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DIRECT PRODUCT DECOMPOSITION OF MV-ALGEBRAS

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The notion of an MV-algebra originally constructed for giving an algebraic structure to the infinite-valued Lukasiewicz propositional logics (Chang [4]), turned out to be related to the theory of linearly ordered groups (Chang [5]), the theory of cyclically ordered groups (Gluschankof [6]), the fuzzy set theory (Belluce [1]), functional analysis and lattice ordered groups (Mundici [10]).

The systems of axioms for defining the notion of an MV-algebra can be formulated in various ways; cf. [2], [4], [6]. We shall apply the notation and axioms from [6].

To each MV-algebra $\mathcal{A} = \langle A; \oplus, *, \neg, 0, 1\rangle$ we can assign a lattice $\mathcal{L}(\mathcal{A}) = \langle A; \vee, \wedge\rangle$, where the operations $\vee$ and $\wedge$ are defined as follows:

(1) $x \vee y = (x * \neg y) \oplus y$,

(2) $x \wedge y = \neg(\neg x \vee \neg y)$

(cf. [4], [5], [6]).

Let us remark that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are MV-algebras such that the lattices $\mathcal{L}(\mathcal{A}_1)$ and $\mathcal{L}(\mathcal{A}_2)$ are isomorphic, then $\mathcal{A}_1$ and $\mathcal{A}_2$ need not be isomorphic. Thus $\mathcal{A}$ cannot be reconstructed from $\mathcal{L}(\mathcal{A})$.

Direct products of MV-algebras have been dealt with in [4] and [2]. If φ is an isomorphism of an MV-algebra $\mathcal{A}$ onto a direct product $\prod\limits_{i\in I} \mathcal{A}_i$, then by means of φ we can construct an internal direct decomposition

$$\varphi_0 \colon \mathcal{A} \longrightarrow \prod_{i\in I} \mathcal{A}_i^0,$$

where for each $i \in I$, $\mathcal{A}_i^0$ is isomorphic to $\mathcal{A}_i$ and the underlying set of $\mathcal{A}_i^0$ is a subset of A containing the element 0. (The method is similar to that which is well-known in the theory of groups; cf. e.g. Kurosh [9], p. 104.) Analogously we can construct internal direct product decompositions of the lattice $\mathcal{L}(\mathcal{A})$.

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In this paper it will be shown that there exists a one-to-one correspondence between the internal product decompositions of an MV-algebra $\mathcal{A}$ and the internal product decompositions of the lattice $\mathcal{L}(\mathcal{A})$. In fact, in a certain sense (specified in 3.3, 3.4 and 3.5) we can say that the internal product decompositions of $\mathcal{A}$ and those of $\mathcal{L}(\mathcal{A})$ are very closely related. As a corollary we obtain that any two internal product decompositions of an MV-algebra have a common refinement. Consequently, any two direct decompositions of an MV-algebra have isomorphic refinements.

By applying some results of [8] on direct product decompositions of a complete lattice ordered group we establish analogous theorems for direct product decompositions of complete MV-algebras. In this way we obtain a generalization of Belluce's theorem [2, Theorem 12] concerning a two-factor direct decomposition of a complete MV-algebra, where the first factor is atomic and the second is atomless.

It is well-known that each polar of a complete lattice ordered group is a direct factor. A question of the relations between polars of an MV-algebra $\mathcal{A}$ and prime ideals of $\mathcal{A}$ which was proposed in [1] will be solved.

1. Preliminaries

We recall the definition of an MV-algebra (cf. [6]).

1.1. Definition. *An MV-algebra is a system $\mathcal{A} = \langle A; \oplus, *, \neg, 0, 1\rangle$, (where $\oplus, *$ are binary operations, $\neg$ is a unary operation and* 0, 1 *are nullary operations) such that the following identities are satisfied:*

(m$_1$) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(m$_2$) $x \oplus 0 = x$;
(m$_3$) $x \oplus y = y \oplus x$;
(m$_4$) $x \oplus 1 = 1$;
(m$_5$) $\neg\neg x = x$;
(m$_6$) $\neg 0 = 1$;
(m$_7$) $x \oplus \neg x = 1$;
(m$_8$) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$;
(m$_9$) $x * y = \neg(\neg x \oplus \neg y)$.

For the following lemma cf. [6] or [2].

1.2. Lemma. *Let $\mathcal{A} = \langle A; \oplus, *, \neg, 0, 1\rangle$ be an MV-algebra. Then the system $\mathcal{L}(\mathcal{A}) = \langle A, \vee, \wedge\rangle$, where $\vee$ and $\wedge$ are binary operations on A defined by* (1) *and* (2) *above, is a distributive lattice with the least element* 0 *and the greatest element* 1.

In what follows, when we consider a partial order on a set A, then it is always the partial order defined by means of the lattice $\mathcal{L}(\mathcal{A})$ from 1.2.

From 1.2 we infer that the above system of axioms is equivalent to that given in [4].

For lattice ordered groups we use the same notation as in [3].

Propositions 1.3 and 1.4 are due to Mundici [10] (Theorem 2.5 and 3.8).

1.3. Proposition. *Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0,u]$ of G. For each a and b in A we put*

$$a \oplus b = (a+b) \wedge u, \quad \neg a = u - a, \quad 1 = u.$$

Next, let the binary operation $*$ on A be defined by (m_9). Then $\mathcal{A} = \langle A; \oplus, *, \neg, 0, 1\rangle$ is an MV-algebra.

If G and $\mathcal{A}$ are as in 1.3 then we denote $\mathcal{A} = \mathcal{A}_0(G,u)$.

1.4. Proposition. *Let $\mathcal{A}$ be an MV-algebra. Then there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \mathcal{A}_0(G,u)$.*

Let us also remark that if $\mathcal{A} = \mathcal{A}_0(G,u)$, then the operations $\vee$ and $\wedge$ as defined by (1) and (2) coincide with the original operations $\vee$ and $\wedge$ on G (reduced to the set A).

The following example shows that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are MV-algebras and if $\mathcal{L}(\mathcal{A}_1)$ is isomorphic to $\mathcal{L}(\mathcal{A}_2)$, then $\mathcal{A}_1$ need not be isomorphic to $\mathcal{A}_2$.

Let G_1 be the additive group of all rationals with the natural linear order and $G_2 = G_1 \circ G_1$, where $\circ$ is the operation of the lexicographic product. Put $u_1 = 1$ and $u_2 = (1,0)$. Then u_i is a strong unit in G_i ($i = 1,2$). The interval $[0,u_1]$ of G_1 is isomorphic to the interval $[0,u_2]$ of G_2. Let the MV-algebra $\mathcal{A}_i$ be constructed from G_i ($i = 1,2$) as in 1.3. Then $\mathcal{L}(\mathcal{A}_1) = [0,u_1]$ and $\mathcal{L}(\mathcal{A}_2) = [0,u_2]$, hence $\mathcal{L}(\mathcal{A}_1)$ is isomorphic to $\mathcal{L}(\mathcal{A}_2)$. It is easy to verify that $\mathcal{A}_1$ is not isomorphic to $\mathcal{A}_2$.

2. Strong units and direct decompositions

In this section some auxiliary results on direct decompositions of a lattice ordered group with a strong unit will be deduced.

Let G be a lattice ordered group and suppose that φ is an isomorphism of G onto the direct product $\prod\limits_{i\in I} G_i$ of lattice ordered groups G_i. For $i(1) \in I$ and $x \in G$ we denote by $x_{i(1)}$ the component of x in $G_{i(1)}$ with the respect to the isomorphism φ. We say that φ is a direct decomposition of G.

Next, let $G^0_{i(1)} = \{g \in G\colon g_i = 0 \text{ for each } i \in I \setminus \{i(1)\}\}$, $x_{i(1)} \in G_{i(1)}$ and let $x^0_{i(1)}$ be the element of $G^0_{i(1)}$ such that $(x^0_{i(1)})_{i(1)} = x_{i(1)}$. Then the map

(1) $\varphi^0\colon G \longrightarrow \prod\limits_{i\in I} G^0_i$

where $\varphi^0(g) = (\ldots, x^0_i, \ldots)_{i\in I}$ is an isomorphism of G onto $\prod\limits_{i\in I} G^0_i$. The direct decomposition φ^0 will be called internal and G^0_i are the internal direct factors of G. All G^0_i's are convex ℓ-subgroups of G.

In what follows we shall deal only with internal direct decompositions and internal direct factors of lattice ordered groups, the word "internal" will therefore be omitted.

A direct factor G^0_i will be called trivial if $G^0_i = \{0\}$. For the case $G \neq \{0\}$ the trivial direct factors G^0_i can be cancelled in (1).

Let (1) be valid and let H be a convex ℓ-subgroup of G such that $G^0_i \subseteq H$ for each $i \in I$. Then H is said to be a completely subdirect product of the lattice ordered groups G^0_i $(i \in I)$; this notion is due to Šik [11].

The following result is well-known.

2.1. Lemma. *A convex ℓ-subgroup K of G is a direct factor of G if and only if for each $x \in G^+$ the set $K \cap [0, x]$ has a greatest element; next, this greatest element is the component of x in K.*

As a corollary we obtain that for each $y \in G$ the component of y in a direct factor K is uniquely determined. More thoroughly: if (1) is valid and if we have another direct decomposition

$$\varphi^{01}\colon G \longrightarrow \prod_{j\in J} G^{01}_j$$

such that there are $i(1) \in I$ and $j(1) \in J$ with $G^0_{i(1)} = G^{01}_{j(1)}$, then for each $y \in G$ the component of y in $G^0_{i(1)}$ (with respect to φ^0) is the same as the component of y in $G^0_{j(1)}$ (with respect to φ^{01}).

Let us remark that an analogous result concerning uniqueness of components does not hold in general for internal direct decompositions of groups.

2.2. Proposition. *Let G be a lattice ordered group with a strong unit. Assume that (1) is valid and that all direct factors G^0_i are nontrivial. Then the set I is finite.*

Proof. By way of contradiction, suppose that the set I is infinite. Thus there are distinct indices $i(n) \in I$ $(n = 1, 2, 3, \ldots)$. Let u be a strong unit in G. There exists $x \in G$ such that for each positive integer n we have $x^0_{i(n)} = nu^0_{i(1)}$. Then for each positive integer m the relation $x \nleq mu$ is valid, which is a contradiction. $\square$

Let L be the interval $[0, u]$ of G. For direct decompositions of the lattice L we shall apply similar notation as in the case of lattice ordered groups. To each direct decomposition

$$\varphi\colon L \longrightarrow \prod_{i\in I} L_i$$

of L we can construct the corresponding internal decomposition (analogously as in the case of lattice ordered groups)

$$\varphi^0\colon L \longrightarrow \prod_{i\in I} L_i^0,$$

where for each $i(1) \in I$, $L_{i(1)}^0$ is the set of all $x \in L$ such that the component of x in L_i under φ is the least element of L_i whenever $i \in I \setminus \{i(1)\}$. Then all L_i^0's are convex sublattices of L with the least element 0. Each L_i^0 possesses a greatest element which will be denoted by z_i and which is the component of u in the direct factor L_i^0 under the isomorphism φ^0. It is easy to verify that for each $x \in L$ and each $i \in I$ the component of x in L_i^0 under φ^0 is the element $x \wedge z_i$.

For each subset X of G let X^δ be the set

$$X^\delta = \{y \in G\colon |y| \wedge |x| = 0 \text{ for each } x \in X\}.$$

2.3. Lemma. *Let u be a strong unit of a lattice ordered group G. Assume that*

$$\psi\colon [0, u] \longrightarrow P \times Q$$

is an internal direct decomposition of the lattice $[0, u]$. Then for each $x \in G$ with $0 \leqslant x$ the set $[0, x] \cap P^{\delta\delta}$ has a largest element, and similarly for $Q^{\delta\delta}$. Further, the join of these largest elements is x.

Proof. For each $x \in G^+$ there exists a positive integer n such that $x \leqslant nu$. We apply induction on n. Let p_0 and q_0 be the components of u in P or Q, respectively (with respect to ψ). Then $u = p_0 \vee q_0$, $p_0 \wedge q_0 = 0$.

Assume that $n = 1$. Then

$[0, x] \cap P^{\delta\delta} = ([0, x] \cap [0, u]) \cap P^{\delta\delta} = [0, x] \cap ([0, u] \cap P^{\delta\delta}) = [0, x] \cap P.$

The component of x in P is the element $x \wedge p_0$; hence this is the largest element of the set $[0, x] \cap P^{\delta\delta}$. The case of $Q^{\delta\delta}$ is analogous. Hence

$$x = x \wedge u = x \wedge (p_0 \vee q_0) = (x \wedge p_0) \vee (x \wedge q_0).$$

Thus the assertion is valid for $n = 1$.

Next, assume that $n > 1$ and that the assertion is valid for $n-1$. It follows from $0 \leqslant x \leqslant nx = (n-1)x + x$ that there are elements x_1 and x_2 in $[0,x]$ such that

$$x = x_1 + x_2, \quad x_1 \leqslant (n-1)x, \quad x_2 \leqslant x.$$

In view of the induction hypothesis there exist elements y_1, y_2, y_3 and y_4 in $[0,u]$ such that

$$y_1 = \sup([0,x_1] \cap P^{\delta\delta}), \quad y_2 = \sup([0,x_1] \cap Q^{\delta\delta}),$$

$$y_3 = \sup([0,x_2] \cap P^{\delta\delta}), \quad y_4 = \sup([0,x_2] \cap Q^{\delta\delta}), \text{ and}$$

$$x_1 = y_1 \vee y_2, \quad x_2 = y_3 \vee y_4.$$

Clearly $a \wedge b = 0$ for each $a \in P^{\delta\delta}$ and each $b \in Q^{\delta\delta}$, thus $a + b = a \vee b$. Then

$x = (y_1 \vee y_2) + (y_3 \vee y_4) = (y_1 + y_2) + (y_3 + y_4) = (y_1 + y_3) + (y_2 + y_4) = (y_1 + y_3) \vee (y_2 + y_4)$.

We have $y_1 + y_3 \in P^{\delta\delta}$, $y_2 + y_4 \in Q^{\delta\delta}$. Let $z \in [0,x] \cap P^{\delta\delta}$. Then $z \wedge (y_2 + y_4) = 0$, hence

$$z = z \wedge x = z \wedge ((y_1 + y_3) \vee (y_2 + y_4)) = z \wedge (y_1 + y_3).$$

Therefore $y_1 + y_3$ is the largest element of the set $[0,x] \cap P^{\delta\delta}$. Similarly, $y_2 + y_4$ is the largest element of the set $[0,x] \cap Q^{\delta\delta}$. The proof is complete. □

2.4. Proposition. *Let G, u, P and Q be as in 2.3. Then there is an internal direct decomposition*

$$\varphi^0 \colon G \longrightarrow P^{\delta\delta} \times Q^{\delta\delta}$$

of the lattice ordered group G.

Proof. In view of 2.1 and 2.3, both $P^{\delta\delta}$ and $Q^{\delta\delta}$ are internal direct factors of G. Next, $(P^{\delta\delta})^{\delta} = Q^{\delta\delta}$. Hence G is an internal direct product of $P^{\delta\delta}$ and $Q^{\delta\delta}$. □

Let us remark that by the obvious induction we can generalize 2.4 to the case of direct decompositions of the lattice $[0,u]$ with any finite number of direct factors; 2.2 shows that this cannot be done for direct decompositions of $[0,u]$ with an infinite number of direct factors.

2.5. Proposition. *Let G and u be as in 2.3. We denote by $F([0,u])$ and $F(G)$ the systems of all internal direct factors of the lattice $[0,u]$ and of the lattice ordered group G, respectively. Both $F([0,u])$ and $F(G)$ are partially ordered by inclusion. For each $P \in F([0,u])$ put $f(P) = P^{\delta\delta}$. Then f is an isomorphism of $F([0,u])$ onto $F(G)$.*

Proof. Let $P_1, P_2 \in F([0,u])$. According to 2.3 and the facts established in the proof of 2.3, $f(P_i) \in F(G)$ for $i = 1, 2$. Moreover, $P_1 \subseteq P_2 \Rightarrow f(P_1) \subseteq f(P_2)$.

Assume that $P_2 \not\subseteq P_1$. Hence there is $x \in P_2 \backslash P_1$. Next there is $P_1' \in F([0,u])$ such that $[0,u]$ is an internal direct product of P_1 and P_1'. Let $x(P_1)$ and $x(P_1')$ be the component of x in P_1 and in P_1', respectively. Then $x(P_1) < x$ and $x = x(P_1) \vee x(P_1')$, hence $x(P_1') > 0$. We have

$$x(P_1') \notin P_1^{\delta\delta}, \quad x(P_1') \in P_2^{\delta\delta},$$

thus $f(P_2) \not\subseteq f(P_1)$. Therefore f is a monomorphism of the partially ordered set $F([u,v])$ into $F(G)$.

Let $X \in F(G)$. Hence there is $Y \in F(G)$ such that there is an internal direct decomposition $\varphi\colon G \longrightarrow X \times Y$ of the lattice ordered group G. Let X^1 be the natural projection of $[0,u]$ into X under φ, and let Y^1 be defined analogously. Then it is easy to verify that

$$X^1 = [0,u] \cap X, \quad Y^1 = [0,u] \cap Y.$$

If we put $\varphi_1(t) = \varphi(t)$ for each $t \in [0,u]$, then

$$\varphi_1\colon [0,u] \longrightarrow X^1 \times Y^1$$

is an internal direct decomposition of the lattice $[0,u]$.

Clearly $Y \subseteq (X^1)^\delta$, hence $X = Y^\delta \supseteq (X^1)^{\delta\delta}$. Let $x \in X, x \geqslant 0$. There is a positive integer n such that $x \leqslant nu$. Let u^1 and u^2 be the components of u in X^1 and in Y^1, respectively (with respect to the isomorphism φ_1). Then $nu = nu^1 + nu^2 = nu^1 \vee nu^2$ and

$$x = x \wedge nu = (x \wedge nu^1) \vee (x \wedge nu^2).$$

Since $nu^2 \in Y$, we get $x \wedge nu^2 = 0$ and thus $x = x \wedge nu^1$. Consequently, $x \in (X^1)^{\delta\delta}$. Hence $X^+ \subseteq (X^1)^{\delta\delta}$ and therefore $X = (X^1)^{\delta\delta}$.

We verified that f is an epimorphism. By summarizing, f is an isomorphism. $\square$

3. Internal direct factors of MV-algebras

When defining an internal direct decomposition of an MV-algebra we proceed analogously as in the case of lattice ordered groups and lattices.

Let $\mathcal{A} = \langle A; \oplus, *, \neg, 0, 1\rangle$ and $\mathcal{A}_i = \langle A_i; \oplus, *, \neg, 0, 1\rangle$ $(i \in I)$ be MV-algebras and let

$$\varphi\colon \mathcal{A} \longrightarrow \prod_{i \in I} \mathcal{A}_i$$

be an isomorphism of $\mathcal{A}$ onto $\prod_{i\in I} \mathcal{A}_i$. For $a \in A$ let a_i be the component of a in A_i with respect to φ.

For each $i(1) \in I$ we denote

$$A^0_{i(1)} = \{a \in A\colon a_i = 0 \text{ for each } i \in I \setminus \{i(1)\}\}.$$

Then $A^0_{i(1)} \subseteq A$ and $0 \in A^0_{i(1)}$. In general, $A^0_{i(1)}$ need not be a subalgebra of $\mathcal{A}$. In a natural way we can introduce the MV-operations on the set $A^0_{i(1)}$; for distinguishing, we shall denote these operations by $\oplus_{i(1)}, *_{i(1)}, \neg_{i(1)}, 0_{i(1)}$ and $1_{i(1)}$.

The operation $\oplus_{i(1)}$ is defined as follows. Let $a, b \in A^0_{i(1)}$ and let $c \in A$ be such that $c_{i(1)} = (a \oplus b)_{i(1)}, c_i = 0$ for each $i \in I \setminus \{i(1)\}$. Then $c \in A^0_{i(1)}$; we put $a \oplus_{i(1)} b = c$.

Analogously we define the operations $*_{i(1)}, \neg_{i(1)}$ and $1_{i(1)}$. Clearly $0_{i(1)} = 0$. Then $\mathcal{A}^0_{i(1)} = \langle A^0_{i(1)}; \oplus_{i(1)}, *_{i(1)}, \neg_{i(1)}, 0, 1_{i(1)}\rangle$ is an MV-algebra.

For each $i \in I$ and each $x^i \in A_i$ let $\varphi_i(x^i)$ be an element of A^0_i such that $(\varphi_i(x^i))_i = x^i$. Then φ_i is an isomorphism of $\mathcal{A}_i$ onto $\mathcal{A}^0_i$.

This yields that the mapping φ^0 of A into $\prod_{i\in I} A_i$ given by

$$\varphi^0(x) = (\ldots, \varphi_i(x_i), \ldots)$$

is an isomorphism of $\mathcal{A}$ onto $\prod_{i\in I} \mathcal{A}^0_i$. We say that

$$\varphi^0\colon \mathcal{A} \longrightarrow \prod_{i\in I} \mathcal{A}^0_i$$

is an internal direct decomposition of $\mathcal{A}$; $\mathcal{A}^0_i$ are called internal direct factors of $\mathcal{A}$.

In the following lemma we assume that $\mathcal{A}$ is an MV-algebra. Then in view of 1.4 we can suppose that $\mathcal{A} = \mathcal{A}_0(G, u)$.

3.1. Lemma. *Let us have an internal direct product decomposition*

$$\varphi\colon G \longrightarrow X \times Y \tag{1}$$

of a lattice ordered group G. Let u_1 and u_2 be the component of u in X and Y, respectively. Then u_1 is a strong unit of X and u_2 is a strong unit in Y.

Proof. This is an immediate consequence of (1). $\square$

In view of 3.1 we can construct the MV-algebras $\mathcal{A}_1 = \mathcal{A}_0(X, u_1)$ and $\mathcal{A}_2 = (Y, u_2)$. The MV-algebra $\mathcal{A}_1$ has the underlying set $X^0 = X \cap [0,u] = [0,u_1]$, and analogously for $\mathcal{A}_2$.

3.2. Lemma. *Let us apply the same assumptions as in 3.1 and let $\mathcal{A}_1$, $\mathcal{A}_2$ be as above. Let ψ be the partial map $\varphi|_{[0,u]}$. Then for each $t \in [0,u]$ we have $\psi(t) \in X^0 \times Y^0$ and the map*

$$\psi\colon [0,u] \longrightarrow X^0 \times Y^0 \tag{2}$$

defines an internal direct decomposition of the MV-algebra $\mathcal{A}$ with direct factors $\mathcal{A}_1$ and $\mathcal{A}_2$.

Proof. For each $t \in G$ let t_1 and t_2 be the components of t in X and in Y, respectively (in view of (1)). Let $t' \in X^0$ and $t'' \in Y^0$. Put $t = t' \vee t''$. Then $t = t_1 \vee t_2$ and $t_1 = t', t_2 = t''$. Hence ψ is an epimorphism.

The operations in $\mathcal{A}_i$ will be denoted by $\oplus_i$, $*_i$, $\neg_i$, 0_i and 1_i ($i = 1,2$). Clearly we have $0_i = 0$ and $1_i = u_i$, hence

$$\psi(0) = (0_1, 0_2), \quad \psi(1) = \psi(u) = (1_1, 1_2).$$

Let $a, b \in [0,u]$. In view of 1.3 we have

$$(a \oplus b)_1 = ((a+b) \wedge u)_1 = (a_1 + b_1) \wedge u_1 = a_1 \oplus_1 b_1,$$

and similarly for $(a \oplus b)_2$, whence

$$\psi(a \oplus b) = (a_1 \oplus_1 b_1, a_2 \oplus_2 b_2).$$

Next, $(\neg a)_1 = (u - a)_1 = u_1 - a_1 = \neg_1 a_1$ and analogously for $(\neg a)_2$, whence

$$\psi(\neg a) = (\neg_1 a_1, \neg_2 a_2).$$

Since the operation $*$ is defined by means of the operations $\oplus$ and $\neg$ (cf. (m_9)) we have also

$$\psi(a * b) = (a_1 *_1 b_1, a_2 *_2 b_2).$$

Therefore (2) defines an internal direct product decomposition of the MV-algebra $\mathcal{A}$ with the direct factors $\mathcal{A}_1$ and $\mathcal{A}_2$. $\square$

3.3. Lemma. *Let $\mathcal{A} = \mathcal{A}_0(G,u)$. Assume that*

$$\chi\colon \mathcal{A} \longrightarrow \prod_{i\in I} \mathcal{A}_i$$

is an internal direct product decomposition of $\mathcal{A}$. For each $i \in I$ let u_i be the component of u in $\mathcal{A}_i$. Then the map

$$\chi\colon [0,u] \longrightarrow \prod_{i\in I} [0,u_i] \tag{3}$$

is, at the same time, an internal direct decomposition of the lattice $[0,u]$.

Proof. This is an immediate consequence of the fact that the lattice operations $\vee$ and $\wedge$ are defined by means of the operations $\oplus, *$ and $\neg$. $\square$

Again, let $\mathcal{A} = \mathcal{A}_0(G,u)$. Suppose that (3) is an internal direct decomposition of the lattice $[0,u]$. Let $i(1)$ be a fixed element of I. In view of (3) there is $u'_{i(1)} \in [0,u]$ such that there is an internal direct decomposition

(4) $\chi_{i(1)}\colon [0,u] \longrightarrow [0,u_{i(1)}] \times [0,u'_{i(1)}]$

of $[0,u]$. Hence according to 2.4 there is an internal direct decomposition

$$\varphi_{i(1)}\colon G \longrightarrow X_{i(1)} \times X'_{i(1)}$$

of the lattice ordered group G such that $u_{i(1)} \in X_{i(1)}$ and $u'_{i(1)} \in X'_{i(1)}$. It is easy to verify that $u_{i(1)}$ and $u'_{i(1)}$ are the components of u in $X_{i(1)}$ and in $X'_{i(1)}$, respectively (in view of $\varphi_{i(1)}$). Then according to 3.1, $u_{i(1)}$ is a strong unit in $X_{i(1)}$; analogously, $u'_{i(1)}$ is a strong unit in $X'_{i(1)}$. Hence we can construct the MV-algebras $\mathcal{A}_{i(1)} = \mathcal{A}_0(X_{i(1)}, u_{i(1)})$ and $\mathcal{A}'_{i(1)} = \mathcal{A}_0(X'_{i(1)}, u'_{i(1)})$. Under this notation we have

3.4. Lemma. *Let $\mathcal{A}$ be as above. Assume that* (3) *is an internal direct decomposition of the lattice $[0,u]$. Then the map*

$$\chi\colon \mathcal{A} \longrightarrow \prod_{i\in I} \mathcal{A}_i$$

determines an internal direct decomposition of $\mathcal{A}$.

Proof. Let $i(1)$ be a fixed element of I. Then (4) is valid. According to 3.2 we have an internal direct decomposition

(5) $\mathcal{A} \longrightarrow \mathcal{A}_{i(1)} \times \mathcal{A}'_{i(1)}$,

where $\mathcal{A}_{i(1)}$ has the underlying set $[0,u_{i(1)}]$ and $\mathcal{A}'_{i(1)}$ has the underlying set $[0,u'_{i(1)}]$.

Consider the map χ as defined above. From (3) and (5) we obtain that $\chi\colon \mathcal{A} \longrightarrow \prod_{i\in I} \mathcal{A}_i$ is an internal direct product decomposition of the MV-algebra $\mathcal{A}$. $\square$

By summarizing, 3.3 and 3.4 yield

3.5. Theorem. *Let $\mathcal{A} = \langle A; \oplus, *, \neg, 0, 1\rangle$ be an MV-algebra and let $\mathcal{L}(\mathcal{A}) = \langle A; \wedge, \vee\rangle$ be the corresponding lattice. Then $\mathcal{A}$ and $\mathcal{L}(\mathcal{A})$ have the same internal direct decompositions (in the sense specified in* 3.3 *and* 3.4*).*

Since any two internal direct decompositions of a lattice with the least element 0 have a common refinement, we obtain

3.6. Corollary. *Any two internal direct decompositions of an MV-algebra $\mathcal{A}$ have a common refinement. Any two direct decompositions of $\mathcal{A}$ have isomorphic refinements.*

4. Complete MV-algebras

An MV-algebra $\mathcal{A}$ is called complete if the corresponding lattice $\mathcal{L}(\mathcal{A})$ is complete. An element $a \in A$ is an atom of $\mathcal{A}$ if it is an atom of $\mathcal{L}(\mathcal{A})$. Next, $\mathcal{A}$ is atomic if for each $y \in A$ with $y > 0$ there is an atom x in $\mathcal{A}$ such that $x \leqslant y$ (we apply the partial order from $\mathcal{L}(\mathcal{A})$). $\mathcal{A}$ is atomless if it has no atom. The set of all atoms of $\mathcal{A}$ will be denoted by At.

4.1. Theorem. ([2], Theorem 9.) *Let $\mathcal{A}$ be a complete MV-algebra. Assume that $At \neq \emptyset$ and that $\mathcal{A}$ is not atomic. Then $\mathcal{A}$ is isomorphic to a direct product $\mathcal{B} \times \mathcal{C}$, where $\mathcal{B}$ is complete and atomic and $\mathcal{C}$ is complete and atomless.*

In the present section we shall prove a generalization of 4.1.

Let L be a lattice and let α be an infinite cardinal. We say that L has the property $p(\alpha)$ if, whenever $x, y \in L$ and $x < y$, then there are $x_1, y_1 \in L$ with $x \leqslant x_1 < y_1 \leqslant y$ such that $\operatorname{card}[x_1, y_1] < \alpha$.

The following two lemmas are easy to verify.

4.2. Lemma. *Let $\mathcal{A}$ be an MV-algebra,* $\operatorname{card} A > 1$. *Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is atomic.*

(ii) *The lattice $\mathcal{L}(\mathcal{A})$ satisfies the condition $p(\aleph_0)$.*

4.3. Lemma. *Let $\mathcal{A}$ be an MV-algebra. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is atomless.*

(ii) *If B is an interval of $\mathcal{L}(\mathcal{A})$,* $\operatorname{card} B > 1$, *then B does not satisfy the condition* $p(\aleph_0)$.

It is easy to verify that each direct factor of a complete MV-algebra must be complete. Hence in view of 4.2 and 4.3, Theorem 4.1 above can be expressed as follows.

4.1′. Theorem. *Let $\mathcal{A}$ be a complete MV-algebra. Then $\mathcal{A}$ is an internal direct product of complete MV-algebras $\mathcal{B}_1$ and $\mathcal{C}_1$ such that*

(a) *either $\mathcal{B}_1$ is a one-element MV-algebra or $\mathcal{B}_1$ is atomic;*

(b) *$\mathcal{C}_1$ satisfies the condition* (ii) *from* 4.3.

Let $\mathcal{A}, G$ and u be as in 1.3 and 1.4. Assume that $\mathcal{A}$ is complete and that G is an internal completely subdirect product of lattice ordered groups G_i $(i \in I)$. Hence each G_i is an internal direct factor of G. For each $i \in I$ let u_i be the component of u in G_i.

Under the above assumptions and notation we have

4.2. Proposition. *$\mathcal{A}$ is an internal direct product of the MV-algebras $\mathcal{A}_i$ $(i \in I)$.*

Proof. Let $i(1)$ be a fixed element of I. Since G is an internal completely subdirect product of the system $\{G_i\}_{i\in I}$ there exists a convex ℓ-subgroup $G'_{i(1)}$ such that G is an internal direct product of lattice ordered groups $G_{i(1)}$ and $G'_{i(1)}$. Let $u'_{i(1)}$ be the component of u in $G'_{i(1)}$. Then the lattice $[0,u]$ is an internal direct product of lattices $[0,u_{i(1)}]$ and $[0,u'_{i(1)}]$. Hence for each $i \in I$, $[0,u_i]$ is a direct factor of the lattice $[0,u]$. Thus according to 3.4 each MV-algebra $\mathcal{A}_i$ is an internal direct factor of $\mathcal{A}$. For each $x \in [0,u]$ and $i \in I$ the component of x in $\mathcal{A}_i$ is $x \wedge u_i$. Consider the mapping $\varphi\colon [0,u] \longrightarrow \prod\limits_{i\in I}[0,u_i]$ defined by $(\varphi(x))_i = x \wedge u_i$ for each $i \in I$. To complete the proof it suffices to verify that φ is an epimorphism.

For each $i \in I$ choose $x^i \in [0,u_i]$. Since $[0,u]$ is a complete lattice there exists $x \in [0,u]$ such that $x = \bigvee\limits_{i\in I} x^i$. Each interval of a lattice ordered group is infinitely distributive; thus for each $i(1) \in I$,

$$u_{i(1)} \wedge x = u_{i(1)} \wedge \left(\bigvee_{i\in I} x^i\right) = \bigvee_{i\in I}(u_{i(1)} \wedge x^i) = u_{i(1)} \wedge x^{i(1)} = x^{i(1)}.$$

Hence $\varphi(x) = (x^i)_{i\in I}$, completing the proof. $\square$

An interval of a lattice is called nontrivial if it has more than one element.

4.3. Theorem. *Let $\mathcal{A}$ be a complete MV-algebra. Then there exists an internal direct decomposition $\varphi\colon \mathcal{A} \longrightarrow \prod_{i\in I} \mathcal{A}_i$ such that for each $i \in I$ one of the following conditions is satisfied:*

(a) *each nontrivial interval of $\mathcal{A}_i$ is finite;*

(b) *there exists an infinite cardinal α_i such that each nontrivial interval of $\mathcal{A}_i$ has cardinality α_i; moreover, $\alpha_i^{\aleph_0} = \alpha_i$.*

Proof. This is a consequence of [5], Theorem 3.7 and of Proposition 4.2 above. □

Let α be an infinite cardinal and let I be as in 4.3. We denote by $I(1)$ the set of all $i \in I$ such that $\alpha_i \geqslant \alpha$; next, we put $I(2) = I \setminus I(1)$. Then $\mathcal{A}$ is an internal direct product of MV-algebras $\mathcal{A}^1$ and $\mathcal{A}^2$, where

(i) $\mathcal{A}^1$ is an internal direct product of MV-algebras $\mathcal{A}_i$ $(i \in I(1))$ if $I(1) \neq \emptyset$, and $\mathcal{A}^1$ is a one-element MV-algebra otherwise,

(ii) $\mathcal{A}^2$ is an internal direct product of MV-algebras $\mathcal{A}_i$ $(i \in I(2))$ if $I(2) \neq \emptyset$, and $\mathcal{A}^2$ is a one-element MV-algebra otherwise.

Then $\mathcal{A}^2$ satisfies the condition $p(\alpha)$ and either $\mathcal{A}^1$ is a one-element MV-algebra or $\mathcal{A}^1$ fails to satisfy the condition $p(\alpha)$. Thus we have

4.4. Theorem. *Let α be an infinite cardinal. Let $\mathcal{A}$ be a complete MV-algebra. Then $\mathcal{A}$ is an internal direct product of MV-algebras $\mathcal{A}^1$ and $\mathcal{A}^2$ such that $\mathcal{A}^2$ satisfies the condition $p(\alpha)$, and either $\mathcal{A}^1$ is a one-element MV-algebra or $\mathcal{A}^1$ fails to satisfy the condition $p(\alpha)$.*

In view of 4.1′, Theorem 4.4 generalizes Theorem 4.1 above.

Let L be a lattice. Let $[a,b]$ be a nontrivial interval of L and let $\mathcal{R}[a,b]$ be the system of all maximal chains of $[a,b]$. We define the length $s[a,b]$ of $[a,b]$ by

$$s[a,b] = \min\{\operatorname{card} R\colon R \in \mathcal{R}[a,b]\}.$$

From 4.2 and from Theorem 2.6 of [8] we obtain

4.5. Theorem. *Let $\mathcal{A}$ be a complete MV-algebra,* $\operatorname{card} A > 1$. *Then $\mathcal{A}$ is an internal direct product of MV-algebras $\mathcal{A}_i$ $(i \in I)$ such that for each $i \in I$ one of the following conditions is satisfied:*

(i) *Every interval in $\mathcal{A}_i$ is finite.*

(ii) *There is an infinite cardinal α_i such that the length of each nontrivial interval in $\mathcal{A}_i$ is α_i.*

By a method analogous to that in 4.4 we can verify that Theorem 4.1 can be deduced from 4.5.

5. Polars in MV-algebras

Again, let $\mathcal{A}$ be an MV-algebra and let the operations $\wedge$ and $\vee$ be defined as in the introduction. For each $X \subseteq A$ we put

$$X^{\perp} = \{a \in A \colon x \wedge a = 0 \text{ for each } x \in X\}.$$

The set $X^{\perp}$ is called a polar in $\mathcal{A}$; it is also called the annihilator of X (cf. [1]).

A subset Y of A is said to be an ideal of $\mathcal{A}$ if it satisfies the following conditions: (i) $0 \in Y$; (ii) if $x, y \in Y$, then $x \oplus y \in Y$, and (iii) if $x \in Y$ and $y \leqslant x$, then $y \in Y$. (Cf. [4], Definition 4.1.)

For each ideal Y in $\mathcal{A}$ we can construct the factor structure $\mathcal{A}/Y$; it is an MV-algebra; cf. [4] (1.18 and 4.3 (ii)).

An ideal Y of $\mathcal{A}$ will be called prime if the factor structure $\mathcal{A}/Y$ is linearly ordered (cf. [1], p. 1360).

5.1. Theorem. ([1], Theorem 26). *If Y is a linearly ordered ideal of $\mathcal{A}$ then $Y^{\perp}$ is a prime ideal.*

In [1] it is remarked that it is not known if all prime ideals of $\mathcal{A}$ can be obtained as annihilators in this manner. We shall answer this question in the negative.

Let B be a Boolean algebra such that B is infinite and has no atom. Hence no nontrivial ideal of B is linearly ordered. The greatest element of B will be denoted by u.

Let E be the vector lattice of all elementary Carathéodory functions on B (cf. [8], Section 3, or Gofman [11]). Each nonzero element f of E can be expressed as

(1) $f = a_1 b_1 + \ldots + a_n b_n$

where $a_i \neq 0$ are reals and $b_i \in B$, $b_i > 0$, $b_{i(1)} \wedge b_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of $\{1, 2, \ldots, n\}$. We can identify the zero element of E with the element 0 of B, and for any $b \in B$ we can put $1b = b$. Let H be the subset of E consisting of the zero element of E and of all elements $f \in F$ that have the form (1) where all a_i's ($i = 1, 2, \ldots, n$) are nonzero integers. Then H is a lattice ordered group; the interval $[0, u]$ of H coincides with the Boolean algebra B. Next, let G be the convex ℓ-subgroup of H which is generated by the element u. Then u is a strong unit of the lattice ordered group G. Let us consider the MV-algebra $\mathcal{A} = \mathcal{A}_0\ (G, u)$. Hence $\mathcal{A}$ has the underlying set B.

From the definition of G we obtain that for each $x \in G$ the relation $2x \wedge u = x$ is valid; hence $x \oplus x = x$. Thus in view of Theorem 1.17, [4]

$$x \oplus y = x \vee y, \quad x * y = x \wedge y$$

for each x, y in A. This yields that for a nonempty subset Y of A the following conditions are equivalent:

(i) Y is an ideal of the Boolean algebra B,

(ii) Y is an ideal of the MV-algebra $\mathcal{A}$.

Hence the notion of a maximal ideal in $\mathcal{A}$ and a maximal ideal in B coincide as well.

There exists a maximal ideal Z of the Boolean algebra B. Hence Z is a maximal ideal of $\mathcal{A}$. Thus according to [4] (Theorems 4.7 and 3.12) the MV-algebra $\mathcal{A}/Z$ is linearly ordered and therefore Z is a prime ideal of $\mathcal{A}$. But Z cannot be represented as $Z = Y^{\perp}$, where Y is a linearly ordered ideal of $\mathcal{A}$; namely, such an ideal Y of $\mathcal{A}$ would be a nonzero linearly ordered ideal of the Boolean algebra B, which is impossible.

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