Article

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Reference

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Direct integration and non-perturbative effects in matrix models

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ABSTRACT: We show how direct integration can be used to solve the closed amplitudes of multi-cut matrix models with polynomial potentials. In the case of the cubic matrix model, we give explicit expressions for the ring of non-holomorphic modular objects that are needed to express all closed matrix model amplitudes. This allows us to integrate the holomorphic anomaly equation up to holomorphic modular terms that we fix by the gap condition up to genus four. There is an one-dimensional submanifold of the moduli space in which the spectral curve becomes the Seiberg-Witten curve and the ring reduces to the non-holomorphic modular ring of the group $\Gamma(2)$. On that submanifold, the gap conditions completely fix the holomorphic ambiguity and the model can be solved explicitly to very high genus. We use these results to make precision tests of the connection between the large order behavior of the $1/N$ expansion and non-perturbative effects due to instantons. Finally, we argue that a full understanding of the large genus asymptotics in the multi-cut case requires a new class of non-perturbative sectors in the matrix model.

Contents

1 Introduction and results

In this paper we propose direct integration as a new method to solve the closed amplitudes for multi-cut matrix models with polynomial potentials. More precisely, we calculate the closed partition function of such matrix models

$$Z(\underline{S}) = \exp\left(\sum_g g_s^{2g-2} F_g(\underline{S}) \right) \tag{1.1}$$

perturbatively in the genus g, but exactly in the 't Hooft parameters $\underline{S}$. Exact means that the $F_g(\underline{S})$ are given in terms of period integrals of the spectral curve and can be written explicitly in terms of modular forms of subgroups of $\mathrm{SP}(2g(\Sigma), \mathbb{Z})$, where $g(\Sigma)$ is the genus of the spectral curve Σ.

Direct integration refers to a method of solving the holomorphic anomaly equation [6] using the modular transformation properties of the amplitudes under the monodromy group of the spectral curve. This method has been developed in the context of topological string theory in [3, 27–29, 33, 53]. The fact that the holomorphic anomaly equations govern such matrix models was suggested by the large N duality of [20]. In this duality, type B topological string amplitudes on certain local Calabi-Yau spaces turn out to be encoded in the $1/N$ expansion of matrix model partition functions. Therefore, the holomorphic anomaly of the topological string naturally carries over to these matrix models as first pointed out in [31]. It has been shown much more generally in [23] that the holomorphic anomaly equation is valid for all matrix models which are solvable by the method of [24].

The holomorphic anomaly equation relates anti-holomorphic derivatives of the closed amplitudes $F_g(\underline{S})$ at genus g to lower genus amplitudes $F_{h<g}(\underline{S})$, in a recursive way. Since only the anti-holomorphic derivative is specified by the equations, the procedure leaves a holomorphic ambiguity, i.e. $F_g(\underline{S}) = F_g^{\mathrm{nh}}(\underline{S}) + f_g(\underline{S})$ splits into a non-holomorphic term $F_g^{\mathrm{nh}}(\underline{S})$, which is determined by the holomorphic anomaly equation, and the holomorphic ambiguity $f_g(\underline{S})$, which must be fixed genus by genus by using modular properties and boundary conditions at special points in the moduli space. The modular transformation properties imply that the amplitudes are generated by a finite ring of modular forms, which have holomorphic as well as non-holomorphic generators. Modularity and the holomorphic anomaly equation imply that the total amplitude $F_g(\underline{S})$ is a polynomial in these generators whose degree grows linearly with the genus. The ambiguity $f_g(\underline{S})$ is a polynomial generated by the smaller ring of holomorphic generators. The finite number of coefficients in this polynomial must be fixed by boundary conditions.

In this paper we find that the gap conditions, which where investigated in non-compact [4, 30–32] and compact Calabi-Yau backgrounds [27–29, 33], provide enough independent boundary conditions to fix the ambiguity (and hence the amplitudes) completely. Following [30] we refer to this property as integrability of the holomorphic anomaly equation.

The large N duality relating matrix models and topological strings gives a natural geometric interpretation to the algebraic objects describing the planar limit of the matrix model [20]. The spectral curve $y(x)$ of the matrix model (which, in the case of polynomial potentials, is a hyperelliptic curve) describes the distribution of eigenvalues in the planar limit, and in the topological string dual it describes the nontrivial part of the Calabi-Yau geometry. We derive the modular ring starting from the Picard-Fuchs equations governing the periods of the form $\Omega = y(x)\mathrm{d}x$. This is a general method,[1] and since we expect that the gap boundary conditions fix the ambiguity, our approach should apply to general multi-cut matrix models with polynomial potential.

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[1]For example, a meromorphic modular form of weight k of $\mathrm{SL}(2, \mathbb{Z})$ or a congruence subgroup fulfills a linear differential equation of order $k + 1$ in the total modular invariant [55].

Of course, the formalism of [24] gives in principle all the genus g free energies of generic multi-cut matrix model in terms of universal formulae on the spectral curve. The price to pay for such a general approach is that its detailed implementation is in practice very involved. Even in two-cut models, going beyond genus two with the methods of [24] is not very feasible. In contrast, direct integration becomes very powerful when the spectral curve and its modular group are simple.

In this paper, in order to illustrate the method of direct integration, we focus on the two-cut matrix model with a cubic potential. In this model the N eigenvalues split in two sets $N = N_1 + N_2$ and condense in sets near the two critical points of the potential. This leads to the cuts in the spectral curve shown in figure 1. There are two independent 't Hooft couplings $S_i = g_s N_i$, $i = 1, 2$, which correspond to the integrals of Ω over the two cuts. As shown in [20], the planar free energy of this matrix model, $F_0(S_1, S_2)$, calculates the exact superpotential W_{eff} of an $\mathcal{N} = 2$ U(M) supersymmetric gauge theory broken down to an $\mathcal{N} = 1$ gauge theory U(M_1) $\times$ U(M_2), by a cubic three-level superpotential in the adjoint [20] (notice that N_i are unrelated to M_i). The higher genus amplitudes $F_g(S_1, S_2)$ in the matrix model arise as generalized couplings in a non-commutative deformation of the $\mathcal{N} = 1$ gauge theory [45, 46].

Certain aspects of the original $\mathcal{N} = 2$ theory can be recovered from the $\mathcal{N} = 1$ theory by breaking the gauge symmetry to the Cartan subgroup and taking the limit in which the superpotential vanishes [13]. When the gauge group is SU(2), a cubic superpotential is enough to go to the Coulomb branch. This implies that various quantities appearing in the Seiberg-Witten solution of pure $\mathcal{N} = 2$ super Yang-Mills theory [49] can be obtained from a matrix model calculation with a cubic potential, and on the slice $S_1 = -S_2$. These include the gauge coupling [18] and the R_+^2 gravitational coupling [19, 35]. In fact, the spectral curve of the cubic matrix model on that slice is identical to the Seiberg-Witten curve [18]. Since the modular group of this curve is particularly simple, direct integration becomes an extremely powerful method to calculate the $F_g(S_1, -S_1)$, as we show in section 4.

On the other hand, in the SU(2), $\mathcal{N} = 2$ gauge theory there is an infinite number of couplings $F_g(a)$, $g \geq 2$, which describe the gauge-gravity couplings $F_+^{2g-2} R_+^2$ involving the graviphoton field strength F_+. These couplings appear naturally in Nekrasov's partition function [43] and they can be also obtained by using the holomorphic anomaly equations. This was shown for the pure gauge theory and SU(2) with matter in [27, 31] and [32] respectively. However, it was noticed in [35] that these higher genus couplings $F_g(a)$ do not agree with the higher genus $F_g(S_1, S_2)$ obtained in the cubic matrix model and then restricted to the slice $S_2 = -S_1$. This disagreement is due to the fact that the Seiberg-Witten differential λ_{SW} differs from the natural differential Ω on the spectral curve of the matrix model. In contrast, τ and F_1 only depend on the spectral curve, and not on the differential, and therefore are the same in both cases. In [36, 51] matrix models are derived which encode all $\mathcal{N} = 2$ gauge theory amplitudes F_g for arbitrary g, however one has to introduce potentials involving polylogarithms and their quantum generalizations.

An interesting application of our computation of the couplings $F_g(S_1, S_2)$ at high genus is the study of non-perturbative effects in matrix models and their connection to the large order behavior of the $1/N$ expansion. It is well-known that, in many quantum systems,

there is a connection between perturbation theory at large orders and instantons (see for example [38]). In matrix models, instanton configurations correspond to the tunneling of eigenvalues between different saddle points [14, 15, 50]. A detailed analysis of these configurations for off-critical, one-cut matrix models can be found in [41], which verified the connection to the large order behavior of the $1/N$ expansion in detail in some nontrivial examples. In this paper we explore this connection in the two-cut matrix model. On the one hand, we find that the large order behavior is controlled at leading order by the action of a single eigenvalue tunneling from one saddle-point to the other, in agreement with the general ideas put forward in [14, 15, 41, 50]. On the the other hand, we argue that a full understanding of this connection requires new non-perturbative sectors which have not been yet identified in the matrix model. The existence of these sectors is also suggested by a recent analysis of the asymptotic behavior of the instanton solutions of the Painlevé I equation [26]. We conjecture that these sectors might involve topological brane-antibrane systems.

The paper is organized as follows. In section 2 we give the general ideas of the direct integration method and of the modular covariant approach of [1]. In section 3 we describe in detail the geometry underlying the two-cut cubic matrix model, we set up the direct integration formalism and we analyze the boundary conditions. In section 4 we point out that the modular covariant formulation is most powerful on the one-dimensional slice of moduli space $S_1 = -S_2$, and we develop direct integration on this submanifold in terms of non-holomorphic modular forms of $\Gamma(2)$. Section 5 is devoted to the analysis of instanton effects in the two-cubic matrix model, and their connection to the large order of the $1/N$ expansion. Finally, the appendices collect some useful information about the cubic matrix model as well as results on modular forms which are used in the paper.

2 Holomorphic anomaly and direct integration

Below we review very briefly the generic aspects of the techniques of direct integration of the holomorphic anomaly equation of [6]

$$\bar{\partial}_{\bar{i}} F_g = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \left(D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_{g-r} D_k F_r \right), \qquad (g > 1), \tag{2.1}$$

which was derived for Calabi-Yau three-folds in [3, 27–29, 33, 53]. In our application to the spectral curve Σ of a matrix model, the D_i are covariant derivatives D_i with respect to the metric G on the moduli space of the Riemann surface Σ.

We note that $\bar{C}_{\bar{i}}^{jk} = \bar{C}_{\bar{i}\bar{j}\bar{k}} G^{\bar{j}j} G^{\bar{k}k}$, where C_{ijk} can be derived from the holomorphic pre-potential F_0 as $C_{ijk} = D_i D_j \partial_k F_0$. The prepotential F_0, the metric $G_{i\bar{j}}$ and flat coordinates $\underline{S}$ can all be derived from the period integrals

$$\left(\int_{a^i} \Omega, \int_{b_i} \Omega \right), \qquad i = 1, \ldots, g(\Sigma) \tag{2.2}$$

over a symplectic basis (a^i, b_i) of $H_1(\Sigma, \mathbb{Z})$. In particular, given a point in the moduli space, one can make a choice of this symplectic basis, so that suitable flat coordinates are

defined by

$$S^i = \int_{a^i} \Omega \tag{2.3}$$

while the b_i periods Π_i fulfill

$$\Pi_i = \frac{\partial F_0}{\partial S^i} . \tag{2.4}$$

These relations determine the prepotential F_0 up to an irrelevant constant. We define the τ matrix of the Riemann surface as

$$\tau_{ij} = \frac{\partial^2 F_0}{\partial S^i \partial S^j}. \tag{2.5}$$

The matrix $\mathrm{Im}(\tau)_{ij}$ is positive definite, and it gives the metric on the moduli space of the model. Equivalently, the metric can be obtained form the Kähler potential

$$K = \frac{1}{2\pi i} \left(\Pi_i \bar{S}^{\bar{\imath}} - \bar{\Pi}_{\bar{\imath}} S^i \right). \tag{2.6}$$

On Riemann surfaces the period integrals can often be directly performed. Alternatively it might be useful to derive the Picard-Fuchs equations and reconstruct the periods as linear combinations of their solutions. Much of the above has been spelled out in the context of the Riemann surfaces for the B-model of topological string theory on non-compact Calabi-Yau in [30]. The relevant compact part of the geometry is given by a Riemann surface and a meromorphic differential, which comes from reducing the holomorphic $(3,0)$-form on the Riemann surface. After identification of the former with the spectral curve Σ and the later with the form Ω, we can use the formalism discussed in [30].

One property of the matrix model geometry is that the periods over the a-cycles do not fulfill the relation $\sum_{i=1}^r S^i = 0$. Usually this relation is inherited by the periods of holomorphic forms due to the homological relation of the cycles. However, in matrix models one has $\sum_{i=1}^r S^i \propto N$, because Ω has one non-vanishing residue outside the cuts. This leads to one algebraic relation between the periods in terms of the r parameters, which for the $r = 2$ case (the cubic matrix model) is expressed in eq. (3.28). The property of a non-vanishing residue is shared with Seiberg-Witten theories with matter [32] and certain non-compact Calabi-Yau geometries with more than one Kähler class [4, 30].

2.1 Direct integration

The so-called propagator plays a decisive role in the solutions of the B-model [6]. For the formalism on the Riemann surface Σ one needs only one type[2] of propagator S^{ij} defined by

$$\bar{\partial}_{\bar{\imath}} S^{ij} = C_{\bar{\imath}}^{ij}, \tag{2.7}$$

where $i, j = 1, \ldots, r$ and r is the number of parameters in the model. Following [6] it can be shown that the F_g can be written as

$$F_g = \sum_{|I|=0}^{3g-3} f_{g,i_1 \ldots i_{|I|}}(\underline{S}) \, S^{i_1 i_2} \ldots S^{i_{|I|-1} i_{|I|}} \tag{2.8}$$

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[2]In the threefold cases there are three types S^{ij}, S^i and S.

where the $f_{g,I}(\underline{S})$ are holomorphic tensors of the moduli. The most important property of the S^{ij} is that

$$\partial_{\bar{\imath}} F_g = C_{\bar{\imath}}^{ij} \frac{\partial F_g}{\partial S^{ij}}. \tag{2.9}$$

If one assumes linear independence of the S^{ij} as functions of $\underline{S}$, it follows from this property that (2.1) can be rewritten as a set of equations

$$\frac{\partial F_g}{\partial S^{ij}} = \frac{1}{2} \left(D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_{g-r} D_k F_r \right), \qquad (g > 1) . \tag{2.10}$$

These equations can be integrated algebraically, provided that the r.h.s. can be expressed in terms of the S^{ij} contracted by holomorphic tensors as in the r.h.s of (2.8). This is possible since the following closing relations are fulfilled due to special geometry [3, 6]

$$D_i S^{kl} = -C_{inm} S^{km} S^{ln} + f_i^{kl}, \tag{2.11}$$

$$\Gamma_{ij}^k = -C_{ijl} S^{kl} + \tilde{f}_{ij}^k, \tag{2.12}$$

$$\partial_i F_1 = \frac{1}{2} C_{ijk} S^{jk} + A_i. \tag{2.13}$$

Here the f_i^{kl}, $\tilde{f}_i^{kl}$ and A_i are holomorphic ambiguities, which must have the same transformation properties as the expressions on the left-hand side. These ambiguities are due to the fact that (2.7) defines S^{ij} only up to an holomorphic tensor. Different choices are possible and lead to a redefinition of the $f_{g,I}$ in (2.8). As we mentioned above the periods are not algebraically independent, see for example (3.28). As a consequence it is possible to make a choice for the above ambiguities so that for a given i one has $S^{ik} = 0$, $\forall k$, i.e. the matrix of propagators has effectively only rank $\rho = r - 1$. We call the auxiliary parameter t . There may be more auxiliary parameters stemming from the independent non-vanishing residua of Ω. If there are κ such residua, the rank is reduced to $\rho = r - \kappa$.

Whether one works with the redundant or the reduced set of propagators the equation (2.10) can easily be integrated w.r.t. S^{ij} and F_g becomes of degree $3g - 3$ in the S^{ij}. This is an efficient way to solve the recursion, but at each step one still has to determine the holomorphic ambiguity.

2.2 Modular covariant formulation

It is possible to relate the non-redundant set of propagators to quasimodular forms. In particular, in the holomorphic polarisation, the following properties derived in [1] hold:

1. $F_g(\underline{S})$ is invariant under the monodromy group Γ of the Riemann surface Σ.

2. $F_g(\underline{S})$ is an *almost*-holomorphic modular function, i.e. its non-holomorphic dependence is encoded solely in $((\tau - \bar{\tau})^{-1})^{IJ}$, where $I, J = 1, \ldots, \rho$ and τ_{IJ} is the standard matrix valued modular parameter living in the Siegel upper half space.[3]

---

[3] $\rho = r - \kappa$, where κ is the number of independent non-vanishing residua of Ω.

3. The non-holomorphic dependence combines always with quasimodular forms E^{IJ} to give almost-holomorphic modular forms

$$\widehat{E}^{IJ} = E^{IJ}(\tau) + ((\tau - \bar{\tau})^{-1})^{IJ}. \qquad (2.14)$$

Here we defined $E^{IJ}(\tau)$ as derivative of $\frac{\partial}{\partial \tau_{IJ}} F_1(\tau)$. The anomaly equation of [6] for $F_1(\tau)$ implies that this is a non-holomorphic modular invariant

$$F_1 = -\log \left[\det^{\frac{1}{2}}(\mathrm{Im}(\tau_{IJ})) \left(\bar{\Phi}_k(\bar{\tau}) \Phi_k(\tau) \right)^a \right] \qquad (2.15)$$

under the monodromy group Γ. $\Phi_k(\tau)$ is a holomorphic Siegel modular cusp form of weight k which vanishes at the discriminant Δ of the Riemann surface. It transforms as $\Phi_k(\tau_\gamma) = \det(C\tau + D)^k \Phi_k(\tau)$, where τ_γ is given by

$$\tau_\gamma = (A\tau + B)(C\tau + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2\rho, \mathbb{Z}) . \qquad (2.16)$$

Such modular forms exist for all genus and can be written as products of even theta functions [37]. The exponent a will make the argument of the log invariant and the vanishing order at the discriminant $\frac{1}{12}\log(\Delta)$. For an elliptic curve $\Phi_k(\tau)$ is typically the Dedekind η-function. However, if the subgroup Γ allows for several cusp forms, $\Phi_k(\tau)$ can be a suitable multiplicative combination of them. In virtue of the definition $\widehat{E}^{IJ}$ transforms as a Siegel modular form

$$\widehat{E}^{IJ}(\tau_\gamma) = (C\tau + D)^I_K (C\tau + D)^J_L \widehat{E}^{KL}(\tau) . \qquad (2.17)$$

4. $F_g(\underline{S})$ can be expanded as

$$F_g = \sum_{|I|=0}^{3g-3} \tilde{f}_{g,I_1,\dots,I_{|I|}} \widehat{E}^{I_1 I_2} \dots \widehat{E}^{I_{|I|-1} I_{|I|}} . \qquad (2.18)$$

Note that $\tilde{f}_{g,I}$ has to compensate for the modular transformation of τ and can in principle be expressed through holomorphic modular forms.

3 The two-cut cubic matrix model

As shown by Dijkgraaf and Vafa in [20], the B-model topological string theory on certain non-compact Calabi-Yau geometries is captured by a matrix model. The matrix model is the n-cut matrix model with potential $W(x)$, while the Calabi-Yau geometry is the following hypersurface in $\mathbb{C}^4$

$$uv = y^2 - (W'(x)^2 + f(x)). \qquad (3.1)$$

Here, $f(x)$ is a polynomial of degree $n-1$ that splits the n double zeroes of $W'(x)^2$, see [39] for a detailed review. In the following we will combine the Dijkgraaf-Vafa correspondence with known results about the holomorphic anomaly equation in order to give a recursive solution of multi-cut matrix models.

3.1 The geometrical setup

In this paper we will be interested in multi-cut, Hermitian matrix models. The partition function is defined by

$$Z = \frac{1}{\text{vol}(\text{U}(N))} \int \mathrm{d}M \, \mathrm{e}^{-\frac{1}{g_s} \text{Tr} W(M)} \tag{3.2}$$

where $W(M)$ is a polynomial of degree $d = n+1$ in the $N \times N$ matrix M. The most general saddle point of this model at large N is a multi-cut solution, in which the eigenvalues of M condense along cuts

$$[a_i^-, a_i^+] \subset \mathbb{C}, \qquad i = 1, \cdots, n, \tag{3.3}$$

in the complex plane. The cuts are centered around the n critical points of $W(x)$. One way of encoding the planar solution of the matrix model is through its resolvent

$$\omega(x) = \frac{1}{N} \Big\langle \text{Tr} \frac{1}{x - M} \Big\rangle. \tag{3.4}$$

The planar limit of this correlator, denoted by $\omega_0(x)$, has the structure (see for example [17])

$$\omega_0(x) = \frac{1}{2t}(W'(x) - y(x)), \tag{3.5}$$

where

$$t = g_s N \tag{3.6}$$

is the total 't Hooft parameter, and

$$y^2(x) = \big(W'(x)\big)^2 + f(x) = c \prod_{i=1}^{n} (x - a_i^-)(x - a_i^+) \tag{3.7}$$

is called the *spectral curve* of the multi-cut matrix model. In the matrix model literature it is customary to write it as

$$y^2(x) = M(x)\sqrt{\sigma(x)}, \tag{3.8}$$

where $\sigma(x)$ is a polynomial in x,

$$\sigma(x) = \prod_{i=1}^{2s} (x - x_i), \tag{3.9}$$

and $s \leq n$. Of course, if all the roots in (3.7) are different, $s = n$ and $M(x)$ is a constant.

Through the large N duality of [20], the spectral curve (3.7) describes the nontrivial part of the target geometry (3.1). The positions of the endpoints are fixed by the asymptotic condition

$$\omega_0(x) \sim \frac{1}{x}, \qquad x \to \infty, \tag{3.10}$$

and by the requirement that there are N_i eigenvalues in each cut,

$$\frac{N_i}{N} = \frac{1}{2} \oint_{\mathcal{C}_i} \frac{\mathrm{d}x}{2\pi \mathrm{i}} \omega_0(x). \tag{3.11}$$

In this equation, $\mathcal{C}_i$ is a contour encircling the cut $[a_i^-, a_i^+]$ counterclockwise. The *partial 't Hooft parameters* S_i are defined by

$$S_i = g_s N_i. \tag{3.12}$$

Notice that

$$t = \sum_{i=1}^{n} S_i. \tag{3.13}$$

In the following we consider a cubic matrix model with potential W given by

$$W(x) = \frac{m}{2}x^2 + \frac{g}{3}x^3. \tag{3.14}$$

Since this model has two critical points $x = a_1$, $x = a_2$, the generic saddle will be a two-cut matrix model. If we write the matrix integral (3.2) in terms of eigenvalues, we have to distinguish two different sets $\{\mu_i\}_{i=1,\cdots,N_1}$, $\{\nu_j\}_{j=1,\cdots,N_2}$, which are expanded around a_1, a_2, respectively, and we obtain

$$Z = \frac{1}{N_1! N_2!} \int \prod_{i=1}^{N_1} \mathrm{d}\mu_i \prod_{j=1}^{N_2} \mathrm{d}\nu_j \prod_{i<j} (\mu_i - \mu_j)^2 (\nu_i - \nu_j)^2 \times$$
$$\times \prod_{i,j} (\mu_i - \nu_j)^2 \, \mathrm{e}^{-\frac{1}{g_s}\left(\sum_i W(\mu_i) + \sum_j W(\nu_j)\right)}. \tag{3.15}$$

Since

$$W'(x) = mx + gx^2 = gx\left(x + \frac{m}{g}\right) = g(x - a_1)(x - a_2), \tag{3.16}$$

$W'(x)^2$ has two double zeroes at $x = a_1$, a_2, that are split by the degree one polynomial

$$f(x) = \lambda x + \mu \tag{3.17}$$

into four roots $a_1^\pm, a_2^\pm$. Hence, the curve for the geometry/matrix model is given by

$$y^2 = W'(x)^2 + f = g^2(x - a_1^-)(x - a_1^+)(x - a_2^-)(x - a_2^+). \tag{3.18}$$

We choose the branch cuts to be along the intervals (a_1^-, a_1^+) and (a_2^-, a_2^+), cf. figure 1. It follows from (3.11) that the 't Hooft parameters for this curve are the periods of the one-form

$$\Omega = y(x)\,\mathrm{d}x \tag{3.19}$$

around the branch cuts. Following the notation of [12], we have

$$S_i = \frac{1}{2\pi\mathrm{i}} \int_{a_i^-}^{a_i^+} \Omega, \qquad \Pi_i = \frac{1}{2\pi\mathrm{i}} \int_{b_i^\Lambda} \Omega. \tag{3.20}$$

These 't Hooft parameters are functions of the couplings in the potential m, g, and of the variables λ, μ. Equivalently, they are functions of the branch points $a_i^\pm$ of the quartic
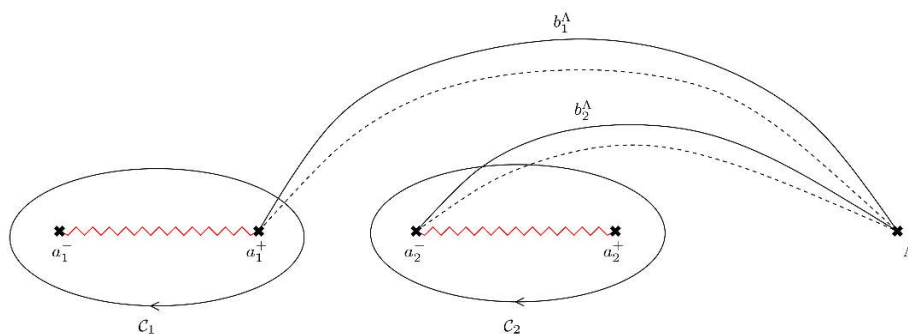
Figure 1. Choice of branch cuts and cycles on the elliptic geometry (3.18).

curve (3.18). It is convenient to define new variables given by

$$
\begin{aligned}
z_1 &= \frac{1}{4}(x_2 - x_1)^2, \\
z_2 &= \frac{1}{4}(x_4 - x_3)^2, \\
Q &= \frac{1}{2}(x_1 + x_2 + x_3 + x_4) = -\frac{m}{g}, \\
I^2 &= \frac{1}{4}\left[(x_3 + x_4) - (x_1 + x_2)\right]^2 = \left(\frac{m}{g}\right)^2 - 2(z_1 + z_2),
\end{aligned}
\tag{3.21}
$$

where we label the cuts more conveniently as

$$
(a_1^-, a_1^+, a_2^-, a_2^+) = (x_1, x_2, x_3, x_4)
\tag{3.22}
$$

and we also have

$$
\sigma(x) = \prod_{i=1}^{4}(x - x_i).
\tag{3.23}
$$

We will use this in order to expand all four periods in powers of z_1 and z_2. Notice that z_i are coordinates that parameterize the complex structure deformations of the local Calabi-Yau geometry (3.1).

Let us consider S_1. For this we change variables to $y = x - \frac{1}{2}(x_1 + x_2)$ and the integral becomes

$$
S_1 = \frac{g}{2\pi} \int_{y_3}^{y_4} \sqrt{(y - y_3)(y - y_4)} \sqrt{y^2 - z_1}\, dy.
$$

Expanding the second square root for z_1 small, each term in the series can be computed explicitly and it is most easily given in terms of a generating function [12],

$$
F(a) = -\pi\sqrt{(y_3 + a)(y_4 + a)} + \frac{\pi}{2}(y_3 + y_4 + 2a)
\tag{3.24}
$$

as follows,

$$
S_1 = \frac{g}{32}(y_3 + y_4)(y_4 - y_3)^2 + \frac{g}{2\pi}\sum_{n=1}^{\infty} c_n \Delta_{21}^{2n} F^{(n)}(0)
$$

where c_n are the coefficients in the expansion of $\sqrt{1-x}$ and $F^{(n)}(a)$ is the n-th derivative with respect to a.

The explicit answer has the following structure,

$$S_1 = \frac{g}{4} z_2 I - \frac{g}{2I} K(z_1, z_2, I^2), \tag{3.25}$$

where

$$K(x, y, z) = \frac{1}{4} xy \left(1 + \frac{1}{4z}(x+y) + \frac{1}{8z^2}(x+y)^2 + \frac{1}{8z^2}xy + \dots \right).$$

It is important to notice that this is symmetric in (x, y), namely, $K(x, y, z) = K(y, x, z)$. This allows us to write,

$$S_2 = -\frac{g}{4} z_1 I + \frac{g}{2I} K(z_1, z_2, I^2). \tag{3.26}$$

In the following we will simplify the expressions by putting $m = g = 1$. It will be useful to change variables to

$$t = S_1 + S_2, \qquad s = \frac{1}{2}(S_1 - S_2) \tag{3.27}$$

where t is the total 't Hooft parameter. Due to (3.25) and (3.26) one immediately obtains

$$t = \frac{1}{4}(z_2 - z_1)\sqrt{1 - 2z_1 - 2z_2}. \tag{3.28}$$

Note, that t can be regarded as a global parameter of the model. Different from t the expression of s in terms of the z_i requires a transcendental function. This more complicated function reflects the dependence of s on the choice of the symplectic basis in (2.2).

As mentioned earlier, there is another possibility to derive the periods as series in z_i which was applied in [31]. There the authors consider a set of Picard-Fuchs differential operators, $\mathcal{L}_1, \mathcal{L}_2$ associated to the spectral curve and differential Ω, which annihilate the periods. Therefore, these can be calculated as solutions to a system of ODEs. The Picard-Fuchs operators, which are given in eq. (A.1) of appendix A, have the following discriminant factors

$$\mathrm{disc} = z_1 z_2 I^2 J = z_1 z_2 (1 - 2(z_1 + z_2))(1 - 6z_1 - 6z_2 + 9z_1^2 + 14z_1 z_2 + 9z_2^2). \tag{3.29}$$

Moreover, their solutions around $z_1 = 0$ and $z_2 = 0$ describe the periods of the elliptic geometry (3.18). Due to the fact that one can find a combination of periods such that the mirror map becomes exact (3.28), it is convenient to introduce adapted coordinates $\tilde{z}_i$, $i = 1, 2$, by

$$\tilde{z}_1 = z_1 + z_2, \quad \tilde{z}_2 = \frac{1}{4}(z_1 - z_2)\sqrt{1 - 2(z_1 + z_2)}, \tag{3.30}$$

as well as coordinates $\tilde{t}_i$, $i = 1, 2$, on the mirror by

$$\tilde{t}_1 = s = \frac{1}{2}(S_1 - S_2), \quad \tilde{t}_2 = t = S_1 + S_2. \tag{3.31}$$

The Yukawa couplings may be found in eq. (A.4) as well as the genus one free energy F_1 in eq. (A.5). Due to the special type of the mirror map

$$\tilde{z}_2 = \tilde{t}_2, \tag{3.32}$$

it is possible to derive a propagator which is of the following special form

$$S = \begin{pmatrix} S^{\tilde{z}_1 \tilde{z}_1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.33}$$

For the technical details as well as for the ambiguities that have to be computed we refer the reader to appendix A. With the help of this input it is easy to implement the direct integration procedure for the cubic matrix model as outlined in section 2.1. It turns out that we can recursively construct the free energies up to genus four. Moreover, we can also evaluate $F_g(S_1, S_2)$ for the cubic matrix model in perturbation theory, as was done in [31, 35]. The expansions of our direct integration analysis read

$$\begin{aligned}
F_2 &= -\frac{1}{240}\left(\frac{1}{S_1^2} + \frac{1}{S_2^2}\right) + \frac{35}{6}(S_1 - S_2) \\
&\quad + 338S_1^2 - 1632S_1S_2 + 338S_2^2 + \mathcal{O}(S^3) \\
F_3 &= \frac{1}{1008}\left(\frac{1}{S_1^4} + \frac{1}{S_2^4}\right) + \frac{5005}{3}(S_1 - S_2) \\
&\quad + \frac{32}{9}\left(52522S_1^2 - 273403S_1S_2 + 52522S_2^2\right) + \mathcal{O}(S^3) \\
F_4 &= -\frac{1}{1440}\left(\frac{1}{S_1^6} + \frac{1}{S_2^6}\right) + \frac{8083075}{6}(S_1 - S_2) \\
&\quad + \frac{880}{3}\left(788369S_1^2 - 4387436S_1S_2 + 788369S_2^2\right) + \mathcal{O}(S^3).
\end{aligned} \tag{3.34}$$

These results agree with the low-order results obtained in [31, 35]. In the following we explain how to parameterize the ambiguity and how to fix the unknowns entering our ansatz.

3.2 Direct integration, boundary conditions and integrability

In the last section we set up the necessary ingredients to perform a direct integration of the holomorphic anomaly equations. As mentioned in section 2.1 the free energies F_g can be written in the following way

$$F_g = \sum_{k=1}^{3g-3} a_k(z_1, z_2) \left(S^{\tilde{z}_1 \tilde{z}_1}\right)^k + f_g(z_1, z_2), \tag{3.35}$$

where a_k are rational functions completely determined by the recursive procedure. f_g is the holomorphic anomaly, which is not constrained by direct integration and must be fixed by supplying further boundary conditions. The amplitudes F_g should be well-defined over the whole moduli space except for points at the boundary of moduli space where the elliptic geometry (3.18) acquires a node, i.e. a cycle of $\mathbb{S}^1$-topology shrinks. Such points are known as conifold points and are given by the zero loci of the discriminant of the Picard-Fuchs system, which we also call conifold divisors.

Thus, regularity and holomorphicity imply that f_g should be a rational function of z_i, where the numerator is at most of the same degree as the denominator. The denominator is

given by the discriminant factors and takes the form $(z_1 z_2 J^2)^{2g-2}$. This gives the following ansatz for the holomorphic ambiguity

$$f_g(z_1, z_2) = \frac{\sum_{k,l} a_{k,l}^{(g)} z_1^k z_2^l}{(z_1 z_2 J^2)^{2g-2}},\qquad (3.36)$$

where the $a_{k,l}^{(g)}$ have to be determined by the boundary conditions. Note that due to the symmetry of the model in z_1 and z_2 it is enough to restrict the numerator to a polynomial which is symmetric in z_1 and z_2. In order to be well defined as $z_i \to \infty$, the degree of this polynomial must be at most $12g - 12$. It turns out that it is sufficient to truncate the degree at $9g - 9$, as long as $g \le 4$. However, this reduced ansatz may not be present at higher genus and one would have to deal with the full ansatz of degree $12g - 12$.

There are two boundary conditions which we will refer to in the following as a cycle gap and b cycle gap. Let us first consider the a cycle gap. Due to the Gaussian contribution to the partition function of a multi-cut matrix model (see for example [35] for more details) it is easily seen that the holomorphic expansion of F_g at small filling fractions is of the form

$$F_g = \frac{B_{2g}}{2g(2g-2)} \left(\frac{1}{S_1^{2g-2}} + \frac{1}{S_2^{2g-2}} \right) + \mathcal{O}(S), \qquad (g > 1). \qquad (3.37)$$

Due to the absence of subleading singular terms in S_i, this property of the expansion is referred to as the gap condition. The coefficients of the subleading singular powers of S_i depend generically on the other S_j, with $j \ne i$ –in fact they are (infinite) series in the S_j. Demanding the vanishing of these series leads in principle to an over-determined system, therefore in the multi-parameter case it is not easy to count the number of independent conditions implied by (3.37).

The gap condition is also present in the expansion of genus g topological string amplitudes near a conifold divisor [4, 29, 30], where we have

$$F_g^c = \frac{B_{2g}}{2g(2g-2)\Pi^{2g-2}} + \mathcal{O}(\Pi^0), \qquad (g > 1). \qquad (3.38)$$

Here, Π is a flat coordinate normal to the divisor. In view of the Dijkgraaf-Vafa correspondence, this behavior should also characterize multi-cut matrix model amplitudes near the divisors of the spectral curve geometry. Again, since the coefficients of the subleading powers of Π depend on the coordinates tangential to the conifold divisor, the counting of conditions in the multi-parameter case is not easily done.

However, it turns out that, when both constraints, (3.37) and (3.38), are taken into account, the holomorphic anomaly f_g is completely and uniquely fixed. We checked this explicitly for genus $g \le 4$. It is then natural to conjecture that the a and b cycle gap conditions are always sufficient to fix all unknowns in the holomorphic ambiguity for general matrix models with polynomial potential. Following [30] we refer to such a property as integrability of the holomorphic anomaly equation.

3.3 Modular covariant formulation

In the last sections we explained how to solve the cubic matrix model with the techniques known from topological string theory. However, we used a somewhat artificial description which does not make the symmetry properties of the geometry completely explicit. Such a formulation is given by writing all quantities in a covariant modular way. Since the geometry is an elliptic curve together with meromorphic differential Ω we expect not only to parameterize the topological amplitudes F_g by the elliptic modulus τ but in addition by an auxiliary parameter. In the following we explore how this can be achieved in detail.

We start by transforming the quartic curve (3.18) to Weierstrass form, where it is easy to read off the j-function. It is given by

$$j(z_1, z_2) = \frac{16\left((1 - 3z_1 - 3z_2)^2 + 12z_1 z_2\right)^3}{z_1 z_2 \left((1 - 3z_1 - 3z_2)^2 - 4z_1 z_2\right)^2}. \tag{3.39}$$

Comparing this modular invariant to its usual Fourier expansion

$$j(\tau) = q^{-1} + 744 + 196884q + \mathcal{O}(q^2), \tag{3.40}$$

we get a relation $\tau = \tau(z_1, z_2)$. Using the definition of j in terms of modular forms yields actually a rational expression.

It is also easy to identify the auxiliary parameter which accompanies τ. Note that the periods/filling fractions are taken with respect to the differential Ω, which is meromorphic. Thus the sum of all cycles is of course homologically trivial, but the sum of the periods does not have to vanish and is rather proportional to the residue of Ω. Since this residue is related to the auxiliary parameter, it is natural to parameterize the topological amplitudes by both, τ and $t = g_s N$. Due to (3.28) we obtain a relation $t = t(z_1, z_2)$.

In principle this allows us to rewrite all quantities in terms of modular forms together with an auxiliary parameter t, by combining the rational expression $\tau = \tau(z_1, z_2)$ with $t = t(z_1, z_2)$. If we do so, we obtain

$$u = \frac{1 - 3z_1 - 3z_2}{2\sqrt{z_1 z_2}}, \qquad 4t = (z_1 - z_2)\sqrt{1 - 2z_1 - 2z_2}, \tag{3.41}$$

where u is given in terms of modular forms b, c and d defined in appendix C by

$$u = \frac{c + d}{b}. \tag{3.42}$$

However, it turns out that, for the general cubic matrix model, the resulting formulae become too complicated. The reason is that the corresponding spectral curve is a generic elliptic curve. However, if we specialize the calculation to the slice $t = 0$, or $S_1 = -S_2$, the curve has $\Gamma(2)$ monodromy (it is the Seiberg-Witten curve of [49]) and it is possible to exploit the formulation in terms of modular forms, as we will see in section 4.

Fortunately, it is possible to give some closed and simple expressions using modular forms for the genus zero and one sectors, which will prove to be useful in due course. We

start by quoting the perturbative calculation of the planar free energy [12]

$$F_0(S_1, S_2) = \frac{1}{2}S_1^2 \log\left(\frac{S_1}{m\Lambda^2}\right) + \frac{1}{2}S_2^2 \log\left(\frac{S_2}{m\Lambda^2}\right) - \frac{3}{4}(S_1^2 + S_2^2) + 2S_1S_2 \log\left(\frac{m}{\Lambda g}\right)$$
$$+ \frac{1}{g\Delta^3}\left(\frac{2}{3}S_1^3 - 5S_1^2S_2 + 5S_1S_2^2 - \frac{2}{3}S_2^3\right) + \mathcal{O}(S^4). \tag{3.43}$$

Of course, $F_0(S_1, S_2)$ is symmetric under the exchange $S_1 \leftrightarrow -S_2$. From the prepotential we can define the tau-coupling (2.5) and also introduce

$$2\pi\mathrm{i}\tau = \frac{\partial^2 F_0}{\partial s^2}. \tag{3.44}$$

It was shown in [42] (see also [9]) that τ can be computed in terms of elliptic functions as

$$\tau = \mathrm{i}\frac{\mathcal{K}'}{\mathcal{K}} = \mathrm{i}\frac{K(k')}{K(k)}, \tag{3.45}$$

where

$$\mathcal{K} = \int_{x_1}^{x_2} \frac{\mathrm{d}z}{\sqrt{|\sigma(z)|}} = \frac{2}{\sqrt{(x_1 - x_3)(x_2 - x_4)}}K(k), \qquad k^2 = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)},$$
$$\mathcal{K}' = \int_{x_2}^{x_3} \frac{\mathrm{d}z}{\sqrt{|\sigma(z)|}} = \frac{2}{\sqrt{(x_1 - x_3)(x_2 - x_4)}}K(k'), \qquad k'^2 = 1 - k^2. \tag{3.46}$$

This modular parameter turns out to match with our definition by the j-function mentioned above. We find, in the full theory

$$\pi\mathrm{i}\tau = \frac{1}{2}\log\left(\frac{-S_1S_2}{m^6}\right) + \frac{17(S_1 - S_2)}{m^3} + \frac{2\left(83S_1^2 - 209S_2S - 1 + 83S_2^2\right)}{m^6} + \cdots \tag{3.47}$$

Let us now consider genus one. Akemann [2] gave a simple expression for F_1, that reads

$$F_1 = -\frac{1}{24}\sum_{i=1}^{4}\ln M_i - \frac{1}{2}\ln K(k) - \frac{1}{12}\ln\Delta + \frac{1}{8}\ln(a_1^- - a_2^-)^2 + \frac{1}{8}\ln(a_1^+ - a_2^+)^2, \tag{3.48}$$

where Δ denotes the discriminant of $\sigma(x)$. Using that $M_i = g$ for the cubic matrix model as well as Thomae's formulae, cf. appendix C, this can be written compactly as

$$F_1 = -\log\eta(\tau) - \frac{1}{24}\log\Delta, \tag{3.49}$$

where η is the Dedekind eta-function.

4   The cubic model for $S_1 = -S_2$

In the following we will specialize the cubic matrix model studied in section 3 to the slice $S_1 = -S_2$. On this slice, $t = 0$, and the direct integration procedure simplifies. Moreover, we are able to write all quantities which are needed for direct integration in terms of simple modular forms. The underlying reason for this is that, when $t = 0$, the spectral curve of the

matrix model becomes the Seiberg-Witten curve, which has simple monodromy properties. Therefore the recursive procedure will be very efficient in obtaining results at high genus.

First of all notice that, by contour deformation,

$$S_1 + S_2 = g \oint_{z=0} \frac{\mathrm{d}z}{z^4} \sqrt{1 + \frac{2m}{g}z + \frac{m^2}{g^2}z^2 + \frac{\lambda}{g^2}z^3 + \frac{\mu}{g^2}z^4} = \frac{\lambda}{2g}. \tag{4.1}$$

Therefore, if the parameter λ in (3.17) vanishes $\lambda = 0$, we have

$$t = S_1 + S_2 = 0. \tag{4.2}$$

In this case one also has [18]

$$\tau_{11} = \tau_{22} = -\tau_{12} = \tau, \tag{4.3}$$

which can be seen from (2.5).

From the point of view of the original matrix model, the slice $S_1 = -S_2$ involves an analytic continuation in the space of 't Hooft parameters. This is because on this slice $S_1/S_2 = N_1/N_2 = -1$, which can not be implemented in the matrix integral (3.15), since $N_{1,2}$ are *a priori* positive integers. In terms of matrix integrals, the slice $S_1 = -S_2$ can be related to a *supermatrix model* [5, 16, 21, 54]. A Hermitian supermatrix has the form

$$\Phi = \begin{pmatrix} A & \Psi \\ \Psi^\dagger & C \end{pmatrix}, \tag{4.4}$$

where A (C) are $N_1 \times N_1$ ($N_2 \times N_2$) Hermitian, Grassmann even matrices, and Ψ is a matrix of complex, Grassmann odd numbers. The supermatrix model is defined by the partition function

$$Z_{\mathrm{s}}(N_1|N_2) = \int \mathcal{D}\Phi \, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Str}W(\Phi)}, \tag{4.5}$$

where we consider a polynomial potential $W(\Phi)$ and Str denotes the supertrace. There are two types of supermatrix models with supergroup symmetry $\mathrm{U}(N_1|N_2)$: the ordinary supermatrix model, and the physical supermatrix model [54]. The ordinary supermatrix model is obtained by requiring A, C to be real Hermitian matrices, while the physical model is obtained by requiring that, after diagonalizing Φ by a superunitary transformation, the resulting eigenvalues are real. The partition function of the physical supermatrix model reads, in terms of eigenvalues [21, 54]

$$Z_{\mathrm{s}}(N_1|N_2) = \frac{1}{N_1!N_2!} \int \prod_{i=1}^{N_1} \mathrm{d}\mu_i \prod_{j=1}^{N_2} \mathrm{d}\nu_j \frac{\prod_{i<j}(\mu_i - \mu_j)^2 (\nu_i - \nu_j)^2}{\prod_{i,j}(\mu_i - \nu_j)^2} \mathrm{e}^{-\frac{1}{g_s}\left(\sum_i W(\mu_i) - \sum_j W(\nu_j)\right)}, \tag{4.6}$$

where the two groups of eigenvalues μ_i, ν_j are expanded around two different critical points of $W(x)$. This partition function is related to (3.15) after changing $N_2 \to -N_2$ [21], therefore it gives a physical realization of the $S_1/S_2 < 0$ slice of the moduli space. Notice, that the moduli space of the local Calabi-Yau for generic complex S_1, S_2 describes both the original matrix integral (3.15) and its supergroup extension (4.6).

4.1 The geometry

In the following we discuss the geometry underlying the curve with $\lambda = 0$. It is easy to see that, up to a shift in the x coordinate, it can be written as

$$y^2 = (x^2 - a^2)(x^2 - b^2), \qquad a > b. \tag{4.7}$$

If we compare this to the Seiberg-Witten curve [49]

$$y^2 = (x^2 - u)^2 - \Lambda_{\mathrm{SW}}^4, \tag{4.8}$$

we find that they are equal once we identify the parameters as

$$u = \frac{a^2 + b^2}{2}, \qquad \Lambda_{\mathrm{SW}}^2 = \frac{a^2 - b^2}{2}. \tag{4.9}$$

We also want to translate these parameters in terms of the cubic matrix model variables. This was already done in [18, 35], and we have

$$\Delta = \frac{m}{g}, \qquad u = \frac{1}{4}\Delta^2. \tag{4.10}$$

We will set

$$g = 1. \tag{4.11}$$

On the other hand, we have the following relation between the Λ parameter appearing in (3.43) and the Seiberg-Witten scale

$$\Lambda = \frac{1}{\sqrt{2}}\Lambda_{\mathrm{SW}}. \tag{4.12}$$

For the simple curve (4.7) one can compute many quantities directly and relate them to modular forms or elliptic integrals. As a starting point the period integrals

$$S = S_1 = -S_2, \qquad \Pi = \partial_s F_0(S, -S) \tag{4.13}$$

can be computed in terms of simple elliptic functions, which was done for S in ref. [18]. Repeating this analysis yields

$$S = \frac{1}{2\pi i} \int_b^a y(x)\mathrm{d}x = \frac{a}{6\pi}\left[(a^2 + b^2)E(k_1) - 2b^2 K(k_1)\right] \tag{4.14}$$

as well as

$$\Pi = \int_{-b}^b y(x)\mathrm{d}x = \frac{2}{3}a\left[(a^2 + b^2)E(k_1') + (b^2 - a^2)K(k_1')\right], \tag{4.15}$$

where the elliptic modulus k_1 and its complementary one k_1' are given by

$$k_1^2 = \frac{a^2 - b^2}{a^2}, \qquad k_1'^2 = 1 - k_1^2 = \frac{b^2}{a^2}. \tag{4.16}$$

The modulus k_1 is related to the usual cross-ratio k^2 introduced in (3.46) as

$$k_1^2 = \frac{4k}{(1 + k)^2}. \tag{4.17}$$

In order to obtain expansions of the periods we introduce the parameters

$$\mu = \frac{\Lambda_{\mathrm{SW}}^2}{u}, \qquad \mu_{\mathrm{D}} = 1 - \frac{\Lambda_{\mathrm{SW}}^2}{u}. \tag{4.18}$$

Small μ corresponds to the semiclassical regime of Seiberg-Witten theory which occurs at $u \to \infty$, whereas small μ_{D} relates to the region near $u \to \Lambda_{\mathrm{SW}}^2$, where a magnetic monopole becomes massless. In these variables the periods read

$$\frac{S}{u^{\frac{3}{2}}} = \frac{\sqrt{1+\mu}}{3\pi}\left[E\left(\frac{2\mu}{1+\mu}\right) + (\mu - 1)K\left(\frac{2\mu}{1+\mu}\right)\right], \tag{4.19}$$

$$\frac{\Pi}{u^{\frac{3}{2}}} = \frac{4\sqrt{2-\mu_{\mathrm{D}}}}{3}\left[E\left(\frac{\mu_{\mathrm{D}}}{2-\mu_{\mathrm{D}}}\right) + (\mu_{\mathrm{D}} - 1)K\left(\frac{\mu_{\mathrm{D}}}{2-\mu_{\mathrm{D}}}\right)\right]. \tag{4.20}$$

Note that $S/u^{3/2}$ and $\Pi/u^{3/2}$ are dimensionless. Further, we expand (4.19) around $\mu = 0$ to obtain

$$\frac{S}{u^{\frac{3}{2}}} = \frac{\mu^2}{8} + \frac{3\mu^4}{256} + \frac{35\mu^6}{8192} + \frac{1155\mu^8}{524288} + \frac{45045\mu^{10}}{33554432} + \cdots, \tag{4.21}$$

which is the expansion (4.19) of [35], after changing to the appropriate variables. The inverse expansion is given by

$$\mu^2 = 8\frac{S}{u^{\frac{3}{2}}} - 6\left(\frac{S}{u^{3/2}}\right)^2 - \frac{17}{2}\left(\frac{S}{u^{3/2}}\right)^3 - \frac{375}{16}\left(\frac{S}{u^{3/2}}\right)^4 - \frac{10689}{128}\left(\frac{S}{u^{3/2}}\right)^5 + \cdots. \tag{4.22}$$

We introduce now the following elliptic modulus τ_0 as

$$\tau_0 = \mathrm{i}\frac{K\left(\frac{1-\mu}{1+\mu}\right)}{K\left(\frac{2\mu}{1+\mu}\right)} = \mathrm{i}\frac{K(k_1')}{K(k_1)} = \frac{\mathrm{i}}{2}\frac{K(k')}{K(k)}, \tag{4.23}$$

which can be expanded in μ. By inverting this series one can derive μ as a function of τ_0. In particular we observe

$$\mu = \frac{b}{c+d}, \tag{4.24}$$

where we follow the notation[4] of [31]. In turn the expression (4.19) defines the variable μ as a function of

$$\frac{S}{u^{3/2}} = 8\frac{S}{m^3} \tag{4.25}$$

as well, and in particular the series (4.23) defines τ_0 as a function of $S/u^{3/2}$:

$$2\pi\mathrm{i}\tau_0 = \log\left(\frac{S}{m^3}\right) + 34\frac{S}{m^3} + 750\left(\frac{S}{m^3}\right)^2 + \frac{71260}{3}\left(\frac{S}{m^3}\right)^3 + \cdots. \tag{4.26}$$

Moreover, comparing with (3.47) yields the identity

$$\tau_0 = \frac{1}{2}\tau(S, -S), \tag{4.27}$$

which is obvious also from (4.23).

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[4]For our conventions on modular forms used in this section, see appendix C.

Consider now the dual elliptic modulus $\tau_{0,\mathrm{D}}$, obtained by a S-transformation on the elliptic modulus,

$$\tau_{0,\mathrm{D}} = -\frac{1}{\tau_0}. \tag{4.28}$$

Following the same lines of thought as before, this defines $\tau_{0,\mathrm{D}}$ as a series in the dual period

$$\frac{S_{\mathrm{D}}}{u^{3/2}} = 8\frac{S_{\mathrm{D}}}{m^3}. \tag{4.29}$$

In the following we will set

$$\Lambda_{\mathrm{SW}} = 1 \tag{4.30}$$

so in particular $\mu = u^{-1} = 4/m^2$. Note that (4.24) therefore defines m as a function of τ_0. Strictly speaking, m is hence a function of S, but in order to establish the relation between τ_0 and $\tau(S, -S)$, i.e. (4.27), we treated m as an independent variable. In all subsequent formulas and expansions we will do so as well.

Next, we compute the Yukawa coupling

$$C_{sss} = \frac{\partial^3 F_0}{\partial s^3}. \tag{4.31}$$

This follows from the general formula for two-cut matrix models given by [42]

$$\frac{\partial^3 F_0}{\partial s^3} = \pi^3 \left[M_1 \cdots M_4 \, \mathcal{K}^3 \prod_{i<j} (x_i - x_j)^2 \right]^{-1} \cdot \sum_{i=1}^{4} \left[\prod_{j \neq i} M_j \cdot \prod_{\substack{k,l \neq i, \\ k<l}} (x_k - x_l)^2 \right] \tag{4.32}$$

where $M_i = M(x_i)$, the spectral curve is written as in (3.8), and $\mathcal{K}$ is given in (3.46) When applied to the Seiberg-Witten curve (4.7) we obtain

$$C_{sss} = \frac{\partial(4\pi\mathrm{i}\tau_0)}{\partial s} = \frac{64\sqrt{2}}{m^3} \frac{(c+d)^{5/2}}{b^2 cd}. \tag{4.33}$$

To see this, one has to apply Thomae's formula, which relates the branch points x_i to ϑ-functions [25] and further one has to express $\mathcal{K}$ in terms of modular forms. This is done as follows. Note that

$$\mathcal{K} = \int_b^a \frac{\mathrm{d}x}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} K(k_1). \tag{4.34}$$

The dimensionless combination $\sqrt{u}\,\mathcal{K}$ can be expanded as a series in τ_0 since

$$\sqrt{u}\,\mathcal{K} = \frac{1}{\sqrt{1+\mu}} K\left(\frac{2\mu}{1+\mu}\right). \tag{4.35}$$

This yields

$$\mathcal{K} = \frac{\pi}{m} \sqrt{\frac{c+d}{2}}. \tag{4.36}$$

We can check the formula (4.31) by calculating this quantity directly from the perturbative result. Evaluating the derivatives at $S_1 = -S_2 = S$ we obtain from (3.43)

$$\frac{\partial^3 F_0}{\partial s^3} = \frac{2}{m^3} \left\{ 34 + \frac{m^3}{S} + 1500\frac{S}{m^3} + 71260\frac{S^2}{m^6} + \cdots \right\}. \tag{4.37}$$

Using (4.26) this coincides with (4.31), if we treat m as an independent variable.

The expression (4.33) for C_{sss} is a modular form of weight -3 on the modular group $\Gamma(2)$ defined in the appendix. We will use as generators for the ring of modular forms on $\Gamma(2)$, $M_*(\Gamma(2))$, the functions

$$K_2 = c + d, \qquad K_4 = b^2, \tag{4.38}$$

which are modular forms of weight two and four, respectively. Note that instead of considering the $\Gamma(2)$ description of the Seiberg-Witten curve (4.7) we could also use the equivalent $\Gamma_0(4)$ description, which amounts to trade τ_0 for $2\tau_0 = \tau(S, -S)$ in all expressions of this section.

4.2 Direct integration and higher genus amplitudes

Having discussed the genus zero sector of the cubic matrix model specialized to the slice $S_1 = -S_2$, let us now turn our attention to the higher genus free energies F_g. According to [23, 31] the matrix model free energies F_g can be promoted to modular invariant, non-holomorphic amplitudes $F_g(\tau_0, \bar\tau_0)$ which satisfy the holomorphic anomaly equations of [6] in the local limit. The matrix model F_g is recovered by formally considering the limit $\bar\tau_0 \to \infty$ while keeping τ_0 fixed.

In order to apply this, we must compute the full non-holomorphic genus one amplitude F_1 and derive the propagator S^{ss}. Using the general formula (3.48) specialized to the Seiberg-Witten curve (4.7), and by following the same argument as for the Yukawa coupling C_{sss}, we obtain

$$F_1(\tau_0, \bar\tau_0) = -\log(\sqrt{\mathrm{Im}\tau_0}\,\eta(\tau_0)\eta(-\bar\tau_0)) + \frac{1}{4}\log\left(\frac{m^2 K_2}{\sqrt{K_4}}\right). \tag{4.39}$$

Indeed, when expanded we find

$$F_1 = -\frac{1}{6}\log S + \frac{S}{3m^3} + 15\left(\frac{S}{m^3}\right)^2 + \frac{6202}{9}\left(\frac{S}{m^3}\right)^3 + 32286\left(\frac{S}{m^3}\right)^4 + \cdots, \tag{4.40}$$

which is precisely the series for F_1 obtained in [35] after setting $S_1 = -S_2 = S$.

Next we turn to the propagator S^{ss}, defined by

$$\overline{C}_{\bar{s}}^{ss} = \bar\partial_{\bar s} S^{ss}, \tag{4.41}$$

where $\overline{C}_{\bar s\bar s\bar s}$ is the complex conjugate of the Yukawa coupling C_{sss} and the indices are raised by means of the metric

$$G_{s\bar s} \sim \mathrm{Im}\tau_0. \tag{4.42}$$

Using the chain rule and the relation (4.31) yields

$$\partial_s F_1(\tau_0, \bar\tau_0) = -\frac{1}{48}C_{sss}\widehat{E}_2(\tau_0, \bar\tau_0) + \partial_s f_1(\tau_0), \tag{4.43}$$

where f_1 is given by

$$f_1(\tau_0) = \frac{1}{4}\log\left(\frac{m^2 K_2}{\sqrt{K_4}}\right). \tag{4.44}$$

Hence, the propagator is identified with

$$S^{ss} = -\frac{1}{24}\widehat{E}_2(\tau_0, \bar{\tau}_0). \tag{4.45}$$

Now we are prepared to apply the method of directly integrating the holomorphic anomaly equations according to [3, 27]. In the conventions of this section the holomorphic anomaly equations can be cast into the following form

$$\frac{\partial F_g}{\partial \widehat{E}_2} = -\frac{1}{192}C_{sss}^2\left[\hat{D}_{\tau_0}^2 F_{g-1} + \frac{\hat{D}_{\tau_0}C_{sss}}{C_{sss}}\hat{D}_{\tau_0}F_{g-1} + \sum_{h=1}^{g-1}\hat{D}_{\tau_0}F_h\hat{D}_{\tau_0}F_{g-h}\right], \quad (g > 1) \tag{4.46}$$

where $\hat{D}_{\tau_0}$ denotes the Maass derivative acting on (almost-holomorphic) modular forms of weight k as

$$\hat{D}_{\tau_0} = \frac{1}{2\pi i}\frac{d}{d\tau_0} - \frac{k}{4\pi \mathrm{Im}\tau_0}. \tag{4.47}$$

Since the ring $\widehat{M}_*(\Gamma(2)) = \mathbb{C}[\widehat{E}_2, K_2, K_4]$ is closed under $\hat{D}_{\tau_0}$, and the F_g's are modular invariant forms, the holomorphic anomaly equation can be integrated with respect to $\widehat{E}_2$. We obtain the following schematic result

$$F_g(\tau_0, \bar{\tau}_0) = \widetilde{\Delta}^{2-2g} \cdot \sum_{k=1}^{3g-3} c_k^{(g)}(\tau_0)\widehat{E}_2^k(\tau_0, \bar{\tau}_0) + f_g(\tau_0), \tag{4.48}$$

where $c_k^{(g)}(\tau_0)$ are modular forms of weight $8(g-1) - 2k$, completely determined by the holomorphic anomaly equation, and $\widetilde{\Delta}$ is just the denominator of C_{sss}. In particular it is a weight eight form given by

$$\widetilde{\Delta} = m^3(K_2^2 - K_4)K_4. \tag{4.49}$$

All the non-trivial information is encoded in the holomorphic ambiguity $f_g(\tau_0)$. It has to be derived genus by genus by supplying further boundary conditions. In the particular case of the cubic matrix model specialized to the slice $S_1 = -S_2$, we will argue in the next subsection that $f_g(\tau_0)$ can be fixed at all genera. Applying this procedure we were able to integrate the holomorphic anomaly equations and obtained the matrix model free energies to genus 52.

Let us at least present the result for the full non-holomorphic genus two amplitude

$$\begin{aligned}F_2(\tau_0, \bar{\tau}_0) = &-\frac{160K_2^5}{81m^6(K_2^2 - K_4)^2K_4^2}\widehat{E}_2^3 - \frac{16K_2^4(5K_2^2 - 7K_4)}{9m^6(K_2^2 - K_4)^2K_4^2}\widehat{E}_2^2\\&-\frac{8K_2^3(77K_2^4 - 132K_2^2K_4 + 63K_4^2)}{27m^6(K_2^2 - K_4)^2K_4^2}\widehat{E}_2 - \frac{4K_2^4(2051K_2^4 - 4005K_2^2K_4 + 1890K_4^2)}{405m^6(K_2^2 - K_4)^2K_4^2}.\end{aligned} \tag{4.50}$$

Here we collect some low genus expansions of the free energy amplitudes of the cubic matrix model on the slice $S_1 = -S_2 = S$:

$$
\begin{aligned}
m^6 F_2 &= -\frac{1}{120}\frac{m^6}{S^2} + \frac{35}{3}\frac{S}{m^3} + 2308\frac{S^2}{m^6} + \frac{1341064}{5}\frac{S^3}{m^9} + 24734074\frac{S^4}{m^{12}} + \cdots \\
m^{12} F_3 &= \frac{1}{504}\frac{m^{12}}{S^4} + \frac{10010}{3}\frac{S}{m^3} + \frac{4036768}{3}\frac{S^2}{m^6} + \frac{1883381692}{7}\frac{S^3}{m^9} + 38608040638\frac{S^4}{m^{12}} + \cdots \\
m^{18} F_4 &= -\frac{1}{720}\frac{m^{18}}{S^6} + \frac{8083075}{3}\frac{S}{m^3} + 1749491040\frac{S^2}{m^6} + \frac{4618613451580}{9}\frac{S^3}{m^9} + \cdots \\
m^{24} F_5 &= \frac{1}{528}\frac{m^{24}}{S^8} + \frac{13013750750}{3}\frac{S}{m^3} + 4038280413440\frac{S^2}{m^6} + \frac{17515677810823140}{11}\frac{S^3}{m^9} + \cdots \\
m^{30} F_6 &= -\frac{691}{163800}\frac{m^{30}}{S^{10}} + 11699361924250\frac{S}{m^3} + \frac{43710230883020800}{3}\frac{S^2}{m^6} + \cdots .
\end{aligned}
\tag{4.51}
$$

We can check some of these results by comparing to the perturbative calculations of [35] specialized to $S_1 = -S_2 = S$. We observe agreement for genus two and three at low order in S/m^3. All higher genus computations are new results.

The direct integration procedure outlined here is by far the most efficient method to calculate higher genus amplitudes in matrix models. It only takes a few minutes to reach e.g. genus 10 on a conventional personal computer.

4.3 Boundary conditions and integrability

According to [1, 23] F_g is an almost-holomorphic modular invariant form under the space-time duality group, in this case $\Gamma(2)$. Hence, F_g is regular except for some points on the boundary of moduli space.

Regularity and holomorphicity imply that f_g should be a rational function, where its denominator is given by an appropriate power of the discriminant of the curve. From the expression (4.48) we see that the denominator of f_g is given by $\widetilde{\Delta}^{2g-2}$, hence a weight $8(g-1)$ form. Modularity now implies that the numerator has to be a form of finite weight, in order to cancel the weight from the denominator. Since the space of weight k forms is finite dimensional, there are only finitely many coefficients to determine. In particular, for $\Gamma(2)$ we have

$$
\dim M_k(\Gamma(2)) = \begin{cases} \frac{k+2}{2}, & k > 2, \quad k \text{ even.} \\ 0, & \text{else.} \end{cases}
\tag{4.52}
$$

In summary this justifies the ansatz

$$
f_g(\tau_0) = \widetilde{\Delta}^{2-2g} \cdot \sum_{k=0}^{4(g-1)} a_k K_2^{2k} K_4^{4(g-1)-k}, \qquad (g > 1)
\tag{4.53}
$$

where $\widetilde{\Delta}$ is given in eq. (4.49). This implies that there are $4g-3$ unknown constants a_k in the ambiguity f_g. These are completely and uniquely fixed by imposing the following two boundary conditions.

First, we know that the holomorphic expansion of F_g at small S has the structure (3.37) specialized to the slice, which imposes $2g-1$ conditions on f_g and leaving $2g-2$ unknowns. Further the holomorphic expansion at conifold divisors is of the form (3.38), where Π is a suitable coordinate transverse to the divisor which vanishes at the conifold. In our case Π is the dual period. Thus, (3.38) imposes $2g-2$ further constraints on the ambiguity, and it determines it completely.

5 Non-perturbative aspects

In this section we address non-perturbative effects in the two-cut matrix model, and its connection to the large order behavior of the $1/N$ expansion. We first review the one-cut case.

5.1 Non-perturbative effects in the one-cut matrix model

For concreteness, we will focus here on the cubic matrix model which we are analyzing in this paper.

In the one-cut cubic matrix model, the large N limit is described by a distribution of eigenvalues around the minimum of the potential at $x = 0$. The eigenvalues fill the interval $[a, b]$. It has been known for some time that instanton sectors in this model are obtained by tunneling a finite, small number of eigenvalues $\ell \ll N$ from this interval to the maximum of the effective potential, located at x_0. The structure of the partition function in the ℓ-instanton sector has been determined in [41, 42], and at one loop it has the form

$$Z^{(\ell)} = \frac{g_s^{\ell^2/2}}{(2\pi)^{\ell/2}} G_2(\ell+1) \mu_1^{\ell^2} \exp\left(-\frac{\ell A}{g_s}\right) \left\{1 + \mathcal{O}(g_s)\right\}. \tag{5.1}$$

In this equation, $G_2(z)$ is the Barnes function. A is the instanton action, and it can be computed in terms of the spectral curve of the one-cut matrix model as

$$A = \int_b^{x_0} \mathrm{d}z\, y(z). \tag{5.2}$$

Finally, μ_1 is the one-loop contribution, and it has the explicit expression

$$\mu_1 = \frac{b-a}{4} \frac{1}{\sqrt{M(x_0)[(a-x_0)(b-x_0)]^{\frac{5}{2}}}}. \tag{5.3}$$

In [41] it was argued, following standard arguments in the large order behavior of perturbation theory [38], that the free energy of the one-instanton amplitude, $F^{(1)}$, should determine the leading asymptotics at large g of the perturbative amplitudes F_g, according to the formula

$$F_g = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \frac{\mathrm{d}z}{z^{g+1}} F^{(1)}(z). \tag{5.4}$$

If we write

$$F^{(1)} = g_s^{1/2} \mathrm{e}^{-A/g_s} \sum_{\ell=1}^\infty \mu_\ell g_s^{\ell-1}, \tag{5.5}$$

we obtain the full $1/g$ asymptotics

$$F_g \sim_g \frac{1}{\pi} A^{-2g-b} \Gamma\left(2g + b\right) \sum_{\ell=1}^{\infty} \frac{\mu_\ell A^{\ell-1}}{\prod_{k=1}^{\ell-1}(2g + b - k)}. \tag{5.6}$$

where

$$b = -\frac{5}{2}. \tag{5.7}$$

The formula (5.6) can be regarded as a generalization of the asymptotics for formal solutions of nonlinear ODEs. The reason is as follows. In the double-scaling limit of the matrix model (see [17]), the total free energy of the matrix model becomes a function of a double-scaled variable z,

$$F(t, g_s) \to F_{\mathrm{ds}}(z), \tag{5.8}$$

and the specific heat $u = -F''_{\mathrm{ds}}(z)$ satisfies the Painlevé I equation

$$u^2 - \frac{1}{6}u'' = z. \tag{5.9}$$

In particular, the genus expansion of the cubic matrix model leads to a formal solution of Painlevé I

$$u(z) = z^{1/2} \sum_{g=0}^{\infty} u_{g,0} z^{-5g/2}. \tag{5.10}$$

On the other hand, the instanton sectors of the matrix model lead to instanton corrections of the form

$$u_\ell(z) = z^{1/2 - 5\ell/8} \mathrm{e}^{-\ell a z^{5/4}} \sum_{n=0}^{\infty} u_{n,\ell} z^{-5n/4} \tag{5.11}$$

where

$$a = \frac{8\sqrt{3}}{5}. \tag{5.12}$$

It can be shown that the coefficients of (5.10) have an asymptotic behavior at large g which is governed by the one-instanton solution $u_1(z)$ in (5.11). The precise formula is,

$$u_{g,0} \sim_g \frac{a^{-2g+\frac{1}{2}}}{\pi} \Gamma\left(2g - \frac{1}{2}\right) \frac{\mathcal{S}_1}{\pi \mathrm{i}} \left\{1 + \sum_{l=1}^{\infty} \frac{u_{l,1} a^l}{\prod_{k=1}^{l}(g - 1/2 - k)}\right\}, \tag{5.13}$$

where $\mathcal{S}_1$ is a Stokes constant. One can explicitly check [14, 15, 41] that (5.13) can be deduced from the double-scaling limit of the asymptotics (5.6). In particular, the constant a is the double-scaling limit of the instanton action.

5.2 Non-perturbative effects in the cubic matrix model

Non-perturbative effects in multi-cut matrix models have been studied in [9, 42]. A multi-cut matrix model with a fixed choice of filling fractions must be regarded as a fixed background, and any other choice of filling fractions leads to an instanton correction to the free energy on the fixed background. To be concrete, let us consider a two-cut matrix model with a fixed background given by N_1, N_2 eigenvalues in the stable and unstable

saddle points, respectively. The partial 't Hooft parameters S_1, S_2 are given as usual by $S_i = g_s N_i$. The total partition function is of the form

$$Z = Z(N_1, N_2) + \sum_{\ell \neq 0} \zeta^\ell \, Z(N_1 - \ell, N_2 + \ell). \tag{5.14}$$

The sum over ℓ corresponds to the tunneling of ℓ eigenvalues from the first cut to the second cut, and at large N, the corresponding partition functions have the form

$$Z^{(\ell)} = \zeta^\ell q^{\ell^2/2} \exp\left(-\frac{\ell A}{g_s}\right) \left\{1 + \mathcal{O}(g_s)\right\}, \qquad \ell \in \mathbb{Z}^* \tag{5.15}$$

where

$$A = \partial_s F_0 \qquad \text{and} \qquad q = \exp\left(\partial_s^2 F_0\right). \tag{5.16}$$

The variable s is given in (3.27). If the cuts of the matrix model are the intervals $[x_1, x_2]$, $[x_3, x_4]$, the instanton action A can be written as

$$A = \int_{x_2}^{x_3} y(x) \mathrm{d}x. \tag{5.17}$$

If $\mathrm{Re}(A) \neq 0$, the instanton contributions are exponentially suppressed if $\mathrm{sgn}(\mathrm{Re}(A)\ell) > 0$, and they are exponentially enhanced if $\mathrm{sgn}(\mathrm{Re}(A)\ell) < 0$. This is just reflecting the fact that the generic background is unstable and if we expand around it we will find tachyonic directions. Notice however that both corrections are non-perturbative in g_s, therefore they are invisible in the genus expansion.

It is generically expected that the existence of these non-perturbative sectors leads to the factorial divergence of the genus expansion around a fixed background. The growth of the perturbative string amplitudes at large genus (and fixed S_1, S_2) should be of the same form as in (5.6), i.e.

$$F_g(S_1, S_2) \sim_g A^{-2g-b} \Gamma(2g + b) + \mathcal{O}(g^{-1}) \tag{5.18}$$

where A is given by (5.17) and b is a constant.

We can test these predictions by numerical methods using our results from direct integration. We start by concentrating on the slice $S_1 = -S_2$. Note that in this case the instanton action A is given by the dual period Π, whose explicit expression is given in eq. (4.15). In order to extract the asymptotics of the sequence $\{F_g\}_{g \geq 0}$ we employ a standard numerical technique known as Richardson extrapolation. The method removes the first terms of the subleading tail and hence accelerates the convergence. Given a sequence $\{S_g\}_{g \geq 0}$ in the form

$$S_g = a_0 + \frac{a_1}{g} + \frac{a_2}{g^2} + \dots \tag{5.19}$$

its Richardson transform is defined by

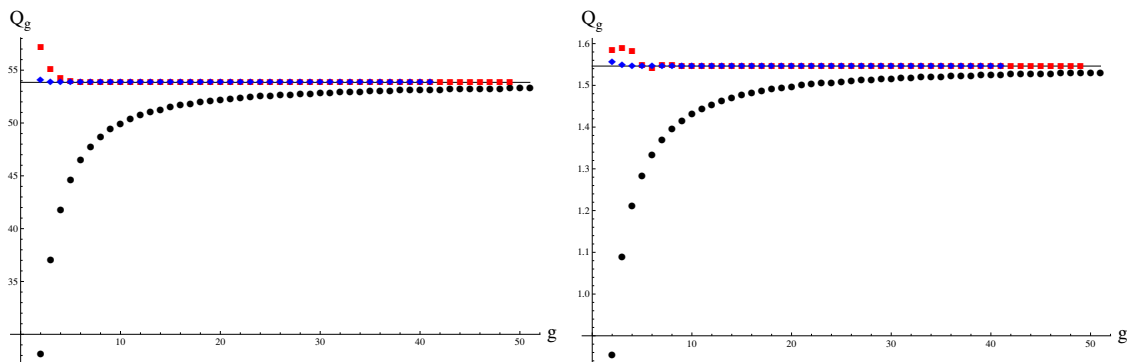$$R_S(g, N) = \sum_{k \geq 0} \frac{(-1)^{k+N}(g+k)^N}{k! \, (N-k)!} S_{g+k}, \tag{5.20}$$

Figure 2. The sequence Q_g ($\bullet$) and two Richardson transforms ($\blacksquare$, $\blacklozenge$) at $\tau_0 = \frac{i}{2}$ (left) and $\tau_0 = \frac{2i}{3} + \frac{1}{9}$ (right) which corresponds to $S \approx 0.139$ and $S \approx 0.117 + 0.016i$, respectively. The leading asymptotics as predicted by the instanton action $|A|^{-2}$ is shown as a straight line. The error for genus 52 is about 10^{-8} % and 10^{-10} %, resp.

such that the sub-leading terms in $\{S_g\}_{g\geq0}$ are cancelled up to order g^{-N}. In fact, it can be shown that if $\{S_g\}_{g\geq0}$ is a finite sequence, the Richardson transform returns exactly the leading term a_0.

Comparing (5.19) with (5.18) one can extract the instanton action by considering the sequence

$$Q_g = \frac{F_{g+1}}{4g^2 F_g} = \frac{1}{A^2}\left(1 + \frac{1+2b}{2g} + \mathcal{O}(g^{-2})\right). \tag{5.21}$$

Once A is confirmed, one can then obtain the parameter b from the new sequence

$$Q'_g = 2g\left(A^2 Q_g - 1\right) = 1 + 2b + \mathcal{O}(g^{-1}). \tag{5.22}$$

In figure 2 and figure 3 we plot the sequences Q_g, Q'_g, together with their Richardson transforms, for two values of S. It is obvious from the numerical calculation that the large genus asymptotics is controlled at leading order by the instanton action. In addition, we find numerically that

$$b = -1. \tag{5.23}$$

This value of b is different from the one characterizing the one-cut model (5.7). In fact, the value (5.7) corresponds to the universality class of pure two-dimensional gravity, while the value (5.23) corresponds rather to the universality class of the $c = 1$ string [47]. It is interesting to see that both behaviors are present in the two-cut cubic matrix model, along different submanifolds of the moduli space (the 2d gravity behavior takes place in the slice $S_2 = 0$, while the $c = 1$ behavior takes place in the slice $S_1 + S_2 = 0$).

Turning our attention to a generic value of the filling fractions (S_1, S_2) in the cubic matrix model, we can try to test our prediction (5.18) by using the results from direct integration of section 3. Since we computed F_g up to genus four, we can only explore the first four elements of the sequence $\{Q_g\}_{g\geq0}$, eq. (5.21). In order to have a better control of the error, we consider a perturbation around the submanifold $S_1 + S_2 = 0$ where the large
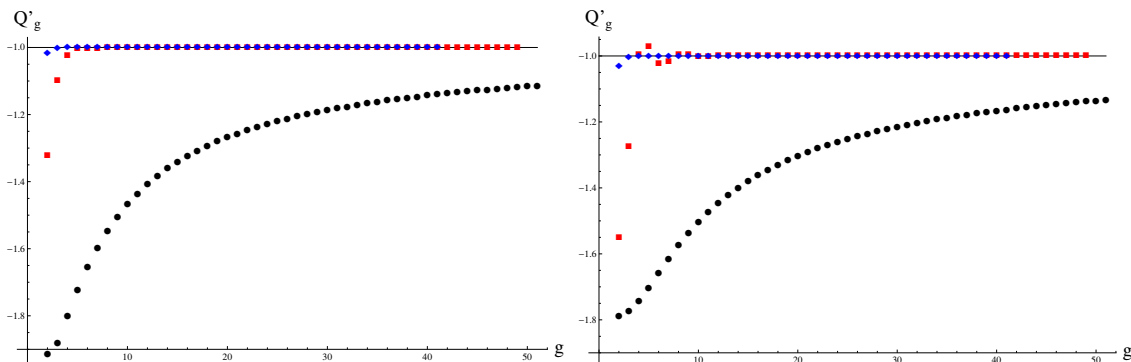
Figure 3. The sequence Q'_g ($\bullet$) and two Richardson transforms ($\blacksquare$, $\blacklozenge$) at $\tau_0 = \frac{i}{2}$ (left) and $\tau_0 = \frac{2i}{3} + \frac{1}{9}$ (right) which corresponds to $S \approx 0.139$ and $S \approx 0.117 + 0.016i$, respectively. The leading asymptotics as predicted by the parameter $b = -1$ is shown as a straight line. The error for genus 52 is about 10^{-8} % in both cases.

order behavior is well established. The instanton action (5.17) is calculated using (A.3) by

$$
\begin{aligned}
A &= \frac{\partial F_0}{\partial S_1} - \frac{\partial F_0}{\partial S_2} = \Pi_1 - \Pi_2 \\
&= \log(S_1)S_1 - \log(S_2)S_2 + \frac{1}{6} - S_1 + S_2 + \mathcal{O}(S^2).
\end{aligned}
\tag{5.24}
$$

Figure 4 shows Q_3, $R_Q(1,2)$ and $|A|^{-2}$ as a function of S_1 in the vicinity of the slice point $S_1 = -S_2 = S = 0.004$, where convergence is ensured. We observe that the behavior of Q_3 and $R_Q(1,2)$ is qualitatively the same as predicted by the instanton action. Moreover, their relative errors stay roughly constant over the complete data set. This seems to indicate that the large order behavior of the genus expansion is also governed by the instanton action in the general two-cut cubic matrix model.

Unfortunately, our numerical results for the generic case are not good enough to determine the value of b reliably. It is an interesting question to know how this value changes as we move in the moduli space. We expect it to be $b = -1$ except in the one-cut slices $S_1 = 0$, $S_2 = 0$, where it takes the value (5.7).

5.3 Asymptotics and non-perturbative sectors

In principle, one should be able to refine the asymptotic formula (5.18) and obtain a generalization of (5.6) involving the g_s expansion of instanton solutions. A natural guess is that the relevant instanton solutions are the closest ones to the given background, i.e. the instanton amplitudes (5.15) with $\ell = \pm 1$. This guess would relate the large genus behavior of $F_g(t_1, t_2)$ to an integral of the form (5.4), involving this time $F^{(1)}$ and $F^{(-1)}$. However, this expectation turns out to be too naive. Indeed, it seems that the asymptotics involves *new non-perturbative sectors* whose matrix model interpretation is yet unknown.

In order to explain this in some detail, we will come back to a simplest case where the asymptotics can be fully determined, namely the Painlevé I equation and its instanton solutions $u_\ell(z)$. It is natural to ask what is the asymptotics of the coefficients $u_{n,\ell}$ appearing in (5.11). Notice that, when ℓ is big, this instanton solution is the double-scaled limit of a
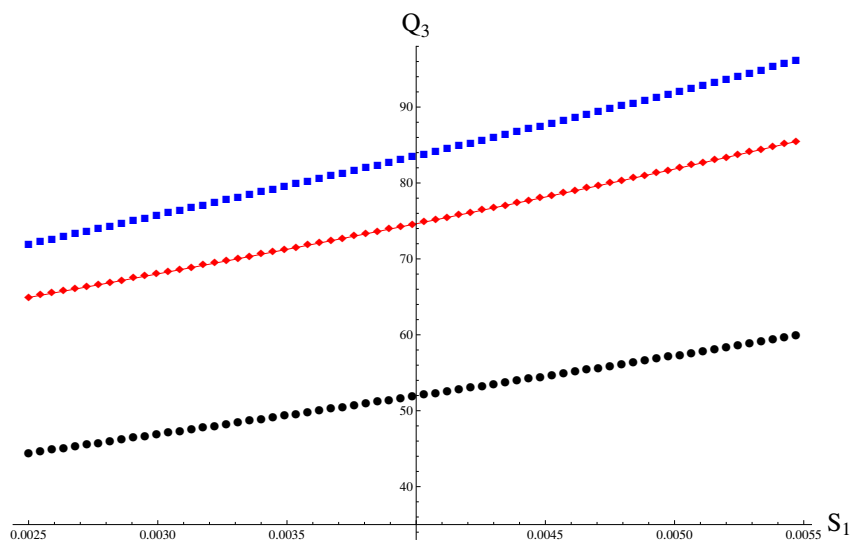
Figure 4. Q_3 ($\bullet$), $R_Q(1,2)$ ($\blacksquare$) and $|A|^{-2}$ ($\blacklozenge$) are plotted for several values of S_1 around the slice point $S_1 = -S_2 = S = 0.004$. Q_3 and $R_Q(1,2)$ have a relative error of about 30 % and 10 %, respectively, as compared to the instanton action $|A|^{-2}$ throughout the data set.
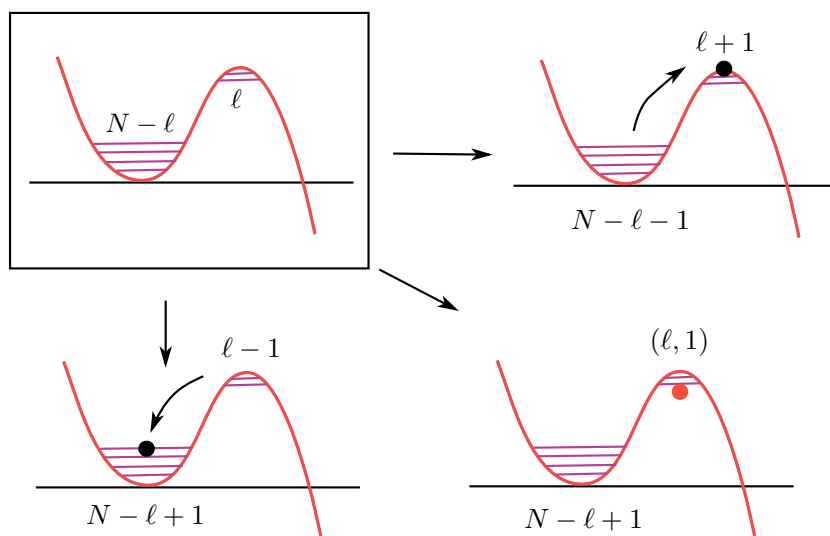

Figure 5. The asymptotics of the coefficients of the ℓ-th instanton solution $u_\ell(z)$ of Painlevé I is determined by the two nearest neighbor instantons, which are obtained by eigenvalue tunneling, and by the generalized instanton amplitude $u_{\ell|1}$, which is represented here by the label $(\ell, 1)$.

two-cut solution, therefore the question of the asymptotics of this sequence is closely related to the original question concerning the asymptotics (5.18). We have seen in (5.13) that the asymptotics of the perturbative solution is governed by the one-instanton solution. In the same way, one would think that the asymptotics of the ℓ-instanton solution is governed

by the $\ell \pm 1$ instanton amplitudes. It has been shown in [26] that this is not the case. In order to understand the asymptotics of a generic instanton sector, one has to consider more general amplitudes, labelled by two non-negative integers:

$$u_{n|m}(z). \tag{5.25}$$

The amplitude where $m = 0$ is the standard instanton amplitude: $u_{\ell|0}(z) = u_\ell(z)$. The other amplitudes can be obtained by requiring

$$u(z, C_1, C_2) = \sum_{n,m \geq 0} u_{n|m}(z) C_1^n C_2^m \tag{5.26}$$

to be a formal solution to the Painlevé I equation, for arbitrary C_1, C_2, and that

$$u_{n|m}(z) \sim e^{-(n-m)az^{5/4}}, \qquad z \to \infty. \tag{5.27}$$

The two-parameter solution of the Painlevé I equation (5.26) is called a *trans-series solution*, and it was introduced by Jean Écalle in the context of resurgent analysis (see for example [48] for an simple introduction to resurgence). It turns out that the asymptotic behavior of the coefficients $u_{n,\ell}$ in the ℓ-th instanton u_ℓ is governed by the solutions $u_{\ell \pm 1}(z)$, but also by the solution $u_{\ell|1}(z)$. This means that the asymptotics of the coefficients $u_{n,\ell}$ as $n \to \infty$ can be obtained by a relation similar to (5.4), but involving $u_{\ell \pm 1}(z)$ as well as $u_{\ell|1}(z)$. For example, let us consider the one-instanton solution, and let us ask what is the asymptotics of the coefficients $u_{n,1}$ appearing in (5.11) with $\ell = 1$. An analysis based on resurgence theory, which can be verified with Riemann-Hilbert techniques, leads to the formula [26]

$$u_{n,1} \sim_n a^{-n+1/2} \frac{S_1}{2\pi i} \Gamma(n - 1/2) \left\{ 2u_{0,2} + (-1)^n \mu_{0,2} + \sum_{l=1}^{\infty} \frac{(2u_{l,2} + (-1)^{n+l}\mu_{l,2})a^l}{\prod_{m=1}^{l}(n - 1/2 - m)} \right\} \tag{5.28}$$

where $u_{n,2}$ are the coefficients of the two-instanton expansion ($\ell = 2$) in (5.11), and $\mu_{n,2}$ are the coefficients of the function

$$u_{1|1}(z) = z^{-3/4} \sum_{n \geq 0} \mu_{n,2} z^{-5n/4}. \tag{5.29}$$

It can be seen, by plugging (5.26) in the Painlevé I equation, that this function satisfies the linear inhomogeneous ODE

$$-\frac{1}{6} u''_{1|1} + 2u_0 u_{1|1} + 2u_1 u_{0|1} = 0. \tag{5.30}$$

There are similar, but more complicated, formulae for the asymptotic behavior of the coefficients $u_{n,\ell}$ for arbitrary ℓ, see [26]. They all involve the trans-series solutions $u_{\ell|1}(z)$.

The instanton amplitude $u_{\ell+1}(z)$ can be obtained from the solution $u_\ell(z)$ by tunneling one extra eigenvalue to the unstable saddle, while the amplitude $u_{\ell-1}(z)$ can be obtained from $u_\ell(z)$ by tunneling one eigenvalue back to the stable saddle. The amplitude $u_{\ell|1}(z)$
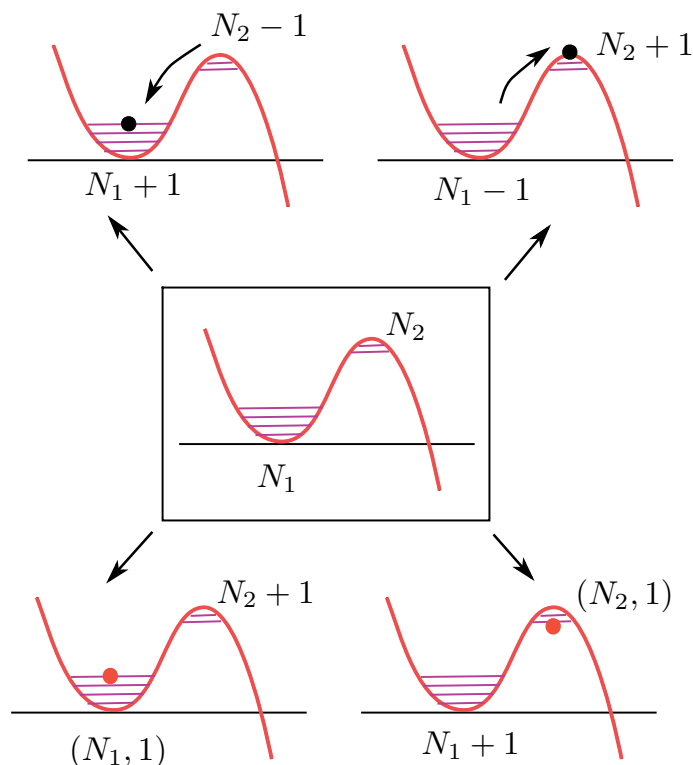
Figure 6. In the two-cut case, given a background with perturbative amplitudes $F_g(t_1, t_2)$, there are two instantons which are obtained by eigenvalue tunneling, and two generalized instanton amplitudes represented by $(N_1, 1)$, $(N_2, 1)$.

does not seem to have, however, an eigenvalue interpretation of this type. The different non-perturbative sectors governing the asymptotics of the ℓ-th instanton solution are depicted in figure 5.

We can now come back to the original problem of determining the large order behavior of $F_g(S_1, S_2)$, corresponding to a two-cut model with N_1, N_2 eigenvalues around the two saddles. There are two instanton configurations which are obtained by eigenvalue tunneling, with fillings $N_1 \pm 1$, $N_2 \mp 1$. It can be seen that the subleading terms in the asymptotics are not reproduced by using just these two configurations. This is not surprising, in view of the result for the instantons of Painlevé I. The above analysis, based on [26], suggests in fact that there should be two other non-perturbative configurations in the two-cut matrix model, that we denote by $(N_1, 1)$ and $(N_2, 1)$, in analogy with our notation in figure 5. These configurations are depicted at the bottom of figure 6. The instantons obtained by eigenvalue tunneling can be easily calculated from the free energy of the generic two-cut matrix model. However, we do not know how to compute the amplitudes involving these new configurations, since there is no analogue of the Painlevé I equation (or the pre-string equation) for the generic two-cut matrix model. In general, it seems that the most general saddle-point of the two-cut matrix model should be labeled by two pairs of integers, (N_1, M_1), (N_2, M_2), associated to the two critical points of the cubic potential.

6 Conclusions and further directions

In this paper, building on [31], we have shown that direct integration of the holomorphic anomaly equations provides a powerful tool to calculate the $1/N$ expansion of multi-cut matrix models. We have seen that, in some circumstances, we can easily fix the holomorphic ambiguity and obtain explicit expressions for the genus g amplitudes. In general, we expect the anomaly equation to be integrable, in the sense that the gap conditions completely fix the holomorphic ambiguity. In the case of the two-cut cubic matrix model in the slice $S_1 = -S_2$, we can use this method to determine the amplitudes to very high genus.

These high genus results have allowed us to obtain quantitative evidence for the connection between large order behavior and eigenvalue tunneling in a multi-cut matrix model. However, our results indicate that the detailed large genus asymptotics of the amplitudes cannot be understood just by considering the non-perturbative sectors associated with eigenvalue tunneling. Indeed, in a similar asymptotic problem analyzed in [26], it was necessary to include new non-perturbative sectors. It is only natural to suggest that a correct understanding of the asymptotic properties, in the multi-cut case, requires also the inclusion of new non-perturbative sectors. In the one-cut case and its double-scaling limit, the amplitudes in these new sectors can be obtained algebraically, as trans-series solutions to the pre-string equation and the Painlevé I equation, respectively. In the multi-cut case there is no analogue of these equations, and therefore the corresponding generalized amplitudes can not be computed with our present tools.

One obvious question is then the following: what is the interpretation of these new non-perturbative sectors in terms of matrix models or topological strings? We will give now some hints which might help in answering this question. Let us first discuss the trans-series solutions $u_{n|m}(z)$ appearing in (5.26). It turns out that $u_{0|\ell}(z)$ can be obtained from $u_{\ell|0}(z)$, the standard instanton amplitude, by changing the sign

$$z^{5/4} \rightarrow -z^{5/4}. \tag{6.1}$$

This corresponds to changing the sign of the string coupling constant $g_s \rightarrow -g_s$. If we think about the $u_\ell(z)$ as describing a set of ℓ D-branes, then the natural interpretation of $u_{0|\ell}(z)$ is as a set of ℓ *anti-D-branes*. Indeed, it has been argued that anti-D-branes are obtained from D-branes in topological string theory just by changing the sign of the string coupling constant [52]. More generally, these should be the ghost D-branes introduced in [44], which reduce to anti-D-branes in the topological string context. It is then natural to interpret the generalized instanton amplitude $u_{n|m}(z)$ as representing a state of n D-branes and m anti-D-branes at the unstable saddle, in the background of $N-n+m$ D-branes in the stable saddle. If this interpretation is correct, the generalized amplitudes in the multi-cut matrix model, which we labeled by two pairs of integers (N_1, M_1), (N_2, M_2), should correspond to a saddle where there are N_i branes and M_i anti-D-branes at the i-th critical point, $i = 1, 2$.

One problem with this interpretation is that, as argued in [20, 52], such a configuration is described in principle by a quiver or supergroup matrix model. If this is the case, the non-perturbative configuration characterized by (N_i, M_i), $i = 1, 2$, would be equivalent to a configuration with only branes or only antibranes at the critical points. More pre-

cisely, we would get $|N_i - M_i|$ branes or $|N_i - M_i|$ anti-branes depending on the sign of $N_i - M_i$. Since explicit calculations show that the amplitude $u_{n|m}(z)$ is not equal to the amplitude $u_{n-m|0}(z)$ [26], the interpretation in terms of brane/anti-brane systems might not be completely appropriate.

We believe that the appearance of these new sectors indicates that we do not fully understand the non-perturbative structure of matrix models and of two-dimensional gravity. Therefore, it would be very important to clarify their meaning and to compute their amplitudes in the multi-cut case.

Acknowledgments

A Data of the two-cut example

In the following we collect the necessary data for our two-cut cubic model of the main body text. We restrict ourselves to the points in moduli space which are relevant for our discussion. For further background on e.g. the monodromy around several divisors in moduli space we refer the reader to [31].

A.1 Large Radius

$C_1 \cap C_2 = \{z_1 = 0\} \cap \{z_2 = 0\}$: the Picard-Fuchs operators governing the periods of the cubic matrix model are given by

$$
\begin{aligned}
\mathcal{L}_1 = {}& (3 - 2z_1 - 6z_2)\partial_1 - 2z_1(1 - 2z_1 - 6z_2)\partial_1^2 + (1 - 10z_1 + 12z_1^2 + 4z_1z_2)\partial_1\partial_2 \\
& + (3 - 6z_1 - 2z_2)\partial_2 + (1 - 10z_2 + 4z_1z_2 + 12z_2^2)\partial_1\partial_2 - 2z_2(1 - 6z_1 - 2z_2)\partial_2^2, \\
\mathcal{L}_2 = {}& -3(1 - 12z_1 + 18z_1^2 + 14z_1z_2) + (-3z_2(1 - 3z_2 + 2z_2^2) \\
& + z_1(7 + 46z_1^2 - 18z_2 + 26z_2^2 + z_1(-39 + 62z_2)))\partial_1 \\
& + (-1 + 2z_1 + 2z_2)(-2z_1(1 + 5z_1^2 - 2z_1z_2 - 3z_2^2 - 4(z_1 + z_2))\partial_1^2 \\
& + (z_1 + z_2)(1 - 8z_1 + 6z_1^2 - 6z_1z_2)\partial_1\partial_2) \\
& - 3(1 - 12z_2 + 14z_1z_2 + 18z_2^2) + (-3z_1(1 - 3z_1 + 2z_1^2) \\
& + z_2(7 - 18z_1 + 26z_1^2 + (-39 + 62z_1)z_2 + 46z_2^2))\partial_2 \\
& + (-1 + 2z_1 + 2z_2)((z_1 + z_2)(1 - 8z_2 - 6z_1z_2 + 6z_2^2)\partial_1\partial_2 \\
& - 2z_2(1 - 3z_1^2 - 2z_1z_2 + 5z_2^2 - 4(z_1 + z_2))\partial_2^2).
\end{aligned}
\tag{A.1}
$$

Its discriminant can be determined to be

$$
\mathrm{disc} = z_1z_2I^2J = z_1z_2(1 - 2(z_1 + z_2))(1 - 6z_1 - 6z_2 + 9z_1^2 + 14z_1z_2 + 9z_2^2), \tag{A.2}
$$

and its solutions around $z_i = 0$, $i = 1, 2$, are given by the following expansions

$$
\begin{aligned}
S_1 &= \frac{z_1}{4} - \frac{1}{8}z_1(2z_1 + 3z_2) + \ldots \\
S_2 &= -\frac{z_1}{4} + \frac{1}{8}z_2(3z_1 + 2z_2) + \ldots \\
\Pi_1 &= S_1 \log\left(\frac{z_1}{4}\right) + \frac{1}{12} - \frac{z_1}{4} - \frac{1}{16}(2z_1^2 - 10z_1z_2 - 5z_2^2) + \ldots \\
\Pi_2 &= S_2 \log\left(-\frac{z_2}{4}\right) - \frac{1}{12} + \frac{z_2}{4} - \frac{1}{16}(5z_1^2 + 10z_1z_2 - 2z_2^2) + \ldots .
\end{aligned}
\tag{A.3}
$$

The Yukawa couplings are given by

$$
\begin{aligned}
C_{z_1z_1z_1} &= \frac{1 - 6z_1 + 9z_1^2 - 5z_2 + 9z_1z_2 + 6z_2^2}{16z_1 I^2} \\
C_{z_1z_1z_2} &= \frac{1 - 3z_1 - 5z_2}{16 I^2} \\
C_{z_1z_2z_2} &= \frac{1 - 5z_1 - 3z_2}{16 I^2} \\
C_{z_2z_2z_2} &= \frac{1 - 5z_1 + 6z_1^2 - 6z_2 + 9z_1z_2 + 9z_2^2}{16z_2 I^2},
\end{aligned}
\tag{A.4}
$$

where all other combinations follow by symmetry. The genus one free energy can be written as

$$
F_1 = -\frac{1}{2}\log\left(\det(G_{i\bar{j}})\right) - \frac{1}{12}\log(z_1z_2) - \frac{1}{2}\log I + \frac{1}{3}\log J. \tag{A.5}
$$

It is convenient to introduce new variables $\tilde{z}_i$, $i = 1, 2$, by

$$
\tilde{z}_1 = z_1 + z_2, \quad \tilde{z}_2 = \frac{1}{4}(z_1 - z_2)\sqrt{1 - 2(z_1 + z_2)}, \tag{A.6}
$$

as well as coordinates $\tilde{t}_i$, $i = 1, 2$, on the mirror by

$$
\tilde{t}_1 = s = \frac{1}{2}(S_1 - S_2), \quad \tilde{t}_2 = t = S_1 + S_2, \tag{A.7}
$$

such that the mirror map becomes as simple as possible. E.g. we have that

$$
\tilde{z}_2 = \tilde{t}_2. \tag{A.8}
$$

This implies that some of the Christoffel symbols vanish:

$$
\Gamma^{\tilde{z}_2}_{\tilde{z}_i \tilde{z}_j} = 0, \quad \text{for} \quad i = 1, 2. \tag{A.9}
$$

There are only four non-vanishing ambiguities $\tilde{f}^k_{ij}$ of equation (2.12), that are given by

$$
\begin{aligned}
\tilde{f}^{\tilde{z}_1}_{\tilde{z}_1\tilde{z}_1} &= -\frac{5 - 28\tilde{z}_1 + 52\tilde{z}_1^2 - 32\tilde{z}_1^3 - 112\tilde{z}_2^2}{2(1 - 2\tilde{z}_1)(1 - 8\tilde{z}_1 + 20\tilde{z}_1^2 - 16\tilde{z}_1^3 + 16\tilde{z}_2^2)}, \\
\tilde{f}^{\tilde{z}_1}_{\tilde{z}_1\tilde{z}_2} &= \frac{24\tilde{z}_2}{1 - 8\tilde{z}_1 + 20\tilde{z}_1^2 - 16\tilde{z}_1^3 + 16\tilde{z}_2^2}, \\
\tilde{f}^{\tilde{z}_1}_{\tilde{z}_2\tilde{z}_2} &= \frac{8 - 16\tilde{z}_1}{1 - 8\tilde{z}_1 + 20\tilde{z}_1^2 - 16\tilde{z}_1^3 + 16\tilde{z}_2^2}.
\end{aligned}
\tag{A.10}
$$

This results in a propagator that has one non-vanishing component in $\tilde{z}$-coordinates, i.e.

$$S^{\tilde{z}_1\tilde{z}_1} = 4\tilde{z}_1^2 - 64\tilde{z}_2^2 + 44\tilde{z}_1^3 - 832\tilde{z}_1\tilde{z}_2^2 + \dots, \qquad S^{\tilde{z}_1\tilde{z}_2} = S^{\tilde{z}_2\tilde{z}_1} = S^{\tilde{z}_2\tilde{z}_2} = 0. \qquad (A.11)$$

The covariant derivative closes on this propagator when one fixes yet another ambiguity f_k^{ij}, cf. eq. (2.11). The only relevant, non-vanishing component is given by

$$f_{\tilde{z}_1}^{\tilde{z}_1\tilde{z}_1} = \frac{8(1 - 2\tilde{z}_1)^3(\tilde{z}_1 - 4\tilde{z}_1^2 + 4\tilde{z}_1^3 - 64\tilde{z}_2^2 + 144\tilde{z}_1\tilde{z}_2^2)}{(1 - 8\tilde{z}_1 + 20\tilde{z}_1^2 - 16\tilde{z}_1^3 + 16\tilde{z}_2^2)^3}. \qquad (A.12)$$

A.2 Conifold

Conifold $J = \{1 - 6z_1 - 6z_2 + 9z_1^2 + 14z_1z_2 + 9z_2^2 = 0\}$: we consider the point $(z_1, z_2) = (\frac{1}{8}, \frac{1}{8}) \in J$. Convenient coordinates are given by

$$z_{c,1} = \frac{1}{\sqrt{2}}(z_1 - z_2), \qquad z_{c,2} = 1 - 4(z_1 + z_2). \qquad (A.13)$$

$z_{c,1}$ parametrizes the tangential direction to the conifold divisor, whereas $z_{c,2}$ the normal one. Transforming the Picard-Fuchs system to these new coordinates the polynomial solutions are given by

$$\begin{aligned}
\omega_1 &= z_{c,1}\sqrt{1 + z_{c,2}} = z_{c,1} + \frac{1}{2}z_{c,1}z_{c,2} - \frac{1}{8}z_{c,1}z_{c,2}^2 + \mathcal{O}(z_c^4), \\
\omega_2 &= z_{c,2}^2 + 8z_{c,1}^2 z_{c,2} + \mathcal{O}(z_c^4).
\end{aligned} \qquad (A.14)$$

We choose as flat coordinates

$$t_{c,i} = \omega_i, \quad i = 1, 2. \qquad (A.15)$$

By Inverting the above relations it is easy to calculate the holomorphic limit of the metric and the Christoffel symbols in z_c coordinates. Transforming the Yukawa couplings C_{ijk} as well as the ambiguities $\tilde{f}_{ij}^k$ yields the propagator at the conifold point. This allows now to expand the free energies F_g in the holomorphic limit at the conifold point.

B The one-cut solution

A special case of the multi-cut matrix model occurs when all the partial 't Hooft parameters are zero except for one, $S_2 = \dots = S_n = 0$. This is called the *one-cut* matrix model. A powerful, recursive solution of the one-cut matrix model at all genera has been known for a long time [7, 8], and it is based on the technique of orthogonal polynomials. In this appendix we list some ingredients of the one-cut solution.

We first recall that the orthogonal polynomials $p_n(\lambda)$ for the potential $V(\lambda)$ are defined by

$$\int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-\frac{V(\lambda)}{g_s}} p_n(\lambda)p_m(\lambda) = h_n \,\delta_{nm}, \qquad n > 0, \qquad (B.1)$$

where p_n are normalized by requiring that $p_n \sim \lambda^n + \cdots$. It is well known (see for example [17]) that the partition function of the matrix model can be expressed as

$$Z_N = \prod_{i=0}^{N-1} h_i = h_0^N \prod_{i=1}^{N} r_i^{N-i}. \qquad (B.2)$$

The coefficients

$$r_n = \frac{h_n}{h_{n-1}} \tag{B.3}$$

satisfy recursion relations depending on the shape of the potential. They also obviously satisfy

$$r_n = \frac{Z_{n-1} Z_{n+1}}{Z_n^2}. \tag{B.4}$$

In the limit $N \to \infty$, n/N becomes a continuous variable that we will denote by z, and r_n is promoted to a function, $R(z, g_s)$. The perturbative g_s expansion is obtained by writing

$$R(z, g_s) = \sum_{\ell=0}^{\infty} R_\ell(z) g_s^{2\ell}. \tag{B.5}$$

We now consider the cubic matrix model with potential

$$\frac{1}{g_s} W(z) = \frac{1}{g_s} \left(-z + \frac{z^3}{3} \right). \tag{B.6}$$

The resulting matrix model is equivalent to the one we considered in the bulk of the paper, up to a linear transformation $z \to az + b$, but leads to a simple recursion relation for the coefficients r_n (see for example [17])

$$r_n \left(\sqrt{1 - r_n - r_{n+1}} + \sqrt{1 - r_n - r_{n-1}} \right) = g_s n. \tag{B.7}$$

The continuum limit of equation (B.7) above is

$$R(t, g_s) \left(\sqrt{1 - R(t, g_s) - R(t + g_s, g_s)} + \sqrt{1 - R(t, g_s) - R(t - g_s, g_s)} \right) = t. \tag{B.8}$$

At lowest order in ℓ and g_s we find

$$2R_0(t) \sqrt{1 - 2R_0(t)} = t. \tag{B.9}$$

It turns out to be convenient to express everything in terms of

$$U(t, g_s) = \sqrt{1 - 2R(t, g_s)} \tag{B.10}$$

This has a g_s expansion of the form

$$U(t, g_s) = \sum_{\ell \geq 0} U_\ell(t) g_s^{2g}. \tag{B.11}$$

Notice that

$$u \equiv U_0(t) \tag{B.12}$$

satisfies

$$u(1 - u^2) = t, \tag{B.13}$$

and one chooses the solution with expansion

$$u = 1 - \frac{t}{2} - \frac{3t^2}{2} - \cdots \tag{B.14}$$

We also have the inverse relation

$$R(t, g_s) = \frac{1}{2}(1 - U^2(t, g_s)). \tag{B.15}$$

This variable eliminates square roots. The recursion becomes

$$U(1 - U^2)\left\{\sqrt{1 + \left(\frac{U^+}{U}\right)^2} + \sqrt{1 + \left(\frac{U^-}{U}\right)^2}\right\} = 2\sqrt{2}t \tag{B.16}$$

where

$$U^{\pm} = U(t \pm g_s, g_s). \tag{B.17}$$

Expanding the equation (B.16) in power series of g_s, and solving it recursively, one can compute in this way the coefficients $U_n(u)$, and from them derive the coefficients R_ℓ. The first few read, when expressed in terms of

$$r = \frac{1 - u^2}{2} \tag{B.18}$$

as

$$R_1(t) = -\frac{(9r - 5)}{32(1 - 3r)^4}, \qquad R_2(t) = -\frac{3\left(162r^3 + 1017r^2 - 1316r + 385\right)}{2048(3r - 1)^9}. \tag{B.19}$$

Once $R(z, g_s)$ are calculated, one can easily calculate the total free energy

$$F(t, g_s) = \sum_{g \geq 0} g_s^{2g-2} F_g(t). \tag{B.20}$$

Let us define

$$\Xi(z, g_s) = \frac{R(z, g_s)}{z} \tag{B.21}$$

One then obtains [7, 8]:

$$g_s^2 F = \int_0^t dz\,(t - z)\log\Xi(z) + \sum_{p=1}^{\infty} g_s^{2p} \frac{B_{2p}}{(2p)!} \frac{d^{2p-1}}{dz^{2p-1}}\left[(t - z)\log\Xi(z, g_s)\right]\Big|_{z=0}^{z=t}$$
$$+ \frac{tg_s}{2}\left[2\log\frac{h_0}{h_0^G} - \log\Xi(0, g_s)\right]. \tag{B.22}$$

Here, h_0/h_G is the integral

$$\frac{h_0}{h_G} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/g_s - x^3/(3g_s)} dx \tag{B.23}$$

understood as a formal power series in g_s. This can be explicitly calculated

$$\frac{h_0}{h_G} = \frac{1}{\pi^{1/2}} \sum_{k \geq 0} \frac{\Gamma\left(\frac{1}{2} + 3k\right)}{3^{2k}(2k)!} g_s^k. \tag{B.24}$$

Notice that, as pointed out in [8] in the case of a quartic potential, the g_s expansion of

$$2\log\frac{h_0}{h_0^G} - \log\Xi(0, g_s) \tag{B.25}$$

contains only odd powers of g_s, as required in order to have an expansion of the free energy in terms of even powers of g_s. Notice that the explicit solution for the F_g in the one cut case provides an additional boundary condition for the holomorphic anomaly of matrix models.

C Modular forms and elliptic integrals

We follow the conventions in [11]. The complete elliptic integral of the first kind is defined as

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{C.1}$$

The parameter k is called the elliptic modulus. Further one defines the complementary modulus as $k'^2 = 1 - k^2$. The complete elliptic integral of the second kind is defined as

$$E(k) = \int_0^1 \mathrm{d}t \sqrt{\frac{1-k^2t^2}{1-t^2}}. \tag{C.2}$$

The complete elliptic integrals of the first and second kind are related to each other by derivation,

$$\frac{\mathrm{d}K}{\mathrm{d}k} = \frac{E(k) - k'^2 K(k)}{kk'^2}, \qquad \frac{\mathrm{d}E}{\mathrm{d}k} = \frac{E(k) - K(k)}{k}. \tag{C.3}$$

Useful transformation formulae are

$$K\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2} K(k),$$

$$E\left(\frac{1-k'}{1+k'}\right) = \frac{1}{1+k'}(E(k) + k'K(k)), \tag{C.4}$$

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)K(k),$$

as well as the Legendre relation

$$E(k)K(k') + E(k')K(k) - K(k)K(k') = \frac{\pi}{2}. \tag{C.5}$$

Consider an elliptic geometry of the form

$$y^2 = \prod_{i=1}^4 (x - x_i), \tag{C.6}$$

where $x_1 < x_2 < x_3 < x_4$ are the branch cuts. Define the half-period ratio of the elliptic geometry, τ, and the elliptic nome as $q = e^{2\mathrm{i}\pi\tau}$. It can be shown that

$$\tau = \mathrm{i}\frac{K(k')}{K(k)}, \tag{C.7}$$

and moreover that

$$K(k) = \frac{\pi}{2}\vartheta_3^2, \quad k^2 = \frac{\vartheta_2^4}{\vartheta_3^4}, \quad k'^2 = \frac{\vartheta_4^4}{\vartheta_3^4}. \tag{C.8}$$

Here ϑ_i are the Jacobi theta-functions defined by

$$\vartheta_2 = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}, \quad \vartheta_3 = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}, \quad \vartheta_4 = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2}. \tag{C.9}$$

The Thomae formula relates the branch cuts of an elliptic curve to theta-functions [25]. For the geometry consider above (C.6) we obtain

$$
\begin{aligned}
\vartheta_2^4(\tau) &= -\mathcal{K}^2(x_1 - x_2)(x_3 - x_4) \\
\vartheta_3^4(\tau) &= -\mathcal{K}^2(x_1 - x_4)(x_2 - x_3) \\
\vartheta_4^4(\tau) &= -\mathcal{K}^2(x_1 - x_3)(x_2 - x_4),
\end{aligned} \tag{C.10}
$$

and thus

$$
\eta^{24}(\tau) = \frac{\mathcal{K}^{12}}{256} \prod_{i<j}(x_i - x_j)^2, \tag{C.11}
$$

where $\mathcal{K}$ and $\mathcal{K}'$ are given in (3.46).

It is convenient to introduce

$$
b = \vartheta_2^4, \quad c = \vartheta_3^4, \quad d = \vartheta_4^4, \tag{C.12}
$$

where either two of them span the ring of $\Gamma(2)$ modular forms. Here the congruence subgroup $\Gamma(2) \subset \mathrm{SL}(2, \mathbb{Z})$ is defined by

$$
\Gamma(2) = \{\gamma \in \mathrm{SL}(2, \mathbb{Z}) \,|\, \gamma \equiv \mathbb{1} \bmod 2\}. \tag{C.13}
$$

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