partial f}{\partial x}(t, x^*(t), u^*(t))$, one gets the inequality $\langle p(s), f(s, x^*(s), v) - f(s, x^*(s), u^*(s))\rangle \leq 0$ which corresponds exactly to the standard *Hamiltonian maximization condition* of the Pontryagin maximum principle given by

$$H(t, x^*(s), p(s), u^*(s)) = \max_{v\in\Omega} H(s, x^*(s), p(s), v), \quad \text{a.e. } s \in [0,T]. \tag{17}$$

We conclude this section by recalling that the *maximized Hamiltonian function* $\mathcal{H} : [0,T] \to \mathbb{R}$ defined by $\mathcal{H}(t) = H(t, x^*(t), p(t), u^*(t))$, can be shown to be absolutely continuous on $[0,T]$ with

$$\dot{\mathcal{H}}(t) = \frac{\partial H}{\partial t}(t, x^*(t), p(t), u^*(t)), \quad \text{a.e. } t \in [0,T].$$

In particular, if the control system (14) is autonomous, then $\mathcal{H}(\cdot)$ is constant over $[0,T]$.

2.2. Sampled-data control case

At the opposite of the permanent control case, if the set $\mathcal{U}$ of admissible controls authorizes the value of the control $u : [0,T] \to \Omega$ to be modified at most $n-1$ times, where $n \in \mathbb{N}^*$ is fixed, the problem (15) is said to be with *sampled-data control*. In such case, for any $u \in \mathcal{U}$, there exists a finite set of n times $0 = t_1 < t_2 < \ldots < t_n < T$ (called *sampling times*) such that $u(t) = u_i \in \Omega$ on each interval $[t_i, t_{i+1})$. We refer to [5,6] for the statement of Pontryagin maximum principle handling sampled-data controls. From now we assume that Ω is convex and our aim is to briefly recall the derivation of the Pontryagin maximum principle for Problem (15) in the case of sampled-data controls. Let x^* be a reference optimal curve associated to the control u^* and let us denote by t_i^* the corresponding sampling times. Consider the convex L^∞-perturbation $u_\varepsilon = u^* + \varepsilon(u - u^*)$, where $u \in \mathcal{U}$ has the same sampling times than u^*. The corresponding variation vector satisfies the affine equation

$$\dot{w}(t) = A(t) \times w(t) + B(t) \times (u(t) - u^*(t)), \quad \text{a.e. } t \in [0,T],$$

with the initial condition $w(0) = 0_{\mathbb{R}^d}$, where $A(t) = \frac{\partial f}{\partial x}(t, x^*(t), u^*(t))$ and $B(t) = \frac{\partial f}{\partial u}(t, x^*(t), u^*(t))$. Introducing the co-state vector p as in the previous section and using the equality

$$w(T) = \int_0^T \Phi(T,s) \times \frac{\partial f}{\partial u}(s, x^*(s), u^*(s)) \times (u(s) - u^*(s))\, \mathrm{d}s,$$

in the inequality (16) give $\int_0^T \left\langle p(s), \frac{\partial f}{\partial u}(s, x^*(s), u^*(s)) \times (u(s) - u^*(s))\right\rangle \mathrm{d}s \leq 0$.
Taking $u(t) = v \in \Omega$ over $[t_i^*, t_{i+1}^*)$ and $u(t) = u^*(t)$ elsewhere, we exactly recover the *nonpositive averaged Hamiltonian gradient conditions* derived in [5,6] given by

$$\left\langle \int_{t_i^*}^{t_{i+1}^*} \frac{\partial H}{\partial u}(s, x^*(s), p(s), u_i^*)\, \mathrm{d}s, v - u_i^* \right\rangle \leq 0, \tag{18}$$

for all $i = 1, \ldots, n-1$ and all $v \in \Omega$.

2.3. New necessary optimality conditions applicable to the Ding et al. model, indirect method

The previous frame is not completely adapted to the force-fatigue model and we briefly recall the necessary conditions, see [2] for more details. Let us introduce the notion of *admissible perturbations*.

Definition 2.1. Let $j \in \{1, \dots, n\}$. We say that $\tilde{\eta}_j \in \mathbb{R}$ is an admissible perturbation of η_j^* if there exists $\alpha_0 > 0$ such that $\eta_j^* + \alpha\tilde{\eta}_j \in [0,1]$ for all $0 \leq \alpha \leq \alpha_0$.

Definition 2.2. Let $j \in \{2, \dots, n-1\}$. We say that $\tilde{t_j} \in \mathbb{R}$ is an admissible perturbation of t_j^* if there exists $\alpha_0 > 0$ such that $(t_j^* + \alpha\tilde{t_j}) - t_{j-1}^* \geq I_m$ and $t_{j+1}^* - (t_j^* + \alpha\tilde{t_j}) \geq I_m$, for all $0 \leq \alpha \leq \alpha_0$.

Definition 2.3. Let $j = n$. We say that $\tilde{t_n} \in \mathbb{R}$ is an admissible perturbation of t_n^* if there exists $\alpha_0 > 0$ such that $(t_n^* + \alpha\tilde{t_n}) - t_{n-1}^* \geq I_m$, for all $0 \leq \alpha \leq \alpha_0$.

The main result is stated as follows.

Theorem 2.4. *Let $\sigma^* = (\eta_1^*, \dots, \eta_n^*, t_2^*, \dots, t_n^*) \in \mathbb{R}^{2n-1}$ be an optimal solution and let x^* stand for the corresponding optimal state. Then, the co-state vector p defined as the unique solution to the backward linear Cauchy problem given by*

$$\dot{p}(t) = -\nabla f(x^*(t))^\top \times p(t), \quad a.e.\ t \in [0,T], \quad p(T) = p_T, \tag{19}$$

is such that:

(1) *the inequality*

$$\left(\int_{t_j^*}^T p_1(s) b(s) ds \right) \tilde{\eta}_j \leq 0,$$

holds true for all $j = 1, \dots, n$ and for all admissible perturbation $\tilde{\eta}_j$ of η_j^*;

(2) *the inequality*

$$\Im(\sigma^*)\ \tilde{t_j} \leq 0, \tag{20}$$

holds true for all $j = 2, \dots, n$ and for all admissible perturbation $\tilde{t_j}$ of t_j^*, where

$$\Im_j(\sigma^*) := (R_0 - 1) \int_{t_{j+1}^*}^T p_1(s) b(s)\ ds\ b(-t_j^*)\eta_{j+1}^* + \int_{t_j^*}^T p_1(s) b(s)\ ds\ b(-t_j^*)\eta_j^* - p_1(t_j^*) b(t_j^*) G(t_{j-1}^*, t_j^*)\eta_j^*.$$

2.4. Numerical results for the indirect method

We consider the problem of maximizing the force at the final time, the amplitudes being fixed to $\eta_i = 1,\ i = 1, \dots, n$, and we take into account the constraints $t_{i+1} - t_i \geq I_m = 10\text{ms},\ i = 1, \dots, n$ (see [2] for an indirect method based on a shooting algorithm with no constraints on the sampling times). The purpose of this indirect method is to show that the transversality conditions (20) are suitable to compute the optimal times $\sigma^* = (t_2^*, \dots, t_n^*)$.

An extremal (x, p, σ), where the state x satisfies the dynamics (14) with a given initial condition, the co-state p satisfies the Cauchy problem (19), both being parameterized by $\sigma = (t_i)_{i=2,\dots,n}$ such that the inequalities (20) are fulfilled, is solution of *a differential variational inequality* problem. We first discretize the problem with respect to the state x and we obtain a finite-dimensional problem, where the unknowns are the sampling times σ (once σ is fixed, so is the initial co-state vector $p(0)$ because $p(T)$ is fixed to $(0,1,0,0,0)^\intercal$).

The inequalities (20) are handled introducing new state variables $y_i,\ i = 2, \dots, n+1$ such that $\dot{y}_i(t) = H(t - t_i)\, p_1(t) b(t),\ y_i(t_i) = 0$ and we obtain $y_i(T) = \int_{t_i}^T p_1(s) b(s)\, \mathrm{d}s$. Then, we rewrite (20) into the inequalities:

$$\Im_j(\sigma)\ H(t_{j+1} - t_j - I_m) \leq 0, \quad \text{and} \quad \Im_j(\sigma)\ H(t_j - t_{j-1} - I_m) \geq 0,\ j = 2, \dots, n,$$

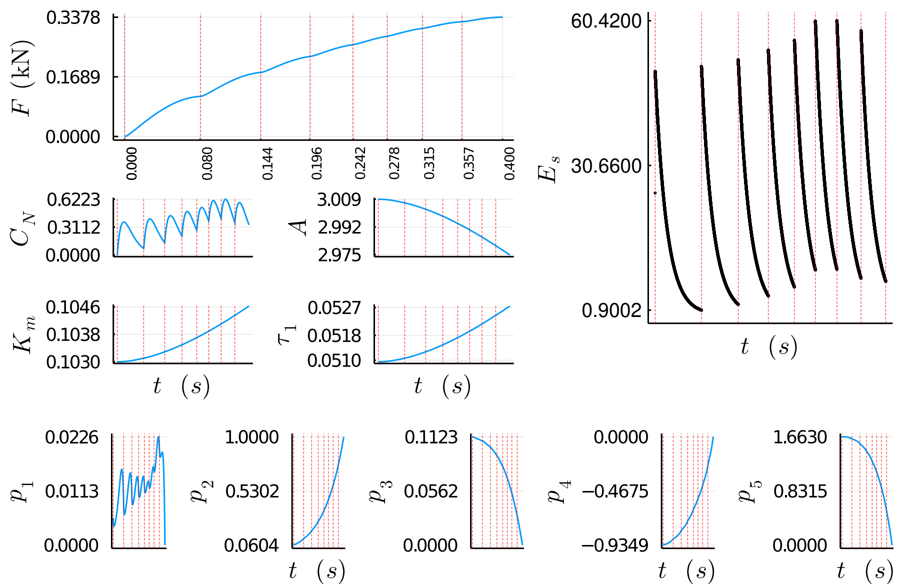

FIGURE 5. *(Indirect method)* Time evolution of the state, co-state variables and FES signal where the t_i's are computed using the necessary conditions (20).

where $\Im_j$ is reformulated as:

$$\Im_j(\sigma^*) := (R_0 - 1)b(-t_j^*)y_{j+1}(T) + b(-t_j^*)y_j(T) - p_1(t_j^*)b(t_j^*)G(t_{j-1}^*, t_j^*),\ j = 2, \dots, n.$$

Our indirect method relies on the finite dimensional optimization problem

$$\min_{\sigma \in \mathbb{R}^{n-1}} \Theta(\sigma),$$

where

$$\Theta(\sigma) := \sum_{i=1}^{n} \log(-\Im_j(\sigma)\ H(t_{j+1} - t_j - I_m)) + \log(\Im_j(\sigma)\ H(t_j - t_{j-1} - I_m))$$

and subject to $x(0) = x_0$, $y_{i+1}(0) = 0$, $p(T) = (0, 1, 0, 0, 0)^\intercal$, $I_m + t_i - t_{i+1} \leq 0$ for $i = 1, \dots, n$. The solution of this problem is computed using a *primal-dual interior point method* coupled with an auto-differentiation method (`CppAD` package) to compute the derivatives. We represent in Fig.5 the solution for $n = 8$ and $T = 400$ms. The final force $F(T)$ =**337.8 N** is close to the one computed by the direct method ($F(T)$ =**339.3 N**) and the optimal sampling times for both methods are given in Table 2.

TABLE 2. Numerical values for the optimal sampling times obtained by the direct approach (Bocop) and the indirect approach and for the necessary conditions Θ_i, $i = 2, \dots, n$ in the case: $n = 8$, $T = 400$ms and $I_m = 10$ms.

i	2	3	4	5	6	7	8
t_i (direct)	0.079	0.144	0.197	0.243	0.284	0.320	0.350
t_i (indirect)	0.080	0.144	0.196	0.242	0.278	0.315	0.357
Θ_i (necessary conditions)	−0.11	−5.45	−62.9	−725	−1779	−26550	−76763

3. Conclusion

In this article we present and compare two methods (direct and indirect schemes) to optimize the FES-input in the Ding et al. force-fatigue model to maximize the force response at the end of the train. We give preliminary numerical simulations proving the ability in both cases to implement the two methods. The same techniques can be used to consider Mayer problems where the cost takes into account the fatigue variables. This approach fits in the frame of open loop optimal controls problems. For the application since only the force is observed one must use an adaptive algorithm like MPC method [1] where we optimize over a larger time with several pulses trains and rest periods and where the fatigues variables are estimated online using an observer. The challenge is to couple an optimization technique with on-line estimation. Our study in this frame allows to compute the maximal response to a train. In the direct case, BOCOP leads to better results than Matlab toolbox. In the indirect case, a nice contribution is to handle the inequalities (20). In the future it is important to investigate first and second order optimality conditions.

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