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# Direct polynomial approach to nonlinear distance (ranging) problems

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In *GPS atmospheric sounding*, *geodetic positioning*, *robotics* and *photogrammetric* (perspective center and intersection) problems, distances (ranges) as observables play a key role in determining the unknown parameters. The measured distances (ranges) are however normally related to the desired parameters via *nonlinear equations* or *nonlinear system of equations* that require *explicit* or *exact* solutions. Procedures for solving such equations are either normally iterative, and thus require linearization or the existing analytical procedures require laborious forward and backward substitutions. We present in the present contribution *direct procedures* for solving *distance nonlinear system of equations* without linearization, iteration, forward and backward substitution. In particular, we exploit the advantage of *faster computers* with large storage capacities and the computer algebraic softwares of Mathematica, Maple and Matlab to test *polynomial based* approaches. These polynomial (algebraic based) approaches turn out to be the key to solving distance nonlinear system of equations. The algebraic techniques discussed here does not however solve all *general types* of nonlinear equations but only those *nonlinear system of equations* that can be converted into *algebraic (polynomial)* form.

**Key words:** Polynomial, Groebner basis, Multipolynomial resultant, nonlinear ranging problem.

## 1. Introduction

In GPS positioning, the use of measured distances from GPS satellites signal senders to receivers on the ground and space has enabled *accurate global sounding* of the atmosphere to retrieve vertical profiles of *temperature*, *pressure* and *density* through the measurement of excess path delay. The geodetic application of GPS focuses on accurate positioning (mm accuracy) from distance measurements while the Navigation approach requires positioning to meters accuracy. The solution of position by measuring distances is normally required for instance in Photogrammetry where the perspective center coordinates have to be evaluated from the measured photo coordinates and the ground coordinates. Another areas of application that may require direct solution of the *nonlinear* distance equations is in Robotics.

The improvement on measuring instruments has also led to Electromagnetic Distance Measuring (EDM) equipments that can give distance measurements to higher accuracies. By measuring two distances from an unknown point to two known points, two *nonlinear* distance equations whose geometrical properties have been studied by Grafarend and Schaffrin (1989, 1991) are normally solved for the coordinates of the unknown point. On the other hand, measuring three distances from an unknown point to three known points, three *nonlinear* distance equations are normally solved for the coordinates of the unknown point.

The conventional analytical approach often used to solve the *nonlinear* distance equations usually makes use of differencing and substitution techniques which are laborious and time consuming. The desire therefore is to have direct procedures akin to the *Gauss elimination technique* applied in the linear case to eliminate the unknowns in a given system of equation.

We propose in this paper the *polynomial approaches* (Awange, 2002; Awange and Grafarend, 2002a, b, c; Cox *et al.*, 1997, 1998; Strumfels, 1998) to solve directly the *nonlinear* distance equations for the coordinates of the unknown station. The polynomial approaches eliminates variables appearing in the *nonlinear* distance equations leaving univariate polynomials that are solvable using Matlab *"solve"* command for the roots. For completeness purpose, we also present the conventional approaches.

In cases where position is required from distance measurements such as for point location or for engineering survey, the procedure becomes handy. The added advantage is that the polynomial approaches are incorporated in modern algebraic software such as Mathematica. In this paper, we solve the nonlinear system of distance equations and express them in terms of *univariate polynomials*. The practitioner needs only to insert the measured and corrected distances and the coordinates of known stations to obtain his or her position. In essence, one does not need to re-invent the wheel by going back to the Mathematica software!

In case of more observations than the unknown, as is often the practise, the present algebraic techniques give way to the *Gauss-Jacobi combinatorial* approach where they are used as the *computing engine* (Awange and Grafarend, 2003). In the coming contribution, we will show you the reader, how the algebraic techniques here applied, solve in the framework of Gauss-Jacobi combinatorial the overdetermined ranging problem.

We begin by first presenting the *two-dimensional ranging*

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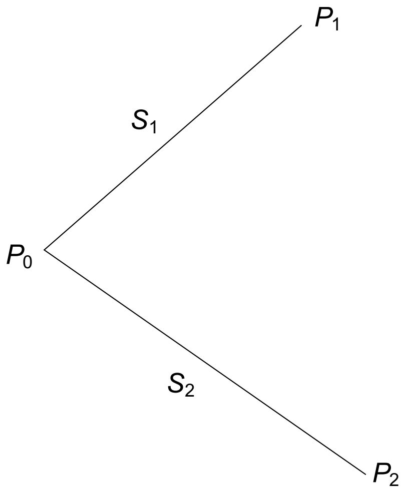


Fig. 1. Distance observations.

problem in Section 2 before presenting the three-dimensional ranging problem in Section 3. In Section 4 we consider the applications of the developed techniques and the conclude in Section 5.

## 2. Two-Dimensional Ranging

### 2.1 Conventional approach

Consider two distances $\{S_1, S_2\}$ measured from the unknown station to two known points $P_1 \in \mathrm{E}^2$ and $P_2 \in \mathrm{E}^2$ as shown in Fig. 1. The two dimensional distance ranging problem involves determining the planar coordinates $\{X_0, Y_0\}_{P_0}$ of the unknown point $P_0 \in \mathrm{E}^2$ from the observed distances $\{S_1, S_2\}$ and the planar coordinates $\{X_1, Y_1\}_{P_1}$ and $\{X_2, Y_2\}_{P_2}$ of known stations $P_1 \in \mathrm{E}^2$ and $P_2 \in \mathrm{E}^2$ respectively. The nonlinear distance equations connecting the observed distances and the planar coordinates of the known stations are expressed as

$$\left[\begin{array}{l}(X_1 - X_0)^2 + (Y_1 - Y_0)^2 = S_1^2\\ (X_2 - X_0)^2 + (Y_2 - Y_0)^2 = S_2^2\end{array}\right. \quad (2\text{-}1)$$

which on expanding leads to

$$\left[\begin{array}{l}X_1^2 + Y_1^2 - 2X_1X_0 - 2Y_1Y_0 + X_0^2 + Y_0^2 = S_1^2\\ \\ X_2^2 + Y_2^2 - 2X_2X_0 - 2Y_2Y_0 + X_0^2 + Y_0^2 = S_2^2.\end{array}\right. \quad (2\text{-}2)$$

The conventional analytic approach to solve (2-2) is to subtract the first equation (2-2i) from the second one (2-2ii) and express one unknown in terms of the other. This would lead to

$$Y_0 = -\left\{\frac{X_1 - X_2}{Y_1 - Y_2}\right\} X_0 + \frac{S_2^2 - S_1^2 + X_1^2 - X_2^2 + Y_1^2 - Y_2^2}{2(Y_1 - Y_2)} \quad (2\text{-}3)$$

which is substituted for $Y_0$ in the first equation of (2-2) to give

$$\left[\begin{array}{l}X_1^2 + Y_1^2 - 2X_1X_0 - 2Y_1\left\{-\left\{\dfrac{X_1 - X_2}{Y_1 - Y_2}\right\} X_0\right.\\ \quad + \left.\dfrac{S_2^2 - S_1^2 + X_1^2 - X_2^2 + Y_1^2 - Y_2^2}{2(Y_1 - Y_2)}\right\} + X_0^2\\ \quad + \left\{-\left\{\dfrac{X_1 - X_2}{Y_1 - Y_2}\right\} X_0 + \dfrac{S_2^2 - S_1^2 + X_1^2 - X_2^2 + Y_1^2 - Y_2^2}{2(Y_1 - Y_2)}\right\}^2\\ \quad - S_1^2 = 0.\end{array}\right. \quad (2\text{-}4)$$

On expanding and factorizing (2-4) one gets

$$\left[\begin{array}{l}(1 + a^2)X_0 + (2ab - 2X_1 - 2Y_1 a)X_0\\ \qquad + b^2 - 2Y_1 b + X_1^2 + Y_1^2 - S_1^2 = 0\\ \text{with}\\ a = -\left\{\dfrac{X_1 - X_2}{Y_1 - Y_2}\right\}\\ \text{and}\\ b = \dfrac{S_2^2 - S_1^2 + X_1^2 - X_2^2 + Y_1^2 - Y_2^2}{2(Y_1 - Y_2)}.\end{array}\right. \quad (2\text{-}5)$$

The quadratic equation (2-5) is solved for $X_0$ and substituted back in (2-3) to give the value of $Y_0$.

### 2.2 Polynomial approaches

The use of Sylvester resultant to solve this problem has already been presented in Awange and Grafarend (2002a). In this section, we present the use of *reduced Groebner basis* approaches discussed in Awange (2002).

Whereas the conventional analytical approach presented in Section 2.1 involves differencing as in (2-3) and substitution as in (2-4), the *reduced Groebner basis* technique (Awange, 2002, Cox *et al.*, 1997, 1998) solves (2-2) directly. In order to achieve this, equation (2-2) is first expressed in the algebraic form as

$$\left[\begin{array}{l}g_1 := X_1^2 + Y_1^2 - 2X_1X_0 - 2Y_1Y_0 + X_0^2 + Y_0^2 - S_1^2 = 0\\ \\ g_2 := X_2^2 + Y_2^2 - 2X_2X_0 - 2Y_2Y_0 + X_0^2 + Y_0^2 - S_2^2 = 0.\end{array}\right. \quad (2\text{-}6)$$

The *reduced Groebner basis* technique is then applied (Awange, 2002) as

$$\left[\begin{array}{l}\text{Groebner Basis}[\{g_1, g_2\}, \{X_0, Y_0\}, \{X_0\}]\\ \text{Groebner Basis}[\{g_1, g_2\}, \{X_0, Y_0\}, \{Y_0\}].\end{array}\right. \quad (2\text{-}7)$$

The first equation of (2-7) ensures that one gets a quadratic equation in $Y_0$ with $X_0$ eliminated while the second equation ensures a quadratic equation in $X_0$ with $Y_0$ eliminated as in Box 2-1.

**2.2.1 Example:** Consider that two distances $\{S_1 = 294.330\ \text{m},\ S_2 = 505.420\ \text{m}\}$ have been measured by an EDM equipment from the unknown station $P_0 \in \mathrm{E}^2$ to two known points $P_1 \in \mathrm{E}^2$ and $P_2 \in \mathrm{E}^2$ whose planar coordinates are $\{X_1 = 1207.850\ \text{m},\ Y_1 = 328.750\ \text{m}\}_{P_1}$ and $\{X_2 = 954.330\ \text{m},\ Y_2 = 925.040\ \text{m}\}_{P_2}$ respectively. The planar distance ranging problem now involves determining the planar coordinates $\{X_0, Y_0\}_{P_0}$ of the unknown point $P_0 \in \mathrm{E}^2$ from the observed distances $\{S_1, S_2\}$ and the planar coordinates of known stations $P_1 \in \mathrm{E}^2$ and

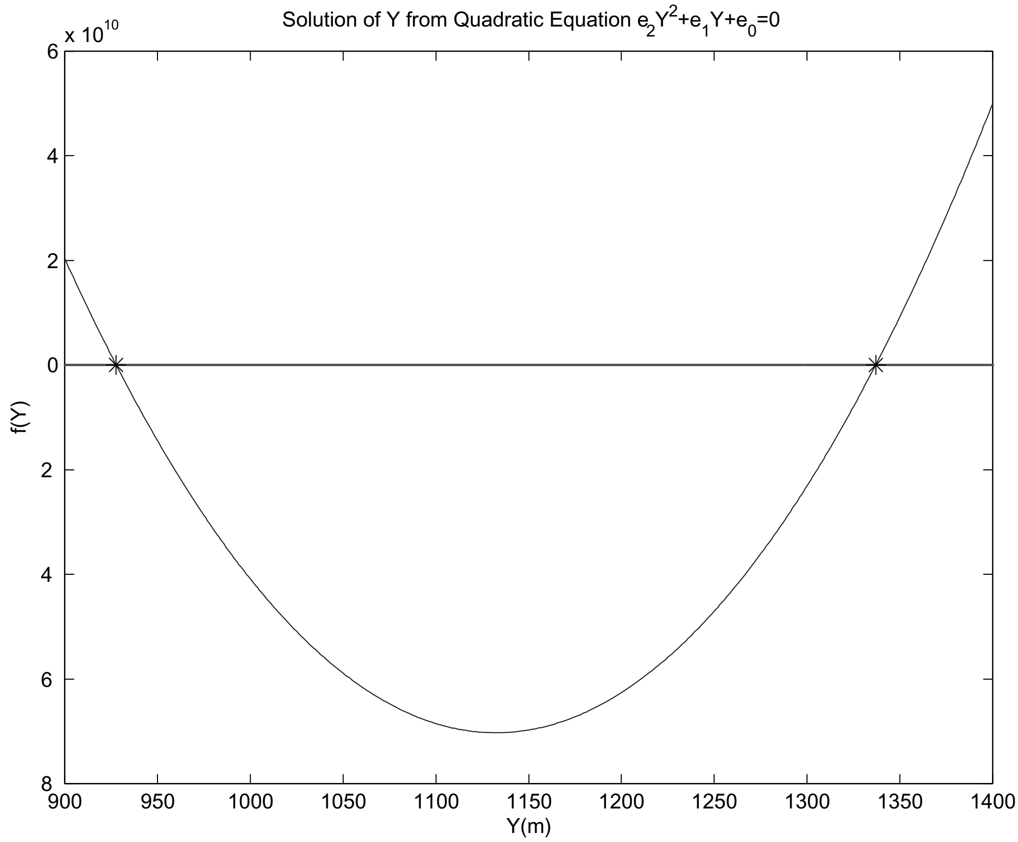


Fig. 2. Solution of the $Y$ coordinates.

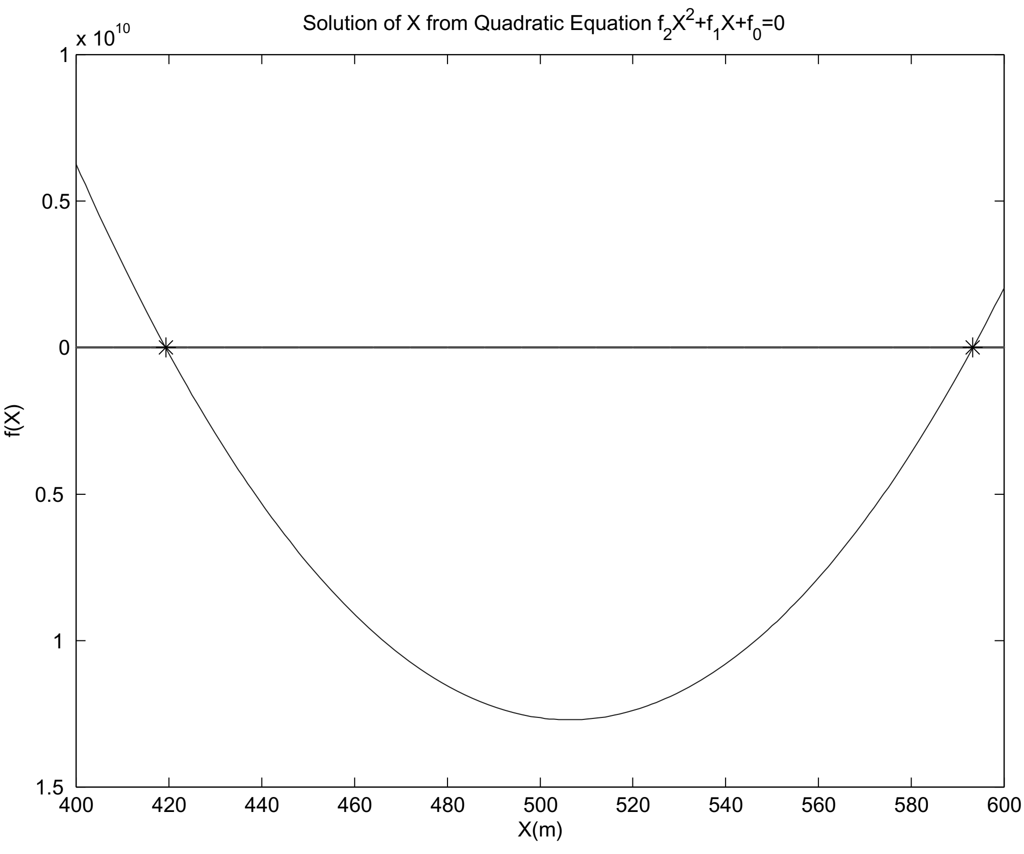


Fig. 3. Solution of the $X$ coordinates.

**Box** 2-1 (*derivation of the univariate polynomials for planar ranging problem*):

$$\boxed{\begin{array}{l} e_2 Y_0^2 + e_1 Y_0 + e_0 = 0 \\ \\ f_2 X_0^2 + f_1 X_0 + f_0 = 0 \end{array}} \tag{2-8}$$

with

$$e_2 = (4X_1^2 - 8X_1X_2 - 8Y_1Y_2 + 4X_2^2 + 4Y_2^2 + 4Y_1^2)$$

$$e_1 = (-4X_1^2Y_1 + 4S_1^2Y_1 - 4Y_1^3 - 4X_2^2Y_2 - 4X_1^2Y_2 + 4Y_1^2Y_2 + 4Y_1Y_2^2 - 4Y_2^3 + 4S_2^2Y_2 - 4S_2^2Y_1 - 4S_1^2Y_2 - 4X_2^2Y_1 + 8X_1X_2Y_1 + 8X_1X_2Y_2)$$

$$\begin{aligned} e_0 = {} & S_2^4 + 2X_1^2Y_2^2 + 4S_1^2X_1X_2 + 4S_2^2X_1X_2 - 2S_2^2X_2^2 - 2Y_1^2Y_2^2 + S_1^4 - 2S_1^2X_1^2 + X_2^4 + 2S_1^2Y_2^2 - 2S_1^2S_2^2 + X_1^4 + Y_1^4 + 2X_2^2Y_1^2 \\ & - 4X_1X_2Y_2^2 + 6X_1^2X_2^2 - 2S_2^2X_1^2 - 2S_2^2Y_2^2 + 2X_1^2Y_1^2 - 2S_1^2X_2^2 + 2S_2^2Y_1^2 - 2S_1^2Y_1^2 + 2X_2^2Y_2^2 - 4X_1X_2^3 - 4X_1X_2Y_1^2 + Y_2^4 - 4X_1^3X_2 \end{aligned}$$

$$f_2 = (4X_1^2 - 8X_1X_2 - 8Y_1Y_2 + 4X_2^2 + 4Y_2^2 + 4Y_1^2)$$

$$f_1 = (-4X_2Y_1^2 - 4X_1Y_1^2 + 4X_1^2X_2 - 4S_2^2X_1 - 4X_2Y_2^2 - 4X_1^3 + 8X_1Y_1Y_2 + 4S_1^2X_1 + 8X_2Y_1Y_2 + 4X_1X_2^2 - 4X_2^3 - 4X_1Y_2^2 + 4S_2^2X_2 - 4S_1^2X_2)$$

$$\begin{aligned} f_0 = {} & S_2^4 + 2X_1^2Y_2^2 - 4X_2^2Y_1Y_2 - 2S_2^2X_2^2 + 6Y_1^2Y_2^2 + 4S_1^2Y_1Y_2 + S_1^4 - 4X_1^2Y_1Y_2 - 2S_1^2X_1^2 + \ldots X_2^4 - 2S_1^2Y_2^2 - 2S_1^2S_2^2 + 4S_2^2Y_1Y_2 \\ & + X_1^4 + Y_1^4 + 2X_2^2Y_1^2 - 2X_1^2X_2^2 + 2S_2^2X_1^2 - 4Y_1^3Y_2 - 2S_2^2Y_2^2 + 2X_1^2Y_1^2 + 2S_1^2X_2^2 - 2S_2^2Y_1^2 - 2S_1^2Y_1^2 + 2X_2^2Y_2^2 - 4Y_1Y_2^3 + Y_2^4 \end{aligned}$$

**Box** 2-2 (*critical configuration of the two-dimensional ranging problem*):

$$\left[\begin{array}{l} f_1(X, Y; X_1, Y_1, S_1) = (X_1 - X)^2 + (Y_1 - Y)^2 - S_1^2 \\ f_2(X, Y; X_2, Y_2, S_2) = (X_2 - X)^2 + (Y_2 - Y)^2 - S_2^2 \end{array}\right. \tag{2-9}$$

$$\left[\begin{array}{l} \dfrac{\partial f_1}{\partial X} = -2(X_1 - X), \quad \dfrac{\partial f_2}{\partial X} = -2(X_2 - X) \\ \\ \dfrac{\partial f_1}{\partial Y} = -2(Y_1 - Y), \quad \dfrac{\partial f_2}{\partial Y} = -2(Y_2 - Y) \end{array}\right. \tag{2-10}$$

$$\left[\begin{array}{l} D = \left|\dfrac{\partial f_i}{\partial X_j}\right| = 4\begin{vmatrix} X_1 - X & X_2 - X \\ Y_1 - Y & Y_2 - Y \end{vmatrix} \\ D \Leftrightarrow \begin{vmatrix} X_1 - X & X_2 - X \\ Y_1 - Y & Y_2 - Y \end{vmatrix} = \begin{vmatrix} X & Y & 1 \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{vmatrix} = 0 \end{array}\right. \tag{2-11}$$

$$\left[\begin{array}{l} \frac{1}{4}D = (X_1 - X)(Y_2 - Y) - (X_2 - X)(Y_1 - Y) \\ \quad = X_1Y_2 - X_1Y - XY_2 + XY - X_2Y_1 + X_2Y + XY_1 - XY \\ \quad = X(Y_1 - Y_2) + Y(X_2 - X_1) + X_1Y_2 - X_2Y_1 \end{array}\right. \tag{2-12}$$

thus

$$\begin{vmatrix} X & Y & 1 \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{vmatrix} = 2 \times \text{Area triangle} P(X, Y), P_1(X_1, Y_1) \text{ and } P_2(X_2, Y_2) \tag{2-13}$$

$$D = \begin{vmatrix} X & Y & 1 \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{vmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \tag{2-14}$$

results in a system of homogeneous equations

$$\left[\begin{array}{l} aX + bY + c = 0 \\ aX_1 + bY_1 + c = 0 \\ aX_2 + bY_2 + c = 0 \end{array}\right. \tag{2-15}$$

$P_2 \in \mathrm{E}^2$ respectively. The conventional analytical approach and the *reduced Groebner basis* approaches both give the desired planar coordinates of the unknown point as $\{Y_0 = \{1336.0025,\ 928.1202\}$ and $X_0 = \{593.7162,\ 420.3000\}\}_{P_0}$ with the *reduced Groebner basis* being direct and faster (i.e. avoids forward and backwards substitutions).

Geometrically, the algebraic curves given by (2-1) would result in a conic intersection of two circles with the centers $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ with radiuses $\{S_1\}$ and $\{S_2\}$ respectively. The applied polynomial approaches decompose this complicated geometry to those of Figs. 2 and 3 which represent univariate polynomials and are simpler to solve. Figures 2 and 3 indicate the solution of (2-8) for the Example presented above. The stars (intersection of the quadratic curves with the zero line) are the solution points. In Box 2-2 we present the critical configuration of the two-dimentional ranging problem. The computed determinants and equations (2-14) and (2-15) indicate the critical configuration to be the case when the points $P(X, Y)$, $P_1(X_1, Y_1)$ and $P_2(X_2, Y_2)$ lie in a line with gradient $-\dfrac{c}{b}$ and intercept $-\dfrac{a}{b}$.

## 3. Three-Dimensional Ranging

This step is commonly referred to in German literature as *"Bogenschnitt"* problem and in English literature as the *"ranging problem"* or *"Arc section"*(Kahmen and Faig, 1988, p. 215) and is the problem of establishing the position of a point given the distances from the unknown point $P \in \mathrm{E}^3$ to three other known stations $P_i \in \mathrm{E}^3 \mid i = 1, 2, 3$. In general the three-dimensional *"Bogenschnitt"* problem can be formulated as follows: Given distances as observations or pseudo-observations from an unknown point $P \in \mathrm{E}^3$ to a minimum of three known points $P_i \in \mathrm{E}^3 \mid i = 1, 2, 3$, determine the position $\{X, Y, Z\}$ of the unknown point $P \in \mathrm{E}^3$ (Fig. 4). When only three known stations are used to determine the position of the unknown station in three-dimension, the problem reduces to that of *three-dimensional closed form solution*. We present below four approaches that can be used to solve the *"three-dimensional Bogenschnitt"* problem in a closed form. This problem is a traditional problem both in Geodesy, Photogrammetry and Robotics. In all the three areas, the determination of the coordinates of the unknown point given the distances from this point to three other known

**Box 3-1** (*differencing of the nonlinear distance equations*):

$$\left[\begin{array}{l} S_1^2=(X_1-X)^2+(Y_1-Y)^2+(Z_1-Z)^2 \\ S_2^2=(X_2-X)^2+(Y_2-Y)^2+(Z_2-Z)^2 \\ S_3^2=(X_3-X)^2+(Y_3-Y)^2+(Z_3-Z)^2 \end{array}\right. \tag{3-1}$$

$$\left[\begin{array}{l} S_1^2=X_1^2+Y_1^2+Z_1^2+X^2+Y^2+Z^2-2X_1X-2Y_1Y-2Z_1Z \\ S_2^2=X_2^2+Y_2^2+Z_2^2+X^2+Y^2+Z^2-2X_2X-2Y_2Y-2Z_2Z \\ S_3^2=X_3^2+Y_3^2+Z_3^2+X^2+Y^2+Z^2-2X_3X-2Y_3Y-2Z_3Z \end{array}\right. \tag{3-2}$$

differencing above

$$\left[\begin{array}{l} S_1^2-S_2^2=X_1^2-X_2^2+Y_1^2-Y_2^2+Z_1^2-Z_2^2 \\ \qquad +2X(X_2-X_1)+2Y(Y_2-Y_1)+2Z(Z_2-Z_1) \\ S_2^2-S_3^2=X_2^2-X_3^2+Y_2^2-Y_3^2+Z_2^2-Z_3^2 \\ \qquad +2X(X_3-X_2)+2Y(Y_3-Y_2)+2Z(Z_3-Z_2) \end{array}\right. \tag{3-3}$$

$$\left[\begin{array}{l} 2X(X_2-X_1)+2Y(Y_2-Y_1)+2Z(Z_2-Z_1)=a \\ 2X(X_3-X_2)+2Y(Y_3-Y_2)+2Z(Z_3-Z_2)=b \end{array}\right. \tag{3-4}$$

$$\left[\begin{array}{l} a=S_1^2-S_2^2-X_1^2+X_2^2-Y_1^2+Y_2^2-Z_1^2+Z_2^2 \\ b=S_2^2-S_3^2-X_2^2+X_3^2-Y_2^2+Y_3^2-Z_2^2+Z_3^2 \end{array}\right. \tag{3-5}$$

**Box 3-2** (*solution of the simultaneous equation by elimination*):

$$\left[\begin{array}{l} 2X(X_2-X_1)+2Y(Y_2-Y_1)=a-2Z(Z_2-Z_1) \\ 2X(X_3-X_2)+2Y(Y_3-Y_2)=b-2Z(Z_3-Z_2) \end{array}\right. \tag{3-6}$$

$$\left[\begin{array}{l} X=\dfrac{a(Y_3-Y_2)-b(Y_2-Y_1)}{2\{(X_2-X_1)(Y_3-Y_2)-(X_3-X_2)(Y_2-Y_1)\}} \\ \qquad -\dfrac{\{(Z_2-Z_1)(Y_3-Y_2)-(Z_3-Z_2)(Y_2-Y_1)\}Z}{\{(X_2-X_1)(Y_3-Y_2)-(X_3-X_2)(Y_2-Y_1)\}} \\ Y=\dfrac{a(X_3-X_2)-b(X_2-X_1)}{2\{(Y_2-Y_1)(X_3-X_2)-(Y_3-Y_2)(X_2-X_1)\}} \\ \qquad -\dfrac{\{(Z_2-Z_1)(X_3-X_2)-(Z_3-Z_2)(X_2-X_1)\}Z}{\{(Y_2-Y_1)(X_3-X_2)-(Y_3-Y_2)(X_2-X_1)\}} \end{array}\right. \tag{3-7}$$

$$\left[\begin{array}{l} X=c-dZ \\ Y=e-fZ \end{array}\right. \tag{3-8}$$

$$\left[\begin{array}{l} c=\dfrac{a(Y_3-Y_2)-b(Y_2-Y_1)}{2\{(X_2-X_1)(Y_3-Y_2)-(X_3-X_2)(Y_2-Y_1)\}} \\ d=\dfrac{\{(Z_2-Z_1)(Y_3-Y_2)-(Z_3-Z_2)(Y_2-Y_1)\}Z}{\{(X_2-X_1)(Y_3-Y_2)-(X_3-X_2)(Y_2-Y_1)\}} \\ e=\dfrac{a(X_3-X_2)-b(X_2-X_1)}{2\{(Y_2-Y_1)(X_3-X_2)-(Y_3-Y_2)(X_2-X_1)\}} \\ f=\dfrac{\{(Z_2-Z_1)(X_3-X_2)-(Z_3-Z_2)(X_2-X_1)\}Z}{\{(Y_2-Y_1)(X_3-X_2)-(Y_3-Y_2)(X_2-X_1)\}} \end{array}\right. \tag{3-9}$$

substituting 3-8 in 3-1i

$$\begin{array}{l}(d^2+f^2+1)Z^2+2(dX_1+fY_1-Z_1-cd-ef)Z \\ \quad +X_1^2+Y_1^2+Z_1^2-2X_1c-2Y_1e-S_1^2+c^2+e^2=0\end{array} \tag{3-10}$$

$$\left\{ gZ^2+hZ+i=0,\ Z_{1,2}=\frac{-h\pm\sqrt{h^2-4gi}}{2g}\right. \tag{3-11}$$

$$\begin{array}{l}(dX_1+fY_1-Z_1-cd-ef)^2=(d^2+f^2+1) \\ \quad \cdot(X_1^2+Y_1^2+Z_1^2-2X_1c-2Y_1e-S_1^2+c^2+e^2)\end{array} \tag{3-12}$$

where

$$\left[\begin{array}{l} g=d^2+f^2+1 \\ h=2(dX_1+fY_1-Z_1-cd-ef) \\ i=X_1^2+Y_1^2+Z_1^2-2X_1c-2Y_1e-S_1^2+c^2+e^2 \end{array}\right. \tag{3-13}$$

points is the key issue.

Starting from three *nonlinear three-dimensional Pythagorus* distance observation equations (3-1) in Box 3-1 relating to the three unknowns $\{X, Y, Z\}$, two equations with three unknowns are derived. Equation (3-1) is expanded in the form given by (3-2) and differenced in (3-3) to eliminate the quadratic terms $\{X^2, Y^2, Z^2\}$. Collecting all the known terms of equation (3-3) to the right hand side and those relating to the unknowns on the left hand side leads to equation (3-4) with the terms $\{a, b\}$ given by (3-5). The solution of the unknown terms $\{X, Y, Z\}$ now involves solving equation (3-4), which has two equations with three unknowns. To circumvent the problem of having more unknowns than the equations, two of the unknowns are sought in terms of the third unknown (e.g. $X = g(Z)$, $Y = g(Z)$).

## 3.1 Conventional approach

**3.1.1 Solution by elimination approach-1** In the elimination approach presented in Box 3-2, equations (3-6) is a simultaneous equation version of equations (3-4) with two equations and two unknowns $\{X, Y\}$ written in terms of the unknown $Z$. By first eliminating $Y$, the value of $X$ is obtained in terms of the unknown value $Z$ and substituted in either of the two equations of (3-6) to give the value of $Y$. The values of $\{X, Y\}$ are as depicted by (3-7) which are expressed in a simplified form (3-8) with the coefficients $\{c, d, e, f\}$ given by (3-9). The values of $\{X, Y\}$ in (3-8) are substituted in the first equation of (3-1) in Box 3-1 to get the quadratic equation (3-10) respectively (3-11) in terms of $Z$ as the unknown. The two solutions of $Z$ are now given by the second equation of (3-11) with the coefficients $\{g, h, i\}$ given in (3-13). Once we solve (3-11) for $Z$, we substitute in (3-8) to obtain the corresponding pair of solutions for $\{X, Y\}$.

**3.1.2 Solution by elimination approach-2** The second approach presented in Box 3-3 involves first writing equation (3-4) in the simultaneous form (3-14), which is expressed in matrix form as (3-15). We now seek the matrix solution of $\{Y, Z\}$ in terms of the unknown element $X$ as expressed by equation (3-16), which is written in a simpler form (3-18) whose the elements are given by (3-17). The solution of equation (3-16) for $\{Y, Z\}$ in terms of $X$ is given by (3-19) respectively (3-20) and (3-21) with the coefficients of (3-21) given by (3-22). Substituting the obtained values of $\{Y, Z\}$ in terms of $X$ in the first equation of (3-1) we obtain a quadratic equation (3-23) in terms of $X$ as the unknown. The two solutions for $X$ are given by the second equation in (3-23) and substituted back in (3-21) to obtain the values of $\{Y, Z\}$. The coefficients $\{l, m, n\}$ in equation (3-23) are given by (3-25).

A pair of solution $(X_1, Y_1, Z_1)$ and $(X_2, Y_2, Z_2)$ are obtained. The correct solution from this pair is obtained with the help of prior information e.g. from an existing map. For GPS positioning, Awange and Grafarend (2002b, c) have already demonstrated that one set of solution will be in space while the other set will be on earth and that the solution can easily be distinguished from the radial distance. Of importance is the problem of *bifurcation*, that is, to identify the

**Box 3-3** (*solution of the simultaneous equation by the matrix approach*):

$$\begin{bmatrix} 2Y(Y_2 - Y_1) + 2Z(Z_2 - Z_1) = a - 2X(X_2 - X_1) \\ 2Y(Y_3 - Y_2) + 2Z(Z_3 - Z_2) = b - 2X(X_3 - X_2) \end{bmatrix} \tag{3-14}$$

$$\begin{bmatrix} Y_2 - Y_1 & Z_2 - Z_1 \\ Y_3 - Y_2 & Z_3 - Z_2 \end{bmatrix}\begin{bmatrix} Y \\ Z \end{bmatrix} = \frac{1}{2}\left\{\begin{bmatrix} a \\ b \end{bmatrix} - 2\begin{bmatrix} X_2 - X_1 \\ X_3 - X_2 \end{bmatrix} X\right\} \tag{3-15}$$

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = \frac{1}{2}d\begin{bmatrix} Z_3 - Z_2 & -(Z_2 - Z_1) \\ -(Y_3 - Y_2) & (Y_2 - Y_1) \end{bmatrix}\left\{\begin{bmatrix} a \\ b \end{bmatrix} - 2\begin{bmatrix} X_2 - X_1 \\ X_3 - X_2 \end{bmatrix} X\right\} \tag{3-16}$$

with $d = \{(Y_2 - Y_1)(Z_3 - Z_2) - (Y_3 - Y_2)(Z_2 - Z_1)\}^{-1}$

$$\begin{bmatrix} a_{11} = Y_2 - Y_1,\ a_{12} = Z_2 - Z_1,\ a_{21} = Y_3 - Y_2,\ a_{22} = Z_3 - Z_2 \\ c_1 = -(X_2 - X_1),\ c_2 = -(X_3 - X_2),\ \ b_1 = \frac{1}{2}a,\ \ b_2 = \frac{1}{2}b \end{bmatrix} \tag{3-17}$$

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = \{a_{11}a_{22} - a_{12}a_{21}\}^{-1}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\left\{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} X\right\} \tag{3-18}$$

$$\begin{bmatrix} Y = \{a_{11}a_{22} - a_{12}a_{21}\}^{-1}\{a_{22}(b_1 + c_1X) - a_{12}(b_2 + c_2X)\} \\ Z = \{a_{11}a_{22} - a_{12}a_{21}\}^{-1}\{a_{11}(b_2 + c_2X) - a_{21}(b_1 + c_1X)\} \end{bmatrix} \tag{3-19}$$

$$\begin{bmatrix} Y = e\,[\{a_{22}b_1 - a_{12}b_2\} + \{a_{22}c_1 - a_{12}c_2\}\,X] \\ Z = e\,[\{a_{11}b_2 - a_{21}b_1\} + \{a_{11}c_2 - a_{21}c_1\}\,X] \end{bmatrix} \tag{3-20}$$

$$\begin{bmatrix} Y = e(f + gX) \\ Z = e(h + iX) \end{bmatrix} \tag{3-21}$$

$$\begin{bmatrix} e = (a_{11}a_{22} - a_{12}a_{21})^{-1},\ \ f = a_{22}b_1 - a_{12}b_2,\ \ g = a_{22}c_1 - a_{12}c_2 \\ h = a_{11}b_2 - a_{21}b_1,\ \ i = a_{11}c_2 - a_{21}c_1,\ \ k = X_1^2 + Y_1^2 + Z_1^2 \end{bmatrix} \tag{3-22}$$

substituting 3-21 in 3-1i

$$\left\{ lX^2 + mX + n = 0,\ X = \frac{-m \pm \sqrt{m^2 - 4ln}}{2l} \right. \tag{3-23}$$

$$(e^2fg + e^2hi - X_1 - egY_1 - eiZ_1)^2 = (e^2i^2 + e^2g^2 + 1)(k - S_1^2 - 2Y_1ef + e^2f^2 - 2Z_1eh + e^2h^2) \tag{3-24}$$

where

$$\begin{bmatrix} l = e^2i^2 + e^2g^2 + 1 \\ m = 2(e^2fg + e^2hi - X_1 - egY_1 - eiZ_1) \\ n = k - S_1^2 - 2Y_1ef + e^2f^2 - 2Z_1eh + e^2h^2 \end{bmatrix} \tag{3-25}$$

point where the quadratic equation has only one solution, i.e. *bifurcates*. Bancroft (1985), Abel and Chaffee (1991), Chaffee and Abel (1994), and Grafarend and Shan (1996) have already treated this problem. In the present study, the *bifurcation* point for (3-11) and (3-23) will be $h^2 = 4gi$ and $m^2 = 4in$ respectively.

### 3.2 Polynomial approaches

**3.2.1 Groebner bases approach** Equation (3-4) is expressed in the algebraic form (3-26) in Box 3-4 with the coefficients given as in (3-27). The *Grooebner basis* is then obtained using the *GroebnerBasis* command in Mathematica 3.0 as illustrated by (3-28) giving the computed *Groebner basis* as in (3-29). The first equation of (3-29) is solved for $Y = g(Z)$ and is as presented in (3-30). This value is substituted in the second equation of (3-29) to give $X = g(Z)$ presented in the first equation of (3-31). The obtained values of $Y$ and $X$ are substituted in the first equation of (3-1) to give a quadratic equation in $Z$. Once this quadratic has been solved for $Z$, The values of $Y$ and $X$ can be obtained from (3-30) and (3-31) respectively. We mention here that the direct solution of $X = g(Z)$ as presented in the second equation of (3-31) could be obtained by computing the *reduced Groebner basis* as explained in Awange (2002) rather than solving for $Y = g(Z)$ and substituting in the second equation of (3-29) to give $X = g(Z)$ presented in first equation of (3-31). Similarly we could obtain $Y = g(Z)$ alone by replacing $Y$ with $X$ in the option section of the *reduced Groebner basis* discussed in Awange (2002).

**3.2.2 Resultant approach** The problem is solved in *four steps* as illustrated in Box 3-5. In the *first step*, we solve for the first variable $X$ in (3-26) by hiding it as a constant and homogenizing the equation using a variable $W$ as in (3-32). In the *second step*, the *Sylvester resultant* (Awange and Grafarend, 2002a) or the *Jacobian determinant* is obtained as in (3-33). The resulting determinant (3-34) is solved for $X = g(Z)$ and presented in (3-35). The procedure is repeated for *steps three* and *four* as in equations (3-36) to (3-39) to solve for $Y = g(Z)$. The obtained values of $X = g(Z)$ and $Y = g(Z)$ are substituted in the first equation of (3-1) to give a quadratic equation in $Z$. Once this quadratic has been solved for $Z$, The values of $X$ and $Y$ can be obtained from (3-35) and (3-39) respectively.

**3.2.3 Example** Consider distances of Figure 4 to be given as $S_1 = 1324.2380$ m, $S_2 = 542.2609$ m and $S_3 =$

**Box** 3-4 (*Groebner basis approach*):

$$a_{02}X + b_{02}Y + c_{02}Z + f_{02} = 0$$
$$a_{12}X + b_{12}Y + c_{12}Z + f_{12} = 0 \quad (3\text{-}26)$$

$$a_{02} = 2(X_1 - X_2), b_{02} = 2(Y_1 - Y_2), c_{02} = 2(Z_1 - Z_2)$$
$$a_{12} = 2(X_2 - X_3), b_{12} = 2(Y_2 - Y_3), c_{12} = 2(Z_2 - Z_3)$$
$$f_{02} = (S_1^2 - X_1^2 - Y_1^2 - Z_1^2) - (S_2^2 - X_2^2 - Y_2^2 - Z_2^2)$$
$$f_{12} = (S_2^2 - X_2^2 - Y_2^2 - Z_2^2) - (S_3^2 - X_3^2 - Y_3^2 - Z_3^2). \quad (3\text{-}27)$$

$$\text{GroebnerBasis}[\{a_{02}X + b_{02}Y + c_{02}Z + f_{02}, a_{12}X + b_{12}Y + c_{12}Z + f_{12}\}, \{X, Y\}] \quad (3\text{-}28)$$

$$g_1 = a_{02}b_{12}Y - a_{12}b_{02}Y - a_{12}c_{02}Z + a_{02}c_{12}Z + a_{02}f_{12} - a_{12}f_{02}$$
$$g_2 = a_{12}X + b_{12}Y + c_{12}Z + f_{12}$$
$$g_3 = a_{02}X + b_{02}Y + c_{02}Z + f_{02}. \quad (3\text{-}29)$$

$$Y = \frac{\{(a_{12}c_{02} - a_{02}c_{12})Z + a_{12}f_{02} - a_{02}f_{12}\}}{(a_{02}b_{12} - a_{12}b_{02})} \quad (3\text{-}30)$$

$$X = \frac{-(b_{12}Y + c_{12}Z + f_{12})}{a_{12}} \text{ or } X = \frac{\{(b_{02}c_{12} - b_{12}c_{02})Z + b_{02}f_{12} - b_{12}f_{02}\}}{(a_{02}b_{12} - a_{12}b_{02})} \quad (3\text{-}31)$$

**Box** 3-5 (*Multipolynomial resultants approach*):

Step 1: Solve for $X$ in terms of $Z$

$$f_1 := (a_{02}X + c_{02}Z + f_{02})W + b_{02}Y$$
$$f_2 := (a_{12}X + c_{12}Z + f_{12})W + b_{12}Y \quad (3\text{-}32)$$

Step 2: Obtain the Sylvester resultant

$$J_X = \det\begin{bmatrix} \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial W} \\ \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial W} \end{bmatrix} = \det\begin{bmatrix} b_{02} & (a_{02}X + c_{02}Z + f_{02}) \\ b_{12} & (a_{12}X + c_{12}Z + f_{12}) \end{bmatrix} \quad (3\text{-}33)$$

$$J_X = b_{02}a_{12}X + b_{02}c_{12}Z + b_{02}f_{12} - b_{12}a_{02}X - b_{12}c_{02}Z - b_{12}f_{02} \quad (3\text{-}34)$$

from (3-34)

$$X = \frac{\{(b_{12}c_{02} - b_{02}c_{12})Z + b_{12}f_{02} - b_{02}f_{12}\}}{(b_{02}a_{12} - b_{12}a_{02})} \quad (3\text{-}35)$$

Step 3: Solve for $Y$ in terms of $Z$

$$f_3 := (b_{02}Y + c_{02}Z + f_{02})W + b_{02}X$$
$$f_4 := (b_{12}Y + c_{12}Z + f_{12})W + a_{12}X \quad (3\text{-}36)$$

Step 4: Obtain the Sylvester resultant

$$J_Y = \det\begin{bmatrix} \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial W} \\ \frac{\partial f_4}{\partial X} & \frac{\partial f_4}{\partial W} \end{bmatrix} = \det\begin{bmatrix} a_{02} & (b_{02}Y + c_{02}Z + f_{02}) \\ a_{12} & (b_{12}Y + c_{12}Z + f_{12}) \end{bmatrix} \quad (3\text{-}37)$$

$$J_Y = a_{02}b_{12}Y + a_{02}c_{12}Z + a_{02}f_{12} - a_{12}b_{02}Y - a_{12}c_{02}Z - a_{12}f_{02} \quad (3\text{-}38)$$

from (3-38)

$$Y = \frac{\{(a_{12}c_{02} - a_{02}c_{12})Z + a_{12}f_{02} - a_{02}f_{12}\}}{(a_{02}b_{12} - a_{12}b_{02})} \quad (3\text{-}39)$$

430.5286 m, the position of $P$ is given by the procedures above as $X = 4157066.1116$ m, $Y = 671429.6655$ m and $Z = 4774879.3704$ m in the Global Reference Frame. Figures 5, 6 and 7 indicate the solution of $X, Y, Z$ respectively. The stars (intersection of the quadratic curves with the zero line) are the solution points. The critical configuration of the three-dimensional ranging problem is presented in Box 3-6. The computed determinants and equations (3-45) and (3-46) indicate the critical configuration to be the case the points $P(X, Y, Z)$, $P_1(X_1, Y_1, Z_1)$, $P_2(X_2, Y_2, Z_2)$ and $P_3(X_3, Y_3, Z_3)$ lie on a plane for the three-dimensional case.

## 4. Applications

In Awange and Grafarend (2002b, c), it was demonstrated how the pseudo-range observations can be solved analytically to obtain the position of a station. This was found particularly handy when the receiver is used for the first time in a new area as the time spend by the receiver for the initial solution is minimized considerably. Algorithms for this had already been presented in GPS Toolbox (e.g. `http://www.ngs.noaa.gov/gps-toolbox/exist.htm`).

**Box** 3-6 (*critical configuration of the three-dimensional ranging*):

$$\begin{bmatrix} f_1(X,Y,Z;X_1,Y_1,Z_1,S_1) = (X_1-X)^2+(Y_1-Y)^2+(Z_1-Z)^2-S_1^2 \\ f_2(X,Y,Z;X_2,Y_2,Z_2,S_2) = (X_2-X)^2+(Y_2-Y)^2+(Z_2-Z)^2-S_2^2 \\ f_3(X,Y,Z;X_3,Y_3,Z_3,S_3) = (X_3-X)^2+(Y_3-Y)^2+(Z_3-Z)^2-S_3^2 \end{bmatrix} \tag{3-40}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial X} = -2(X_1-X), \ \frac{\partial f_2}{\partial X} = -2(X_2-X), \ \frac{\partial f_3}{\partial X} = -2(X_3-X) \\ \frac{\partial f_1}{\partial Y} = -2(Y_1-Y), \ \frac{\partial f_2}{\partial Y} = -2(Y_2-Y), \ \frac{\partial f_3}{\partial Y} = -2(Y_3-Y) \\ \frac{\partial f_1}{\partial Z} = -2(Z_1-Z), \ \frac{\partial f_2}{\partial Z} = -2(Z_2-Z), \ \frac{\partial f_3}{\partial Zv} = -2(Z_3-Z) \end{bmatrix} \tag{3-41}$$

$$\begin{bmatrix} D = \left|\frac{\partial f_i}{\partial X_j}\right| = -8 \begin{vmatrix} X_1-X & Y_1-Y & Z_1-Z \\ X_2-X & Y_2-Y & Z_1-Z \\ X_3-X & Y_3-Y & Z_1-Z \end{vmatrix} \\ D \Leftrightarrow \begin{vmatrix} X_1-X & Y_1-Y & Z_1-Z \\ X_2-X & Y_2-Y & Z_1-Z \\ X_3-X & Y_3-Y & Z_1-Z \end{vmatrix} = \begin{vmatrix} X & Y & Z & 1 \\ X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \end{vmatrix} = 0 \end{bmatrix} \tag{3-42}$$

$$\begin{bmatrix} -\frac{1}{8}D = \{-Z_1Y_3+Y_1Z_3-Y_2Z_3+Y_3Z_2-Y_1Z_2+Y_2Z_1\}X \\ +\{-Z_1X_2-X_1Z_3+Z_1X_3+X_1Z_2-X_3Z_2+X_2Z_3\}Y \\ +\{Y_1X_2-Y_1X_3+Y_3X_1-X_2Y_3-X_1Y_2+Y_2X_3\}Z \\ +X_1Y_2Z_3-X_1Y_3Z_2-X_3Y_2Z_1+X_2Y_3Z_1-X_2Y_1Z_3+X_3Y_1Z_2 \end{bmatrix} \tag{3-43}$$

thus

$$\begin{vmatrix} X & Y & Z & 1 \\ X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \end{vmatrix} \tag{3-44}$$

describes six times volume of the tetrahedron formed by the points $P(X,Y,Z)$, $P_1(X_1,Y_1,Z_1)$, $P_2(X_2,Y_2,Z_2)$ and $P_3(X_3,Y_3,Z_3)$ . Therefore

$$D = \begin{vmatrix} X & Y & Z & 1 \\ X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \end{vmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0, \tag{3-45}$$

results in a system of homogeneous equations

$$\begin{bmatrix} aX+bY+cZ+d=0 \\ aX_1+bY_1+cY_1+d=0 \\ aX_2+bY_2+cZ_2+d=0 \\ aX_3+bY_3+cZ_3+d=0 \end{bmatrix} \tag{3-46}$$

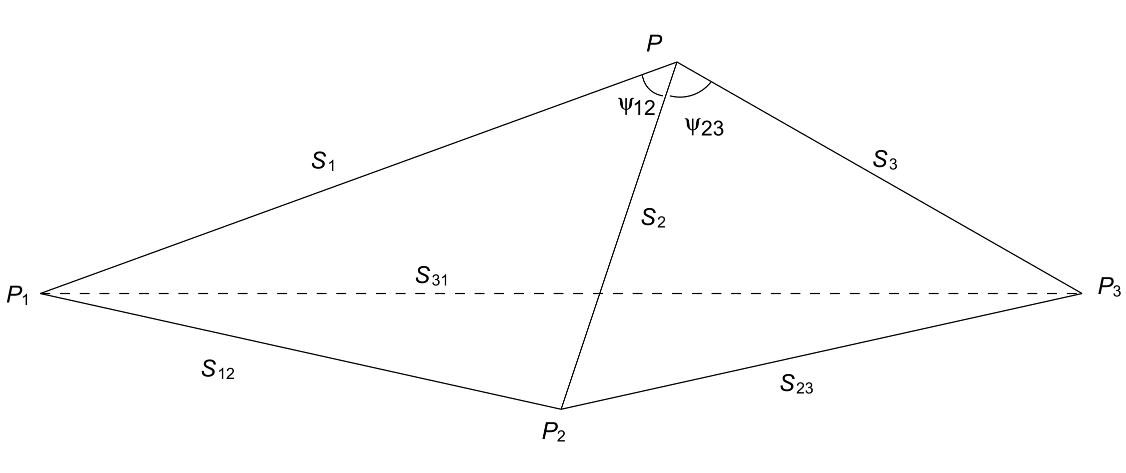


Fig. 4. Three-dimensional distance observations.

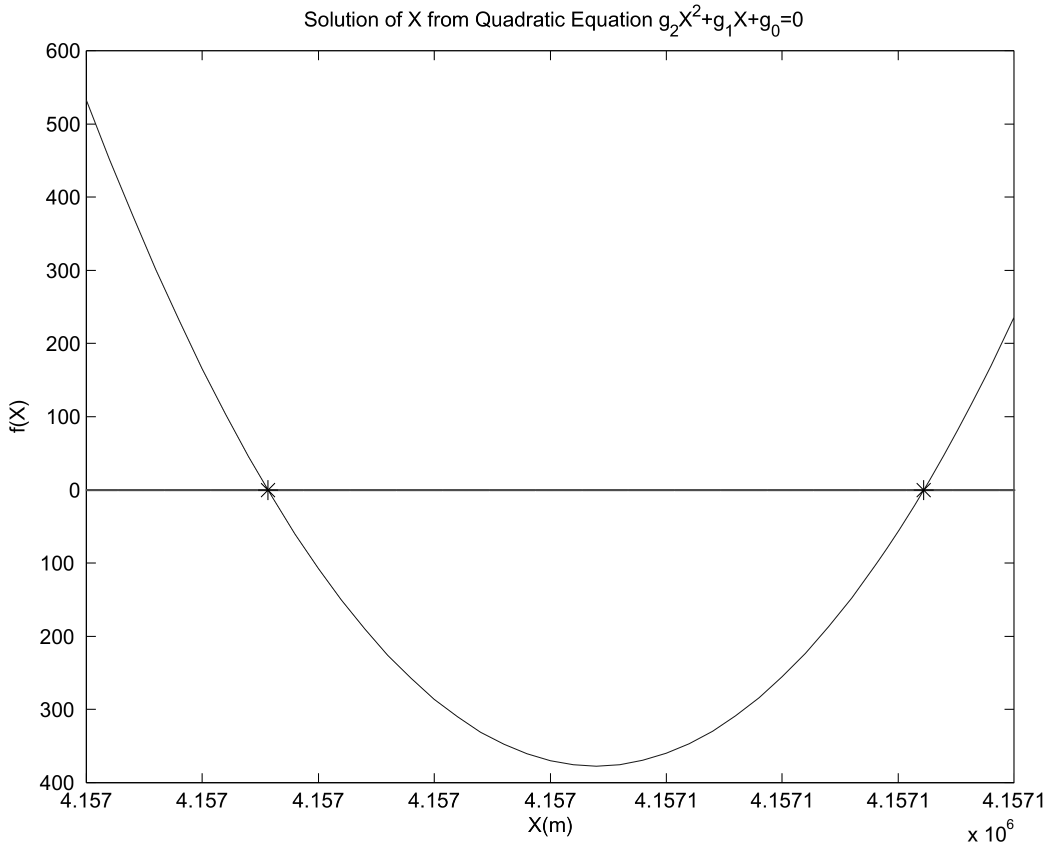


Fig. 5. Solution of the $X$ coordinates.

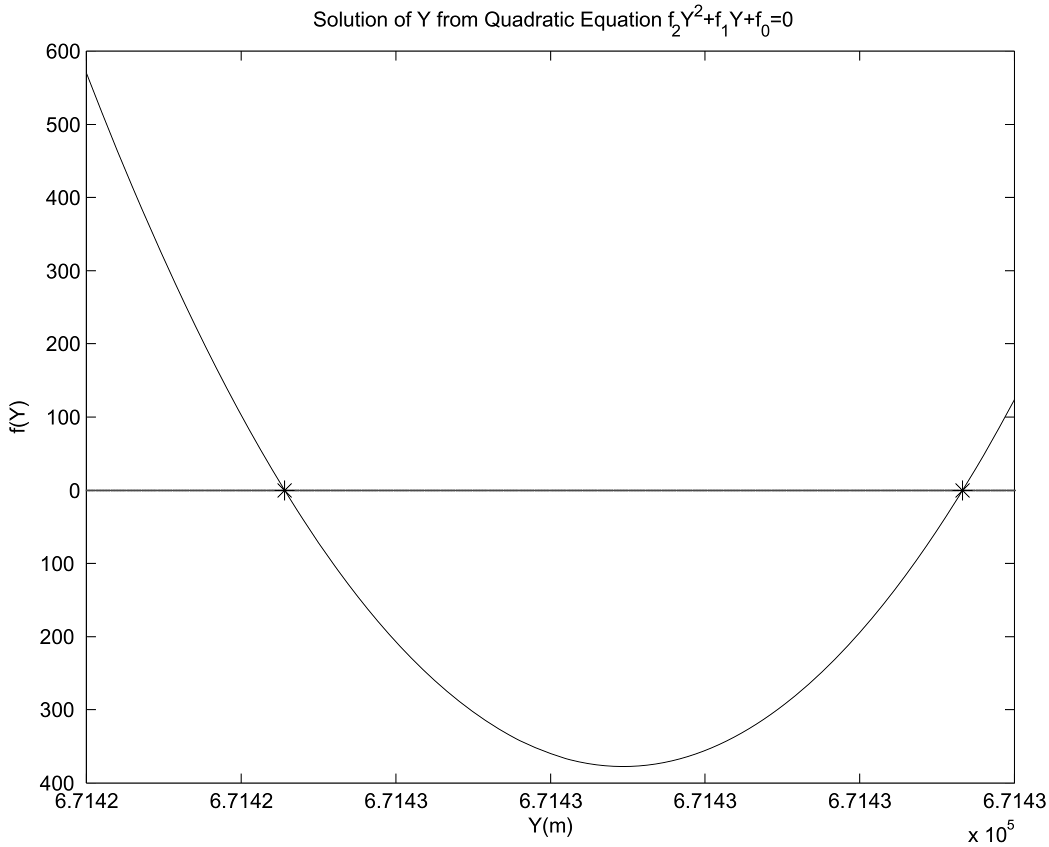


Fig. 6. Solution of the $Y$ coordinates.

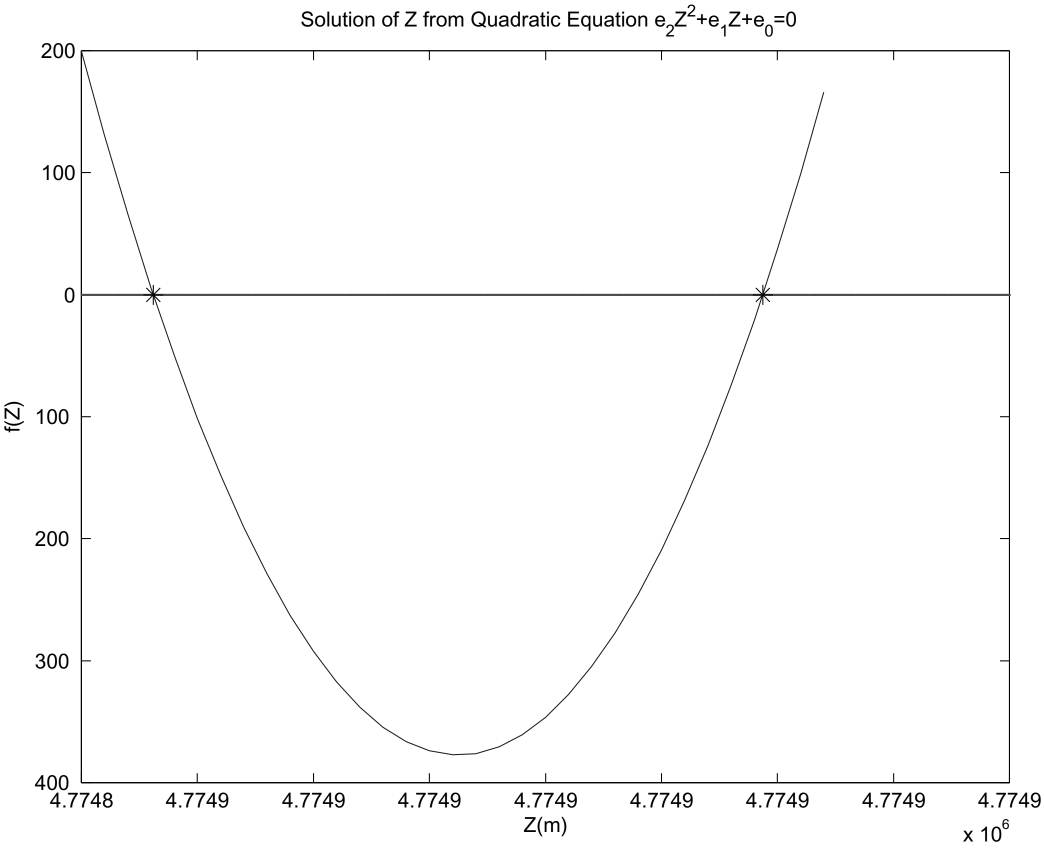


Fig. 7. Solution of the $Z$ coordinates.

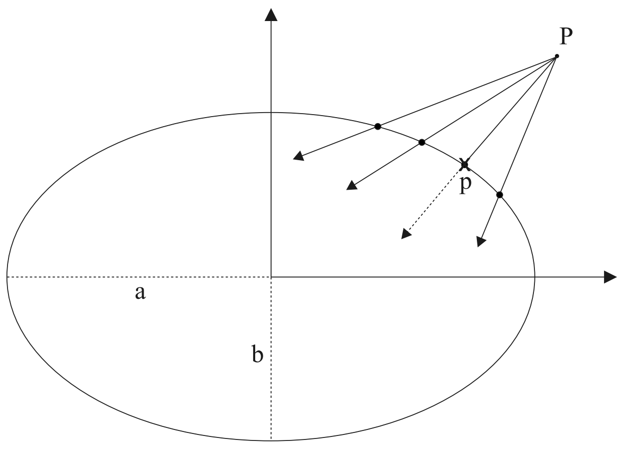


Fig. 8. Minimum distance mapping of a point $P$ on the Earth's topographic surface to a point $p$ on the *International Reference Ellipsoid* $\mathbb{E}^2_{a,a,b}$.

Besides the GPS positioning, the technique is applicable to the *Minimum Distance Mapping (MDM)*. In order to relate a point $P$ on the Earth's topographic surface to a point $p$ on the *International Reference Ellipsoid* $\mathbb{E}^2_{a,a,b}$ we work with a bundle of half straight lines so called *projection lines* which depart from $P$ and intersect $\mathbb{E}^2_{a,a,b}$ either not at all or in two points. There is *one projection line* which is at minimum distance relating $P$ to $p$. *Figure* 8 is an illustration of such *MDM*. Such an *optimization problem* is formulated by means of the Lagrangean $\pounds(x_1, x_2, x_3, x_4)$ which can be solved using the described algebraic techniques.

In the ongoing studies to come in later publications, the three-dimensional ranging will be useful in the second step of the $P4P$ solution needed in the perspective center problem and triple three-dimensional intersection problem (needed for block triangulation) in Photogrammetry. This becomes vital once the dimensionless spatial angles in terms of the horizontal and vertical directions $T_i$ and $B_i$ respectively given by

$$\cos\psi_{12} = \cos B_1 \cos B_2 \cos(T_2 - T_1) + \sin B_1 \sin B_1 \quad (4\text{-}47)$$

and the *space angles in terms of image coordinates/perspective coordinates* $(x_1, y_1), (x_2, y_2)$ *and the* *focal length f"* given by

$$\cos\psi_{12} = \frac{x_1x_2 + y_1y_2 + f^2}{\sqrt{x_1^2 + y_1^2 + f^2}\sqrt{x_2^2 + y_2^2 + f^2}}. \quad (4\text{-}48)$$

are converted to distances and used to solve the closed form solution of the *three-dimensional intersection* problem where a triple of three points $P_1, P_2, P_3$ are given by their three-dimensional Cartesian coordinates $X_1$, $Y_1$, $Z_1$, $X_2$, $Y_2$, $Z_2$, $X_3$, $Y_3$, $Z_3$, but the coordinates of the zero point $X_0, Y_0, Z_0$ are unknown. The *dimensionless quantities* $\psi_{12}, \psi_{23}, \psi_{31}$ are space angles $\psi_{12} = \angle P_0P_1P_2$, $\psi_{23} = \angle P_0P_2P_3$, $\psi_{31} = \angle P_1P_3P_0$ which are derived from the measurements.

As a modern application, currently the polynomial approach applied here to solve the ranging problem are being subjected to obtain exact solution for the *bending angles* of the path delay problem of *atmospheric sounding* which are required to determine the atmospheric profiles of temperature, pressure and geopotential heights. This will be vital for the retrieval of the atmospheric parameters relevant for environmental monitoring.

## 5. Conclusion

The *polynomial approach* offers a direct approach to solving *nonlinear system of equations* for the ranging problem giving *univariate polynomials* that directly gives the position of the unknown station once we insert the measured distances and the coordinates of the known stations. This is in contrast to the conventional analytical approaches which has to do with differencing and substitution to arrive at the same result. The *polynomial approaches* thus adds to the treasury of approaches to solving the *nonlinear system of equations* ranging problem with an advantage of being direct and faster. Another advantage is that the *polynomial approaches* are incorporated in most computer algebra algorithms such as *Mathematica* and *Maple*. Once again, we point out that the algebraic technique requires the *nonlinear system of equations* to be converted first into algebraic.

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