# DIRECT ASYMPTOTIC INTEGRATION OF THE EQUATIONS OF TRANSVERSELY ISOTROPIC ELASTICITY FOR A PLATE NEAR CUT-OFF FREQUENCIES

*by* J. D. KAPLUNOV

(*Institute for Problems in Mechanics, Russian Academy of Science, Prospect Vernadskogo* 101-1*, Moscow* 117526*, Russia*)

L. YU KOSSOVICH

(*Department of Mathematical Elasticity and Bio-Mechanics, Saratov State University, Astrakhankaya Street, Saratov* 410026*, Russia*)

*and* G. A. ROGERSON

(*Department of Computer and Mathematical Sciences, University of Salford, Salford* M5 4WT)

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**Summary**

Direct asymptotic integration of the equations of transversely isotropic elasticity, for a layer with in-plane axis of transverse isotropy and zero surface traction, is carried out in the vicinity of the cut-off frequencies. In direct contrast to the corresponding isotropic case, in which there is only one family of thickness shear resonance frequencies, two such families are observed to exist. Consequently, the two-dimensional equations for the associated long-wave amplitudes are each scalar equations, rather than the single vector equation which arises in the isotropic case. These equations, together with that corresponding to thickness stretch resonance, are obtained. The exact dispersion relation is also derived and asymptotic expansions, giving frequency as a function of scaled wave number, are obtained in the neighbourhood of the cut-off frequencies. This both reveals the appropriate asymptotic orders of stress and displacement components to help facilitate the direct asymptotic integration and retrospectively acts as a check on the coefficients of the two-dimensional equations. The mathematical equivalence of approximations derived from the exact solutions and exact solutions derived from asymptotically approximate equations is therefore verified.

## 1. Introduction

The use of fibre-reinforced composites is prevalent in modern structures, especially those for which a high strength-to-weight ratio is of primary concern. For such materials a preferred direction, usually termed the fibre direction, exists and a transversely isotropic model is most commonly employed. For many applications fibre-reinforced materials are formed into bonded layers, each with a specific fibre orientation relative to some fixed reference direction. The practical importance of such structures has resulted in a high number of publications aimed at elucidating the mechanical and dynamic properties of layered media, many including layers composed of fibre-reinforced material. For a detailed list of recent publications within this area the reader is referred to the review article by Chimenti (**1**). In dynamic problems concerning fibre-reinforced media it is common to use a continuum model, whereby the fibres are assumed to be an inherent material property, rather

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than some form of inclusion, see for example, Rogerson (**2**). A detailed account of the constitutive theory used in modeling fibre-reinforced composites, and the implications of using a continuum theory, may be found in Spencer (**3**).

In this paper we seek to generalize asymptotic methods previously used for isotropic thin-walled elastic bodies to the transversely isotropic case; see Kaplunov *et al.* (**4**). Specifically, that book develops an approach to analysing long-wave high frequency, long-wave low frequency, short-wave low frequency and short-wave high frequency motions. In the context of the present paper our concern is long-wave high frequency motions of a transversely isotropic elastic layer with in-plane axis of transverse isotropy and zero surface tractions. It is readily established, except for fibres of small extensibility, that only for such motions is the asymptotic behaviour of the stress–strain state (SSS) significantly different from the corresponding isotropic case, with the existence of two distinct families of thickness shear resonance frequencies. The analysis of long-wave high frequency motions is highly relevant for stationary thickness vibration of, or transient response to high frequency shock loading in, thin-walled bodies. There is also application of the analysis of such motions to fluid–structure interaction, this being particularly pertinent to jumps in radiation power and first-order resonances of high frequency Lamb waves in scattering, phenomena known to occur in the long-wave high frequency region; see for example, Kaplunov (**5**) and Kaplunov *et al.* (**6**). A further noteworthy point is the possible dominance of motions of the type we consider in problems involving fixed faces, such problems being characterized by the absence of fundamental modes; see for example, (**7**).

The method of direct asymptotic integration in the vicinity of thickness resonance (cut-off) frequencies is adopted. This enables two-dimensional second-order equations for the long-wave amplitudes to be derived. The first attempt to analyse long-wave vibrations, in the vicinity of cut-off frequencies, using direct asymptotic integration was seemingly carried out within the framework of isotropic linear elasticity and in the context of stationary vibrations; see Achenbach (**8**). However, that work does not involve derivation of the appropriate two-dimensional equations and the link between the direct asymptotic integration of static problems, see, for example, Goldenveiser (**9**, **10**), and their dynamic counterparts was not made. The close connection between the direct asymptotic integration techniques needed for both static and dynamic problems has been established and exploited, and appropriate two-dimensional equations derived, in the isotropic case, see Kaplunov *et al.* (**4**), Kaplunov (**11**, **12**). It is noted that similar two-dimensional equations have also been derived by asymptotic analysis of energy functionals, Berdichevskii (**13**) and Le (**14**). All of the previously mentioned investigations are concerned with the application of direct asymptotic integration to isotropic bodies. In the case of anisotropy seemingly little work within this area has been carried out, one exception, however, being an investigation of trapped modes (at cut-off frequencies) in a transversely isotropic elastic plate with surface elevation; see Tovstik (**15**).

We begin this paper in section 2 with a brief derivation of the exact dispersion relation associated with small amplitude vibrations of a transversely isotropic elastic layer. In view of the anisotropy this is done for the full three-dimensional equations and with a fibre direction parallel to the surface of the layer. This specific fibre direction is both mathematically expedient and the most common in industrial applications. In section 3 long-wave high frequency approximations of the dispersion relations associated with flexural and extensional vibrations are derived. A similar investigation in the context to linear isotropic elasticity was seemingly first carried out by Nigul (**16**). A consequence of the approximations is that the asymptotic order of the displacement and stress components in the vicinity of thickness stretch and thickness shear resonance frequencies is readily established. This is done by analysing the asymptotic form of the associated eigenfunctions. A knowledge of the asymptotic order of stress and displacement components motivates appropriate scalings and thereby facilitates direct asymptotic integration in the vicinity of thickness shear and thickness stretch resonance frequencies in sections 4 and 5, respectively. In the case of thickness shear there exist two scalar equations, which is in contrast to the single vector equation previously obtained in the isotropic case. The asymptotic integration is carried out using a systematic perturbation scheme which could readily be extended both to higher order and layered media. The result of the integration process, in each case, yields a two-dimensional equation for the long-wave amplitudes, the coefficients of which agree with the corresponding approximations of the dispersion relations derived in section 3. It should be emphasized that our primary concern in this paper is to construct a two-dimensional asymptotic model to accurately describe the dynamic response of a layered structure. However, the fact that approximations based on the exact solution are mathematically equivalent to the exact solution of approximate governing equations, something not shared by some other plate theories which are based on *ad hoc* assumptions, serves both as a validation of the model and as an indication of its potential.

## 2. Governing equations and derivation of the exact dispersion relation

Our concern in this paper is a layer, composed of linear transversely isotropic elastic material, of thickness $2h$ and infinite lateral extent. The stress–strain relation for such an elastic solid may be expressed in the component form

$$\begin{aligned}\sigma_{ij} &= \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + a_i a_j e_{kk})\\ &\quad +2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta a_k a_m e_{km} a_i a_j,\end{aligned} \tag{2.1}$$

in which $\alpha$, $\beta$, $\lambda$, $\mu_T$ and $\mu_L$ are material parameters, $\mathbf{a}$ is a unit vector defining the direction of transverse isotropy, $\sigma_{ij}$ denote the components of the Cauchy stress tensor and $e_{ij}$ the associated infinitesimal strain tensor components; see, for example, Green (**17**). Attention is restricted to the case in which the direction of transverse isotropy lies in the plane of the plate, this being both mathematically expedient and common in engineering applications. A Cartesian coordinate system of axes $Ox_1x_2x_3$ is therefore chosen with origin $O$ in the mid-plane, $Ox_3$ normal to the plane of the plate and $Ox_1$ coincident with the direction of transverse isotropy, a direction usually termed the *fibre direction*. The equations of motion are assumed in their usual form

$$\frac{\partial\sigma_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2}, \tag{2.2}$$

in which $\rho$ is the material density, $u_i(\mathbf{x}, t)$ the components of displacement for a particle at position $\mathbf{x}$ at time $t$ and summation over the suffix $j$ is assumed. It is remarked that the summation convention will apply in this paper *only* in respect of Latin subscripts. Equations (2.2) are to be solved subject to traction free boundary conditions on the upper and lower faces of the plate, namely

$$\sigma_{i3} = 0, \quad i = 1, 2, 3, \text{ at } x_3 = \pm h. \tag{2.3}$$

The stress–displacement relations for a transversely isotropic elastic material are deduced from (2.1), for a fibre direction parallel to $Ox_1$ taking the explicit form

$$\sigma_{11} = \rho c_5^2\frac{\partial u_1}{\partial x_1} + \rho c_4^2\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right), \tag{2.4}$$

$$\sigma_{22} = \rho c_4^2 \frac{\partial u_1}{\partial x_1} + \rho c_1^2 \frac{\partial u_2}{\partial x_2} + \rho (c_1^2 - 2c_2^2) \frac{\partial u_3}{\partial x_3}, \tag{2.5}$$

$$\sigma_{33} = \rho c_4^2 \frac{\partial u_1}{\partial x_1} + \rho (c_1^2 - 2c_2^2) \frac{\partial u_2}{\partial x_2} + \rho c_1^2 \frac{\partial u_3}{\partial x_3}, \tag{2.6}$$

$$\sigma_{12} = \rho c_3^2 \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right), \quad \sigma_{13} = \rho c_3^2 \left( \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right), \quad \sigma_{23} = \rho c_2^2 \left( \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right), \tag{2.7}$$

within which $c_1^2$, $c_2^2$, $c_3^2$, $c_4^2$ and $c_5^2$ are defined by

$$\begin{aligned} \rho c_1^2 &= \lambda + 2\mu_T, \quad \rho c_2^2 = \mu_T, \quad \rho c_3^2 = \mu_L, \\ \rho c_4^2 &= \lambda + \alpha, \quad \rho c_5^2 = \lambda + 4\mu_L - 2\mu_T + 2\alpha + \beta. \end{aligned} \tag{2.8}$$

Solutions of the equations of motion are now sought in the form of the travelling-wave solutions

$$\mathbf{u} = (U, V, W) e^{kpx_3} e^{i(k \cos\theta x_1 + k \sin\theta x_2 - \omega t)}, \tag{2.9}$$

in which $k$ is the wave number and $\omega$ the frequency, of a wave travelling with phase speed $v = \omega / k$ in a direction within the plane of the plate at an angle $\theta$ with the $x_1$ axis and $p$ is to be determined, in terms of $k$, $\theta$, $\omega$ and material constants, from the equations of motion. Substituting equation (2.9) into the equations of motion (2.2) yields

$$\{\bar{c}_3^2 p^2 - (\cos^2\theta \bar{c}_5^2 + \bar{c}_3^2 \sin^2\theta - \bar{v}^2)\} U - (\bar{c}_3^2 + \bar{c}_4^2) \cos\theta \sin\theta V + ip(\bar{c}_3^2 + \bar{c}_4^2) \cos\theta W = 0, \tag{2.10}$$

$$-\cos\theta \sin\theta (\bar{c}_3^2 + \bar{c}_4^2) U + \{\bar{c}_2^2 p^2 - (\cos^2\theta \bar{c}_3^2 + \sin^2\theta - \bar{v}^2)\} V + i \sin\theta p (1 - \bar{c}_2^2) W = 0, \tag{2.11}$$

$$ip \cos\theta (\bar{c}_3^2 + \bar{c}_4^2) U + i \sin\theta p (1 - \bar{c}_2^2) V + \{p^2 - (\cos^2\theta \bar{c}_3^2 + \sin^2\theta \bar{c}_2^2 - \bar{v}^2)\} W = 0, \tag{2.12}$$

in which $\bar{v}$ is defined by $c_1 k \bar{v} = \omega$ and $\bar{c}_i = c_i / c_1$. Equations (2.10) to (2.12) have non-trivial solutions (known as partial waves) provided that either

$$p^2 = p_2^2 = \left( \frac{\cos^2\theta \bar{c}_3^2 + \sin^2\theta \bar{c}_2^2 - \bar{v}^2}{\bar{c}_2^2} \right) \tag{2.13}$$

or

$$\begin{aligned} &\bar{c}_3^2 p^4 + \{(\bar{c}_3^2 + \bar{c}_4^2)^2 \cos^2\theta - \bar{c}_3^2 (\cos^2\theta \bar{c}_3^2 + \sin^2\theta - \bar{v}^2) - (\cos^2\theta \bar{c}_5^2 + \sin^2\theta \bar{c}_3^2 - \bar{v}^2)\} p^2 \\ &\quad + (\cos^2\theta \bar{c}_5^2 + \sin^2\theta \bar{c}_3^2 - \bar{v}^2)(\cos^2\theta \bar{c}_3^2 + \sin^2\theta - \bar{v}^2) - \cos^2\theta \sin^2\theta (\bar{c}_3^2 + \bar{c}_4^2)^2 = 0. \end{aligned} \tag{2.14}$$

Denoting the two roots of (2.14) by $p_1^2$ and $p_3^2$ and then superposing the six resulting solutions of the form (2.13) gives a general form for a travelling-wave solution $u_i = u_i^* e^{i(k \cos\theta x_1 + k \sin\theta x_2 - \omega t)}$ as

$$u_1^* = F(p_1)(A_1^+ e^{kp_1 x_3} + A_1^- e^{-kp_1 x_3}) + F(p_3)(A_3^+ e^{kp_3 x_3} + A_3^- e^{-kp_3 x_3}), \tag{2.15}$$

$$\begin{aligned} u_2^* = &-\{\sin\theta (A_1^+ e^{kp_1 x_3} + A_1^- e^{-kp_1 x_3}) + p_2 (A_2^+ e^{kp_2 x_3} - A_2^- e^{-kp_2 x_3})\} \\ &- \sin\theta (A_3^+ e^{kp_3 x_3} + A_3^- e^{-kp_3 x_3}), \end{aligned} \tag{2.16}$$

$$\begin{aligned} u_3^* = &\, ip_1 (A_1^+ e^{kp_1 x_3} - A_1^- e^{-kp_1 x_3}) + i \sin\theta (A_2^+ e^{kp_2 x_3} + A_2^- e^{-kp_2 x_3}) \\ &+ ip_3 (A_3^+ e^{kp_3 x_3} - A_3^- e^{-kp_3 x_3}), \end{aligned} \tag{2.17}$$

in which

$$F(p) = \left\{ \frac{\cos^2\theta \bar{c}_3^2 + \sin^2\theta - \bar{v}^2 - p^2}{\cos^2\theta(\bar{c}_3^2 + \bar{c}_4^2)} \right\}$$

and $A_m^+$, $A_m^-$, $m = 1, 2, 3$ are constants.

The six homogeneous equations in six unknowns arising from the boundary conditions (2.3) may be represented as two sets of three equations in three unknowns, to yield

$$p_1\eta_1(p_1)\hat{A}_1^- C_1 - \cos^2\theta\sin\theta(\bar{c}_3^2 + \bar{c}_4^2)\hat{A}_2^+ C_2 + p_3\eta_1(p_3)\hat{A}_3^- C_3 = 0, \tag{2.18}$$

$$2\sin\theta p_1\hat{A}_1^- C_1 + (p_2^2 + \sin^2\theta)\hat{A}_2^+ C_2 + 2\sin\theta p_3\hat{A}_3^- C_3 = 0, \tag{2.19}$$

$$\eta_2(p_1)\hat{A}_1^- S_1 + 2\bar{c}_2^2(\bar{c}_3^2 + \bar{c}_4^2)p_2\sin\theta\hat{A}_2^+ S_2 + \eta_2(p_3)\hat{A}_3^- S_3 = 0, \tag{2.20}$$

or

$$p_1\eta_1(p_1)\hat{A}_1^+ S_1 - \cos^2\theta\sin\theta(\bar{c}_3^2 + \bar{c}_4^2)\hat{A}_2^- S_2 + p_3\eta_1(p_3)\hat{A}_3^+ S_3 = 0, \tag{2.21}$$

$$2\sin\theta p_1\hat{A}_1^+ S_1 + (p_2^2 + \sin^2\theta)\hat{A}_2^- S_2 + 2\sin\theta p_3\hat{A}_3^+ S_3 = 0, \tag{2.22}$$

$$\eta_2(p_1)\hat{A}_1^+ C_1 + 2\bar{c}_2^2(\bar{c}_3^2 + \bar{c}_4^2)p_2\sin\theta\hat{A}_2^+ C_2 + \eta_2(p_3)\hat{A}_3^+ C_3 = 0, \tag{2.23}$$

in which $\hat{A}_m^+ = A_m^+ + A_m^-$, $\hat{A}_m^- = A_m^+ - A_m^-$, $C_m = \cosh(kp_m h)$, $S_m = \sinh(kp_m h)$, $m = 1, 2, 3$ and within which

$$\eta_1(p) = (k_2^2 - \bar{v}^2 - p^2 - k_1^2\bar{c}_4^2), \quad \eta_2(p) = \bar{c}_3^2(p^2 - k_2^2) + \bar{c}_4^2(\bar{c}_3^2 k_1^2 - \bar{v}^2) + 2\bar{c}_2^2 k_2^2(\bar{c}_3^2 + \bar{c}_4^2). \tag{2.24}$$

It is noted that when the determinant of coefficients associated with (2.18) to (2.20) vanishes, then $A_1^+ + A_1^- = A_2^+ - A_2^- = A_3^+ + A_3^- = 0$ and from the solutions (2.15) to (2.17) it is deduced that $W$ is symmetric about $x_3 = 0$, with $U$ and $V$ anti-symmetric. Setting the determinant of the coefficients of (2.18) to (2.20) to zero will therefore yield the dispersion relation associated with flexural waves, given explicitly by

$$\begin{aligned}
&-\eta_2(p_3)\{(p_2^2 + \sin^2\theta)(\sin^2\theta - \bar{v}^2 - p_1^2 - \cos^2\theta\bar{c}_4^2) + 2\cos^2\theta\sin^2\theta(\bar{c}_3^2 + \bar{c}_4^2)\}p_1 T_3\\
&+4\bar{c}_2^2(\bar{c}_3^2 + \bar{c}_4^2)p_1 p_2 p_3\sin^2\theta(p_3^2 - p_1^2)T_2\\
&+\eta_2(p_1)\{(p_2^2 + \sin^2\theta)(\sin^2\theta - \bar{v}^2 - p_3^2 - \cos^2\theta\bar{c}_4^2)\\
&+2\cos^2\theta\sin^2\theta(\bar{c}_3^2 + \bar{c}_4^2)\}p_3 T_1 = 0.
\end{aligned} \tag{2.25}$$

Similarly the determinant of coefficients of (2.21) to (2.23) yields the dispersion relation associated with extensional waves, namely

$$\begin{aligned}
&-\eta_2(p_3)\{(p_2^2 + \sin^2\theta)(\sin^2\theta - \bar{v}^2 - p_1^2 - \cos^2\theta\bar{c}_4^2)\\
&+2\cos^2\theta\sin^2\theta(\bar{c}_3^2 + \bar{c}_4^2)\}p_1 T_2 T_1 + 4\bar{c}_2^2(\bar{c}_3^2 + \bar{c}_4^2)p_1 p_2 p_3\sin^2\theta(p_3^2 - p_1^2)T_1 T_3\\
&+\eta_2(p_1)\{(p_2^2 + \sin^2\theta)(\sin^2\theta - \bar{v}^2 - p_3^2 - \cos^2\theta\bar{c}_4^2)\\
&+2\cos^2\theta\sin^2\theta(\bar{c}_3^2 + \bar{c}_4^2)\}p_3 T_3 T_2 = 0,
\end{aligned} \tag{2.26}$$

in which $T_m = \tanh(kp_m x_3)$, $m = 1, 2, 3$. The dispersion relations (2.25) and (2.26) were seemingly first derived by Green (**17**) and Green and Milosavljevic (**18**) in the context of studies of flexural and extensional waves, respectively. In both of these studies attention was largely focused on the fundamental modes, with no asymptotic analysis of the type to be carried out in the present paper.

We end this section by noting that $\hat{A}_1^{-,+}$, $\hat{A}_2^{-,+}$ and $\hat{A}_3^{-,+}$ may be represented in terms of the one parameter $\tilde{A}$ as

$$\hat{A}_1^- = \left(\frac{p_3[(p_2^2+\sin^2\theta)\eta_1(p_3)+2\cos^2\theta\sin^2\theta\sin^2\theta]C_3}{p_1C_1}\right)\tilde{A}, \tag{2.27}$$

$$\hat{A}_1^+ = \left(\frac{p_3[(p_2^2+\sin^2\theta)\eta_1(p_3)+2\cos^2\theta\sin^2\theta]S_3}{p_1S_1}\right)\tilde{A}, \tag{2.28}$$

$$\hat{A}_2^- = \left(\frac{2\sin\theta p_3[\eta_1(p_3)-\eta_1(p_1)]S_3}{S_2}\right)\tilde{A},$$

$$\hat{A}_2^+ = \left(\frac{2\sin\theta p_3[\eta_1(p_3)-\eta_1(p_1)]C_3}{C_2}\right)\tilde{A}, \tag{2.29}$$

$$\hat{A}_3^- = (\eta_1(p_1)(p_2^2+\sin^2\theta)+2\cos^2\theta\sin^2\theta)\tilde{A},$$

$$\hat{A}_3^+ = (\eta_1(p_1)(p_2^2+\sin^2\theta)+2\cos^2\theta\sin^2\theta)\tilde{A}. \tag{2.30}$$

## 3. Long-wave high frequency approximations

We shall now seek long-wave high frequency approximations of the two dispersion relations (2.25) and (2.26) to give scaled frequency as a function of scaled wave number $kh$. Specifically, the approximations sought are valid for all harmonics of the dispersion relations in the vicinity of cut-off frequencies, that is, in the low wave number regime. To begin it is noted that in the long-wave region $\bar{v}$ is large, although $\omega$ remains $O(n)$, the mode number. Accordingly (2.13) and (2.14) are used to establish that

$$p_1^2 = -\frac{\bar{v}^2}{\bar{c}_3^2}+\bar{p}_1^2+O(\bar{v}^{-2}),\quad p_2^2 = -\frac{\bar{v}^2}{\bar{c}_2^2}+\bar{p}_2^2,\quad p_3^2 = -\bar{v}^2+\bar{p}_3^2+O(\bar{v}^{-2}), \tag{3.1}$$

in which

$$\bar{c}_3^2\bar{p}_1^2 = p_1^{(1)}\cos^2\theta+p_1^{(2)}\sin^2\theta,\quad \bar{c}_2^2\bar{p}_2^2 = p_2^{(1)}\cos^2\theta+p_2^{(2)}\sin^2\theta,$$
$$\bar{p}_3^2 = p_3^{(1)}\cos^2\theta+p_3^{(2)}\sin^2\theta,$$

with

$$p_1^{(1)} = \frac{(\bar{c}_3^2+\bar{c}_4^2)^2}{(\bar{c}_3^2-1)}+\bar{c}_5^2,\quad p_1^{(2)}=\bar{c}_3^2,\quad p_2^{(1)}=\bar{c}_3^2,\quad p_2^{(2)}=\bar{c}_2^2,\quad p_3^{(1)} = \frac{(\bar{c}_3^2+\bar{c}_4^2)^2}{(1-\bar{c}_3^2)}+\bar{c}_3^2,$$
$$p_3^{(2)} = 1.$$

The approximations (3.1) may now be inserted into equation (2.25) to establish that in the long-wave high frequency approximation

$$\Gamma_1\tan(k\hat{p}_1h)+\Gamma_2\tan(k\hat{p}_2h)+\bar{v}^2\tan(k\hat{p}_3h)\sim 0, \tag{3.2}$$

in which $p_m = i\hat{p}_m$, $m = 1, 2, 3$ and

$$\Gamma_1 = a_1\cos^2\theta,\quad \Gamma_2 = a_2\sin^2\theta,\quad a_1 = \frac{(1+\bar{c}_4^2)^2\bar{c}_3^3}{(1-\bar{c}_3^2)^2},\quad a_2 = 4\bar{c}_2^3.$$

From equation (3.2) it is deduced that three distinct cases exist:

$$\tan(k\hat{p}_1h) \sim \bar{v}^2 \quad \text{or} \quad \tan(k\hat{p}_2h) \sim \bar{v}^2 \quad \text{or} \quad \tan(k\hat{p}_3h) \sim \bar{v}^{-2}. \tag{3.3}$$

We begin by considering the first case from which we are able to deduce that

$$k\hat{p}_1h = \left(n+\tfrac{1}{2}\right)\pi + \phi(kh)^2 + O(kh)^3 \Longrightarrow \tan(k\hat{p}_1h) = \frac{-1}{\phi(kh)^2} + O(1),$$
$$\tan(k\hat{p}_2h) = \tan\left(\Lambda^a_{sh1}/\bar{c}_2\right) + O(kh)^2, \quad \tan(k\hat{p}_3h) = \tan(\Lambda^a_{sh1}) + O(kh)^2, \tag{3.4}$$

in which $\phi$ is an $O(1)$ quantity to be determined. It is also noted that $kh \sim \bar{v}^{-1}$ and $\Lambda^a_{sh1}$ is the associated cut-off frequency, defined by

$$\Lambda^a_{sh1} = \left(n+\tfrac{1}{2}\right)\pi\bar{c}_3.$$

Upon inserting the leading-order terms in equation (3.4) into (3.2), and equating leading-order terms to zero, $\phi$ is obtained explicitly. This may then be used in equation $(3.1)_1$ to obtain

$$\bar{\omega}^2 = (\Lambda^a_{sh1})^2 + \left(\bar{p}_1^2\bar{c}_3^2 + \frac{2\bar{c}_3\Gamma_1}{\Lambda^a_{sh1}\tan(\Lambda^a_{sh1})}\right)(kh)^2 + O(kh)^4, \tag{3.5}$$

in which the non-dimensional frequency $\bar{\omega}$ is defined by $\bar{\omega} = \omega h/c_1$. In a similar way approximations appropriate to the two other cases indicated by (3.3) are also obtainable, yielding

$$\bar{\omega}^2 = (\Lambda^a_{sh2})^2 + \left(\bar{p}_2^2\bar{c}_2^2 + \frac{2\Gamma_2\bar{c}_2}{\Lambda^a_{sh2}\tan(\Lambda^a_{sh2})}\right)(kh)^2 + O(kh)^4 \tag{3.6}$$

and

$$\bar{\omega}^2 = (\Lambda^a_{st})^2 + \left(\bar{p}_3^2 - 2\left(\frac{\Gamma_1\tan(\Lambda^a_{st}/\bar{c}_3) + \Gamma_2\tan(\Lambda^a_{st}/\bar{c}_2)}{\Lambda^a_{st}}\right)\right)(kh)^2 + O(kh)^4, \tag{3.7}$$

within which the cut-off frequencies $\Lambda^a_{sh2}$ and $\Lambda^s_{st}$ are defined by

$$\Lambda^a_{sh2} = \left(n+\tfrac{1}{2}\right)\pi\bar{c}_2, \quad \Lambda^a_{st} = n\pi.$$

We now consider the corresponding extensional-wave problem for which the analogous form of (3.2) is given by

$$\Gamma_1\tan(k\hat{p}_2h)\tan(k\hat{p}_3h) + \Gamma_2\tan(k\hat{p}_1h)\tan(k\hat{p}_3h) + \bar{v}^2\tan(k\hat{p}_1h)\tan(k\hat{p}_2h) \sim 0. \tag{3.8}$$

From equation (3.8) it is deduced that

$$\tan(k\hat{p}_2h) \sim \bar{v}^{-2} \quad \text{or} \quad \tan(k\hat{p}_3h) \sim \bar{v}^2 \quad \text{or} \quad \tan(k\hat{p}_1h) \sim \bar{v}^2. \tag{3.9}$$

In a very similar fashion to that previously used for the flexural case it is deduced that the three corresponding expansions associated with extensional waves are given by

$$\bar{\omega}^2 = (\Lambda^s_{sh1})^2 + \left(\bar{p}_1^2\bar{c}_3^2 - \frac{2\bar{c}_3\Gamma_1\tan(\Lambda^s_{sh1})}{\Lambda^s_{sh1}}\right)(kh)^2 + O(kh)^4, \tag{3.10}$$

$$\bar{\omega}^2 = (\Lambda^s_{sh2})^2 + \left(\bar{p}_2^2\bar{c}_2^2 - \frac{2\Gamma_2\bar{c}_2\tan(\Lambda^s_{sh2})}{\Lambda^s_{sh2}}\right)(kh)^2 + O(kh)^4, \tag{3.11}$$

$$\bar{\omega}^2 = (\Lambda^s_{st})^2 + \left(\bar{p}_3^2 + 2\left(\frac{\Gamma_1\cot(\Lambda^s_{st}/\bar{c}_3) + \Gamma_2\cot(\Lambda^s_{st}/\bar{c}_2)}{\Lambda^s_{st}}\right)\right)$$
$$\times(kh)^2 + O(kh)^4, \tag{3.12}$$

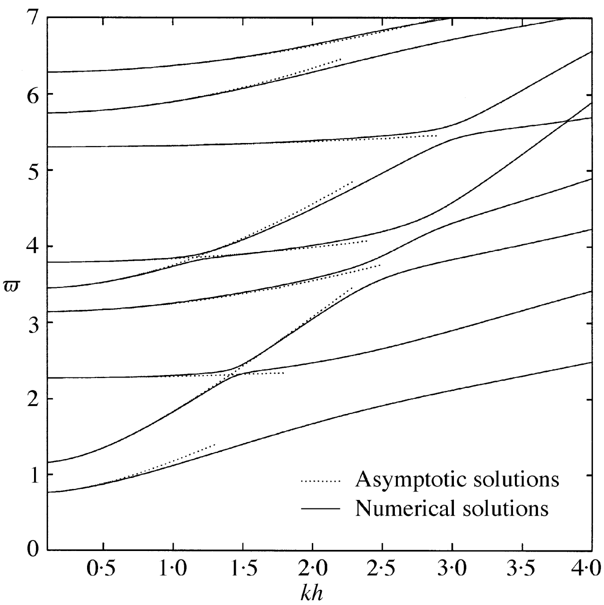


**Fig. 1** Comparison of numerical and asymptotic solutions of the dispersion relation for the first nine non-zero cut-off frequencies for flexural modes

within which the cut-off frequencies $\Lambda^s_{sh1}$, $\Lambda^s_{sh2}$ and $\Lambda^s_{st}$ are defined by

$$\Lambda^s_{sh2} = n\pi\bar{c}_2, \quad \Lambda^s_{st} = \left(n + \tfrac{1}{2}\right)\pi, \quad \Lambda^s_{sh1} = n\pi\bar{c}_3. \tag{3.13}$$

In Figs 1 and 2 numerical solutions of the first nine harmonics of flexural and extensional motions, showing $\bar{\omega}$ as a function of $kh$, are shown with their appropriate asymptotic expansions. These figures relate to a direction of propagation $\theta = 70°$ and employ material parameters measured by Markham (**19**) for a carbon fibre-epoxy resin composite, for which $c_1$, $c_2$, $c_3$, $c_4$ and $c_5$ are given by

$$\rho c_1^2 = 10{\cdot}57 \times 10^9\mathrm{Nm}^{-2}, \quad \rho c_2^2 = 2{\cdot}46 \times 10^9\mathrm{Nm}^{-2}, \quad \rho c_3^2 = 5{\cdot}66 \times 10^9\mathrm{Nm}^{-2},$$
$$\rho c_4^2 = 4{\cdot}37 \times 10^9\mathrm{Nm}^{-2}, \quad \rho c_5^2 = 241{\cdot}71 \times 10^9\mathrm{Nm}^{-2}.$$

Although the asymptotic solutions are strictly valid only for $kh \ll 1$, they never the less provide a good approximation in many cases until the harmonic reaches the so-called ghost line associated with a wave front. These are lines in the $(\bar{\omega}, kh)$ plane along which the harmonics have a near-constant gradient over a specific wave number range. Over this range the group velocity has close to zero gradient, indicating a wave front. A point of interest associated with these plots concerns the first harmonic of extensional motion, for which their clearly exists an associated mode *below* the cut-off frequency for small $kh$. Moreover, before $\bar{\omega}$ reaches its minimum value the group velocity is negative. We shall discuss this point in more detail later in the paper.

It is noted that the cut-off frequencies $\Lambda^a_{st}$ and $\Lambda^s_{st}$ are the so-called thickness stretch resonance frequencies associated with anti-symmetric and symmetric motions. These represent the natural frequencies of stretch (in the $Ox_3$ direction) motions of an infinitesimally thin transverse fibre of

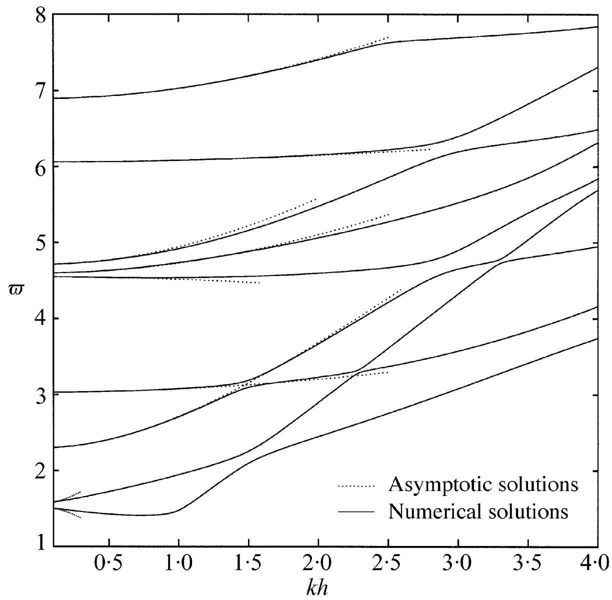


**Fig. 2** Comparison of numerical and asymptotic solutions of the dispersion relation for the first nine non-zero cut-off frequencies for extensional modes

the layer. The stretch thickness resonance frequencies are the eigenvalues of

$$\frac{\partial^2 u_3}{\partial x_3^2} + \omega^2 u_3 = 0, \qquad \frac{\partial u_3}{\partial x_3} = 0 \quad \text{at } x_3 = \pm h. \tag{3.14}$$

Similarly, $\Lambda^{a,s}_{sh1}$ and $\Lambda^{a,s}_{sh2}$ are the first and second shear resonance frequencies, which satisfy the eigenvalue problems

$$\frac{\partial^2 u_1}{\partial x_3^2} + \bar{c}_3^{-2}\omega^2 u_1 = 0, \qquad \frac{\partial u_1}{\partial x_3} = 0 \quad \text{at } x_3 = \pm h, \tag{3.15}$$

$$\frac{\partial^2 u_2}{\partial x_3^2} + \bar{c}_2^{-2}\omega^2 u_2 = 0, \qquad \frac{\partial u_2}{\partial x_3} = 0 \quad \text{at } x_3 = \pm h, \tag{3.16}$$

respectively. It is also noted that in the present transversely isotropic case two distinct families of thickness shear resonance frequencies exist, contrasting with only one in the corresponding isotropic case; see for example, Kaplunov *et al.* (**4**). The asymptotic structure therefore may deviate significantly from the corresponding isotropic case in respect of specific frequencies of vibration. In particular, it is now possible that two different families of shear resonance frequencies may be excited independently.

## 4. Asymptotically approximate equations in the vicinity of the thickness shear resonance frequencies

In order to derive asymptotically approximate equations we first introduce the non-dimensional parameter $\eta$, defined as the ratio of plate half-thickness to wavelength, thus

$$\eta = \frac{h}{l} \Longrightarrow \eta = kh, \tag{4.1}$$

in which $l$ is the wavelength. For long-wave motions we may assume that $h \ll l$ and we therefore regard $\eta$ as a small non-dimensional parameter. It is now possible to insert the approximations (3.1), in conjunction with the both definitions of $(\Lambda^a_{sh\alpha})$ $(\Lambda^s_{sh\alpha})$, into (2.27) to (2.30) to establish that in the vicinity of either the first ($\alpha = 1$) or second ($\alpha = 2$) symmetric (anti-symmetric) shear resonance frequencies,

$$\hat{A}^-_\alpha, \hat{A}^+_\alpha \sim O(\bar{v})^4\tilde{A}, \quad \hat{A}^+_\beta, \hat{A}^-_\beta \sim O(\bar{v})^3\tilde{A}, \quad \hat{A}^-_3, \hat{A}^+_3 \sim O(\bar{v})^4\tilde{A}, \tag{4.2}$$

in which $\alpha,\ \beta \in \{1, 2 : \alpha \neq \beta\}$. Utilization of (2.15) to (2.17) is now made to motivate the introduction of scaled variables appropriate to long-wave high frequency approximations, in the vicinity of the thickness shear resonance frequencies, thus

$$u_\alpha = l\eta u^0_\alpha, \quad u_\beta = l\eta^3 u^0_\beta, \quad u_3 = l\eta^2 u^0_3, \tag{4.3}$$

in which a superscript 0 indicates an $O(1)$ quantity. The corresponding orders of stress components are readily obtainable and we therefore note that

$$\sigma_{\alpha\alpha} = \rho c_1^2 \eta \sigma^0_{\alpha\alpha}, \quad \sigma_{\alpha\beta} = \rho c_1^2 \eta \sigma^0_{\alpha\beta}, \tag{4.4}$$

$$\sigma_{\alpha 3} = \rho c_1^2 \sigma^0_{\alpha 3}, \quad \sigma_{\beta 3} = \rho c_1^2 \eta^2 \sigma^0_{\beta 3}, \quad \sigma_{33} = \rho c_1^2 \eta \sigma^0_{33}. \tag{4.5}$$

In passing it is remarked that the asymptotic structure indicated by (4.3) to (4.5) differs from that associated with the isotropic case, in which $u_\alpha \sim u_\beta$ and $\sigma_{\alpha 3} \sim \sigma_{\beta 3}$. Motivated by (3.5), (3.6), (3.10), (3.11), (3.15) and (3.16), and in view of the fact that we are considering long-wave high frequency approximations in the vicinity of the first or second thickness shear resonance frequencies, it is assumed that

$$\frac{\partial^2 u_k}{\partial \tau^2} + \Lambda^2_{sh\alpha} u_k \sim \eta^2 u_k, \quad k = 1, 2, 3. \tag{4.6}$$

Before we begin to consider approximations of the equations of motion we introduce appropriate scales for spatial coordinates and time, thus

$$x_\alpha = l\xi_\alpha, \quad x_3 = l\eta\zeta, \quad t = l\eta c_1^{-1}\tau. \tag{4.7}$$

Equations (4.3) to (4.7) may now be used with the three equations of motion to establish that

$$\frac{\partial \sigma^0_{\alpha 3}}{\partial \zeta} + \Lambda^2_{sh\alpha} u^0_\alpha + \eta^2 \left( \frac{\partial \sigma^0_{\alpha\alpha}}{\partial \xi_\alpha} + \frac{\partial \sigma^0_{\alpha\beta}}{\partial \xi_\beta} \right) - \left( \frac{\partial^2 u^0_\alpha}{\partial \tau^2} + \Lambda^2_{sh\alpha} u^0_\alpha \right) = 0, \tag{4.8}$$

$$\frac{\partial \sigma^0_{\alpha\beta}}{\partial \xi_\alpha} + \frac{\partial \sigma^0_{\beta\beta}}{\partial \xi_\beta} + \frac{\partial \sigma^0_{\beta 3}}{\partial \zeta} + \Lambda^2_{sh\alpha} u^0_\beta - \left( \frac{\partial^2 u^0_\beta}{\partial \tau^2} + \Lambda^2_{sh\alpha} u^0_\beta \right) = 0, \tag{4.9}$$

$$\frac{\partial \sigma^0_{\alpha 3}}{\partial \xi_\alpha} + \frac{\partial \sigma^0_{33}}{\partial \zeta} + \Lambda^2_{sh\alpha} u^0_3 + \eta^2 \frac{\partial \sigma^0_{\beta 3}}{\partial \xi_\beta} - \left( \frac{\partial^2 u^0_3}{\partial \tau^2} + \Lambda^2_{sh\alpha} u^0_3 \right) = 0. \tag{4.10}$$

If equations (4.3) are inserted into the stress components (2.4) to (2.7) the stress–displacement relations are obtained in the form

$$\sigma^0_{\alpha\alpha} = \kappa_{\alpha\alpha}\frac{\partial u^0_\alpha}{\partial \xi_\alpha} + m_\alpha \frac{\partial u^0_3}{\partial \zeta} + \eta^2 \kappa_{\alpha\beta}\frac{\partial u^0_\beta}{\partial \xi_2}, \quad \sigma^0_{\beta\beta} = \kappa_{\beta\alpha}\frac{\partial u^0_\alpha}{\partial \xi_\alpha} + m_\beta \frac{\partial u^0_3}{\partial \zeta} + \eta^2 \kappa_{\beta\beta}\frac{\partial u^0_\beta}{\partial \xi_\beta}, \tag{4.11}$$

$$\sigma^0_{33} = m_\alpha \frac{\partial u^0_\alpha}{\partial \xi_\alpha} + \frac{\partial u^0_3}{\partial \zeta} + \eta^2 m_\beta \frac{\partial u^0_\beta}{\partial \xi_\beta}, \qquad \sigma^0_{\alpha\beta} = \bar{c}^2_3 \frac{\partial u^0_\alpha}{\partial \xi_\beta} + \eta^2 \bar{c}^2_3 \frac{\partial u^0_\beta}{\partial \xi_\alpha}, \tag{4.12}$$

$$\sigma^0_{\alpha 3} = \mathcal{C}_\alpha \frac{\partial u^0_\alpha}{\partial \zeta} + \eta^2 \mathcal{C}_\alpha \frac{\partial u^0_3}{\partial \xi_\alpha}, \qquad \sigma_{\beta 3} = \mathcal{C}_\beta\left(\frac{\partial u^0_3}{\partial \xi_\beta} + \frac{\partial u^0_\beta}{\partial \zeta}\right), \tag{4.13}$$

in which

$$\kappa_{11} = \bar{c}^2_5, \quad \kappa_{12} = \kappa_{21} = m_1 = \bar{c}^2_4, \quad \kappa_{22} = 1, \quad \mathcal{C}_1 = \bar{c}^2_3, \quad \mathcal{C}_2 = \bar{c}^2_2, \quad m_2 = 1 - 2\bar{c}^2_2. \tag{4.14}$$

Solutions to the governing equations (4.8) to (4.10) are now sought in the form

$$u^0_i = u^{0(1)}_i + \eta^2 u^{0(2)} + O(\eta^4), \quad i = 1, 2, 3. \tag{4.15}$$

Equations (4.11) to (4.13) may now be used in equations (4.8) to (4.10) to establish the leading-order system of equations

$$\mathcal{C}_\alpha \frac{\partial^2 u^{0(1)}_\alpha}{\partial \zeta^2} + \Lambda^2_{sh\alpha} u^{0(1)}_\alpha = 0, \tag{4.16}$$

$$\mathcal{C}_\beta \frac{\partial^2 u^{0(1)}_\beta}{\partial \zeta^2} + \Lambda^2_{sh\alpha} u^{0(1)}_\beta = -(\bar{c}^2_3 + \kappa_{\beta\alpha})\frac{\partial^2 u^{0(1)}_\alpha}{\partial \xi_\alpha \partial \xi_\beta} + (m_\beta + \mathcal{C}_\beta)\frac{\partial^2 u^{0(1)}_3}{\partial \xi_\beta \partial \zeta}, \tag{4.17}$$

$$\frac{\partial^2 u^{0(1)}_3}{\partial \zeta^2} + \Lambda^2_{sh\alpha} u^{0(1)}_3 = -(m_\alpha + \mathcal{C}_\alpha)\frac{\partial^2 u^{0(1)}_\alpha}{\partial \zeta \partial \xi_\alpha}. \tag{4.18}$$

Equations (4.16) to (4.18) must be solved subject to the leading-order boundary conditions

$$\frac{\partial u^{0(1)}_\alpha}{\partial \zeta} = 0, \quad \frac{\partial u^{0(1)}_3}{\partial \xi_\beta} + \frac{\partial u^{0(1)}_\beta}{\partial \zeta} = 0, \quad m_\alpha \frac{\partial u^{0(1)}_\alpha}{\partial \xi_\alpha} + \frac{\partial u^{0(1)}_3}{\partial \zeta} = 0 \quad \text{at } \zeta = \pm 1. \tag{4.19}$$

*The anti-symmetric solution*

In order to proceed further the anti-symmetric and symmetric cases must be considered separately. First the anti-symmetric solution is considered for which the leading-order equations (4.16) to (4.18), subject to the boundary conditions (4.19), are readily solved to obtain the leading-order displacement components

$$u^{0(1)}_\alpha = U^{(0,0)}_\alpha(\xi_1, \xi_2, \tau)\sin\left(\frac{\Lambda^a_{sh\alpha}\zeta}{\sqrt{\mathcal{C}_\alpha}}\right), \tag{4.20}$$

$$u^{0(1)}_\beta = U^{(0,0)}_\beta \sin\left(\frac{\Lambda^a_{sh\alpha}\zeta}{\sqrt{\mathcal{C}_\alpha}}\right) + U^{(1,0)}_\beta \sin(\Lambda^a_{sh\alpha}\zeta) + U^{(2,0)}_\beta \sin\left(\frac{\Lambda^a_{sh\alpha}\zeta}{\sqrt{\mathcal{C}_\beta}}\right), \tag{4.21}$$

$$u^{0(1)}_3 = W^{(0,0)}\cos\left(\frac{\Lambda^a_{sh\alpha}\zeta}{\sqrt{\mathcal{C}_\alpha}}\right) + W^{(1,0)}\cos(\Lambda^a_{sh\alpha}\zeta), \tag{4.22}$$

in which

$$U_\beta^{(0,0)} = \frac{-\mathcal{C}_\alpha}{(\mathcal{C}_\alpha - \mathcal{C}_\beta)(\Lambda^a_{sh\alpha})^2}\left(\bar{c}_3^2 + \kappa_{\beta\alpha} + \frac{(m_\alpha + \mathcal{C}_\alpha)(m_\beta + \mathcal{C}_\beta)}{\mathcal{C}_\alpha - 1}\right)\frac{\partial^2 U_\alpha^{(0,0)}}{\partial\xi_\alpha\partial\xi_\beta}, \tag{4.23}$$

$$U_\beta^{(1,0)} = -\left(\frac{(m_\alpha + 1)(m_\beta + \mathcal{C}_\beta)\mathcal{C}_\alpha}{(1-\mathcal{C}_\alpha)(1-\mathcal{C}_\beta)(\Lambda^a_{sh\alpha})^2}\right)\frac{\sin(\Lambda^a_{sh\alpha}/\sqrt{\mathcal{C}_\alpha})}{\sin(\Lambda^a_{sh\alpha})}\frac{\partial^2 U_\alpha^{(0,0)}}{\partial\xi_\alpha\partial\xi_\beta}, \tag{4.24}$$

$$U_\beta^{(2,0)} = \frac{\sqrt{\mathcal{C}_\beta}\sin(\Lambda^a_{sh\alpha}/\sqrt{\mathcal{C}_\alpha})\mathcal{C}_\alpha(m_\alpha + 1)(m_\beta + 1)\cot(\Lambda^a_{sh\alpha})}{(\Lambda^a_{sh\alpha})^2(1-\mathcal{C}_\alpha)(1-\mathcal{C}_\beta)\cos(\Lambda^a_{sh\alpha}/\sqrt{\mathcal{C}_\beta})}\frac{\partial^2 U_\alpha^{(0,0)}}{\partial\xi_\alpha\partial\xi_\beta}, \tag{4.25}$$

$$W^{(0,0)} = -\frac{(m_\alpha + \mathcal{C}_\alpha)\sqrt{\mathcal{C}_\alpha}}{\Lambda^a_{sh\alpha}(\mathcal{C}_\alpha - 1)}\frac{\partial U_\alpha^{(0,0)}}{\partial\xi_\alpha}, \quad W^{(1,0)} = \frac{\sin(\Lambda^a_{sh\alpha}/\sqrt{\mathcal{C}_\alpha})}{\Lambda^a_{sh\alpha}\sin(\Lambda^a_{sh\alpha})}\left(\frac{\mathcal{C}_\alpha(m_\alpha + 1)}{\mathcal{C}_\alpha - 1}\right)\frac{\partial U_\alpha^{(0,0)}}{\partial\xi_\alpha}, \tag{4.26}$$

and $U_\alpha^{(0,0)}$ is the long-wave amplitude. In order to obtain a two-dimensional governing equation for $U_\alpha^{(0,0)}$, correct up to $O(\eta^2)$, we need only consider equation (4.8) at second order, which takes the form

$$\begin{aligned}\mathcal{C}_\alpha\frac{\partial^2 u_\alpha^{0(2)}}{\partial\zeta^2} + (\Lambda^a_{sh\alpha})^2 u_\alpha^{0(2)} = &-(\mathcal{C}_\alpha + m_\alpha)\frac{\partial^2 u_3^{0(1)}}{\partial\zeta\partial\xi_\alpha} - \kappa_{\alpha\alpha}\frac{\partial^2 u_\alpha^{0(1)}}{\partial\xi_\alpha^2}\\ &-\bar{c}_3^2\frac{\partial^2 u_\alpha^{0(1)}}{\partial\xi_\beta^2} + \eta^{-2}\left(\frac{\partial^2 u_\alpha^{0(1)}}{\partial\tau^2} + \Lambda^2_{sh\alpha}u_\alpha^{0(1)}\right),\end{aligned} \tag{4.27}$$

and which must be solved subject to the appropriate second-order boundary condition

$$\frac{\partial u_\alpha^{0(2)}}{\partial\zeta} + \frac{\partial u_3^{0(1)}}{\partial\xi_\alpha} = 0 \text{ at } \zeta = \pm 1. \tag{4.28}$$

The solution of equation (4.27) subject to the boundary condition (4.28) is given by

$$u_\alpha^{0(2)} = V_\alpha^{(0,0)}\sin\left(\frac{\Lambda^a_{sh\alpha}\zeta}{\sqrt{\mathcal{C}_\alpha}}\right) + V_\alpha^{(0,1)}\zeta\cos\left(\frac{\Lambda^a_{sh\alpha}\zeta}{\sqrt{\mathcal{C}_\alpha}}\right) + V_\alpha^{(1,0)}\sin(\Lambda^a_{sh\alpha}\zeta), \tag{4.29}$$

within which

$$V_\alpha^{(1,0)} = -\frac{(m_\alpha + \mathcal{C}_\alpha)(m_\alpha + 1)\mathcal{C}_\alpha}{(1-\mathcal{C}_\alpha)^2(\Lambda^a_{sh\alpha})^2}\frac{\sin(\Lambda^a_{sh\alpha}/\sqrt{\mathcal{C}_\alpha})}{\sin(\Lambda^a_{sh\alpha})}\frac{\partial^2 U_\alpha^{(0,0)}}{\partial\xi_\alpha^2}, \tag{4.30}$$

$$\begin{aligned}V_\alpha^{(0,1)} = \frac{1}{2\sqrt{\mathcal{C}_\alpha}\Lambda^a_{sh\alpha}}\Bigg\{&\left(\kappa_{\alpha\alpha} + \frac{(\mathcal{C}_\alpha + m_\alpha)^2}{\mathcal{C}_\alpha - 1}\right)\frac{\partial^2 U_\alpha^{(0,0)}}{\partial\xi_\alpha^2} + \bar{c}_3^2\frac{\partial^2 U_\alpha^{(0,0)}}{\partial\xi_\beta^2}\\ &-\eta^{-2}\left(\frac{\partial^2 U_\alpha^{(0,0)}}{\partial\tau^2} + (\Lambda^a_{sh\alpha})^2 U_\alpha^{(0,0)}\right)\Bigg\}.\end{aligned} \tag{4.31}$$

Without resorting to higher-order approximations it is not possible to obtain an expression for $V_\alpha^{(0,0)}$. However, it not not necessary to do this in order to obtain an equation for the long-wave amplitude $U_\alpha^{(0,0)}$. Such an equation is obtained by satisfying the boundary condition (4.28) to yield

$$\left(\frac{\partial^2}{\partial\tau^2} + (\Lambda^a_{sh\alpha})^2\right)U_\alpha^{(0,0)} - \eta^2\left(\sum_{\gamma=\alpha,\beta} P_{sh}^{(\gamma)} + \frac{T_{sh}^{(\gamma)}\cot(\Lambda^a_{sh\gamma})}{\Lambda^a_{sh\gamma}}\frac{\partial^2 U_\alpha^{(0,0)}}{\partial\xi_\gamma^2}\right) = 0, \tag{4.32}$$

in which

$$T_{sh}^{(\alpha)} = \frac{2(m_\alpha + 1)^2 \mathcal{C}_\alpha^2}{(1 - \mathcal{C}_\alpha)^2}, \quad P_{sh}^{(\alpha)} = \kappa_{\alpha\alpha} - \frac{(\mathcal{C}_\alpha + m_\alpha)^2}{1 - \mathcal{C}_\alpha}, \quad T_{sh}^{(\beta)} = 0, \quad P_{sh}^{(\beta)} = \bar{c}_3^2. \tag{4.33}$$

In the vicinity of the first thickness shear resonance frequency, for which $\alpha = 1$, (4.14) may be used in conjunction with those given directly after (3.1) and (3.2) to establish that

$$P_{sh}^{1} = p_1^{(1)}, \quad P_{sh}^{(2)} = p_1^{(2)}, \quad T_{sh}^{(1)} = 2\bar{c}_3 a_1, \tag{4.34}$$

enabling the appropriate two-dimensional equation to be deduced from (4.32) in the form

$$\left(\frac{\partial^2}{\partial \tau^2} + (\Lambda_{sh1}^a)^2\right) U_1^{(0,0)} - \eta^2 \left\{ \left( p_1^{(1)} + \frac{2a_1 \bar{c}_3 \cot(\Lambda_{st1}^a)}{\Lambda_{st1}^a} \right) \frac{\partial^2 U_1^{(0,0)}}{\partial \xi_1^2} + p_1^{(2)} \frac{\partial^2 U_1^{(0,0)}}{\partial \xi_2^2} \right\} = 0. \tag{4.35}$$

The corresponding equation associated with vibrations in the vicinity of the second shear resonance frequencies is similarly given by

$$\left(\frac{\partial^2}{\partial \tau^2} + (\Lambda_{sh2}^a)^2\right) U_2^{(0,0)} - \eta^2 \left\{ p_2^{(1)} \frac{\partial^2 U_2^{(0,0)}}{\partial \xi_1^2} + \left( p_2^{(2)} + \frac{2\bar{c}_2 a_2 \cot(\Lambda_{sh2}^a)}{\Lambda_{sh2}^a} \right) \frac{\partial^2 U_2^{(0,0)}}{\partial \xi_2^2} \right\} = 0. \tag{4.36}$$

It is now possible to confirm, by assuming solutions of the form

$$U_\alpha^{(0,0)} = \hat{V}_\alpha e^{i(\cos\theta \xi_1 + \sin\theta \xi_2 - \bar{\omega}\tau)}, \tag{4.37}$$

that the dispersion relation approximations obtained from the exact relations, see (3.5) and (3.6), are exactly the same as those obtained from (4.35) and (4.36), respectively.

*The symmetric solution*

In the symmetric case the solution (4.20) is replaced by

$$u_\alpha^{0(1)} = U_\alpha^{(0,0)}(\xi_1, \xi_2, \tau) \cos\left(\frac{\Lambda_{sh\alpha}^a \zeta}{\sqrt{\mathcal{C}_\alpha}}\right). \tag{4.38}$$

A similar analysis may then be followed to obtain the analogous equations to (4.36) and (4.37) in the forms

$$\left(\frac{\partial^2}{\partial \tau^2} + (\Lambda_{sh1}^a)^2\right) U_1^{(0,0)} - \eta^2 \left\{ \left( p_1^{(1)} - \frac{2a_1 \bar{c}_3 \tan(\Lambda_{st1}^a)}{\Lambda_{st1}^a} \right) \frac{\partial^2 U_1^{(0,0)}}{\partial \xi_1^2} + p_1^{(2)} \frac{\partial^2 U_1^{(0,0)}}{\partial \xi_2^2} \right\} = 0. \tag{4.39}$$

$$\left(\frac{\partial^2}{\partial \tau^2} + (\Lambda_{sh2}^a)^2\right) U_2^{(0,0)} - \eta^2 \left\{ p_2^{(1)} \frac{\partial^2 U_2^{(0,0)}}{\partial \xi_1^2} + \left( p_2^{(2)} - \frac{2\bar{c}_2 a_2 \tan(\Lambda_{sh2}^a)}{\Lambda_{sh2}^a} \right) \frac{\partial U_2^{(0,0)}}{\partial^2 \xi_2^2} \right\} = 0, \tag{4.40}$$

where again consistency with the relevant approximations in section 3, namely (3.10) and (3.11), is easily verified.

The possible existence of negative group velocity associated with certain dispersion curve branches at low wave number was noted earlier in the paper. In order to investigate this further, and its consequence for the governing equation for the long-wave amplitude, we consider equation (4.40). For a given angle of propagation new variables $\bar{\xi}_1$ and $\bar{\xi}_2$ may be chosen, with $\bar{\xi}_1 = \cos\theta\xi_1 + \sin\theta\xi_2$ and $O\bar{\xi}_1$ along the direction of propagation and $O\bar{\xi}_2$ normal to it. The governing equation for $U_2^{(0,0)}$ associated with propagation along $O\bar{\xi}_1$ may now be written as

$$\frac{\partial^2 U_2^{(0,0)}}{\partial\tau^2} + (\Lambda^a_{sh2})^2 U_2^{(0,0)} - \eta^2 A_n \frac{\partial U_2^{(0,0)}}{\partial\bar{\xi}_1^2} = 0, \tag{4.41}$$

in which $n$ denotes the harmonic number and

$$A_n = p_2^{(1)}\cos^2\theta + \left(p_2^{(2)} - \frac{2\bar{c}_2 a_2 \tan(\Lambda^a_{sh2})}{\Lambda^a_{sh2}}\right)\sin^2\theta. \tag{4.42}$$

As might have been expected the term $A_n$ in equation (4.41) is exactly the same as the multiplier of $(kh)^2$ in the corresponding expansion (3.11). When $A_n < 0$ the group velocity will be negative at low wave number. In any region for which this occurs the governing equation for $U_2^{(0,0)}$ will not be hyperbolic, as it is when $A_n > 0$, but rather elliptic. It is then the case that any initial-value problem cannot be properly posed.

In Fig. 3 $A_n$ is plotted against $\theta$ for the first six modes of the associated family of shear resonance frequencies. In this case when $\theta = 0$, $A_n = p_2^{(1)}$ and is independent of $n$. Also for the specific material constants $\Lambda^a_{sh2}$ is close to $n\pi/2$ and therefore the contribution from the second term within the larger brackets in (4.42) is negligible for all even modes. For the odd modes, $A_1$ becomes zero at the lowest value of $\theta$, with $A_3$ also becoming zero but at a higher value of $\theta$. In contrast for $A_5$, and all even modes, $A_n$ is always positive with $A_2$, $A_4$ and $A_6$ indistinguishable. It would appear that the low wave number models derived might well have some application in elucidating the effect of negative group velocity on dynamic response.

A highly noteworthy point is that in contrast to the isotropic case, in which only a single vector equation exists, we now have through (4.39) and (4.40) two scalar equations. The implication is that two distinct families of shear resonance frequencies now exist, increasing the range of potential material response. In particular both families may be excited independently for vibration in the vicinity of the appropriate frequency. This is in direct contrast with the isotropic case for which both types to shear vibration are in general always excited simultaneously. Moreover, each type may only ever be excited independently for highly specific forms of dynamic load.

## 5. Asymptotically approximate equations in the vicinity of the thickness stretch resonance frequency

In the vicinity of the stretch resonance frequency it is deduced from equations (2.27) to (2.30) that

$$\hat{A}_1^-, \hat{A}_1^+ \sim O(\bar{v}^2)\tilde{A}, \quad \hat{A}_2^-, \hat{A}_2^+ \sim O(\bar{v}^3)\tilde{A}, \quad \hat{A}_3^-, \hat{A}_3^+ \sim O(\bar{v}^4)\tilde{A}, \tag{5.1}$$

which by using (2.15) to (2.17) motivates introduction of the scaling

$$u_\alpha = l\eta^2 u_\alpha^0, \quad u_3 = l\eta u_3^0. \tag{5.2}$$

The corresponding order of the stress components is given by

$$\sigma_{\alpha\alpha} = \rho c_1^2 \sigma_{\alpha\alpha}^0, \quad \sigma_{33} = \rho c_1^2 \sigma_{33}^0, \quad \sigma_{12} = \rho c_1^2 \eta^2 \sigma_{12}^0, \quad \sigma_{\alpha 3} = \rho c_1^2 \eta^2 \sigma_{\alpha 3}^0. \tag{5.3}$$

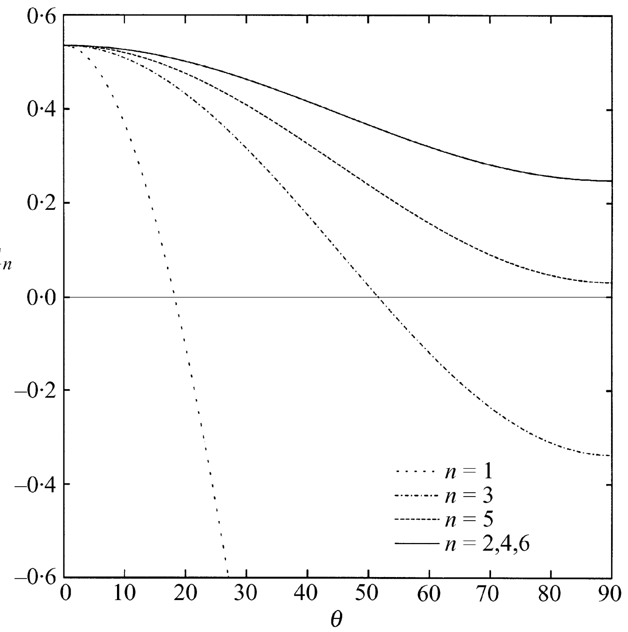


**Fig. 3** Plot of $A_n$ against $\theta$ for the second shear resonance frequencies of extensional motion, $n = 1, 2, \ldots, 6$

In view of the fact that we are considering long-wave high frequency approximations in the vicinity of the stretch resonance frequencies, it follows from (3.7), (3.12) and (3.14) that we may assume

$$\frac{\partial^2 u_k}{\partial \tau^2} + \Lambda_{st}^2 u_k \sim \eta^2 u_k, \quad k = 1, 2, 3. \tag{5.4}$$

Equations (5.2) and (5.3) may now be used in conjunction with (5.4) to establish that the first two components of the equations of motion may be cast in the form

$$\frac{\partial \sigma_{\alpha\alpha}^0}{\partial \xi_\alpha} + \frac{\partial \sigma_{\alpha 3}^0}{\partial \zeta} + \Lambda_{st}^2 u_\alpha^0 - \left(\frac{\partial^2 u_\alpha^0}{\partial \tau^2} + \Lambda_{st}^2 u_\alpha^0\right) + \eta^2 \frac{\partial \sigma_{\alpha\beta}^0}{\partial \xi_\beta} = 0, \tag{5.5}$$

with the third given by

$$\frac{\partial \sigma_{33}^0}{\partial \zeta} + \Lambda_{st}^2 u_3^0 + \eta^2\left(\frac{\partial \sigma_{13}^0}{\partial \xi_1} + \frac{\partial \sigma_{23}^0}{\partial \xi_2}\right) - \left(\frac{\partial^2 u_3^0}{\partial \tau^2} + \Lambda_{st}^2 u_3^0\right) = 0. \tag{5.6}$$

The stress-displacement relations are now expressible in the form

$$\sigma_{\alpha\alpha}^0 = m_\alpha \frac{\partial u_3^0}{\partial \zeta} + \eta^2\left(\kappa_{\alpha\alpha}\frac{\partial u_\alpha^0}{\partial \xi_\alpha} + \kappa_{\alpha\beta}\frac{\partial u_\beta^0}{\partial \xi_\beta}\right), \quad \sigma_{\alpha 3} = \mathcal{C}_\alpha\left(\frac{\partial u_\alpha^0}{\partial \zeta} + \frac{\partial u_3^0}{\partial \zeta_\alpha}\right), \tag{5.7}$$

$$\sigma_{\alpha\beta} = \bar{c}_3^2\left(\frac{\partial u_\alpha^0}{\partial \xi_\beta} + \frac{\partial u_\beta^0}{\partial \xi_\alpha}\right), \quad \sigma_{33} = \frac{\partial u_3^0}{\partial \zeta} + \eta^2\left(m_\alpha \frac{\partial u_\alpha^0}{\partial \xi_\alpha} + m_\beta \frac{\partial u_\beta^0}{\partial \xi_\beta}\right). \tag{5.8}$$

If a series solution of the form (4.15) is sought the leading-order equations are given by

$$\mathcal{C}_\alpha \frac{\partial^2 u_\alpha^{0(1)}}{\partial \zeta^2} + \Lambda_{st}^2 u_\alpha^{0(1)} = -(m_\alpha + \mathcal{C}_\alpha) \frac{\partial^2 u_3^{0(1)}}{\partial \zeta \partial \xi_\alpha}, \tag{5.9}$$

$$\frac{\partial^2 u_3^{0(1)}}{\partial \zeta^2} + \Lambda_{st}^2 u_3^{0(1)} = 0, \tag{5.10}$$

which must be solved in conjunction with the leading-order boundary conditions

$$\frac{\partial u_\alpha^{0(1)}}{\partial \zeta} + \frac{\partial u_3^{0(1)}}{\partial \xi_\alpha} = 0, \quad \frac{\partial u_3^{0(1)}}{\partial \zeta} = 0, \quad \text{at } \zeta = \pm 1. \tag{5.11}$$

*The anti-symmetric solution*

We begin by considering the anti-symmetric solution of equations (5.9) and (5.10), subject to the boundary conditions (5.11), this solution taking the form

$$u_\alpha^{0(1)} = U_\alpha^{(0,0)} \sin(\Lambda_{st}^a \zeta) + U_\alpha^{(\alpha,0)} \sin\left(\frac{\Lambda_{st}^a \zeta}{\sqrt{\mathcal{C}_\alpha}}\right), \quad \alpha = 1, 2, \tag{5.12}$$

$$u_3^{0(1)} = W^{(0,0)} \cos(\Lambda_{st}^a \zeta), \tag{5.13}$$

in which

$$U_\alpha^{(0,0)}(\xi_1, \xi_2, \tau) = \frac{\mathcal{C}_\alpha + m_\alpha}{(1 - \mathcal{C}_\alpha)\Lambda_{st}^a} \frac{\partial W^{(0,0)}}{\partial \xi_\alpha}, \qquad U_\alpha^{(\alpha,0)} = \frac{\sqrt{\mathcal{C}_\alpha}(1 + m_\alpha)\cos(\Lambda_{st}^a)}{\Lambda_{st}^a \cos(\Lambda_{st}^a/\sqrt{\mathcal{C}_\alpha})(1 - \mathcal{C}_\alpha)} \frac{\partial W^{(0,0)}}{\partial \xi_\alpha}, \tag{5.14}$$

with $W^{(0,0)}$ the associated long-wave amplitude. In order to derive the governing equation for $W^{(0,0)}$ we need to consider the equation associated with (5.6) at second order, which is given by

$$\frac{\partial^2 u_3^{0(2)}}{\partial \zeta^2} + (\Lambda_{st}^a)^2 u_3^{0(2)} = \eta^{-2}\left(\frac{\partial^2 u_3^{0(1)}}{\partial \tau^2} + (\Lambda_{st}^a)^2 u_3^{0(1)}\right) - \sum_{\alpha=1}^{2}\left((m_\alpha + \mathcal{C}_\alpha)\frac{\partial u_\alpha^{0(1)}}{\partial \zeta \partial \xi_\alpha} + \mathcal{C}_\alpha \frac{\partial^2 u_3^{0(1)}}{\partial \xi_\alpha^2}\right) = 0, \tag{5.15}$$

and which must be solved in conjunction with the appropriate second-order boundary condition

$$\frac{\partial u_3^{0(1)}}{\partial \zeta} + m_1 \frac{\partial u_1^{0(1)}}{\partial \xi_1} + m_2 \frac{\partial u_2^{0(1)}}{\partial \xi_2} = 0 \text{ at } \zeta = \pm 1. \tag{5.16}$$

The inhomogeneous second-order differential equation (5.15) may be solved to yield the solution

$$u_3^{0(2)} = V^{(0,0)} \cos(\Lambda_{st}^a) + V^{(0,1)} \zeta \sin(\Lambda_{st}^a \zeta) + \sum_{\alpha=1}^{2} V^{(\alpha,0)} \cos\left(\frac{\Lambda_{st}^a \zeta}{\sqrt{\mathcal{C}_\alpha}}\right), \tag{5.17}$$

in which

$$V^{(\alpha,0)} = \left(\frac{\mathcal{C}_\alpha(1 + m_\alpha)(\mathcal{C}_\alpha + m_\alpha)}{(\Lambda_{st}^a)^2(1 - \mathcal{C}_\alpha)^2}\right) \frac{\cos(\Lambda_{st}^a \zeta)}{\cos(\Lambda_{st}^a/\sqrt{\mathcal{C}_\alpha})} \frac{\partial^2 W^{(0,0)}}{\partial \xi_\alpha^2}, \tag{5.18}$$

$$V^{(0,1)} = \frac{-1}{2(\Lambda_{st}^a)^2}\sum_{\alpha=1}^{2}\left(\left\{\frac{\mathcal{C}_\alpha + 2m_\alpha\mathcal{C}_\alpha + m_\alpha^2}{1-\mathcal{C}_\alpha}\right\}\frac{\partial^2 W^{(0,0)}}{\partial\xi_\alpha^2} + \eta^{-2}\Lambda_{st}^a\left\{\frac{\partial^2 W^{(0,0)}}{\partial\tau^2} + (\Lambda_{st}^a)^2 W^{(0,0)}\right\}\right). \tag{5.19}$$

It is noted that it is not possible to obtain $V^{(0,0)}$ at this order of approximation. The next order of approximation is required therefore if the stress–strain state is to be completely determined at this order. However, as $V^{(0,0)}$ is the coefficient of $\cos(\Lambda_{st}^a\zeta)$, which vanishes at $\zeta = \pm 1$, this is of no consequence in the derivation of a two-dimensional equation for $W^{(0,0)}$. Such an equation is derived by satisfying the boundary condition (5.16), yielding

$$\frac{\partial^2 W^{(0,0)}}{\partial\tau^2} + (\Lambda_{st}^a)^2 W^{(0,0)} + \eta^2\sum_{\alpha=1}^{2}\left(T_{st}^{(\alpha)} + \frac{P_{st}^{(\alpha)}}{\Lambda_{st}^a}\tan\left(\frac{\Lambda_{st}^a}{\sqrt{\mathcal{C}_\alpha}}\right)\right)\frac{\partial^2 W^{(0,0)}}{\partial\xi_\alpha^2} = 0, \tag{5.20}$$

in which

$$T_{st}^{(\alpha)} = -\left(\frac{\mathcal{C}_\alpha + 2m_\alpha\mathcal{C}_\alpha + m_\alpha^2}{1-\mathcal{C}_\alpha}\right), \quad P_{st}^{(\alpha)} = \frac{2\mathcal{C}_\alpha^{\frac{3}{2}}(1+m_\alpha)^2}{(1-\mathcal{C}_\alpha)^2}. \tag{5.21}$$

The definitions (4.14) may now be used to establish that

$$T_{st}^{(1)} = -p_3^{(1)}, \quad T_{st}^{(2)} = -p_3^{(2)}, \quad P_{st}^{(1)} = \frac{2a_1}{\Lambda_{st}^a}, \quad P_{st}^{(2)} = \frac{4a_2}{\Lambda_{st}^a}, \tag{5.22}$$

enabling (5.20) to be cast into the form

$$\frac{\partial^2 W^{(0,0)}}{\partial\tau^2} + (\Lambda_{st}^a)^2 W^{(0,0)} - \eta^2\left\{\left(p_3^{(1)} + \frac{2a_1}{\Lambda_{st}^a}\tan\left(\frac{\Lambda_{st}^a}{\bar{c}_3}\right)\right)\frac{\partial^2 W^{(0,0)}}{\partial\xi_1^2} + \left(p_3^{(2)} + \frac{4a_2}{\Lambda_{st}^a}\tan\left(\frac{\Lambda_{st}^a}{\bar{c}_2}\right)\right)\frac{\partial^2 W^{(0,0)}}{\partial\xi_2^2}\right\} = 0. \tag{5.23}$$

It is a straightforward matter now to assume a solution of the form (4.37) and verify that scaled frequency is exactly that derived as an appropriate approximation of the exact anti-symmetric dispersion relation; see equation (3.7).

*The symmetric solution*

In the case of the symmetric solution the counterpart of (5.13) is

$$u_3^{0(1)} = W^{(0,0)}\cos(\Lambda_{st}^s\zeta). \tag{5.24}$$

Following the analysis for the anti-symmetric case, the analogous equation to (5.20) is given by

$$\frac{\partial^2 W^{(0,0)}}{\partial\tau^2} + (\Lambda_{st}^s)^2 W^{(0,0)} + \eta^2\sum_{\alpha=1}^{2}\left(T_{st}^{(\alpha)} - \frac{P_{st}^{(\alpha)}}{\Lambda_{st}^s}\cot\left(\frac{\Lambda_{st}^s}{\sqrt{\mathcal{C}_\alpha}}\right)\right)\frac{\partial^2 W^{(0,0)}}{\partial\xi_\alpha^2} = 0, \tag{5.25}$$

which upon using the definitions (5.22) may be cast in the form

$$\frac{\partial^2 W^{(0,0)}}{\partial\tau^2} + (\Lambda_{st}^s)^2 W^{(0,0)} - \eta^2\left\{\left(p_3^{(1)} - \frac{2a_1}{\Lambda_{st}^s}\cot\left(\frac{\Lambda_{st}^s}{\bar{c}_3}\right)\right)\frac{\partial^2 W^{(0,0)}}{\partial\xi_1^2} + \left(p_3^{(2)} - \frac{4a_2}{\Lambda_{st}^s}\cot\left(\frac{\Lambda_{st}^s}{\bar{c}_2}\right)\right)\frac{\partial^2 W^{(0,0)}}{\partial\xi_2^2}\right\} = 0. \quad (5.26)$$

It is now possible to assume a solution of the form (4.37) and verify that the dispersion relation obtained from (5.26) is identical to the expansion (3.12).

## 6. Concluding remarks

In this paper the equations associated with a layer of transversely isotropic elastic material have been integrated in the vicinity of cut-off frequencies. Appropriate two-dimensional equations have been derived for the long-wave amplitudes, the coefficients of which agree with the similar ones obtained through approximation of the corresponding exact dispersion relation. It should be reiterated that the construction of a two-dimensional theory is the motivation for this work. The fact that the appropriate approximation of the dispersion relation may be recovered acts both as a check on the governing equation and an indication of its potential. In particular, the derived governing equation may be useful for problems involving structures of complex geometry for which the dispersion relation is not obtainable in closed form. The methods used in this paper may readily be generalized to consider long-wave high frequency approximations of different linear wave guides and problems involving surface or edge loading.


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