

**DIRECT LOCAL AND GLOBAL APPROXIMATION  
THEOREM FOR BERNSTEIN - TYPE OPERATORS**

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ABSTRACT. Direct local and global estimates are established for Bernstein - type operators using Ditzian - Totik modulus of smoothness of second order.

**1. Introduction**

In [2] it was shown, among others, that for the Bernstein operator

$$B_n(f, x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

the estimate

$$(1) \quad |B_n(f, x) - f(x)| \leq C_1 \cdot \omega_\phi^2\left(f, n^{-1/2} \frac{\varphi(x)}{\phi(x)}\right)$$

holds true for  $x \in [0, 1]$ ,  $\varphi(x) = \sqrt{x(1-x)}$  and  $f \in C[0, 1]$ , where the Ditzian - Totik modulus of second order is given by

$$\omega_\phi^2(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \pm h\phi(x) \in [0, 1]} |f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x))|,$$

in which  $\phi : [0, 1] \rightarrow \mathbf{R}$  is an admissible step - weight function of the Ditzian - Totik modulus ( see [1, p. 8] ) such that  $\phi^2$  is a concave function.

The aim of the paper is to give a direct local and global approximation theorem for Bernstein - type operators, similar to ( 1 ), replacing the condition  $\phi^2$  is concave function.

In what follows we suppose that  $\phi : [0, 1] \rightarrow \mathbf{R}$  is an admissible step - weight function ( see [1, Section 1.2] ) such that  $\beta = \beta(0) = \beta(1) \in [0, 1/2]$  and  $\phi(x) \sim x^\beta$  as  $x \rightarrow 0+$ ,  $\phi(x) \sim (1-x)^\beta$  as  $x \rightarrow 1-$ . Then, by [1, p. 24, Theorem 3.1.2 ] we have that the K - functional  $\bar{K}_{2,\phi}(f, \delta^2)$  and the Ditzian - Totik modulus  $\omega_\phi^2(f, \delta)$  are equivalent. Here

$$\bar{K}_{2,\phi}(f, \delta^2) = \inf_{g' \in A.C._{loc}[0,1]} \left\{ \|f - g\| + \delta^2 \|\phi^2 g''\| + \delta^{2/(1-\beta)} \|g''\| \right\}$$

and  $\|\cdot\|$  denotes the uniform norm on  $[0, 1]$ .

2000 *Mathematics Subject Classification.* 41A25, 41A36.

*Key words and phrases.* Quasi - convex function, the second modulus of smoothness of Ditzian - Totik, Bernstein - type operators.

<sup>1</sup>Received: May 1, 2003

Furthermore, we need the following definition : the function  $\psi : [0, 1] \rightarrow \mathbf{R}$  is *quasi - convex* if for every  $x_1, x_2 \in [0, 1]$  and  $v$  between  $x_1$  and  $x_2$  we have  $\psi(v) \leq \max \{ \psi(x_1), \psi(x_2) \}$ .

## 2. The direct approximation theorem

Our main result is the following

**Theorem 1.** *Let  $L_n : C[0, 1] \rightarrow C[0, 1]$  be a sequence of uniformly bounded positive linear operators such that  $L_n(u - x, x) \equiv 0$  and  $L_n((u - x)^2, x) \leq C_2 \cdot \frac{\varphi^2(x)}{n}$ ,  $x \in [0, 1]$ ,  $n = 1, 2, \dots$ . If there exists  $\lambda \in [0, 1]$  with the properties*

- (i) *the function  $\psi(x) = \frac{x^\lambda(1-x)^\lambda}{\phi^2(x)}$  is quasi - convex on  $[0, 1]$  and*
- (ii)  *$L_n \left( \frac{(u-x)^2}{x^\lambda(1-x)^\lambda} \cdot \left( \frac{x^\lambda(1-x)^\lambda}{\phi^2(x)} + \frac{u^\lambda(1-u)^\lambda}{\phi^2(u)} \right), x \right) \leq \frac{C_3}{n} \cdot \frac{\varphi^2(x)}{\phi^2(x)}$  for  $x \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right]$*

*then the pointwise approximation*

$$(2) \quad |L_n(f, x) - f(x)| \leq C \omega_\phi^2 \left( f, n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right)$$

*holds true for  $x \in [0, 1]$ ,  $n = 1, 2, \dots$  and  $f \in C[0, 1]$ , where the constant  $C$  depends only on  $\phi$ .*

**Remark 1.**  $C_1, C_2, C_3, C_4, C_5, C_6$  denote absolute positive constants and  $C > 0$  is an absolute constant which depends only on  $\phi$ . The value of  $C$  may vary with each occurrence.

To prove our theorem we need the following two lemmas :

**Lemma 1.** *For  $x, u \in (0, 1)$ ,  $\lambda \in [0, 1]$  and  $v$  between  $x$  and  $u$  we have*

$$\frac{|u - v|}{v^\lambda(1 - v)^\lambda} \leq \frac{|u - x|}{x^\lambda(1 - x)^\lambda}.$$

*Proof.* If  $x \leq v \leq u$  then  $1/v^\lambda \leq 1/x^\lambda$  and the function  $h(v) = (u - v)/(1 - v)^\lambda$  is decreasing on  $[x, u]$ . So  $h(v) \leq h(x)$ . Hence  $h(v)/v^\lambda \leq h(x)/x^\lambda$ . Analogously, if  $u \leq v \leq x$  then  $1/(1 - v)^\lambda \leq 1/(1 - x)^\lambda$  and the function  $h(v) = (v - u)/v^\lambda$  is increasing on  $[u, x]$ . Therefore  $h(v) \leq h(x)$  and  $h(v)/(1 - v)^\lambda \leq h(x)/(1 - x)^\lambda$ , which was to be proved.

**Lemma 2.** *For  $x \in [0, 1/n) \cup (1 - 1/n, 1]$  and  $\alpha \in [0, 1]$  we have*

$$\frac{\varphi^2(x)}{n} \leq \left( \frac{\varphi^2(x)}{x^\alpha(1 - x)^\alpha} \cdot \frac{1}{n} \right)^{\frac{2}{2-\alpha}}.$$

*Proof.* We have to prove that

$$\frac{x(1 - x)}{n} \leq \left( x^{1-\alpha}(1 - x)^{1-\alpha} \cdot \frac{1}{n} \right)^{\frac{2}{2-\alpha}}.$$

This is equivalent with

$$\left(x(1-x) \cdot \frac{1}{n}\right)^{1-\frac{\alpha}{2}} \leq x^{1-\alpha}(1-x)^{1-\alpha} \cdot \frac{1}{n}$$

or  $x^{\frac{\alpha}{2}}(1-x)^{\frac{\alpha}{2}} \leq (1/n)^{\frac{\alpha}{2}}$  or  $x(1-x) \leq 1/n$ . But the last estimate is true for  $x \in [0, 1/n] \cup (1-1/n, 1]$ .

*Proof of the theorem.* Let  $x \in [1/n, 1-1/n]$  and  $g' \in A.C.loc[0, 1]$ . In view of Taylor's expansion

$$g(u) = g(x) + (u-x)g'(x) + \int_x^u (u-v)g''(v)dv$$

and  $L_n(u-x, x) \equiv 0$  we obtain

$$\begin{aligned} |L_n(g, x) - g(x)| &\leq L_n\left(\left|\int_x^u (u-v)g''(v)dv\right|, x\right) \\ &\leq L_n\left(\left|\int_x^u \frac{|u-v|}{\phi^2(v)} \cdot \phi^2(v)|g''(v)|dv\right|, x\right) \\ &\leq L_n\left(\left|\int_x^u \frac{|u-v|}{\phi^2(v)}dv\right|, x\right) \cdot \|\phi^2g''\| \\ (3) \quad &= L_n\left(\left|\int_x^u \frac{|u-v|}{v^\lambda(1-v)^\lambda} \cdot \frac{v^\lambda(1-v)^\lambda}{\phi^2(v)}dv\right|, x\right) \cdot \|\phi^2g''\| \end{aligned}$$

By (i), we have for  $v$  between  $x$  and  $u$  that  $\psi(v) \leq \max\{\psi(x), \psi(u)\} \leq \psi(x) + \psi(u)$ . Hence, in view of (3), Lemma 1 and (ii) we have

$$\begin{aligned} |L_n(g, x) - g(x)| &\leq \\ &\leq L_n\left(\left|\int_x^u \frac{|u-v|}{v^\lambda(1-v)^\lambda} \cdot \left(\frac{x^\lambda(1-x)^\lambda}{\phi^2(x)} + \frac{u^\lambda(1-u)^\lambda}{\phi^2(u)}\right)dv\right|, x\right) \cdot \|\phi^2g''\| \\ &\leq L_n\left(\frac{(u-x)^2}{x^\lambda(1-x)^\lambda} \cdot \left(\frac{x^\lambda(1-x)^\lambda}{\phi^2(x)} + \frac{u^\lambda(1-u)^\lambda}{\phi^2(u)}\right), x\right) \cdot \|\phi^2g''\| \\ (4) \quad &\leq \frac{C_3}{n} \cdot \frac{\varphi^2(x)}{\phi^2(x)} \cdot \|\phi^2g''\| \end{aligned}$$

Furthermore, if  $x \in [0, 1/n] \cup (1-1/n, 1]$  and  $g' \in A.C.loc[0, 1]$  then from

$$g(u) = g(x) + (u-x)g'(x) + \int_x^u (u-v)g''(v)dv$$

we obtain  $|L_n(g, x) - g(x)| \leq L_n((u-x)^2, x) \cdot \|g''\|$ . Using  $L_n((u-x)^2, x) \leq C_2 \cdot \frac{\varphi^2(x)}{n}$  and Lemma 2 for  $\alpha = 2\beta \in [0, 1]$  we get

$$\begin{aligned} |L_n(g, x) - g(x)| &\leq C_2 \cdot \frac{\varphi^2(x)}{n} \cdot \|g''\| \leq C_2 \cdot \left(\frac{\varphi^2(x)}{x^{2\beta}(1-x)^{2\beta}} \cdot \frac{1}{n}\right)^{\frac{2}{2-2\beta}} \cdot \|g''\| \\ (5) \quad &= C_2 \cdot \left(\frac{\varphi^2(x)}{\phi^2(x)} \cdot \frac{\phi^2(x)}{x^{2\beta}(1-x)^{2\beta}} \cdot \frac{1}{n}\right)^{\frac{2}{2-2\beta}} \cdot \|g''\| \end{aligned}$$

But for  $\phi$  we have  $\beta = \beta(0) = \beta(1)$ . So there exists  $C = C(\phi) > 0$  such that

$$\frac{\phi^2(x)}{x^{2\beta}(1-x)^{2\beta}} \leq C \quad \text{for } x \in \left[0, \frac{1}{n}\right) \cup \left(1 - \frac{1}{n}, 1\right].$$

Hence, by (5) we have

(6)

$$|L_n(g, x) - g(x)| \leq C \cdot \left(\frac{\varphi^2(x)}{\phi^2(x)} \cdot \frac{1}{n}\right)^{\frac{2}{2-2\beta}} \cdot \|g''\| = C \cdot \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right)^{\frac{2}{1-\beta}} \cdot \|g''\|$$

In conclusion (4) and (6) imply

$$|L_n(g, x) - g(x)| \leq$$

$$(7) \quad \leq C \cdot \left\{ \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right)^2 \cdot \|\phi^2 g''\| + \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right)^{\frac{2}{1-\beta}} \cdot \|g''\| \right\}$$

for every  $x \in [0, 1]$ .

On the other hand,  $\{L_n\}$  is a sequence of uniformly bounded operators, therefore  $\|L_n f\| \leq C_4 \|f\|$  for  $n = 1, 2, \dots$  and  $f \in C[0, 1]$ . Hence, by (7) we obtain

$$|L_n(f, x) - f(x)| \leq$$

$$\leq |L_n(f - g, x) - (f - g)(x)| + |L_n(g, x) - g(x)|$$

$$\leq (C_4 + 1) \|f - g\| + |L_n(g, x) - g(x)|$$

$$\leq C \left\{ \|f - g\| + \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right)^2 \cdot \|\phi^2 g''\| + \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right)^{\frac{2}{1-\beta}} \cdot \|g''\| \right\}$$

Using the equivalence between  $\bar{K}_{2, \phi} \left(f, \left(n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right)^2\right)$  and  $\omega_\phi^2 \left(f, n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)}\right)$  we get the assertion of our theorem.

**Remark 2.** There are functions  $\phi : [0, 1] \rightarrow \mathbf{R}$  such that

$$\psi(x) = \frac{x(1-x)}{\phi^2(x)}, \quad x \in [0, 1]$$

is quasi - convex and  $\phi^2$  is non - concave.

Indeed, the function

$$\psi(x) = \begin{cases} 1, & \text{if } x \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right] \\ -2x + \frac{3}{2}, & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right) \\ 2x - \frac{1}{2}, & \text{if } x \in \left(\frac{1}{2}, \frac{3}{4}\right) \end{cases}$$

is quasi - convex on  $[0, 1]$  and

$$\phi^2(x) = \begin{cases} x(1-x), & \text{if } x \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right] \\ \frac{x(1-x)}{-2x + \frac{3}{2}}, & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right) \\ \frac{x(1-x)}{2x - \frac{1}{2}}, & \text{if } x \in \left(\frac{1}{2}, \frac{3}{4}\right) \end{cases}$$

is not a concave function on  $[0, 1]$ , because

$$\frac{d^2}{dx^2} \left( \frac{x(1-x)}{3-4x} \right) = \frac{6(3-4x)}{(3-4x)^4} > 0$$

for  $x \in (1/4, 1/2)$ .

### 3. Applications

In this section we shall apply our theorem for some well - known positive linear operators, when  $\phi^2(x) = x^\lambda(1-x)^\lambda$ ,  $0 \leq \lambda \leq 1$ . So we obtain estimates similar to ( 2 ), which combine the classical estimate ( $\lambda = 0$ ) and the estimate developed by Ditzian - Totik ( $\lambda = 1$ ).

Let us observe that for  $\phi^2(x) = x^\lambda(1-x)^\lambda$  the conditions (i) and (ii) are satisfied. Indeed, in this case  $\psi(x) = 1$ ,  $x \in [0, 1]$ , which is obviously quasi - convex and

$$\begin{aligned} L_n \left( \frac{(u-x)^2}{x^\lambda(1-x)^\lambda} \cdot \left( \frac{x^\lambda(1-x)^\lambda}{\phi^2(x)} + \frac{u^\lambda(1-u)^\lambda}{\phi^2(u)} \right), x \right) &= \\ &= \frac{2}{x^\lambda(1-x)^\lambda} \cdot L_n((u-x)^2, x) \\ &\leq \frac{2C_2}{n} \cdot \frac{\varphi^2(x)}{x^\lambda(1-x)^\lambda} = \frac{2C_2}{n} \cdot \frac{\varphi^2(x)}{\phi^2(x)}, \end{aligned}$$

$x \in [1/n, 1 - 1/n]$ , respectively. Therefore we have the following results:

a) *Stancu operator* [5] is defined by

$$S_n^t(f, x) = \sum_{k=0}^n w_{n,k}(x, t) f \left( \frac{k}{n} \right),$$

where  $x \in [0, 1]$ ,  $f \in C[0, 1]$  and

$$w_{n,k}(x, t) = \binom{n}{k} \frac{\prod_{i=0}^{k-1} (x+it) \prod_{j=0}^{n-k-1} (1-x+jt)}{\prod_{r=0}^{n-1} (1+rt)},$$

$t$  is a positive parameter which may depends only on the natural number  $n$ . We have  $S_n^t(u-x, x) \equiv 0$  and

$$S_n^t((u-x)^2, x) = \frac{1+nt}{n(1+t)} \cdot x(1-x) \leq \frac{C_5}{n} \cdot x(1-x)$$

if  $t = t(n) = O(n^{-1})$ . So

$$|S_n^t(f, x) - f(x)| \leq C \omega_{\varphi^\lambda}^2 \left( f, n^{-1/2} \varphi^{1-\lambda}(x) \right),$$

where  $x \in [0, 1]$ ,  $f \in C[0, 1]$ ,  $t = t(n) = O(n^{-1})$ ,  $n = 1, 2, \dots$

b) *Goodman - Sharma operator* [4] is defined by

$$\begin{aligned} U_n(f, x) &= \\ &= f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 (n-1) p_{n-2,k-1}(t) f(t) dt, \end{aligned}$$

where  $x \in [0, 1]$  and  $f \in C[0, 1]$ . We have  $U_n(u - x, x) \equiv 0$  and

$$U_n((u - x)^2, x) = \frac{2}{n+1} \cdot x(1-x) \leq \frac{2}{n} \cdot x(1-x).$$

Then

$$|U_n(f, x) - f(x)| \leq C \omega_{\varphi^\lambda}^2 \left( f, n^{-1/2} \varphi^{1-\lambda}(x) \right),$$

$x \in [0, 1]$ ,  $f \in C[0, 1]$ ,  $n = 1, 2, \dots$

c) The generalized Goodman - Sharma operator was defined in [3] as follows :

$$U_n^t(f, x) =$$

$$= f(0)w_{n,0}(x, t) + f(1)w_{n,n}(x, t) + \sum_{k=1}^{n-1} w_{n,k}(x, t) \int_0^1 (n-1) p_{n-2,k-1}(t) f(t) dt,$$

where  $x \in [0, 1]$ ,  $f \in C[0, 1]$  and  $w_{n,k}(x, t)$  is given in a). We have  $U_n^t(u - x, x) \equiv 0$  and

$$U_n^t((u - x)^2, x) = \frac{2 + (n+1)t}{(n+1)(1+t)} \cdot x(1-x) \leq \frac{C_6}{n} \cdot x(1-x)$$

if  $t = t(n) = O((n+1)^{-1})$ . So

$$|U_n^t(f, x) - f(x)| \leq C \omega_{\varphi^\lambda}^2 \left( f, n^{-1/2} \varphi^{1-\lambda}(x) \right),$$

where  $x \in [0, 1]$ ,  $f \in C[0, 1]$ ,  $t = t(n) = O((n+1)^{-1})$ ,  $n = 1, 2, \dots$

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