

DIRECT PRODUCT OF DIVISION RINGS AND A PAPER OF ABIAN

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ABSTRACT. It is shown that the rings under the title admit an order-theoretical characterization as in the commutative case studied by Abian.

Introduction. Let R be an associative ring equipped with the binary relation (\leq) defined by $x \leq y$ if and only if $xy = x^2$ in R . In this paper, it is shown that (\leq) is an order relation on R if and only if, R has no nilpotent elements ($\neq 0$). Conditions on the binary relation (\leq) in order that R split into a direct product of division rings are then studied in the light of Abian's result [1, Theorem 1]. Using similar argumentation and using certain subdirect representation of rings with no nilpotent elements, one obtains a complete similarity with the commutative case (yet, no extra complication in the computations).

CONVENTIONS. R is an associative ring which is, unless otherwise stated, with no nilpotent elements (other than 0). As a result of [2], R can be embedded into a direct product of skewdomains, $R \rightarrow \prod_{i \in I} R_i$ (that is to say, rings R_i having no one-sided divisors of zero). The former embedding is fixed throughout the paper. It is therefore legitimate to identify any element x in R with the family consisting of all its projections $(x_i)_{i \in I}$. Finally, all definitions in [1] are extended (verbatim) to the present case (of a noncommutative ring R) and are freely used throughout.

In this paper we offer the following generalization to the noncommutative case of Abian's result [1, Theorem 1].

THEOREM. *Any ring R equipped with its binary relation (\leq) defined by $a \leq b$, if and only if $ab = a^2$, is isomorphic to a direct product of division rings if and only if (\leq) is an order relation on R such that R is hyperatomic and orthogonally complete (in the sense of Abian).*

The 'only if' is just a combination of the forthcoming Lemma 2

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and a partial duplication of Abian's proof. (See first part of the proof of [1, Theorem 1].) The 'if' breaks into several lemmas some which are of independent interest.

LEMMA 1. *If $a \leq b$ is an order relation on a ring R , then R has no nilpotent elements ($\neq 0$).*

PROOF. Assume $x^2=0$ in R . Then $x \leq 0$. However 0 is the least element under \leq . Therefore $x=0$.

LEMMA 2. *If R has no nilpotent elements, then (\leq) is a (multiplicatively) permissible (that is to say, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $c \in R$) order relation on R .*

PROOF. Using the mentioned embedding of R (see Conventions), it suffices to show the result for a skewdomain. In this case the occurrence $ab=a^2$, is equivalent with $a=0$ or $a=b$. Now the latter occurrence defines a binary relation which is obviously a permissible order relation on R . Lemma 2 is established.

Some known properties of rings without nilpotents R which follow at once from the considered embedding of R , are collected without proof in the next lemma.

LEMMA 3. *If (\leq) orders R , then*

(1) *Any idempotent element e ($=e^2$) of R is in the center of R , and $ex=xe \leq x$ for all $x \in R$.*

(2) *For any $a, x \in R$ (i) $x^2a=x^2$ implies $xa=x$; (ii) $x^2a=x$ implies $xax=x$; (iii) $x^2a=0$ implies $xa=0$.*

We assume henceforth that R is ordered, or equivalently (Lemmas 1 and 2), that R has no nilpotent elements. Along the lines of the proof of [1, Theorem 1] let us now show

LEMMA 4. *Let a be an hyperatom of R . For any r such that $ar \neq 0$, ar is an hyperatom.*

PROOF. Let $x \leq ar$ in R . By definition [1, Definition 1, (6)], there is s so that $xs \leq ars = a$. As $x \leq ar$, $x^2 = xar$ implies $x^2s = xars = xa$, and $x^2 = (xa)r = (x^2s)r = x^2(sr)$. Then (Lemma 3, (2)(i)) $x = x(sr) = (xs)r$. As a is an hyperatom, either $xs=0$ and so, $x = (xs)r = 0$, or $xs = a$ in which case $x = ar$. Finally, assume $(ar)y \neq 0$ for some given y . Then, for $t = t'r$ with t' so that $a(ry)t' = a$, we get $(ar)yt = a(ry)t'r = ar$ proving thereby that ar is an hyperatom.

LEMMA 5. *Let $x \neq 0$ in R . Assume that $q \leq x$ for some hyperatom $q \neq 0$. Then there is an idempotent hyperatom e such that $ex \neq 0$.*

PROOF. As $q \neq 0$, $q^2 \neq 0$. Then $q^2s = q$ for some s . It follows (Lemma 3, (2)(iii)) that $qsq = q$. Set $e = qs = sq$. As $q \neq 0$, $e \neq 0$ and by Lemma 4, e is an idempotent hyperatom. Evaluating ex , we get $ex = qsx = sqx = sq^2 = qsq = q \neq 0$.

LEMMA 6. *The set $E = \{e_i, i \in I\}$ of all idempotent hyperatoms of R is an orthogonal set and each of its elements e_i generates a division ring $D_i = e_iR$.*

PROOF (SKETCHED). By definition of an hyperatom e such that $e = e^2$, and by property (1) in Lemma 3.

LEMMA 7. *If R is hyperatomic, then $f = x \rightarrow (e_i x)_{i \in I}$ is a monomorphism from R into a direct product of division rings $D_i = e_iR$.*

PROOF. It is an immediate consequence of Lemmas 5 and 6.

LEMMA 8. *The embedding in Lemma 7 has the following properties:*

- (1) *Each factor $D_i = e_iR$ is a skewdomain.*
- (2) *If $(a^{(\alpha)})_{\alpha \in A}$ is any family of elements of R having a supremum a in R then for any fixed $i \in I$, $a_i^{(\alpha)} = 0$ for all $\alpha \in A$ implies $a_i = 0$.*

PROOF. Let $(a^{(\alpha)})_{\alpha \in A}$ be a family of elements in R admitting supremum a in R . Let $i \in I$ such that $a_i^{(\alpha)} = 0$ for all $\alpha \in A$. To prove that $a_i = 0$. Here $a_i = e_i a$. Set $a' = a - e_i a$. As E is orthogonal,

$$\begin{aligned} a'_\mu &= a_\mu & \text{if } \mu \neq i, \\ &= 0 & \text{if } \mu = i. \end{aligned}$$

Then $a_\mu^{(\alpha)} \leq a'_\mu$ for all $\mu \in I$. Consequently, $a^{(\alpha)} \leq a'$ for all $\alpha \in A$. Then $a \leq a'$, that is to say, $a(a - e_i a) = a^2$, if and only if, $a^2 e_i = 0$, equivalently $a e_i = 0$, equivalently, $a_i = 0$.

LEMMA 9. *If R admits at least one imbedding as in Lemma 8, then R has the following property:*

- (A) *For any family $(a^{(\alpha)})_{\alpha \in A}$ of elements of R admitting a supremum in R , and any $b \in R$, $(ba^{(\alpha)})_{\alpha \in A}$ admits a supremum in R equal to $b(\sup_{\alpha} a^{(\alpha)})$.*

PROOF. As (\leq) is left permissible, $a^{(\alpha)} \leq a$ implies $ba^{(\alpha)} \leq ba$ for any $\alpha \in A$, and ba is an upper bound of $\{ba^{(\alpha)}, \alpha \in A\}$. Also, ba is the least upper bound. For let $ba^{(\alpha)} \leq u$ in R . If $ba \leq u$ were not true, then for some $i \in I$, $b a_i \neq u_i$. Then $a_i \neq 0$, and $a_i^{(\alpha_0)}$ for some $\alpha_0 \in A$ follows. As $a_i^{(\alpha_0)} \leq a_i$, in the skewdomain D_i we must have $a_i^{(\alpha_0)} = a_i$ (see Lemma 2 and its proof). As $b a_i^{(\alpha)} \leq u_i$ for all α , in particular, $b a_i (= b a_i^{(\alpha_0)}) \leq u_i$, a contradiction.

LEMMA 10. Let R be an orthogonally complete ring satisfying (A). Let $F = \{e_\lambda, \lambda \in \Lambda\}$ be an orthogonal set of idempotents in R such that $\phi = x \rightarrow (e_\lambda x)_{\lambda \in \Lambda}$ is a monomorphism. Then ϕ is an isomorphism.

PROOF. For let $x^{(\lambda)} \in e_\lambda R$, λ ranging over Λ . As $X = \{x^{(\lambda)}, \lambda \in \Lambda\}$ is orthogonal it admits a supremum x in R . Evaluating the λ th projection of x on $e_\lambda R$ we get

$$x_\lambda = e_\lambda x = e_\lambda \left(\sup_{\mu \in \Lambda} x^{(\mu)} \right) = \sup_{\mu} (e_\lambda x^{(\mu)}) = \sup \{0, e_\lambda x^{(\lambda)}\} = e_\lambda x^{(\lambda)} = x^{(\lambda)}$$

for all $\lambda \in \Lambda$. Thus ϕ is epi. The lemma is established and implies immediately together with Lemmas 7, 8 and 9 the required splitting of R .

REMARK. As shown by Abian, Property (A) holds for any commutative ring R without nilpotent elements [1, Lemma 8]. Note also that Abian's proof of (A) does not use subdirect representation of R .

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