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Direct Position Determination of Multiple Radio Signals

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The most common methods for position determination of radio signal emitters such as communications or radar transmitters are based on measuring a specified parameter such as angle of arrival (AOA) or time of arrival (TOA) of the signal. The measured parameters are then used to estimate the transmitter's location. Since the measurements are done at each base station independently, without using the constraint that the AOA/TOA estimates at different base stations should correspond to the same transmitter's location, this is a suboptimal location determination technique. Further, if the number of array elements at each base station is M, and the signal waveforms are unknown, the number of cochannel simultaneous transmitters that can be localized by AOA is limited to $M - 1$. Also, most AOA algorithms fail when the sources are not well angularly separated. We propose a technique that uses exactly the same data as the common AOA methods but the position determination is direct. The proposed method can handle more than $M - 1$ cochannel simultaneous signals. Although there are many stray parameters, only a two-dimensional search is required for a planar geometry. The technique provides a natural solution to the measurements sources association problem that is encountered in AOA-based location systems. In addition to new algorithms, we provide analytical performance analysis, Cramér-Rao bounds and Monte Carlo simulations. We demonstrate that the proposed approach frequently outperforms the traditional AOA methods for unknown as well as known signal waveforms.

Keywords and phrases: angle of arrival, time of arrival, array processing, emitter localization.

1. INTRODUCTION

The problem of emitter location attracts much interest in the signal processing, communications, and underwater acoustics literature. Defense oriented location systems have been reported since World War I. Civilian systems are now in use for the localization of cellular phone callers, spectrum monitoring, and law enforcement. Perhaps the first paper on the mathematics of emitter location, using angle of arrival (AOA), is due to Stansfield [1]. Many other publications followed including a fine review paper by Torrieri [2]. The papers by Krim and Viberg [3] and Wax [4] are comprehensive review papers on antenna array processing for location by AOA. Recently, Van Trees [5] published a book that is fully devoted to array processing. Positioning by time of arrival (TOA) and its derivatives (DTOA, EOTD) is used extensively in cellular phone localization [6], radar systems [7], and underwater acoustics [8]. In underwater acoustics, matched-field processing (MFP) is often proposed for source localization [9]. MFP can be interpreted as the maximum a posteriori (MAP) estimate of location given the observed signal at the output of an array of sensors [9, 10]. Other interpretation of MFP is the well-known beamforming extended to wide bandwidth signals, nonplanar wave fields, and unknown environmental parameters [11]. The majority of the literature on MFP focuses on single source localization.

In this correspondence, we discuss a method that solves the localization problem using the data collected at all sensors at all base stations *together*, in contradiction to the traditional AOA/TOA approach that is composed of two separate steps: (1) AOA/TOA independent estimates and (2) triangulation based on the results of the first step. The traditional techniques can be classified as *decentralized* processing methods [12, 13, 14, 15]. Wax and Kailath [12] discussed eigenstructure algorithms for narrowband signals observed by multiple arrays, assuming perfect spatial coherence across each array but no coherence between arrays. In [13], Stoica et al. proposed variants of the method of direction estimation (MODE) algorithm for decentralized processing. In [14], Weinstein discussed pairwise processing as an alternative for centralized processing of a wideband single array.

The results indicate that pairwise processing may be used at high signal-to-noise ratio (SNR) without significant loss of performance. Recently, Kozick and Sadler [15] presented performance analysis for localization of a single wideband source using multiple distributed arrays. They assume perfect spatial coherence over each array and frequency-selective coherence between arrays. The proposed method is based on bearing estimation at each array and delay estimation using the signal observed by a single element at each array.

As indicated in [16], it is rather obvious that measuring AOA/TOA at each base station separately and independently is suboptimal since this approach ignores the constraint that the measurements must correspond to the same source position. Moreover, the base stations are geographically separated, and therefore the desired signal often appears weak or absent in some of the base stations. Thus, the system must somehow ensure that all AOA/TOA measurements used to locate a specific source correspond to the same source. In the case of cochannel simultaneous sources, the localization system confronts an association problem of deciding which of the multiple AOA/TOA estimates reported by the base stations correspond to which source.

The direct position determination (DPD) method that we propose takes advantage of the rather simple propagation assumptions that are usually used for radio frequency (RF) signals. We assume a line of sight propagation with unknown complex attenuation at each base station. We also assume that all base stations are time synchronized to the level provided by GPS (approximately 50 nanoseconds). The proposed method may implicitly use both the array response at each station and the TOA at each station. We derive the maximum likelihood estimate (MLE) of the sources position. However, the cost function associated with MLE requires a multidimensional search in the multiple sources case. Thus, for multiple signals with *unknown* waveforms we resort to a method based on the ideas of Schmidt [17] also known as the MUSIC (multiple signals classification) algorithm. For multiple signals with *known* waveforms, we use the ideas of [18] to simplify the cost function. We show that for a planar geometry of sources and base stations, a two-dimensional search is sufficient to localize all sources. For a general geometry, only a three-dimensional search is needed. A side benefit of the DPD is its ability to determine the positions of more sources than the number of sensors at each base station in contrast to AOA. The DPD technique requires the transmission of the received signals (possibly sampled) to a central processing location, in contrast to AOA and TOA that require only the transmission of the measured parameters to the central processing location. This is the cost of employing DPD. The paper focuses on the multiple signals case.

In Section 2, we define the models that we use, Section 3 describes potential algorithms for known and unknown waveforms, Section 4 includes some numerical examples that demonstrate the potential of the advocated approach, and Section 5 contains our conclusions. The appendices include derivations of the Cramér-Rao bounds, the performance analysis of the algorithms, and a discussion of the frequency-domain model.

2. PROBLEM FORMULATION

Consider Q transmitters and L base stations intercepting the transmitted signals. Each base station is equipped with an antenna array consisting of M elements. The bandwidth of the signal is small compared to the inverse of the propagation time over the array aperture. Denote the qth transmitter's position by the $D\times 1$ vector of coordinates $\mathbf{p}_q$ (obviously, for planar geometry $D = 2$ and for the general case $D = 3$). The complex envelopes of the signals observed by the ℓth base station array are given by

$$\mathbf{r}_\ell(t) = \sum_{q=1}^{Q} b_{\ell q}\mathbf{a}_\ell(\mathbf{p}_q)s_q\left(t-\tau_\ell(\mathbf{p}_q)-t_q^{(0)}\right)+\mathbf{n}_\ell(t), \quad 0 \le t \le T, \tag{1}$$

where $\mathbf{r}_\ell(t)$ is a time-dependent $M \times 1$ vector, $b_{\ell q}$ is an *unknown* complex scalar representing the channel attenuation between the qth transmitter and the ℓth base station, $\mathbf{a}_\ell(\mathbf{p}_q)$ is the ℓth array response to a signal transmitted from position $\mathbf{p}_q$, and $s_q(t - \tau_\ell(\mathbf{p}_q) - t_q^{(0)})$ is the qth signal waveform, transmitted at time $t_q^{(0)}$ and delayed by $\tau_\ell(\mathbf{p}_q)$. The vector $\mathbf{n}_\ell(t)$ represents noise and interference observed by the array. The observed signal can be partitioned into K sections, each of length $T/K \gg \max\{\tau_\ell(\mathbf{p}_q)\}$. The maximum propagation time of interest is the propagation time between the base stations. For example, if the largest separation between the base stations is 10 Km then $T/K \gg 34$ microseconds and T/K of about 7 milliseconds will satisfy the requirement (see Van Trees [5, chapter 5] and Appendix D for further discussion of this model). When the total observation time T is long, the sources are assumed to be stationary. Otherwise, the location accuracy might be degraded. Each section can be Fourier transformed and the result of this process is given by the following equation:

$$\mathbf{r}_\ell(j,k) = \sum_{q=1}^{Q} b_{\ell q}\mathbf{a}_\ell(\mathbf{p}_q)s_q(j,k)e^{-i\omega_j[\tau_\ell(\mathbf{p}_q)+t_q^{(0)}]} + \mathbf{n}_\ell(j,k), \qquad j = 1,2,\dots,J;\ k = 1,2,\dots,K, \tag{2}$$

where $\mathbf{r}_\ell(j,k)$ is the Fourier coefficient of the kth section of the observed signal corresponding to frequency ω_j, $s(j,k)$ is the jth Fourier coefficient of the kth section of the signal, and $\mathbf{n}_\ell(j,k)$ represents the jth Fourier coefficient of the kth section of the noise waveform.

For easy exhibition, we define the following vectors and scalars:

$$\begin{aligned}\bar{s}_q(j,k) &\triangleq s_q(j,k)e^{-i\omega_j t_q^{(0)}},\\ \bar{\mathbf{a}}_\ell(j,\mathbf{p}_q,b_{\ell q}) &\triangleq b_{\ell q}\mathbf{a}_\ell(\mathbf{p}_q)e^{-i\omega_j\tau_\ell(\mathbf{p}_q)}.\end{aligned} \tag{3}$$

We observe that all information about the transmitter's position is embedded in the vector $\bar{\mathbf{a}}_\ell(j,\mathbf{p}_q,b_{\ell q})$. This leads to the following representation of (2):

$$\mathbf{r}_\ell(j,k) = \sum_{q=1}^{Q} \bar{\mathbf{a}}_\ell(j,\mathbf{p}_q,b_{\ell q})\bar{s}_q(j,k) + \mathbf{n}_\ell(j,k). \tag{4}$$

In matrix notation, (4) becomes

$$\begin{aligned}\mathbf{r}_\ell(j,k) &= \mathbf{A}_\ell \bar{\mathbf{s}}(j,k) + \mathbf{n}_\ell(j,k),\\ \mathbf{A}_\ell(j) &\triangleq [\bar{\mathbf{a}}_\ell(j,\mathbf{p}_1,b_{\ell 1}),\dots,\bar{\mathbf{a}}_\ell(j,\mathbf{p}_Q,b_{\ell Q})],\\ \bar{\mathbf{s}}(j,k) &\triangleq [\bar{s}_1(j,k),\dots,\bar{s}_Q(j,k)]^{\mathrm{T}}.\end{aligned} \tag{5}$$

Since the vector $\bar{\mathbf{s}}(j,k)$ is the same at all base stations, we can concatenate the observed vectors at all base stations and form the following equation that encompasses all the data and all the information of the location system at hand:

$$\begin{aligned}\mathbf{r}(j,k) &= \mathbf{A}(j)\bar{\mathbf{s}}(j,k) + \mathbf{n}(j,k),\\ \mathbf{r}(j,k) &\triangleq [\mathbf{r}_1^{\mathrm{T}}(j,k),\dots,\mathbf{r}_L^{\mathrm{T}}(j,k)]^{\mathrm{T}},\\ \mathbf{n}(j,k) &\triangleq [\mathbf{n}_1^{\mathrm{T}}(j,k),\dots,\mathbf{n}_L^{\mathrm{T}}(j,k)]^{\mathrm{T}},\\ \mathbf{A}(j) &\triangleq [\mathbf{A}_1^{\mathrm{T}}(j),\dots,\mathbf{A}_L^{\mathrm{T}}(j)]^{\mathrm{T}}.\end{aligned} \tag{6}$$

Remarks

(1) Note in passing that (6) is based on the assumption that the envelopes of the signals are the same at all base stations, up to delay and amplitude caused by the propagation channel. Although this assumption is realistic in most cases of interest, we can solve the L sets of equations represented by (5) without relying on this assumption, and still get improved localization with respect to (w.r.t.) traditional AOA. We do not proceed along this line in this work. This approach was adopted in [12].

(2) If we assume that the signals' waveforms are unknown, then it is impossible to uniquely determine the complex attenuation coefficients at all base stations and the signal waveforms. We therefore assume that the attenuation coefficients at one of the arrays (e.g., the first array) are all real and $\sum_{\ell=1}^{L}|b_{\ell q}|^2 = 1$.

The problem that we address now is how to efficiently estimate the locations of the emitters under various assumptions.

3. LOCATION ALGORITHMS

In this section, we discuss potential location algorithms for the common case of signals with unknown waveforms and for the less common case of signals with known waveforms. We make use of results presented in the literature for the AOA case by introducing modifications as necessary.

3.1. *Signals with unknown waveforms*

In this section, we assume that the receivers do not know the waveforms a priori. This is the case in most of the applications. It is straightforward to write the probability density function, under appropriate assumptions, of the observations presented in (6), as a function of the unknown parameters.

The unknown parameters include the QJK snapshots of the complex signals $\{\bar{\mathbf{s}}(j,k)\}$, the $(L-1)Q$ complex attenuation factors of the signals at the base stations $\{b_{\ell q}\}$, and the two-dimensional real-valued location vector of each transmitter $\{\mathbf{p}_q\}$ overall $2QJK + 2(L-1)Q + 2Q$ real parameters. The MLE will therefore require a complex multidimensional search over the parameter space.

For a single source an MLE was presented in [19], where the multidimensional search is replaced by a D-dimensional search. In order to avoid the multidimensional search for the multisource case, we can follow the steps leading to the MUSIC algorithm [17]. First note that

$$\begin{aligned}\mathbf{R}(j) &\triangleq E\{\mathbf{r}(j,k)\mathbf{r}^H(j,k)\} = \mathbf{A}(j)\mathbf{\Lambda}(j)\mathbf{A}^H(j) + \eta\mathbf{I},\\ \mathbf{\Lambda}(j) &\triangleq E\{\bar{\mathbf{s}}(j,k)\bar{\mathbf{s}}^H(j,k)\},\\ E\{\mathbf{n}(j,k)\mathbf{n}^H(j,k)\} &= \eta\mathbf{I},\end{aligned} \tag{7}$$

where we assumed that the noise is temporally and spatially white, uncorrelated between sensors and between frequencies, uncorrelated with the signals and is zero-mean with variance η. The column vectors of $\mathbf{A}$ are orthogonal to the noise subspace of $\mathbf{R}(j)$ and contained in the signal subspace.

Following the MUSIC algorithm, we propose the cost function

$$\begin{aligned}F(\mathbf{p},\mathbf{b}) &\triangleq \sum_j \bar{\mathbf{a}}^H(j,\mathbf{p},\mathbf{b})\mathbf{U}_s(j)\mathbf{U}_s^H(j)\bar{\mathbf{a}}(j,\mathbf{p},\mathbf{b}),\\ \bar{\mathbf{a}}(j,\mathbf{p},\mathbf{b}) &\triangleq [\bar{\mathbf{a}}_1^{\mathrm{T}}(j,\mathbf{p},b_1),\dots,\bar{\mathbf{a}}_L^{\mathrm{T}}(j,\mathbf{p},b_L)]^{\mathrm{T}},\\ \mathbf{b} &\triangleq [b_1,\dots,b_L]^{\mathrm{T}},\end{aligned} \tag{8}$$

where $\mathbf{U}_s(j)$ is an $ML \times Q$ matrix consisting of the eigenvectors of $\mathbf{R}(j)$ corresponding to the Q largest eigenvalues. Here, $\mathbf{p}$ and $\mathbf{b}$ are variable vectors representing the unknown position and unknown attenuations. Recall that the vectors $\bar{\mathbf{a}}(j,\mathbf{p},\mathbf{b})$ contain the L unknown complex attenuation coefficients in addition to the unknown location. The minimum points of $F(\mathbf{p},\mathbf{b})$ depend on all unknowns and therefore require a $2(L-1)+D$ dimensional search.

In order to reduce this search, we propose to represent $\bar{\mathbf{a}}(j,\mathbf{p},\mathbf{b})$ as follows:

$$\begin{aligned}\bar{\mathbf{a}}(j,\mathbf{p},\mathbf{b}) &= \mathbf{\Gamma}(j)\mathbf{H}\mathbf{b},\\ \mathbf{\Gamma}(j) &\triangleq \operatorname{diag}\left\{\mathbf{a}_1^{\mathrm{T}}(\mathbf{p})e^{-i\omega_j\tau_1(\mathbf{p})},\dots,\mathbf{a}_L^{\mathrm{T}}(\mathbf{p})e^{-i\omega_j\tau_L(\mathbf{p})}\right\},\\ \mathbf{H} &\triangleq \mathbf{I}_L \otimes \mathbf{1}_M,\end{aligned} \tag{9}$$

where $\mathbf{\Gamma}(j)$ is a diagonal matrix whose elements are the elements of the response vectors of the arrays at all base stations, $\mathbf{I}_L$ stands for the $L \times L$ identity matrix, $\mathbf{1}_M$ stands for an $M \times 1$ column vector of ones, and finally $\otimes$ stands for the Kronecker product. Substituting (9) in (8), we get

$$F(\mathbf{p},\mathbf{b}) = \mathbf{b}^H\mathbf{H}^H\left[\sum_j \mathbf{\Gamma}^H(j)\mathbf{U}_s(j)\mathbf{U}_s^H(j)\mathbf{\Gamma}(j)\right]\mathbf{H}\mathbf{b}. \tag{10}$$

Recall that we assumed that the norm of $\mathbf{b}$ is one in order to facilitate a unique solution. Hence, for any assumed position $\mathbf{p}$, the maximum of $F(\mathbf{p},\mathbf{b})$ corresponds to the maximal eigenvalue of the matrix $\mathbf{D}(\mathbf{p})$ defined by

$$\mathbf{D}(\mathbf{p}) \triangleq \mathbf{H}^H\left[\sum_j \mathbf{\Gamma}^H\mathbf{U}_s(j)\mathbf{U}_s^H(j)\mathbf{\Gamma}\right]\mathbf{H}. \tag{11}$$

Therefore, (10) reduces to

$$F(\mathbf{p}) = \lambda_{\max}[\mathbf{D}(\mathbf{p})], \tag{12}$$

where the right-hand side of (12) denotes the largest eigenvalue of $\mathbf{D}(\mathbf{p})$, and the matrix $\mathbf{D}(\mathbf{p})$ is a function of the observed data (i.e., $\mathbf{U}_s(j)$) and the array response at each base station to an emitter located at $\mathbf{p}$. It is clear that the maximization of (12) requires only a D-dimensional search. It is interesting to note that the dimensions of the matrix $\mathbf{D}(\mathbf{p})$ are $L \times L$ which are usually rather small.

Obviously, one can adjust many known algorithms to handle the problem at hand including conventional beamforming, Capon's method, min-norm, and so forth.

3.2. *Signals with known waveforms*

In certain applications, the transmitted waveforms are known to the location system. For example, in cellular systems synchronization and training sequences are transmitted periodically and are known *a-priori*. Moreover, it is possible to detect the data sequence of a digitally modulated signal and then restore the complex signal envelope based on the known modulation scheme. In this section, we examine the position determination problem for known waveforms. We start by following the algebraic steps taken in [18].

We assume that the noise $\mathbf{n}(j,k)$ is circularly symmetric complex Gaussian random vector with zero-mean and second-orders statistic given by

$$\begin{aligned} E\{\mathbf{n}(i,k)\mathbf{n}^H(j,\ell)\} &= \eta\mathbf{I}\delta_{ij}\delta_{k\ell},\\ E\{\mathbf{n}(i,k)\mathbf{n}^{\mathrm{T}}(j,\ell)\} &= 0. \end{aligned} \tag{13}$$

Define the signal sample covariance:

$$\hat{\mathbf{R}}_{ss}(j) \triangleq \frac{1}{K}\sum_{k=1}^{K}\bar{\mathbf{s}}(j,k)\bar{\mathbf{s}}^H(j,k). \tag{14}$$

We further assume that asymptotically as $K \to \infty$, the signals sample covariance is diagonal.

The log-likelihood function of the array output vectors $\mathbf{r}(j,k)$ is proportional to

$$\begin{aligned} F &= \sum_{j=1}^{J}\frac{1}{K}\sum_{k=1}^{K}\left|\left|\mathbf{r}(j,k)-\mathbf{A}(j)\bar{\mathbf{s}}(j,k)\right|\right|^2\\ &= \sum_{j=1}^{J}\frac{1}{K}\sum_{k=1}^{K}\left[\mathbf{r}(j,k)-\mathbf{A}(j)\bar{\mathbf{s}}(j,k)\right]^H\left[\mathbf{r}(j,k)-\mathbf{A}(j)\bar{\mathbf{s}}(j,k)\right]\\ &= \sum_{j=1}^{J}\frac{1}{K}\sum_{k=1}^{K}\mathbf{r}^H(j,k)\mathbf{r}(j,k)-\mathbf{r}^H(j,k)\mathbf{A}(j)\bar{\mathbf{s}}(j,k)\\ &\qquad -\bar{\mathbf{s}}^H(j,k)\mathbf{A}^H(j)\mathbf{r}(j,k)\\ &\qquad +\bar{\mathbf{s}}^H(j,k)\mathbf{A}^H(j)\mathbf{A}(j)\bar{\mathbf{s}}(j,k). \end{aligned} \tag{15}$$

Define

$$\hat{\mathbf{R}}_{sr}(j) \triangleq \frac{1}{K}\sum_{k=1}^{K}\bar{\mathbf{s}}(j,k)\mathbf{r}^H(j,k). \tag{16}$$

Substituting (14) and (16) in (15) and ignoring the first term, which is constant, we get

$$\begin{aligned} F_1 &= \operatorname{tr}\Bigg\{\sum_{j=1}^{J}\frac{1}{K}\sum_{k=1}^{K}-\bar{\mathbf{s}}(j,k)\mathbf{r}^H(j,k)\mathbf{A}(j)\\ &\qquad -\mathbf{A}^H(j)\mathbf{r}(j,k)\bar{\mathbf{s}}^H(j,k)\\ &\qquad +\mathbf{A}^H(j)\mathbf{A}(j)\bar{\mathbf{s}}(j,k)\bar{\mathbf{s}}^H(j,k)\Bigg\}\\ &= \operatorname{tr}\left\{\sum_{j=1}^{J}-\hat{\mathbf{R}}_{sr}(j)\mathbf{A}(j)-\mathbf{A}^H(j)\hat{\mathbf{R}}_{sr}^H(j)+\mathbf{A}^H(j)\mathbf{A}(j)\hat{\mathbf{R}}_{ss}(j)\right\}\\ &= \operatorname{tr}\Bigg\{\sum_{j=1}^{J}-\hat{\mathbf{R}}_{ss}(j)\hat{\mathbf{R}}_{ss}^{-1}(j)\hat{\mathbf{R}}_{sr}(j)\mathbf{A}(j)\\ &\qquad -\mathbf{A}^H\hat{\mathbf{R}}_{sr}^H(j)\hat{\mathbf{R}}_{ss}^{-1}(j)\hat{\mathbf{R}}_{ss}(j)+\mathbf{A}^H(j)\mathbf{A}(j)\hat{\mathbf{R}}_{ss}(j)\Bigg\}. \end{aligned} \tag{17}$$

Note that

$$\operatorname{tr}\{\hat{\mathbf{R}}_{ss}(j)\hat{\mathbf{R}}_{ss}^{-1}(j)\hat{\mathbf{R}}_{sr}(j)\mathbf{A}(j)\} = \operatorname{tr}\{\hat{\mathbf{R}}_{ss}^{-1}(j)\hat{\mathbf{R}}_{sr}(j)\mathbf{A}(j)\hat{\mathbf{R}}_{ss}(j)\}. \tag{18}$$

And therefore (17) can be displayed as follows:

$$\begin{aligned} F_1 &= \operatorname{tr}\Bigg\{\sum_{j=1}^{J}\Big[-\hat{\mathbf{R}}_{ss}^{-1}(j)\hat{\mathbf{R}}_{sr}(j)\mathbf{A}(j)-\mathbf{A}^H(j)\hat{\mathbf{R}}_{sr}^H(j)\hat{\mathbf{R}}_{ss}^{-1}(j)\\ &\qquad +\mathbf{A}^H(j)\mathbf{A}(j)\Big]\hat{\mathbf{R}}_{ss}(j)\Bigg\}. \end{aligned} \tag{19}$$

Define

$$\hat{\mathbf{A}}(j) = \hat{\mathbf{R}}_{sr}^H(j)\hat{\mathbf{R}}_{ss}^{-1}(j). \tag{20}$$

Then, minimizing F_1 is equivalent to minimizing

$$F_2 = \operatorname{tr}\left\{\sum_{j=1}^{J}\left[\widehat{\mathbf{A}}^H(j)\widehat{\mathbf{A}}(j) - \widehat{\mathbf{A}}^H(j)\mathbf{A}(j) - \mathbf{A}^H(j)\widehat{\mathbf{A}}(j) + \mathbf{A}^H(j)\mathbf{A}(j)\right]\widehat{\mathbf{R}}_{ss}(j)\right\} \tag{21}$$

$$= \operatorname{tr}\left\{\sum_{j=1}^{J}[\mathbf{A}(j) - \widehat{\mathbf{A}}(j)]^H[\mathbf{A}(j) - \widehat{\mathbf{A}}(j)]\widehat{\mathbf{R}}_{ss}(j)\right\}.$$

Obviously, the term $\widehat{\mathbf{A}}^H(j)\widehat{\mathbf{A}}(j)$ is constant and can be added to the cost function.

Since we assumed that the signals are uncorrelated, $\widehat{\mathbf{R}}_{ss}(j)$ is asymptotically diagonal and the cost function can be decoupled:

$$F_3 = \sum_{q=1}^{Q} F_3(q),$$
$$F_3(q) \triangleq \sum_{j=1}^{J}\|\bar{\mathbf{a}}(j,\mathbf{p}_q,\mathbf{b}_q) - \hat{\mathbf{a}}_q(j)\|^2, \tag{22}$$

where $\bar{\mathbf{a}}(j,\mathbf{p}_q,\mathbf{b}_q)$ and $\hat{\mathbf{a}}_q(j)$ represent the qth column of $\mathbf{A}(j)$ and $\widehat{\mathbf{A}}(j)$, respectively.

The minimization of $F_3(q)$ can be done as follows:

$$F_3(q) = \sum_{j=1}^{J}\|\bar{\mathbf{a}}(j,\mathbf{p}_q,\mathbf{b}_q)-\hat{\mathbf{a}}_q(j)\|^2 = \sum_{j=1}^{J}\|\mathbf{\Gamma}_q(j)\mathbf{H}\mathbf{b}_q-\hat{\mathbf{a}}_q(j)\|^2. \tag{23}$$

The vector $\mathbf{b}$ that minimizes the cost function is given by

$$\hat{\mathbf{b}}_q = \left(\sum_j \mathbf{H}^H\mathbf{\Gamma}_q^H(j)\mathbf{\Gamma}_q(j)\mathbf{H}\right)^{-1}\mathbf{H}^H\sum_j\mathbf{\Gamma}_q^H(j)\hat{\mathbf{a}}_q(j). \tag{24}$$

Replacing $\mathbf{b}$ with $\hat{\mathbf{b}}$ in (23), we get a cost function that depends on $\mathbf{p}$ only. The cost function can be simplified by relying on the assumption that

$$\|\mathbf{a}_\ell(j)\| = 1 \quad \forall j,\ell,\mathbf{p}, \tag{25}$$

which immediately leads to

$$\sum_j \mathbf{H}^H\mathbf{\Gamma}_q^H(j)\mathbf{\Gamma}_q(j)\mathbf{H} = \mathbf{I}_L. \tag{26}$$

Define the vector

$$\mathbf{w}_q \triangleq \sum_j \mathbf{\Gamma}_q^H(j)\hat{\mathbf{a}}_q(j). \tag{27}$$

Substituting (27) and(26) in (24) and, in turn, in (23) yields

$$F_3(q) = \sum_{j=1}^{J}\|\mathbf{\Gamma}_q(j)\mathbf{H}\mathbf{H}^H\mathbf{w}_q - \hat{\mathbf{a}}_q(j)\|^2. \tag{28}$$

It is easy to verify that minimizing (28) is equivalent to maximizing

$$F_4(q) = \mathbf{w}_q^H\mathbf{H}\mathbf{H}^H\mathbf{w}_q = \|\mathbf{H}^H\mathbf{w}_q\|^2$$
$$= \sum_{\ell=1}^{L}\left|\sum_{j=1}^{J} e^{i\omega_j\tau_\ell(\mathbf{p})}\mathbf{a}_\ell^H(\mathbf{p})\hat{\mathbf{a}}_q^{(\ell)}(j)\right|^2, \tag{29}$$

where $\hat{\mathbf{a}}_q^{(\ell)}(j)$ is the ℓth subvector of $\hat{\mathbf{a}}_q(j)$ (i.e., the subvector associated with the ℓth base station).

Note that (29) indicates that the cost function is in fact a sum of L distinct cost functions, each associated with a distinct base station. This is the reason that DPD outperforms methods that maximize the cost function at each base station independently.

Thus, we can locate each of the emitters by a simple D-dimensional search.

4. NUMERICAL RESULTS

In order to examine the performance of the advocated methods and compare it with the traditional AOA approach, we performed extensive Monte Carlo simulations. Some examples are shown here.

We applied two different techniques in order to locate the transmitter:

(1) AOA estimation using MUSIC or Beamforming at each base station independently;
(2) DPD according to the algorithms described in the previous section.

The performance evaluation is based on the RMS error defined as follows:

$$\text{RMS} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - x_t)^2 + (\hat{y}_i - y_t)^2}, \tag{30}$$

where (x_t, y_t) is the emitter location, $(\hat{x}_i, \hat{y}_i)$ is the ith location estimate, and N is the number of experiments.

Test Case 1

Consider three base stations placed at coordinates $(2,-2)$, $(2,0)$, and $(2,2)$ and a single emitter located at $(0,1.5)$. All coordinates are in km. The transmitted signal is a carrier amplitude modulated by a narrowband random Gaussian waveform. The signal is unknown to the receivers. Each base station is equipped with a uniform linear array (ULA) of only three antenna elements. Each location determination is based on 200 snapshots each of 4.5 milliseconds at a single frequency (i.e., $K = 200$, $J = 1$). The snapshot length ensures that the errors introduced by the finite-length FFT are 30 dB below the signal level. The SNR (at the base station receiving the strongest signal) is varied between 3 dB and 23 dB using 2 dB steps. At each SNR value,

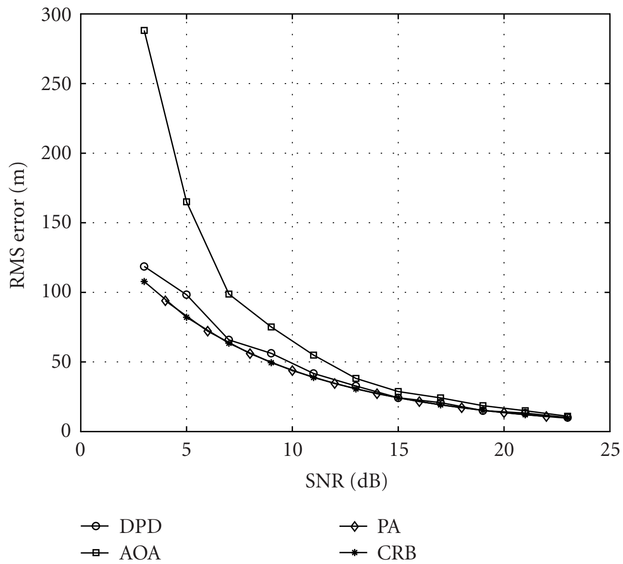

Figure 1: RMS error of DPD and traditional AOA, Cramér-Rao bound, and performance analysis results for three base stations, a single source, and unknown waveforms ($L = 3$, $M = 3$, $Q = 1$, $K = 200$, $J = 1$, and $\mathbf{b} = [1, 0.8, 0.4]$).

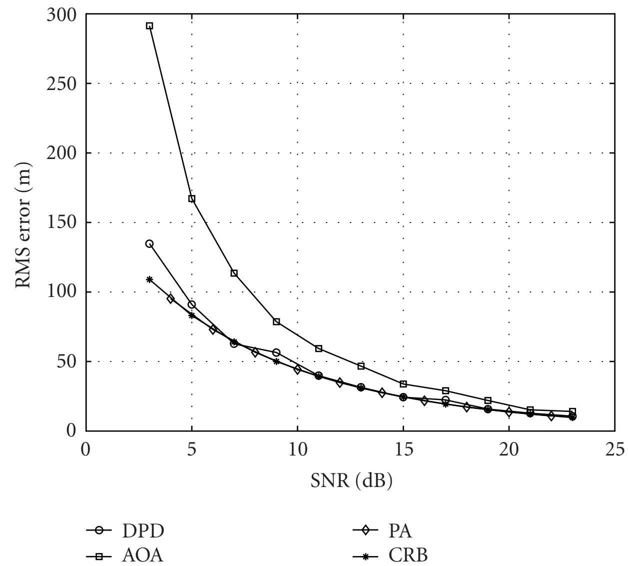

Figure 2: RMS error of DPD and traditional AOA, Cramér-Rao bound, and performance analysis results for three base stations, two sources, and unknown waveforms ($L = 3$, $M = 3$, $Q = 2$, $K = 200$, $J = 1$, and $\mathbf{b} = [1, 0.8, 0.4]$).

we performed 100 experiments in order to obtain the statistical properties of the performance. The attenuation vector is selected as $\mathbf{b} = [1, 0.8, 0.4]^T$.

Figure 1 shows the experimental results, using (12), the Cramér-Rao bound, derived in Appendix B, and the performance analysis results described in Appendix C. The plots indicate that DPD is superior to the traditional approach of independent AOA estimates at each base station. The advantage of DPD is at low SNR. At high SNR, both methods give results that coincide with the theoretical bound.

Test Case 2

In a second experiment, we kept the base stations at the same locations and we used two emitters placed at $(0, +1.5)$ and $(0, -1.5)$. The results for each of the sources are shown in Figure 2. It is again clear that the DPD outperforms AOA at low SNR while both methods are equivalent at high SNR. At very high SNR, modeling errors will dominate the performance. Modeling errors include finite-length FFT, calibration errors, synchronizations errors, propagation errors, and so forth.

Test Case 3

In a third experiment, we kept the base stations at the same locations and we used two emitters placed at $(0, +Y)$ and $(0, -Y)$ and 100 snapshots, and SNR $=$ 20 dB. The channel attenuation to all base stations is equal. We changed Y from 200 meters to 1200 meters and the results are plotted in Figure 3. It can be seen, as expected, that the traditional AOA accuracy is very sensitive to sources that are not well separated as opposed to the DPD method.

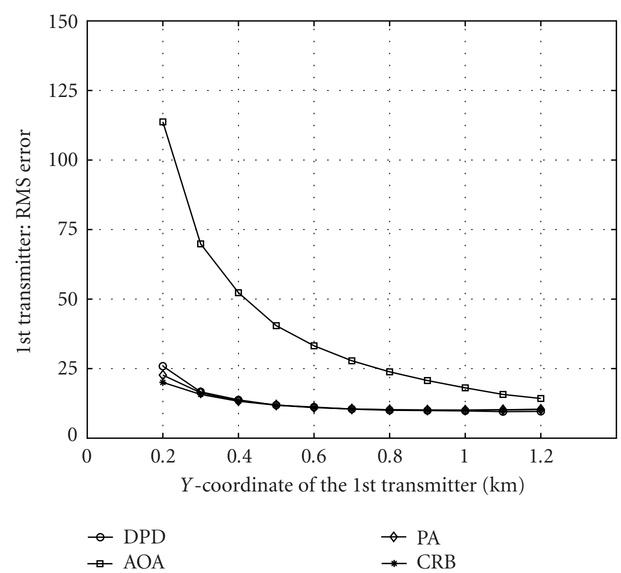

Figure 3: RMS error of DPD and traditional AOA, Cramér-Rao bound, and performance analysis results for three base stations, two sources with increasing separation, and unknown waveforms ($L = 3$, $M = 3$, $Q = 2$, $K = 200$, $J = 1$, SNR $=$ 20 dB, and $\mathbf{b} = [1, 1, 1]$).

Test Case 4

In a fourth experiment, we kept the base stations at the same location and placed three transmitters at $(0, 1.5)$, $(0, -1.5)$, and $(-1, 0)$. Each base station collects 1000 snapshots and the attenuation is equal at all base stations. Since each base station is equipped with an array of only three elements of traditional AOA based on MUSIC fails. However, DPD works fine as shown in Figure 4.

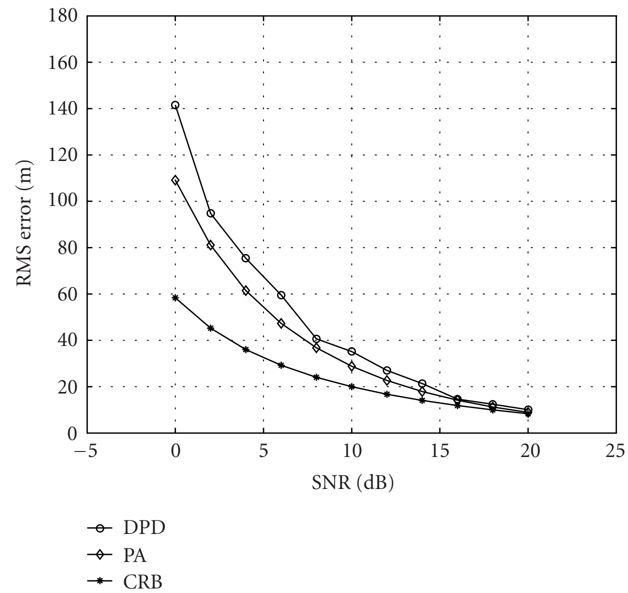

FIGURE 4: RMS error of DPD, Cramér-Rao bound, and performance analysis results for three base stations, three sources, and unknown waveforms ($L = 3$, $M = 3$, $Q = 3$, $K = 1000$, $J = 1$, and $\mathbf{b} = [1, 1, 1]$).

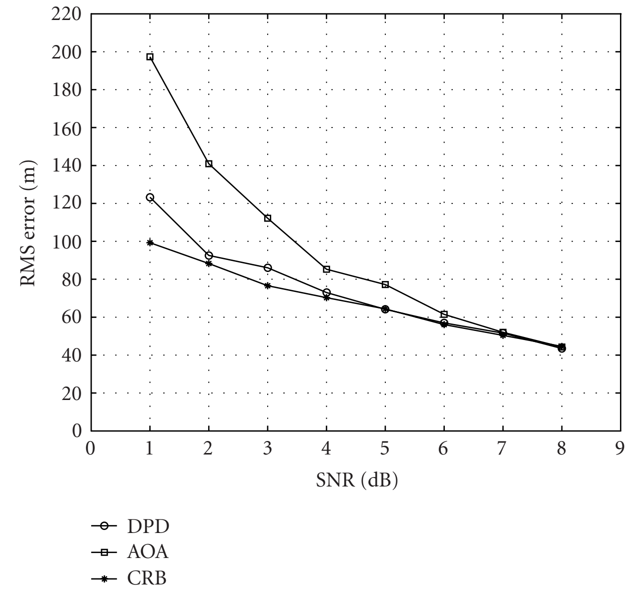

FIGURE 5: RMS error of DPD and traditional AOA for *known* waveforms. Four base stations each equipped with five-element circular arrays are used ($L = 4$, $M = 5$, $Q = 1$, $K = 1000$, $J = 1$, and $\mathbf{b} = [1, 0.1, 0.1, 1]$).

Test Case 5

In a fifth experiment, we used four base stations located at $(-2, -2)$, $(-2, +2)$, $(2, -2)$, and $(2, +2)$ and a single source at $(1, 1)$. Each base station is equipped with a circular array of five elements. The waveforms are known. The number of snapshots is 1000. The attenuation to two base stations is 0 dB and for the other two is -10 dB. The accuracy results are plotted in Figure 5.

In Figure 6, we show how the cost function looks for unknown waveforms, four base stations, each equipped with an array of only three elements, and three transmitters. Common AOA methods cannot handle three transmitters with three-element array contrary to the advocated method.

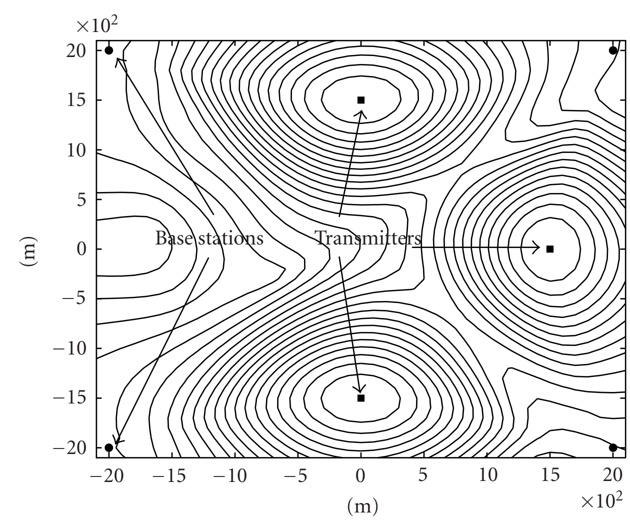

FIGURE 6: Contour description of the cost function for unknown waveforms, four base stations, and three transmitters.

5. CONCLUSIONS

We have proposed a direct position determination technique for localizing multiple narrowband radio frequency sources. The technique can locate more sources than the traditional AOA approach. Moreover, DPD provides better accuracy than traditional AOA and it does not encounter the association problem of independent AOA measurements at each base station. The proposed technique uses the MUSIC approach in order to reduce the complexity of the algorithm in the case of unknown waveforms. The advantages of DPD do not come without a price. While in traditional methods only AOA estimates must be transferred to a central processing location for triangulation, the DPD requires raw signal data to be transferred to a common processor.

APPENDICES

A. CRB FOR KNOWN SIGNALS

We start with (6) repeated here for easy reference:

$$\mathbf{r}(j, k) = \mathbf{A}(j)\bar{\mathbf{s}}(j, k) + \mathbf{n}(j, k). \tag{A.1}$$

The unknown parameters are the entries of $\mathbf{P}$ and $\mathbf{B}$ defined as follows:

$$\begin{aligned}\mathbf{P} &\triangleq [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_Q],\\ \mathbf{B} &\triangleq [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_Q],\\ \mathbf{b}_q &\triangleq [b_{1,q}, b_{2,q}, \ldots, b_{L,q}]^{\mathrm{T}}.\end{aligned} \tag{A.2}$$

The log-likelihood function is given by

$$\ln L = \text{const} - \frac{1}{\eta}\sum_{j,k}\left|\left|\mathbf{r}(j,k) - \mathbf{A}(j)\bar{\mathbf{s}}(j,k)\right|\right|^2. \tag{A.3}$$

It is easy to verify the derivatives

$$\frac{\partial \ln L}{\partial P_{nq}} = \frac{2}{\eta}\,\mathrm{Re}\left\{\sum_{j,k}\bar{\mathbf{s}}_q^*(j,k)\frac{\partial \mathbf{a}_q^H(j)}{\partial P_{nq}}\mathbf{n}(j,k)\right\}, \tag{A.4}$$

where P_{nq} is the n, q element of the matrix $\mathbf{P}$, and $\mathbf{a}_q(j)$ is the qth column of $\mathbf{A}(j)$. Define the matrices

$$\begin{aligned}\mathbf{D}_n(j) &\triangleq \left[\frac{\partial \mathbf{a}_1(j)}{\partial P_{n1}}, \frac{\partial \mathbf{a}_2(j)}{\partial P_{n2}}, \ldots, \frac{\partial \mathbf{a}_Q(j)}{\partial P_{nQ}}\right],\\ \mathbf{X}(j,k) &\triangleq \mathrm{diag}\left\{\bar{\mathbf{s}}(j,k)\right\}.\end{aligned} \tag{A.5}$$

Now (A.4) can be written as follows:

$$\frac{\partial \ln L}{\partial \mathbf{P}_n} = \frac{2}{\eta}\,\mathrm{Re}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\mathbf{D}_n^H(j)\mathbf{n}(j,k)\right\}, \tag{A.6}$$

where $\mathbf{P}_n$ is the nth row of $\mathbf{P}$.

Define now the following variables:

$$\begin{aligned}\bar{\mathbf{C}}_n(j) &\triangleq \left[\frac{\partial \mathbf{a}_1(j)}{\partial \bar{\mathbf{b}}_{n1}}, \frac{\partial \mathbf{a}_2(j)}{\partial \bar{\mathbf{b}}_{n2}}, \ldots, \frac{\partial \mathbf{a}_Q(j)}{\partial \bar{\mathbf{b}}_{nQ}}\right],\\ \tilde{\mathbf{C}}_n(j) &\triangleq \left[\frac{\partial \mathbf{a}_1(j)}{\partial \tilde{\mathbf{b}}_{n1}}, \frac{\partial \mathbf{a}_2(j)}{\partial \tilde{\mathbf{b}}_{n2}}, \ldots, \frac{\partial \mathbf{a}_Q(j)}{\partial \tilde{\mathbf{b}}_{nQ}}\right] = i\bar{\mathbf{C}}_n(j).\end{aligned} \tag{A.7}$$

We can now write

$$\begin{aligned}\frac{\partial \ln L}{\partial \bar{\mathbf{B}}_n} &= \frac{2}{\eta}\,\mathrm{Re}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\bar{\mathbf{C}}_n^H(j)\mathbf{n}(j,k)\right\},\\ \frac{\partial \ln L}{\partial \tilde{\mathbf{B}}_n} &= \frac{2}{\eta}\,\mathrm{Im}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\bar{\mathbf{C}}_n^H(j)\mathbf{n}(j,k)\right\},\end{aligned} \tag{A.8}$$

where $\bar{\mathbf{B}}_n$, $\tilde{\mathbf{B}}_n$ are the real and imaginary parts of the nth row of $\mathbf{B}$, respectively.

The blocks of the Fisher information matrix (FIM) are given by

$$\begin{aligned}\mathrm{FIM}_{11}(n,m) &\triangleq E\left[\frac{\partial \ln L}{\partial \bar{\mathbf{B}}_n}\right]\left[\frac{\partial \ln L}{\partial \bar{\mathbf{B}}_m}\right]^{\mathrm{T}}\\ &= \frac{2}{\eta}\,\mathrm{Re}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\bar{\mathbf{C}}_n^H(j)\bar{\mathbf{C}}_m(j)\mathbf{X}(j,k)\right\},\\ \mathrm{FIM}_{12}(n,m) &\triangleq E\left[\frac{\partial \ln L}{\partial \bar{\mathbf{B}}_n}\right]\left[\frac{\partial \ln L}{\partial \tilde{\mathbf{B}}_m}\right]^{\mathrm{T}}\\ &= -\frac{2}{\eta}\,\mathrm{Im}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\bar{\mathbf{C}}_n^H(j)\bar{\mathbf{C}}_m(j)\mathbf{X}(j,k)\right\},\\ \mathrm{FIM}_{22}(n,m) &\triangleq E\left[\frac{\partial \ln L}{\partial \tilde{\mathbf{B}}_n}\right]\left[\frac{\partial \ln L}{\partial \tilde{\mathbf{B}}_m}\right]^{\mathrm{T}}\\ &= \frac{2}{\eta}\,\mathrm{Re}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\bar{\mathbf{C}}_n^H(j)\bar{\mathbf{C}}_m(j)\mathbf{X}(j,k)\right\},\end{aligned} \tag{A.9}$$

where $\mathrm{FIM}_{i,j}(n,m)$ stands for the n, m subblock of the $\mathrm{FIM}_{i,j}$ block.

$$\begin{aligned}\mathrm{FIM}_{31} &\triangleq E\left[\frac{\partial \ln L}{\partial \mathbf{P}_n}\right]\left[\frac{\partial \ln L}{\partial \bar{\mathbf{B}}_m}\right]^{\mathrm{T}}\\ &= \frac{2}{\eta}\,\mathrm{Re}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\mathbf{D}_n^H(j)\bar{\mathbf{C}}_m(j)\mathbf{X}(j,k)\right\},\\ \mathrm{FIM}_{32} &\triangleq E\left[\frac{\partial \ln L}{\partial \mathbf{P}_n}\right]\left[\frac{\partial \ln L}{\partial \tilde{\mathbf{B}}_m}\right]^{\mathrm{T}}\\ &= -\frac{2}{\eta}\,\mathrm{Im}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\mathbf{D}_n^H(j)\bar{\mathbf{C}}_m(j)\mathbf{X}(j,k)\right\},\\ \mathrm{FIM}_{33} &\triangleq E\left[\frac{\partial \ln L}{\partial \mathbf{P}_n}\right]\left[\frac{\partial \ln L}{\partial \mathbf{P}_m}\right]^{\mathrm{T}}\\ &= \frac{2}{\eta}\,\mathrm{Re}\left\{\sum_{j,k}\mathbf{X}^H(j,k)\mathbf{D}_n^H(j)\mathbf{D}_m(j)\mathbf{X}(j,k)\right\}.\end{aligned} \tag{A.10}$$

The CRB bound is obtained by inverting the FIM.

B. CRB FOR UNKNOWN GAUSSIAN SIGNALS

It is well known that the FIM for zero-mean Gaussian signals is given by

$$[\mathrm{FIM}]_{i,j} = \mathrm{tr}\left\{\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial \theta_i}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial \theta_j}\right\}, \tag{B.1}$$

where $\mathbf{R}$ is the observations covariance and θ_i is the ith parameter. The covariance matrix for a given frequency is

$$\mathbf{R} = \mathbf{A}\mathbf{\Lambda}\mathbf{A}^H + \eta\mathbf{I}. \tag{B.2}$$

The unknown parameters are the entries of $\mathbf{P}$, $\mathbf{B}$, $\mathbf{\Lambda}$ defined in (A.2). We will frequently use the notation $\mathbf{e}_n$ for nth column vector of the identity matrix. First note that

$$\begin{aligned}
\operatorname{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\Lambda_i}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\Lambda_j}\right\} &= \operatorname{tr}\left\{\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_j\mathbf{e}_j^{\mathrm{T}}\mathbf{A}^H\right\} \\
&= \left(\mathbf{e}_j^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\right)\left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_j\right) \\
&= \left(\mathbf{e}_j^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\right)\left(\mathbf{e}_j^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\right)^*.
\end{aligned} \tag{B.3}$$

Thus, the FIM related to the diagonal elements of the signal covariance is given by

$$\mathrm{FIM}_{\Lambda\Lambda} = \left(\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\right)\times\left(\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\right)^*, \tag{B.4}$$

where $\times$ denotes element-by-element multiplication.

We can write

$$\begin{aligned}
&\operatorname{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{q\ell}}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{km}}\right\} \\
&\quad= \operatorname{tr}\Big\{\mathbf{R}^{-1}\left[\bar{\mathbf{C}}_\ell\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H+\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}\bar{\mathbf{C}}_\ell^H\right] \\
&\qquad\times\mathbf{R}^{-1}\left[\bar{\mathbf{C}}_m\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H+\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\right]\Big\}.
\end{aligned} \tag{B.5}$$

Rearranging the terms, we get

$$\begin{aligned}
&\operatorname{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{q\ell}}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{km}}\right\} \\
&\quad= \operatorname{tr}\left\{\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\right\} \\
&\qquad+ \operatorname{tr}\left\{\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\right\} \\
&\qquad+ \operatorname{tr}\left\{\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}\bar{\mathbf{C}}_\ell^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\right\} \\
&\qquad+ \operatorname{tr}\left\{\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}\bar{\mathbf{C}}_\ell^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\right\}.
\end{aligned} \tag{B.6}$$

Taking advantage of the trace operator, we can write

$$\begin{aligned}
&\operatorname{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{q\ell}}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{km}}\right\} \\
&\quad= \left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\right)\left(\mathbf{e}_q^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\right) \\
&\qquad+ \left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\right)\left(\mathbf{e}_q^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\right) \\
&\qquad+ \left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\right)\left(\mathbf{e}_q^{\mathrm{T}}\bar{\mathbf{C}}_\ell^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\right) \\
&\qquad+ \left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\right)\left(\mathbf{e}_q^{\mathrm{T}}\bar{\mathbf{C}}_\ell^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\right).
\end{aligned} \tag{B.7}$$

Rearranging the terms again, we get

$$\begin{aligned}
&\operatorname{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{q\ell}}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{km}}\right\} \\
&\quad= \left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\right)\left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\right)^* \\
&\qquad+ \left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\right)\left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\right)^* \\
&\qquad+ \left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\right)\left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\right)^* \\
&\qquad+ \left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\right)\left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\right)^*.
\end{aligned} \tag{B.8}$$

Finally,

$$\begin{aligned}
&\operatorname{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{q\ell}}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{km}}\right\} \\
&\quad= 2\operatorname{Re}\left\{\left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\right)\left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\right)^*\right\} \\
&\qquad+ 2\operatorname{Re}\left\{\left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\mathbf{e}_q\right)\left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_q\right)^*\right\}.
\end{aligned} \tag{B.9}$$

Thus, the corresponding blocks of the FIM are given by

$$\begin{aligned}
\mathrm{FIM}_{\bar{\mathbf{B}}_m\bar{\mathbf{B}}_\ell} &= 2\operatorname{Re}\Big\{\left(\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\right)\times\left(\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\right)^* \\
&\qquad+ \left(\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\right)\times\left(\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\right)^*\Big\} \\
&= 2\operatorname{Re}\left\{\left(\mathbf{A}_1^H\bar{\mathbf{C}}_\ell\right)\times\left(\bar{\mathbf{C}}_m^H\mathbf{A}_1\right)^*+\left(\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\right)\times\mathbf{A}_2^*\right\}, \\
\mathrm{FIM}_{\bar{\mathbf{B}}_m\tilde{\mathbf{B}}_\ell} &= 2\operatorname{Re}\left\{\left(\mathbf{A}_1^H\tilde{\mathbf{C}}_\ell\right)\times\left(\bar{\mathbf{C}}_m^H\mathbf{A}_1\right)^*+\left(\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\tilde{\mathbf{C}}_\ell\right)\times\mathbf{A}_2^*\right\}, \\
\mathrm{FIM}_{\tilde{\mathbf{B}}_m\tilde{\mathbf{B}}_\ell} &= 2\operatorname{Re}\left\{\left(\mathbf{A}_1^H\tilde{\mathbf{C}}_\ell\right)\times\left(\tilde{\mathbf{C}}_m^H\mathbf{A}_1\right)^*+\left(\tilde{\mathbf{C}}_m^H\mathbf{R}^{-1}\tilde{\mathbf{C}}_\ell\right)\times\mathbf{A}_2^*\right\}, \\
\mathrm{FIM}_{\mathbf{P}_m\mathbf{P}_\ell} &= 2\operatorname{Re}\left\{\left(\mathbf{A}_1^H\mathbf{D}_\ell\right)\times\left(\mathbf{D}_m^H\mathbf{A}_1\right)^*+\left(\mathbf{D}_m^H\mathbf{R}^{-1}\mathbf{D}_\ell\right)\times\mathbf{A}_2^*\right\}, \\
\mathrm{FIM}_{\mathbf{P}_m\bar{\mathbf{B}}_\ell} &= 2\operatorname{Re}\left\{\left(\mathbf{A}_1^H\bar{\mathbf{C}}_\ell\right)\times\left(\mathbf{D}_m^H\mathbf{A}_1\right)^*+\left(\mathbf{D}_m^H\mathbf{R}^{-1}\bar{\mathbf{C}}_\ell\right)\times\mathbf{A}_2^*\right\}, \\
\mathrm{FIM}_{\mathbf{P}_m\tilde{\mathbf{B}}_\ell} &= 2\operatorname{Re}\left\{\left(\mathbf{A}_1^H\tilde{\mathbf{C}}_\ell\right)\times\left(\mathbf{D}_m^H\mathbf{A}_1\right)^*+\left(\mathbf{D}_m^H\mathbf{R}^{-1}\tilde{\mathbf{C}}_\ell\right)\times\mathbf{A}_2^*\right\},
\end{aligned} \tag{B.10}$$

where

$$\begin{aligned}
\mathbf{A}_1 &\triangleq \mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}, \\
\mathbf{A}_2 &\triangleq \boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}.
\end{aligned} \tag{B.11}$$

We can also obtain

$$\begin{aligned}
&\operatorname{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\Lambda_i}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\bar{b}_{km}}\right\} \\
&\quad= \operatorname{tr}\left\{\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\left(\bar{\mathbf{C}}_m\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H+\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\right)\right\} \\
&\quad= \operatorname{tr}\Big\{\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H \\
&\qquad\qquad+ \mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\Big\} \\
&\quad= \left(\mathbf{e}_k^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\right)\left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\right) \\
&\qquad+ \left(\mathbf{e}_k^{\mathrm{T}}\bar{\mathbf{C}}_m^H\mathbf{R}^{-1}\mathbf{A}\mathbf{e}_i\right)\left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\right) \\
&\quad= \left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\right)^*\left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\right) \\
&\qquad+ \left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\right)\left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\right)^* \\
&\quad= 2\operatorname{Re}\left\{\left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\mathbf{e}_k\right)^*\left(\mathbf{e}_i^{\mathrm{T}}\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\mathbf{e}_k\right)\right\}.
\end{aligned} \tag{B.12}$$

Thus, the associated blocks of the FIM are given by

$$\begin{aligned}
\mathrm{FIM}_{\boldsymbol{\Lambda}\bar{\mathbf{B}}_m} &= 2\operatorname{Re}\left\{\left(\mathbf{A}^H\mathbf{R}^{-1}\mathbf{A}\boldsymbol{\Lambda}\right)^*\times\left(\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\right)\right\} \\
&= 2\operatorname{Re}\left\{\left(\mathbf{A}^H\mathbf{A}_1\right)^*\times\left(\mathbf{A}^H\mathbf{R}^{-1}\bar{\mathbf{C}}_m\right)\right\}, \\
\mathrm{FIM}_{\boldsymbol{\Lambda}\tilde{\mathbf{B}}_m} &= 2\operatorname{Re}\left\{\left(\mathbf{A}^H\mathbf{A}_1\right)^*\times\left(\mathbf{A}^H\mathbf{R}^{-1}\tilde{\mathbf{C}}_m\right)\right\}, \\
\mathrm{FIM}_{\boldsymbol{\Lambda}\mathbf{P}_m} &= 2\operatorname{Re}\left\{\left(\mathbf{A}^H\mathbf{A}_1\right)^*\times\left(\mathbf{A}^H\mathbf{R}^{-1}\mathbf{D}_m\right)\right\}.
\end{aligned} \tag{B.13}$$

The CRB is obtained by inverting the complete FIM.

C. PERFORMANCE ANALYSIS FOR DPD FOR UNKNOWN SIGNALS

In this appendix, we introduce a small error analysis of the proposed algorithm for unknown signals. Although the algorithm does not impose any requirements on the signals' statistics, in order to facilitate the analysis, we assume that the transmitted signals are statistically independent, zero-mean, jointly Gaussian, and therefore satisfy

$$E\{\bar{s}_q(j,k)\bar{s}_p^H(m,\ell)\} = \mathbf{\Lambda}_{q,q}(j)\delta_{q,p}\delta_{j,m}\delta_{k,\ell}. \tag{C.1}$$

For analysis convenience, we review briefly some of the definitions of the proposed algorithm.

Recall that the algorithm is based on maximizing the largest eigenvalue of $\mathbf{D}$, where

$$\begin{gathered}\mathbf{D} = \mathbf{H}^H\mathbf{T}\mathbf{H},\\ \mathbf{T} \triangleq \sum_{j=1}^{J}\sum_{q=1}^{Q}\mathbf{\Gamma}^H(j)\mathbf{u}_q(j)\mathbf{u}_q^H(j)\mathbf{\Gamma}(j),\end{gathered} \tag{C.2}$$

and $\mathbf{u}_q(j)$ denotes the qth eigenvector of the sample covariance matrix $\widehat{\mathbf{R}}(j)$, corresponding to the qth eigenvalue γ_q, where we assume that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{ML}$. Using the eigendecomposition of the Hermitian matrix $\mathbf{D}$, we have

$$\begin{gathered}\mathbf{D} = \mathbf{W}\mathbf{\Phi}\mathbf{W}^H,\\ \mathbf{W} \triangleq [\mathbf{w}_1,\ldots,\mathbf{w}_L],\\ \mathbf{\Phi} \triangleq \operatorname{diag}\{\lambda_1,\ldots,\lambda_L\},\end{gathered} \tag{C.3}$$

where $\mathbf{w}_j$, λ_j are the jth eigenvector/eigenvalue pair and $\lambda_1 \geq \cdots \geq \lambda_L$. Thus, we have $\lambda_{\max} \equiv \lambda_1$.

The computation of the covariance of $\mathbf{p}_q$ depends on the estimated eigenvectors $\{\hat{\mathbf{u}}_q,\ 1 \leq q \leq Q\}$. In the ideal case, $\hat{\mathbf{u}}_q = \mathbf{u}_q$, $1 \leq q \leq Q$, and the local maxima of the cost function occur are at the true source locations $\{\mathbf{p}_q\}_{q=1}^{Q}$. In reality, the eigenvectors $\hat{\mathbf{u}}_q$ are perturbed, and therefore the local maxima occur at locations $\widehat{\mathbf{p}}_q$ that are not identical to the true locations. The covariance of the perturbations of $\widehat{\mathbf{p}}_q$ is related to the covariance perturbations of the $\hat{\mathbf{u}}_q$ by the following equation taken from [20, page 534, equation (15)]:

$$\begin{gathered}\operatorname{cov}\{\widehat{\mathbf{p}}_q\} = \left[\frac{\partial^2\lambda_1}{\partial\mathbf{p}^2}\right]^{-1}\left[\frac{\partial^2\lambda_1}{\partial\mathbf{p}\partial\boldsymbol{\xi}}\right]\operatorname{cov}\{\widehat{\boldsymbol{\xi}}\}\left[\frac{\partial^2\lambda_1}{\partial\mathbf{p}\partial\boldsymbol{\xi}}\right]^{\mathrm{T}}\left[\frac{\partial^2\lambda_1}{\partial\mathbf{p}^2}\right]^{-\mathrm{T}},\\ \boldsymbol{\xi} \triangleq \left[\boldsymbol{\xi}_1^{\mathrm{T}},\ldots,\boldsymbol{\xi}_Q^{\mathrm{T}}\right]^{\mathrm{T}},\\ \boldsymbol{\xi}_q \triangleq \left[\boldsymbol{\xi}_q^{\mathrm{T}}(0),\ldots,\boldsymbol{\xi}_q^{\mathrm{T}}(J)\right]^{\mathrm{T}},\\ \boldsymbol{\xi}_q(j) \triangleq \left[\bar{\mathbf{u}}_q^{\mathrm{T}}(j),\tilde{\mathbf{u}}_q^{\mathrm{T}}(j)\right]^{\mathrm{T}},\end{gathered} \tag{C.4}$$

where $\bar{\mathbf{u}}_k(j) \triangleq \operatorname{Re}\{\mathbf{u}_k(j)\}$; $\tilde{\mathbf{u}}_k(j) \triangleq \operatorname{Im}\{\mathbf{u}_k(j)\}$.

We note that the right-hand side of (C.4) has to be evaluated at the true values of $\mathbf{p}_q$, $\mathbf{u}_q(j)$.

The (k,ℓ) submatrix $\operatorname{cov}\{\widehat{\boldsymbol{\xi}}_k,\widehat{\boldsymbol{\xi}}_\ell\}$ of the covariance matrix $\operatorname{cov}\{\widehat{\boldsymbol{\xi}}\}$ is given by [20, equation (22)]:

$$\operatorname{cov}\{\widehat{\boldsymbol{\xi}}_k,\widehat{\boldsymbol{\xi}}_\ell\} = \frac{1}{2}\begin{bmatrix}\operatorname{Re}\left(\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell\}+\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell^*\}\right) & -\operatorname{Im}\left(\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell\}-\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell^*\}\right)\\ \operatorname{Im}\left(\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell\}+\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell^*\}\right) & \operatorname{Re}\left(\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell\}-\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell^*\}\right)\end{bmatrix}, \tag{C.5}$$

where $\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell\}$ and $\operatorname{cov}\{\hat{\mathbf{u}}_k,\hat{\mathbf{u}}_\ell^*\}$ are block diagonal matrices, where the jth block is given by [20, equations (23)-(24)]:

$$\begin{gathered}\operatorname{cov}\{\hat{\mathbf{u}}_k(j),\hat{\mathbf{u}}_\ell(j)\} = \frac{\gamma_k(j)}{K}\delta_{k\ell}\sum_{\substack{i=1\\ i\neq k}}^{ML}\frac{\gamma_i(j)}{\left(\gamma_k(j)-\gamma_i(j)\right)^2}\mathbf{u}_i(j)\mathbf{u}_i^H(j),\\ \operatorname{cov}\{\hat{\mathbf{u}}_k(j),\hat{\mathbf{u}}_\ell^*(j)\} = -(1-\delta_{k\ell})\frac{\gamma_k(j)\gamma_\ell(j)}{K\left(\gamma_k(j)-\gamma_\ell(j)\right)^2}\mathbf{u}_\ell(j)\mathbf{u}_k^{\mathrm{T}}(j).\end{gathered} \tag{C.6}$$

For convenience, we use the definitions

$$\begin{gathered}\mathbf{\Omega} \triangleq \left[\frac{\partial^2\lambda_1}{\partial\mathbf{p}^2}\right],\\ \mathbf{\Psi} \triangleq \left[\frac{\partial^2\lambda_1}{\partial\mathbf{p}\partial\boldsymbol{\xi}}\right]\operatorname{cov}\{\widehat{\boldsymbol{\xi}}\}\left[\frac{\partial^2\lambda_1}{\partial\mathbf{p}\partial\boldsymbol{\xi}}\right]^{\mathrm{T}}.\end{gathered} \tag{C.7}$$

Expressions for $\mathbf{\Omega}$

We first note that

$$\frac{\partial\lambda_1}{\partial p_k} = \sum_{m,n=1}^{L}\frac{\bar{\partial}\lambda_1}{\bar{\partial}\mathbf{D}_{m,n}}\frac{\partial\mathbf{D}_{m,n}}{\partial p_k} = \sum_{m,n=1}^{L}\mathbf{W}_{n,1}\mathbf{W}_{m,1}^*\frac{\partial\mathbf{D}_{m,n}}{\partial p_k}, \tag{C.8}$$

where p_k is the kth element of the vector $\mathbf{p}$, and $\bar{\partial}f/\bar{\partial}\alpha$ is the Brandwood complex derivative [21] (i.e., the derivative of a real-valued function of a complex variable and its conjugate $f(\alpha,\alpha^*)$ is taken w.r.t. α regarding α^* as a constant) and we used the results obtained in [22, equation (A.12)] to express $\bar{\partial}\lambda_1/\bar{\partial}\mathbf{D}_{m,n}$.

Using (C.8), we can express the (k,ℓ) element of $\mathbf{\Omega}$ as

$$\frac{\partial^2\lambda_1}{\partial p_\ell\partial p_k} = \sum_{m,n=1}^{L}\frac{\partial\mathbf{W}_{n,1}\mathbf{W}_{m,1}^*}{\partial p_\ell}\frac{\partial\mathbf{D}_{m,n}}{\partial p_k} + \mathbf{W}_{n,1}\mathbf{W}_{m,1}^*\frac{\partial^2\mathbf{D}_{m,n}}{\partial p_\ell\partial p_k}. \tag{C.9}$$

Using (C.2) yields the following expressions for the derivatives:

$$\begin{aligned}\frac{\partial \mathbf{D}_{m,n}}{\partial p_k} &= 2\,\mathrm{Re}\left\{(\mathbf{e}_m \otimes \mathbf{1}_M)^{\mathrm{T}} \frac{\partial \mathbf{T}}{\partial p_k}(\mathbf{e}_n \otimes \mathbf{1}_M)\right\},\\ \frac{\partial^2 \mathbf{D}_{m,n}}{\partial p_\ell \partial p_k} &= 2\,\mathrm{Re}\left\{(\mathbf{e}_m \otimes \mathbf{1}_M)^{\mathrm{T}} \frac{\partial \mathbf{T}}{\partial p_\ell \partial p_k}(\mathbf{e}_n \otimes \mathbf{1}_M)\right\},\end{aligned} \tag{C.10}$$

$$\begin{aligned}\frac{\partial \mathbf{W}_{\ell,1}}{\partial p_k} &= \sum_{m,n} \frac{\bar{\partial} \mathbf{W}_{\ell,1}}{\bar{\partial} \mathbf{D}_{m,n}} \frac{\partial \mathbf{D}_{m,n}}{\partial p_k}\\ &= \sum_{m,n}\left[\sum_{j=2}^{L} \frac{\mathbf{W}_{n,1}\mathbf{W}_{m,j}^{*}\mathbf{W}_{\ell,j}}{\lambda_1-\lambda_j}\right]\frac{\partial \mathbf{D}_{m,n}}{\partial p_k},\end{aligned} \tag{C.11}$$

where $\bar{\partial}\mathbf{W}_{\ell,1}/\bar{\partial}\mathbf{D}_{m,n}$ was expressed using the results of [22, equation (A.18)].

Substituting (C.10) and (C.11) in (C.9) yields an explicit expression for $\partial^2\lambda_1/\partial p_\ell \partial p_k$.

Expressions for $\boldsymbol{\Psi}$

We first note that

$$\frac{\partial^2 \lambda_1}{\partial \boldsymbol{\xi}_\ell \partial p_k} = \left[\left(\frac{\partial^2 \lambda_1}{\partial \bar{\mathbf{u}}_\ell \partial p_k}\right)^{\mathrm{T}}, \left(\frac{\partial^2 \lambda_1}{\partial \tilde{\mathbf{u}}_\ell \partial p_k}\right)^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{C.12}$$

Using (C.8), we get

$$\begin{aligned}\frac{\partial^2 \lambda_1}{\partial \bar{\mathbf{u}}_\ell \partial p_k} &= \sum_{m,n} \frac{\partial \mathbf{W}_{n,1}\mathbf{W}_{m,1}^{*}}{\partial \bar{\mathbf{u}}_\ell}\frac{\partial \mathbf{D}_{mn}}{\partial p_k} + \mathbf{W}_{n,1}\mathbf{W}_{m,1}^{*}\frac{\partial^2 \mathbf{D}_{mn}}{\partial \bar{\mathbf{u}}_\ell \partial p_k},\\ \frac{\partial^2 \lambda_1}{\partial \tilde{\mathbf{u}}_\ell \partial p_k} &= \sum_{m,n} \frac{\partial \mathbf{W}_{n,1}\mathbf{W}_{m,1}^{*}}{\partial \tilde{\mathbf{u}}_\ell}\frac{\partial \mathbf{D}_{mn}}{\partial p_k} + \mathbf{W}_{n,1}\mathbf{W}_{m,1}^{*}\frac{\partial^2 \mathbf{D}_{mn}}{\partial \tilde{\mathbf{u}}_\ell \partial p_k},\end{aligned} \tag{C.13}$$

where the derivatives in (C.13) are given by

$$\begin{aligned}\frac{\partial \mathbf{W}_{\ell,1}}{\partial \bar{\mathbf{u}}_k} &= \sum_{m,n} \frac{\bar{\partial} \mathbf{W}_{\ell,1}}{\bar{\partial} \mathbf{D}_{m,n}} \frac{\partial \mathbf{D}_{m,n}}{\partial \bar{\mathbf{u}}_k},\\ \frac{\partial \mathbf{W}_{\ell,1}}{\partial \tilde{\mathbf{u}}_k} &= \sum_{m,n} \frac{\bar{\partial} \mathbf{W}_{\ell,1}}{\bar{\partial} \mathbf{D}_{m,n}} \frac{\partial \mathbf{D}_{m,n}}{\partial \tilde{\mathbf{u}}_k},\\ \frac{\partial \mathbf{D}_{m,n}}{\partial \bar{\mathbf{u}}_\ell} &= \left[\frac{\partial \mathbf{D}_{m,n}}{\partial \bar{\mathbf{u}}_\ell(0)}, \frac{\partial \mathbf{D}_{m,n}}{\partial \bar{\mathbf{u}}_\ell(1)}, \dots, \frac{\partial \mathbf{D}_{m,n}}{\partial \bar{\mathbf{u}}_\ell(J)}\right],\\ \frac{\partial \mathbf{D}_{m,n}}{\partial \tilde{\mathbf{u}}_\ell} &= \left[\frac{\partial \mathbf{D}_{m,n}}{\partial \tilde{\mathbf{u}}_\ell(0)}, \frac{\partial \mathbf{D}_{m,n}}{\partial \tilde{\mathbf{u}}_\ell(1)}, \dots, \frac{\partial \mathbf{D}_{m,n}}{\partial \tilde{\mathbf{u}}_\ell(J)}\right],\end{aligned} \tag{C.14}$$

and $\bar{\partial}\mathbf{W}_{\ell,1}/\bar{\partial}\mathbf{D}_{m,n}$ can be found in (C.11).

We also have

$$\begin{aligned}\frac{\partial \mathbf{D}_{mn}}{\partial \bar{\mathbf{u}}_\ell(j)} &= \mathbf{u}_\ell^{H}(j)\boldsymbol{\Gamma}(j)\mathbf{E}_{n,m}\boldsymbol{\Gamma}^{H}(j) + \mathbf{u}_\ell^{\mathrm{T}}(j)\boldsymbol{\Gamma}^{*}(j)\mathbf{E}_{m,n}\boldsymbol{\Gamma}^{\mathrm{T}}(j),\\ \frac{\partial \mathbf{D}_{mn}}{\partial \tilde{\mathbf{u}}_\ell(j)} &= i\left[\mathbf{u}_\ell^{H}(j)\boldsymbol{\Gamma}(j)\mathbf{E}_{n,m}\boldsymbol{\Gamma}^{H}(j) - \mathbf{u}_\ell^{\mathrm{T}}(j)\boldsymbol{\Gamma}^{*}(j)\mathbf{E}_{m,n}\boldsymbol{\Gamma}^{\mathrm{T}}(j)\right],\\ \mathbf{E}_{nm} &\triangleq (\mathbf{e}_n \otimes \mathbf{1}_M)(\mathbf{e}_m \otimes \mathbf{1}_M)^{\mathrm{T}},\\ \frac{\partial^2 \mathbf{D}_{m,n}}{\partial \bar{\mathbf{u}}_\ell \partial p_k} &= \left[\frac{\partial^2 \mathbf{D}_{mn}}{\partial \bar{\mathbf{u}}_\ell(0)\partial p_k}, \dots, \frac{\partial^2 \mathbf{D}_{mn}}{\partial \bar{\mathbf{u}}_\ell(J)\partial p_k}\right],\\ \frac{\partial^2 \mathbf{D}_{mn}}{\partial \tilde{\mathbf{u}}_\ell \partial p_k} &= \left[\frac{\partial^2 \mathbf{D}_{mn}}{\partial \tilde{\mathbf{u}}_\ell(0)\partial p_k}, \dots, \frac{\partial^2 \mathbf{D}_{mn}}{\partial \tilde{\mathbf{u}}_\ell(J)\partial p_k}\right].\end{aligned} \tag{C.15}$$

Using (C.10), we can express each of the terms in the above equation as

$$\begin{aligned}\frac{\partial^2 \mathbf{D}_{m,n}}{\partial \bar{\mathbf{u}}_\ell(j)\partial p_k} &= \mathbf{u}_\ell^{H}(j)\left[\boldsymbol{\Gamma}(j)\mathbf{E}_{nm}\frac{\partial \boldsymbol{\Gamma}^{H}(j)}{\partial p_k} + \frac{\partial \boldsymbol{\Gamma}(j)}{\partial p_k}\mathbf{E}_{nm}\boldsymbol{\Gamma}^{H}(j)\right]\\ &\quad + \mathbf{u}_\ell^{\mathrm{T}}(j)\left[\boldsymbol{\Gamma}^{*}(j)\mathbf{E}_{mn}\frac{\partial \boldsymbol{\Gamma}^{\mathrm{T}}(j)}{\partial p_k} + \frac{\partial \boldsymbol{\Gamma}^{*}(j)}{\partial p_k}\mathbf{E}_{mn}\boldsymbol{\Gamma}^{\mathrm{T}}(j)\right],\\ \frac{\partial^2 \mathbf{D}_{mn}}{\partial \tilde{\mathbf{u}}_\ell(j)\partial p_k} &= i\mathbf{u}_\ell^{H}(j)\left[\boldsymbol{\Gamma}(j)\mathbf{E}_{nm}\frac{\partial \boldsymbol{\Gamma}^{H}(j)}{\partial p_k} + \frac{\partial \boldsymbol{\Gamma}(j)}{\partial p_k}\mathbf{E}_{nm}\boldsymbol{\Gamma}^{H}(j)\right]\\ &\quad - \mathbf{u}_\ell^{\mathrm{T}}(j)\left[\boldsymbol{\Gamma}^{*}(j)\mathbf{E}_{mn}\frac{\partial \boldsymbol{\Gamma}^{\mathrm{T}}(j)}{\partial p_k} + \frac{\partial \boldsymbol{\Gamma}^{*}(j)}{\partial p_k}\mathbf{E}_{mn}\boldsymbol{\Gamma}^{\mathrm{T}}(j)\right].\end{aligned} \tag{C.16}$$

Substituting (C.14) and (C.16) in (C.13) and using straightforward algebraic manipulations yields

$$\begin{aligned}\frac{\partial^2 \lambda_1}{\partial \bar{\mathbf{u}}_\ell(j)\partial p_k} &= \mathbf{u}_\ell^{H}(j)\mathbf{Z}_k + \mathbf{u}_\ell^{\mathrm{T}}(j)\mathbf{Z}_k^{*} = 2\,\mathrm{Re}\left\{\mathbf{u}_\ell^{H}(j)\mathbf{Z}_k\right\},\\ \frac{\partial^2 \lambda_1}{\partial \tilde{\mathbf{u}}_\ell(j)\partial p_k} &= i\left(\mathbf{u}_\ell^{H}(j)\mathbf{Z}_k - \mathbf{u}_\ell^{\mathrm{T}}(j)\mathbf{Z}_k^{*}\right) = -2\,\mathrm{Im}\left\{\mathbf{u}_\ell^{H}(j)\mathbf{Z}_k\right\},\\ \mathbf{Z}_k &\triangleq \sum_{m=1}^{L}\sum_{n=1}^{L} \mathbf{X}_{n,m}^{k} + \mathbf{W}_{n1}(n)\mathbf{W}_{m1}^{*}\mathbf{Y}_{n,m}^{k},\\ \mathbf{X}_{nm}^{k} &\triangleq \sum_{r=1}^{L}\sum_{\ell=1}^{L}\left[\sum_{j=2}^{L}\frac{\mathbf{W}_{\ell 1}\mathbf{W}_{rj}^{*}\left[\mathbf{W}_{nj}\mathbf{W}_{m1}^{*} + \mathbf{W}_{mj}\mathbf{W}_{n1}\right]}{\lambda_1-\lambda_j}\right]\\ &\quad \times \boldsymbol{\Gamma}\mathbf{E}_{lr}\boldsymbol{\Gamma}^{H}(\mathbf{e}_m \otimes \mathbf{J}_M)\frac{\partial \mathbf{T}}{\partial p_k}(\mathbf{e}_n \otimes \mathbf{J}_M)^{\mathrm{T}},\\ \mathbf{Y}_{nm}^{k} &\triangleq \boldsymbol{\Gamma}\mathbf{E}_{nm}\frac{\partial \boldsymbol{\Gamma}^{H}}{\partial p_k} + \frac{\partial \boldsymbol{\Gamma}}{\partial p_k}\mathbf{E}_{nm}\boldsymbol{\Gamma}^{H}.\end{aligned} \tag{C.17}$$

Recalling that $\gamma_k(j) = \eta$ for all $Q+1 \leq k \leq ML$, we can express the (m,n) element of $\boldsymbol{\Psi}$ using [20, equation (27)] as

$$\begin{aligned}\boldsymbol{\Psi}_{m,n} &= \frac{\partial^2\lambda_1}{\partial\boldsymbol{\xi}_m\partial p_k}\operatorname{cov}\{\hat{\boldsymbol{\xi}}_k,\hat{\boldsymbol{\xi}}_\ell\}\frac{\partial^2\lambda_1}{\partial\boldsymbol{\xi}_n\partial p_\ell}\\ &= \frac{2\eta}{N}\operatorname{Re}\Bigg\{\sum_{j=1}^{J}\sum_{k=1}^{Q}\sum_{j=Q+1}^{ML}\frac{\gamma_k(j)}{(\gamma_k(j)-\eta)^2}\\ &\qquad\qquad \times\mathbf{u}_k^H(j)\mathbf{Z}_m\mathbf{u}_\ell(j)\mathbf{u}_\ell^H(j)\mathbf{Z}_n\mathbf{u}_k(j)\Bigg\}.\end{aligned} \tag{C.18}$$

In summary, using the result of (C.18) and (C.9), we can express the matrix $\operatorname{cov}\{\hat{\mathbf{p}}_i\}$ in (C.4).

Special case: uniform linear array

So far, we have considered a general array configuration. In this section, we obtain the required expressions for a ULA with M elements. The coordinates of the mth element, at the ℓth base station, are $x = (m-1)\Delta\cos\phi_\ell$; $y = (m-1)\Delta\sin\phi_\ell$, where Δ is the elements spacing and ϕ_ℓ is the counterclockwise rotation of the array baseline with respect to the x-axis.

The steering vector of the ℓth base station takes the form

$$\mathbf{a}_\ell(\mathbf{p}) = \left[1 \quad e^{jk\Delta\cos\theta'_\ell} \quad \cdots \quad e^{jk(M-1)\Delta\cos\theta'_\ell}\right]^{\mathrm{T}}, \tag{C.19}$$

where $k = 2\pi/\lambda$ is the signal wave number, λ is the wave length, and θ'_ℓ is the direction of arrival (relative to the array baseline) of a signal emitted from (x_t, y_t) to the ℓth base station located at (x_ℓ, y_ℓ). In addition, the delay between the emitter and the base station is $\tau_\ell = d_\ell/c$, where $d_\ell = \sqrt{(x_t-x_\ell)^2+(y_t-y_\ell)^2}$ is the distance between the emitter and the ℓth base station and c is the speed of propagation.

Note that the AOA with respect to the x-axis at the ℓth base station is $\theta_\ell = \theta'_\ell + \phi_\ell$.

The steering vector derivatives with respect to the axis are

$$\frac{d\mathbf{a}_\ell}{dx} = -jk\Delta\sin\theta'_\ell\frac{d\theta'_\ell}{dx}\mathbf{a}_\ell^{\mathrm{T}}\bar{\mathbf{M}}, \qquad \frac{d\mathbf{a}_\ell}{dy} = -jk\Delta\sin\theta'_\ell\frac{d\theta'_\ell}{dy}\mathbf{a}_\ell^{\mathrm{T}}\bar{\mathbf{M}}, \tag{C.20}$$

where $\bar{\mathbf{M}} \triangleq \operatorname{diag}(0,1,\ldots,M-1)$ and

$$\frac{d\theta'_\ell}{dx} = \frac{d\theta_\ell}{dx} = -\frac{\sin\theta_\ell}{d_\ell}, \qquad \frac{d\theta'_\ell}{dy} = \frac{d\theta_\ell}{dy} = \frac{\cos\theta_\ell}{d_\ell}. \tag{C.21}$$

We also have

$$\begin{aligned}\frac{d^2\mathbf{a}_\ell}{dx^2} &= -j\frac{k\Delta}{d_\ell^2}\mathbf{a}_\ell^{\mathrm{T}}\bar{\mathbf{M}}\Big[\sin(\theta'_\ell+\theta_\ell)\sin(\theta_\ell)\mathbf{I}\\ &\qquad - jk\Delta\sin^2\theta'_\ell\sin^2\theta_\ell\bar{\mathbf{M}}\Big],\\ \frac{d^2\mathbf{a}_\ell}{dy^2} &= -j\frac{k\Delta}{d_\ell^2}\mathbf{a}_\ell^{\mathrm{T}}\bar{\mathbf{M}}\Big[\cos(\theta'_\ell+\theta_\ell)\cos(\theta_\ell)\mathbf{I}\\ &\qquad - jk\Delta\sin^2\theta'_\ell\cos^2\theta_\ell\bar{\mathbf{M}}\Big],\\ \frac{d^2\mathbf{a}_\ell}{dx\,dy} &= j\frac{k\Delta}{d_\ell^2}\mathbf{a}_\ell^{\mathrm{T}}\bar{\mathbf{M}}\Big[\sin(\theta'_\ell+\theta_\ell)\cos(\theta_\ell)\mathbf{I}\\ &\qquad - j\frac{1}{2}k\Delta\sin^2\theta'_\ell\sin2\theta_\ell\bar{\mathbf{M}}\Big].\end{aligned} \tag{C.22}$$

This concludes the performance analysis.

D. ON THE FREQUENCY-DOMAIN MODEL FOR FINITE-LENGTH OBSERVATIONS

Consider the observation $s(t)$, $0 \leq t \leq T_1$, at a given base station and the observation $s(t-D)$, $0 \leq t \leq T_1$, at a different base station, where D denotes the time difference of arrival. The Fourier transform of these signals are given by

$$\begin{aligned}S_1 &= \int_0^{T_1} s(t)e^{-i\omega t}dt,\\ S_2 &= \int_0^{T_1} s(t-D)e^{-i\omega t}dt = e^{-i\omega D}\int_{-D}^{T_1-D} s(\sigma)e^{-i\omega\sigma}d\sigma.\end{aligned} \tag{D.1}$$

The relation between S_1 and S_2 is given by

$$\begin{aligned}S_2 &= e^{-i\omega D}(S_1+\Delta),\\ \Delta &\triangleq \int_{-D}^{0} s(\sigma)e^{-i\omega\sigma}d\sigma - \int_{T_1-D}^{T_1} s(\sigma)e^{-i\omega\sigma}d\sigma.\end{aligned} \tag{D.2}$$

In the main text, we used the approximation $S_2 \cong e^{-i\omega D}S_1$ under the assumption that $\Delta \ll S_1$. It is easy to verify that the energy relation between S_1 and Δ is given by

$$\frac{E\{|S_1|^2\}}{E\{|\Delta|^2\}} = \frac{T_1}{2D} \tag{D.3}$$

for a random signal with flat spectral density. Thus, in order to get a ratio of 20 dB, the required observation length T_1 should be $200D$. In the body of the paper, each snapshot is of length $T_1 = T/K$.

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