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DIRECT SUMS OF LOCAL TORSION-FREE ABELIAN GROUPS

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Abstract. The category of local torsion-free abelian groups of finite rank is known to have the cancellation and n-th root properties but not the Krull-Schmidt property. It is shown that 10 is the least rank of a local torsion-free abelian group with two non-equivalent direct sum decompositions into indecomposable summands. This answers a question posed by M.C.R. Butler in the 1960's.

1. Introduction

Let TF denote the category of local torsion-free abelian groups of finite rank, where an abelian group G is *local* if there is a fixed prime p with $qG = G$ for each prime $q \neq p$. Each M in TF has the *cancellation property* (if $M \oplus N$ is isomorphic to $M \oplus K$ in TF, then N is isomorphic to K), and the *n-th root property* (if the direct sum M^n of n copies of M is isomorphic to N^n for some N in TF, then M is isomorphic to N) [Lady 75]. An M in TF is a *Krull-Schmidt group* if any two direct sum decompositions of M into indecomposable summands are *equivalent*, i.e. unique up to isomorphism and order of summands.

M.C.R. Butler, in an unpublished note dating from the 1960's, constructed an example of a local torsion-free abelian group of rank 16 that is not a Krull-Schmidt group (see [Arnold 82]) and asked for the smallest such rank. An example of a rank-10 local torsion-free abelian group that is not a Krull-Schmidt group is given in [Arnold 01].

This paper is devoted to showing that 10 is the minimum such rank, i.e. if $M \in TF$ with rank $M \leq 9$, then M is a Krull-Schmidt group. Many arguments in this paper carry over directly to torsion-free modules of finite rank over valuation domains, keeping in mind that the existence and minimal rank of a non-Krull-Schmidt module depends on the structure of the valuation domain; see [Goldsmith May 99] and references.

The *quasi-isomorphism* category $TF_{\mathbb{Q}}$ of TF is an additive category with objects those of TF but with morphism sets $\mathbb{Q} \otimes \mathrm{Hom}(M, N)$ for M, $N \in TF$ and $\mathbb{Q}$ the rational numbers. The category $TF_{\mathbb{Q}}$ is a *Krull-Schmidt category* in that each object in $TF_{\mathbb{Q}}$ can be written uniquely, up to isomorphism in $TF_{\mathbb{Q}}$ and order, as a finite direct sum of indecomposable objects in $TF_{\mathbb{Q}}$; see [Walker 64]. This is because

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an indecomposable object M in $TF_{\mathbb{Q}}$ has a local endomorphism ring $\mathbb{Q}\mathrm{End}M$ in $TF_{\mathbb{Q}}$. Indecomposable objects in $TF_{\mathbb{Q}}$ are called *strongly indecomposable groups*, isomorphism in $TF_{\mathbb{Q}}$ is called *quasi-isomorphism,* and summands of groups in $TF_{\mathbb{Q}}$ are called *quasi-summands.*

Each $M \in TF$ is a torsion-free $\mathbb{Z}_{(p)}$-module, where $\mathbb{Z}_{(p)}$ is the localization of the integers at the prime p. The rank of M as a group is equal to the rank of M as a $\mathbb{Z}_{(p)}$-module, and $\mathrm{Hom}_{\mathbb{Z}}(M,N) = \mathrm{Hom}_{\mathbb{Z}_{(p)}}(M,N)$ for each $M, N \in TF$. Hence, rank $M = 1$ if and only if M is isomorphic to either $\mathbb{Z}_{(p)}$ or $\mathbb{Q}$. Define *p-rank* M to be the $\mathbb{Z}/p\mathbb{Z}$-dimension of M/pM, a finite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space. Notice that p-rank $M \leq$ rank M, M is divisible if and only if p-rank $M = 0$, and M is isomorphic to a free $\mathbb{Z}_{(p)}$-module if and only if p-rank $M =$ rank M. Moreover, if N is a $\mathbb{Z}_{(p)}$-submodule of M, then the p-rank of the pure submodule of M generated by N is less than or equal to the p-rank of N and if M is quasi-isomorphic to $N \oplus K$, then p-rank $M = p$-rank N + p-rank K [Arnold 72]. If M is *reduced* (no proper divisible subgroups), then M is isomorphic to a pure subgroup of M^*, the completion of M in the p-adic topology. Moreover, M^* is a free $\mathbb{Z}^*$-module with rank equal to the p-rank of M, where $\mathbb{Z}^*$ is the p-adic completion of $\mathbb{Z}_{(p)}$. Each endomorphism of M lifts to a unique $\mathbb{Z}^*$-endomorphism of M^*, whence End M is a pure subring of $\mathrm{End}_{\mathbb{Z}^*} M^*$.

2. Uniqueness of direct sums

A group $M \in TF$ has the *one-sided UDS property* if whenever $M \oplus N$ is isomorphic to $K_1 \oplus \ldots \oplus K_n \in TF$ with M quasi-isomorphic to each K_j, then M is isomorphic to some K_j. The group M has the *UDS property* if whenever $N_1 \oplus \ldots \oplus N_m$ is isomorphic to $K_1 \oplus \ldots \oplus K_n \in TF$ with M quasi-isomorphic to each N_i and K_j, then $m = n$ and there is a relabelling of indices with each N_i isomorphic to K_i.

Given a strongly indecomposable M in TF, there is a faithful $G_M \in TF$ quasi-isomorphic to M such that End G_M/NEnd G_M is a maximal order in the division algebra $\mathbb{Q}$End $M/J\mathbb{Q}$End M, where $J\mathbb{Q}$End M is the Jacobson radical of the finite dimensional $\mathbb{Q}$-algebra $\mathbb{Q}$End M, NEnd $G_M =$ End $G_M \cap J\mathbb{Q}$End M is a nilpotent ideal of End M, and G_M is *faithful* if $IG_M \neq G_M$ for each maximal right ideal I of End G_M [Arnold 01]. A maximal right ideal J of End G_M/pEnd G_M has the *unique maximal condition* if whenever I is a non-zero right ideal and J is a unique maximal right ideal of End G_M/pEnd G_M containing I, then $I = J$.

The first lemma is the local version of [Arnold 01, Theorem 1.5].

Lemma 1. *The following statements are equivalent for a strongly indecomposable N in TF, where G_N is as defined above*:

(i) *N has the UDS property.*

(ii) *Each group in TF quasi-isomorphic to N has the one-sided UDS property.*

(iii) *Either End G_N is a local ring or else End G_N has exactly two maximal right ideals M_1 and M_2 such that M_1 is a principal right ideal of End G_N, $G_N/M_1G_N \cong \mathbb{Z}/p\mathbb{Z}$, and M_1/pEnd G_N has the unique maximal condition in End G_N/pEnd G_N.*

Following are non-trivial examples of groups in TF with the UDS property.

Example 1. If $N \in$ TF is strongly indecomposable with p-rank $N \leq 2$, then N has the (one-sided) UDS property.

Proof. It suffices to confirm the conditions of Lemma 1(iii). If p-rank $N \leq 1$, then p-rank $G_N \leq 1$, since G_N is quasi-isomorphic to N. Thus, either p-rank $G_N = 0$ and $G_N \cong \mathbb{Q}$ or else p-rank $G_N = 1, G_N^* \cong \mathbb{Z}^*$, and End G_N is isomorphic to a pure subring of $\mathbb{Z}^* \cong \mathrm{End}_{\mathbb{Z}^*}\mathbb{Z}^*$. In either case, End G_N is a local ring, as desired.

Now assume that p-rank $N = p$-rank $G_N = 2$ and End G_N is not a local ring. Let $M_1, \ldots, M_n$ be distinct maximal right ideals of End G_N with $n \geq 2$. Then $G_N/(M_1 \cap ... \cap M_n)G_N \cong G_N/M_1G_N \oplus ...\oplus G_N/M_nG_N$ and pEnd $G_N \subseteq J$End $G_N \subseteq M_1 \cap ... \cap M_n$. Since p-rank $\mathrm{G}_N = 2$ and G_N is faithful, it follows that $n = 2$, pEnd $G_N = M_1 \cap M_2 = J$End G_N, each $G_N/M_iG_N \cong \mathbb{Z}/p\mathbb{Z}$, and each M_i/pEnd G_N has the unique maximal condition. Finally, each M_i is principal as an application of Nakayama's Lemma, because pEnd $G_N = J$End G_N and End G_N/pEnd G_N is finite. □

The next lemma is used for an induction step in the proof of the main theorem.

Lemma 2. *Assume that $M = N \oplus N' = K_1 \oplus ... \oplus K_n \in TF$. There are subgroups $K_i^{'}$ of K_i with $N \oplus N' = N\oplus\ K_1^{'} \oplus ... \oplus K_n^{'}$ if either*
(a) [Warfield 72] *End N is a local ring or*
(b) [Arnold Lady 75] *N and N' have no quasi-summands in common.*
In this case, N' is isomorphic to $K_1^{'} \oplus ... \oplus K_n^{'}$.

An indecomposable $M \in TF$ is *purely indecomposable* if p-rank $M = 1$. In this case, End M is a local ring, being a pure subring of $\mathbb{Z}^* \cong \mathrm{End}_{\mathbb{Z}^*}\mathbb{Z}^*$. Dually, M is *co-purely indecomposable if* M is indecomposable with rank $M = p$-rank $M + 1$. There is a contravariant duality F on $TF_{\mathbb{Q}}$ sending a purely indecomposable group M to a co-purely indecomposable group $F(M)$ [Arnold 72] (see [Lady 77] for an alternate definition of the duality). Hence, $\mathbb{Q}$End $F(M)$ is isomorphic to $\mathbb{Q}$End M, a subring of the p-adic rationals $\mathbb{Q}^*$.

Following are some elementary properties of purely indecomposable and co-purely indecomposable groups that are consequences of the definitions and the duality F.

Proposition 1 ([Arnold 72]). *Let $M \in TF$.*

(a) *If M is purely indecomposable, then:*
 (i) *End M is a pure subring of Z^*;*
 (ii) *each pure subgroup of M is strongly indecomposable;*
 (iii) *if $K \in TF$ is a homomorphic image of M with rank $K <$ rank M, then K is divisible; and*
 (iv) *two purely indecomposable groups M and N in TF are isomorphic if and only if rank M = rank N and Hom$(M, N) \neq 0$; equivalently M and N are quasi-isomorphic.*

(b) *If M is co-purely indecomposable, then:*
 (i) *End M is isomorphic to a subring of $\mathbb{Q}^*$, hence an integral domain;*
 (ii) *each torsion-free homomorphic image of M is strongly indecomposable;*
 (iii) *if K is a pure subgroup of M with rank $K <$ rank M, then M is a free $\mathbb{Z}_{(p)}$-module; and*
 (iv) *two co-purely indecomposable groups M and N are quasi-isomorphic if and only if rank M = rank N and Hom$(M, N) \neq 0$.*

(c) *If M is indecomposable with rank ≥ 2 and N is co-purely indecomposable with rank $M <$ rank N, then Hom$(M, N) = 0$.*

(d) *If M is purely indecomposable with rank ≥ 3 and N is co-purely indecomposable with rank M = rank N, then Hom$(M, N) = 0$.*

Remark 1. There is a co-purely indecomposable $M \in TF$ with p-rank 3 and rank 4 that does not have either UDS property. In this case End M has 3 maximal right ideals and M is the summand of a non-Krull-Schmidt group of rank 12 [Arnold 01, Remark]. In view of the following lemma, this group cannot be a summand of a non-Krull-Schmidt group of rank $8 = 2$(rank M). On the other hand, if $M \in TF$ and End M has at least 4 maximal right ideals, then M is a summand of a non-Krull Schmidt group of rank equal to 2(rank M).

The next lemma is used in the proof of the main theorem. In view of Proposition 1(b)(i), the hypotheses are satisfied if N is co-purely indecomposable.

Lemma 3. *Assume that $N \in TF$ with End N an integral domain and $M = N \oplus N' = K_1 \oplus K_2 \in TF$ with each K_i indecomposable. If p-rank $N \leq 3$, then N is isomorphic to some K_i.*

Proof. The proof is a variation on a proof given in [Arnold 01]. Let π be a projection of M onto N with kernel N', and π_i a projection of M onto K_i for each i with $1_M = \pi_1 + \pi_2$. Then $1_N = \beta_1 + \beta_2$, where $\beta_i \in$ End N is the restriction of $\pi\pi_i$ to N and $\beta_i(N)$ is contained in a subgroup $\pi(K_i)$ of N. Since End N is an integral domain, $\mathbb{Q}$End N is a field and each β_i is a unit in $\mathbb{Q}$End N.

For each $1 \leq i \leq 2$, let $I_i = \beta_i$End N, a right ideal of End N. Then End $N = I_1 + I_2$, since $1_N = \beta_1 + \beta_2$. Each (End $N)/I_i$ is bounded by a power of p since β_i is a unit in $\mathbb{Q}$End N. Moreover, I_iN is contained in $A_i = \pi(K_i)$ so that $[N : A_i]$ is finite. It now suffices to prove that $N \cong A_i$ for some i, in which case $N \cong K_i$.

If some $[N : A_i] = 1$, then $N = A_i$ and the proof is complete. The next step is to assume that each $[N : A_i] \neq 1$ and reduce to the case that each $[N : A_i] = p$. Suppose, by way of induction, that $[N : A_i] \neq p$. Choose $x \in N \backslash A_i$ such that $px \in A_i$. Then $A_i \subset A_i + \mathbb{Z}x$. If N and $A_i + \mathbb{Z}x$ are not isomorphic, then replace A_i by $A'_i = A_i + \mathbb{Z}x$. If $N \cong A_i + \mathbb{Z}x$, say $f \in$ End N with $f(N) = A_i + \mathbb{Z}x$, then replace A_i by $A'_i = f^{-1}(A_i)$. In either case, $[N : A'_i]$ is a proper divisor of $[N : A_i]$.

The substitution of A'_i for A_i doesn't change the hypothesis that End $N = I_1 + I_2$ for right ideals I_i of bounded index with I_iN contained in A_i. In particular, $I'_i = f^{-1}I_i$ is an ideal of End N (since I_iN is a subgroup of $f(N)$), I'_iN is contained in A'_i, and $I'_i + \Sigma\{I_j : j \neq i\} =$ End N (since fEnd $N = \Sigma fI_i$ is contained in $I_1 + \Sigma\{fIj : j \neq i\}$). If $N \cong A'_i$, then, by the construction of A'_i, $N \cong A_i$. By induction, and the fact that $[N : A'_i]$ is a proper divisor of $[N : A_i]$, the A_i's can be chosen with each $[N : A_i] = p$.

At this stage, End $N = I_1 + I_2$ for right ideals I_i of finite index in End N with I_iN contained in a subgroup A_i of N and $[N : A_i] = p$ for each i. Replace I_i by $I_i + p$End N, if necessary, to guarantee that pEnd N is contained in I_i for each i. But p-rank $N \leq 3$, pEnd $N \subseteq J$End N, End N is an integral domain, and $N/(M_1 \cap ... \cap M_n)N \cong N/M_1N \oplus ... \oplus N/M_nN$ for maximal ideals M_i of End N. Hence, End N has at most 3 maximal right ideals M_1, M_2, and M_3 and pEnd $N = M_1^{i_1}M_2^{i_2}M_3^{i_3}$ with $i_1 + i_2 + i_3 \leq 3$. Furthermore, $pN \subseteq (I_1 \cap I_2)N$, and $N/(I_1 \cap I_2)N \cong N/I_1N \oplus N/I_2N$. After relabelling subscripts, if necessary, $I_1 = M_1, N/I_1N = \mathbb{Z}/p\mathbb{Z}$, and $I_1N = A_1$. Finally, I_1 is principal by Nakayama's

Lemma, since $p\text{End } N \subseteq J\text{End } N$ and $I_1/p\text{End } N$ is principal. This shows that A_1 is isomorphic to N, as desired. □

The point of the next lemma, as used in the proof of the main theorem, is that Lemma 2(a) applies to a group quasi-isomorphic to a direct sum of two purely indecomposable groups of the same rank.

Lemma 4. *If $N \in TF$ is indecomposable and quasi-isomorphic to $A \oplus B$ for purely indecomposable groups A and B in TF with rank A = rank B, then $Hom(A, B) = 0 = Hom(B, A)$ and End N is a local ring.*

Proof. Choose purely indecomposable pure subgroups A and B of N and some least positive integer i with $p^i N \subset A \oplus B \subset N$. Since p-rank $N = 2$, $N/p^i N \cong \mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z}$ for some $1 \leq j \leq i$. Because A and B are purely indecomposable pure subgroups of N, $N/(A \oplus B) \cong \mathbb{Z}/p^j\mathbb{Z}$, say $N = A \oplus B + \mathbb{Z}(a, b)(1/p^j)$ for some $a \in A \backslash pA$ and $b \in B \backslash pB$.

If $\text{Hom}(A, B) \neq 0$ or $\text{Hom}(B, A) \neq 0$, then A and B are isomorphic by Proposition 1(iv). Moreover, $C = N/A$ is purely indecomposable and quasi-isomorphic to B. Hence, $C \cong A \cong B$ and $\text{Hom}(C, N)C = N$. By Baer's Lemma [Arnold 82], A is a summand of N, a contradiction to the assumption that N is indecomposable.

Now assume that $\text{Hom}(A, B) = 0 = \text{Hom}(B, A)$. Then A and B are fully invariant subgroups of $N = A \oplus B + \mathbb{Z}(a, b)(1/p^j)$. Thus, End N is the pullback of a homomorphism $A \rightarrow \mathbb{Z}/p^j\mathbb{Z}$ with kernel $p^j A$ and a homomorphism $B \rightarrow \mathbb{Z}/p^j\mathbb{Z}$ with kernel $p^j B$. It follows that $\text{End } N/p^j\text{End } N \cong \mathbb{Z}/p^j\mathbb{Z}$, whence End N is a local ring. □

3. The main theorem

Theorem 1. *If $M \in TF$ and* rank $M \leq 9$*, then M is a Krull-Schmidt group.*

Proof. Let N be an indecomposable summand of M of minimal rank and $M = N \oplus N_1 \oplus \ \dots \oplus N_m = K_1 \oplus \ \dots \oplus K_n$ with each N_i and K_j indecomposable. Then rank $N \leq 4$, rank $N \leq$ rank N_j, and rank $N \leq$ rank K_i for each i and j, since rank $M \leq 9$ and N is an indecomposable summand of M of minimal rank.

If p-rank $N \leq 1$, then End N is a local ring, as noted above. In this case, by Lemma 2(a), $N_1 \oplus \ \dots \oplus N_m$ is isomorphic to $K_1' \oplus \ \dots \oplus K_n'$ for subgroups K_i' of K_i. It follows, by an induction on the rank of M, that M is a Krull-Schmidt group. In particular, if p-rank N = rank N, then N is free and cyclic, hence of p-rank 1.

If N and $N_1 \oplus \ \dots \oplus N_m$ have no quasi-summands in common, then, by Lemma 2(b), the proof is completed by an induction on the rank of M.

In view of the preceding remarks, it is now sufficient to assume that M is reduced, $2 \leq p$-rank $N <$ rank $N \leq 4$ for each indecomposable summand N of minimal rank, and if $M = N \oplus N'$, then N and $N' = N_1 \oplus \ \dots \oplus N_m$ have a quasi-summand in common. Under these assumptions, M has no rank-1 quasi-summands. This is because the only rank-1 groups in TF are $\mathbb{Z}_{(p)}$ and $\mathbb{Q}$ and, since M is reduced, any rank-1 quasi-summand must actually be a summand isomorphic to $\mathbb{Z}_{(p)}$. The strategy of the remainder of the proof is to show that N must be isomorphic to some K_i, in which case the cancellation property for $N \in TF$ and an induction on the rank of M shows that M is a Krull-Schmidt group.

First assume that rank $N = 4$, p-rank $N = 3$. Then N, being indecomposable, is co-purely indecomposable, hence strongly indecomposable by Proposition 1. Thus,

$N_1 \oplus \ ... \oplus N_m$ is quasi-isomorphic to $N \oplus L$ for some L of rank ≤ 1. To see this, recall that N and $N_1 \oplus \ ... \oplus N_m$ have a quasi-summand in common, N is strongly indecomposable, rank $N \geq$ rank N_i, and rank $N + \sum_i$rank $N_i = 4 + \sum_i$rank $N_i \leq 9$. Since M has no rank-1 quasi-summands, $L = 0$, $m = 1$, and $n = 2$. But $TF_{\mathbb{Q}}$ is a Krull-Schmidt category so that $M = N \oplus N_1 = K_1 \oplus K_2$ has rank 8 with N quasi-isomorphic to N_1, K_1, and K_2. By Lemma 3, N is isomorphic to either K_1 or K_2, as desired.

Next, consider the case that rank $N = 4$ and p-rank $N = 2$. If N is strongly indecomposable, then, as above, $M = N \oplus N_1 = K_1 \oplus K_2$ has rank 8 and N is quasi-isomorphic to N_1, K_1, and K_2. By Example 1, N has the UDS property so that N is isomorphic to either K_1 or K_2, as desired. If N is not strongly indecomposable, then N is quasi-isomorphic to $A \oplus B$, where A and B are purely indecomposable groups with p-rank 1 and rank 2. This is because M has no rank-1 quasi-summands. Now apply Lemmas 2 and 4 and induction on the rank of M to see that M is a Krull-Schmidt group.

The only remaining case is that p-rank $N = 2$, rank $N = 3$. In this case N is co-purely indecomposable, hence strongly indecomposable by Proposition 1. Since N and N_1 have a quasi-summand in common, N_1 is quasi-isomorphic to $N \oplus A$ for some pure subgroup A of N_1 with $1 \leq p$-rank $A <$ rank $A \leq 3$. This is because M has no rank-1 quasi-summands and rank $M \leq 9$.

If A has p-rank 1, then $\mathrm{Hom}(A, N) = 0$ by Proposition 1(c) and (d), since A is purely indecomposable with $2 \leq$ rank $A \leq 3 =$ rank N, and N is co-purely indecomposable. In this case, $\mathrm{Hom}(A, M) = \mathrm{Hom}(A, A)$. It follows that A is a pure fully invariant subgroup, hence equal to a subgroup of some K_i, say K_1. Thus, $N \oplus (N_1/A)$ is isomorphic to $(K_1/A) \oplus K_2 \oplus K_3$ and induction on the rank of M completes the proof.

Finally, assume that A has p-rank 2. Then rank $A = 3 =$ rank N and A and N are both co-purely indecomposable. If $\mathrm{Hom}(A, N) = 0$, then, as above, M is a Krull-Schmidt group. Finally, if $\mathrm{Hom}(A, N) \neq 0$, then A is quasi-isomorphic to N, since A and N are both co-purely indecomposable modules with the same rank Hence, $M = N \oplus N_1 = K_1 \oplus K_2 \oplus ... \oplus K_n$ has rank 9 with $n \leq 3$. If $n = 3$, then N is quasi-isomorphic to K_1, K_2 and K_3 by the minimality of the rank of N. In this case, Example 1 yields N isomorphic to some K_i. If $n = 2$, then, by Lemma 3, N is isomorphic to some K_i, as desired. □

Example 2 ([Arnold 01])**.** There is a rank-10 group in TF that is not a Krull-Schmidt group.

Proof. The argument is briefly outlined. There is $M \in TF$ of p-rank 4 and rank 5 such that $M \cong$ End M, a subring of an algebraic number field with exactly four maximal ideals M_1, M_2, M_3,and M_4, and $pM = p$End $M = M_1 \cap M_2 \cap M_3 \cap M_4$. Furthermore, there are subgroups A_1 and A_2 of M not isomorphic to M with $(M_1 \cap M_2)M \subset A_1$ and $(M_3 \cap M_4)M \subset A_2$. It follows that there is $B \in TF$ with $M \oplus B = A_1 \oplus A_2$, a rank 10 group in TF that is not a Krull-Schmidt group. □

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